J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Bubbling string solutions for the self-dual Einstein–Maxwell–Higgs equation

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ARTICLE INFO

Article history:
Received 12 July 2018
Available online xxxx
Submitted by M. Musso

Keywords:
Self-dual Einstein–Maxwell–Higgs equation
Bubbling solutions

ABSTRACT

In this article, we establish the existence of type I nontopological bubbling solutions for the Einstein–Maxwell–Higgs equation in $\mathbb{R}^2$. Our solutions bubble at some string points and the local profile near those points comes from the radially symmetric solutions. We also prove that such bubbling solutions cannot exist on compact surfaces.

1. Introduction

Let $\mathcal{S}$ be a two dimensional smooth manifold with a symmetric metric tensor $h = (h_{ij})$, $i, j = 1, 2$. In this paper, we study the following elliptic system on $\mathcal{S}$:

$$\begin{cases} \Delta_h \tilde{u} = \dfrac{1}{\tilde{\varepsilon}^2}(e^{\tilde{u}} - \tau^2) + 4\pi \displaystyle\sum_{j=1}^{b} n_j \delta_{p_j}, \\ 2K_h = -a\Big[\dfrac{1}{\tilde{\varepsilon}^2}(e^{\tilde{u}} - \tau^2) - \dfrac{1}{\tau^2}\Delta_h e^{\tilde{u}}\Big]. \end{cases} \tag{1.1}$$

Here, K_h is the Gaussian curvature for $(\mathcal{S}, h)$ and Δ_h is the Laplace–Beltrami operator induced from the metric h. The set $\mathcal{P} = \{p_1, p_2, \cdots, p_b\}$ consists of distinct points in $\mathcal{S}$ called string points, and δ_{p_j} denotes the Dirac measure concentrated at the point p_j. The coefficient $n_j \in \mathbb{N}$ is the multiplicity of p_j and the total string number is defined by

$$N = n_1 + \cdots + n_b. \tag{1.2}$$

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https://doi.org/10.1016/j.jmaa.2018.11.011
0022-247X/

The unknowns are not only $\tilde{u} : \mathcal{S}\backslash\mathcal{P} \to \mathbb{R}$ but also the metric h. There are three constants $\tilde{\varepsilon} > 0$, $\tau > 0$ and $a \geq 0$.

The system (1.1) originates from the self-dual Einstein–Maxwell–Higgs model describing an interaction of an Abelian gauge field and a scalar field on a $(3+1)$ dimensional spacetime manifold $\mathcal{M}$. This model is an extension of the classical Ginzburg–Landau model for superconductivity in the frame of general relativity. The metric of $\mathcal{M}$ is unknown and should obey the Einstein equations. If we assume $\mathcal{M} = \mathbb{R}^{1,1} \times \mathcal{S}$ and that the gauge fields and the scalar field have two dimensional nature, then we are led to (1.1) as a special type of the static Euler–Lagrange equations. In this situation, the Einstein equations are reduced to the second equation of (1.1) and the metric h on $\mathcal{S}$ appears as the unknown metric component of $\mathcal{M}$. We call (1.1) the self-dual Einstein–Maxwell–Higgs equation. The constant $\tilde{\varepsilon}$ represents the strength the electromagnetic interaction, τ is the symmetry breaking parameter, and a is the scaled gravitational constant. If $a = 0$, there is no gravitational effect on the model. Solutions of (1.1), called string solutions, are related to an explanation of some issues in cosmology such as galaxy formation. See [6,9,17,19] for the physical background and the derivation of (1.1).

We have four unknowns $\tilde{u}, h_{11}, h_{12}, h_{22}$ for (1.1) and it is not easy to solve in its form. So, we assume that the metric is conformal to a given metric h_0 and set $h = e^{\eta} h_0$ for some smooth function η. Then, (1.1) is rewritten as

$$\begin{cases} \Delta_{h_0} \tilde{u} = \dfrac{1}{\tilde{\varepsilon}^2} e^{\eta}(e^{\tilde{u}} - \tau^2) + 4\pi \displaystyle\sum_{j=1}^{b} n_j \delta_{p_j}, \\ \Delta_{h_0}(-\eta - \dfrac{a}{\tau^2} e^{\tilde{u}}) = -2K_{h_0} - \dfrac{a}{\tilde{\varepsilon}^2} e^{\eta}(e^{\tilde{u}} - \tau^2). \end{cases} \tag{1.3}$$

Now, we have two unknown functions $\tilde{u}$ and η. The purpose of this paper is to construct solutions for small $\tilde{\varepsilon}$ which have bubbles at string points. To solve (1.3), we split the situation into two cases: $\mathcal{S}$ is either compact or noncompact. In the following, we reduce the system into a single elliptic equation for u in each case and state the main results.

First, let us assume that $\mathcal{S}$ is noncompact. In particular, we suppose that $\mathcal{S} = \mathbb{R}^2$ and h_0 is the standard Euclidean metric. Then $K_{h_0} = 0$ and by adding the first and the second equations of (1.3), we deduce that

$$-\eta - \frac{a}{\tau^2} e^{\tilde{u}} + a\tilde{u} - \sum_{j=1}^{b} 2an_j \ln|x - p_j|$$

is a harmonic function. By setting this function to be zero, we can represent η in terms of $\tilde{u}$:

$$\eta = -\frac{a}{\tau^2} e^{\tilde{u}} + a\tilde{u} - \sum_{j=1}^{b} 2an_j \ln|x - p_j|.$$

By substituting η in the first equation of (1.3) by this representation and setting $\tilde{u} - \ln\tau^2 = u$ and $\tilde{\varepsilon}\tau^{-1-a} = \varepsilon$, we obtain the final reduction

$$\Delta u = \frac{1}{\varepsilon^2} \Big(\prod_{j=1}^{b} |x - p_j|^{2n_j} \Big)^{-a} e^{a(u - e^u)}(e^u - 1) + 4\pi \sum_{j=1}^{b} n_j \delta_{p_j}. \tag{1.4}$$

Due to the physical motivation, there are three kinds of boundary conditions for (1.4): as $|x| \to \infty$,

$$\text{topological conditions} : u(x) \;\to\; \sigma \in \mathbb{R},$$

$$\text{nontopological conditions of type I} : u(x) \;\to\; -\infty,$$

$$\text{nontopological conditions of type II} : u(x) \quad \to \quad \infty.$$

Solutions for each boundary condition are called topological solutions, type I nontopological solutions and type II nontopological solutions, respectively. In the rest of this paper, we simply call a type I (resp. type II) nontopological solution as a type I (resp. type II) solution.

If $a = 0$, then (1.4) becomes

$$\Delta u = \frac{1}{\varepsilon^2}(e^u - 1) + 4\pi \sum_{j=1}^{b} n_j \delta_{p_j}. \tag{1.5}$$

In the physical literature, this case implies that there is no gravitational effect on the Maxwell–Higgs model. The equation (1.5) possesses a unique topological solution with $\sigma = 0$ ([15]). Moreover, there exists one parameter family of type I solutions u_α of (1.5) satisfying

$$\frac{1}{2\pi} \int\limits_{\mathbb{R}^2} e^{u_\alpha} dx = \alpha > 0.$$

See [1,16].

If $a > 0$, the situation is quite sophisticated. The existence and properties of solutions are heavily dependent on the values of a and N. As mentioned in [9], there is a big difference between the case $0 < aN < 1$ and other cases. In various places of computation, one may encounter terms which behave like $|x|^{2-2aN}$ near zero or at infinity. Then, the integrability issues occur and one may face severe obstructions for the case $aN \geq 1$. If we add some additional conditions on (1.4), then such difficulty may overcome. For instance, one may consider the simplest case that $p_1 = \cdots = p_b = 0$. Under this hypothesis, (1.4) is transformed into

$$\Delta u = \frac{1}{\varepsilon^2}|x|^{-2aN} e^{a(u-e^u)}(e^u - 1) + 4\pi N \delta_0. \tag{1.6}$$

If we further assume that $u(x) = u(r)$ with $r = |x|$, then we obtain the radial version:

$$\begin{cases} u'' + \dfrac{1}{r}u' + \dfrac{1}{\varepsilon^2} r^{-2a\lambda} e^{a(u-e^u)}(1 - e^u) = 0, \quad r > 0, \\ u(r) = 2\lambda \ln r + O(1) \quad \text{near} \quad r = 0. \end{cases} \tag{1.7}$$

Here, we replace N in (1.6) by λ in (1.7) to emphasize that one may consider (1.7) for any positive real number λ. The first result for (1.7) was given in [6,14], which say that if $0 < a\lambda < 1$, there exist a unique topological solution and one parameter family of type I solutions. This result was improved by [9] in the sense that solutions for (1.7) are classified for all possible ranges of $a\lambda$. When either $0 < a\lambda < 1$ or $1 < a\lambda < 2$, (1.7) possesses a unique topological solution and one parameter families of type I and II solutions. For topological solutions, $\sigma = 0$ for $0 < a\lambda < 1$ and $\sigma < 0$ for $1 < a\lambda < 2$. If $a\lambda = 1$ or $a\lambda \geq 2$, then there are only type II solutions. For later use, we state the result for the type I solutions.

Theorem A *([9]).*

(i) *Suppose that $0 \leq a\lambda < 1$. Then for each $\beta > 4/a$, there exists a unique type I nontopological solution $U_{\lambda,\beta,\varepsilon}$ of (1.7) which satisfies*

Please cite this article in press as: J. Han, J. Sohn, Bubbling string solutions for the self-dual Einstein–Maxwell–Higgs equation, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.11.011

$$\begin{cases} \dfrac{1}{\varepsilon^2}\displaystyle\int_0^\infty r^{1-2a\lambda} e^{a(U_{\lambda,\beta,\varepsilon}-e^{U_{\lambda,\beta,\varepsilon}})}(1-e^{U_{\lambda,\beta,\varepsilon}})dr = \beta, \\ U_{\lambda,\beta,\varepsilon}(r) = (2\lambda-\beta)\ln r + \kappa_{\lambda,\beta,\varepsilon} + O(r^{2-a\beta}) \quad as \quad r \to \infty. \end{cases} \tag{1.8}$$

Here, $\kappa_{\lambda,\beta,\varepsilon}$ is a constant which depends on λ, β and ε.

(ii) *Suppose that $1 < a\lambda < 2$. Then for each $2\lambda < \beta < 4/a$, there exists a unique type I nontopological solution $U_{\lambda,\beta,\varepsilon}$ of (1.7) satisfying (1.8).*

(iii) *If $a\lambda = 1$ or $a\lambda \geq 2$, then (1.7) does not possess any type I nontopological solutions.*

In this article, we focus on the multistring case of (1.4), i.e., $b > 1$. In this case, there have been results only for $0 < aN < 1$ due to the technical difficulty mentioned as above. Topological multistring solutions were constructed in [8,18] by the super- and sub-solution method. In [4], the author obtained type I solutions which are perturbed from the solutions of the Liouville type equations on the plane. In this paper, we will show the existence of nontopological solutions known as bubbling solutions like [5,7,11–13]. Using the method of [5,7], we construct nontopological solutions by patching radially symmetric solutions of (1.7) at small neighborhoods of some $p_j \in \mathcal{P}_1 = \{p_1, \cdots, p_{b_1}\} \subset \mathcal{P}$. Such solutions are different from the perturbation of the solutions of the Liouville type equations.

There are some differences between our solutions and the bubbling solutions of [5,7,11]. We pick radial profiles $U_{n_j,\beta_j,\varepsilon_j}$ at each blowup point p_j and try to find a solution u_ε such that

$$u_\varepsilon(x) = U_{n_j,\beta_j,\varepsilon_j}\Big(\frac{|x-p_j|}{\varepsilon^{\alpha_j}}\Big) + O(1) \ \text{ near } x = p_j \ \text{ as } \ \varepsilon \to 0.$$

Generically, it is reasonable to have different scale index $\alpha_j = (1-an_j)^{-1}$ at each $p_j \in \mathcal{P}_1$, while we have a uniform scale index $\alpha_j = 1$ for bubbling solutions in [5,7,11]. In this paper, we will take $n_j = 1$ at each bubble by the technical restriction of our method (see Theorem 2.1 and Proposition 4.1 (iii)) and so we get the uniform scale ε^α with $\alpha = (1-a)^{-1}$. Furthermore, when we choose radial profiles of (1.7) associated with a bubble p_j, we have to take different ε, say ε_j, at each bubble. This is due to the weight function $g(x)$ and does not happen for bubbling solutions of [5,7,11]. Such a property is necessary for the analysis of the linearized operators at the approximate solutions.

To state the first main result of this paper, we set up necessary hypothesis in the following. We want to find bubbling solutions u_ε of (1.4) satisfying the following: as $\varepsilon \to 0$,

- $$u_\varepsilon(x) = U_{n_j,\beta_j,\varepsilon_j}\Big(\frac{|x-p_j|}{\varepsilon^{\alpha_j}}\Big) + O(1) \ \text{ near } x = p_j, \tag{1.9}$$
- $$u_\varepsilon(x) = (2N-\beta)\ln|x| + O(1) \ \text{ as } \ |x| \to \infty, \tag{1.10}$$
- $$\frac{1}{2\pi\varepsilon^2} g(x) f(u_\varepsilon) \to \sum_{j=1}^{b_1} \beta_j \delta_{p_j} \ \text{ in the sense of measures,} \tag{1.11}$$
- $$\sum_{j=1}^{b_1} \beta_j = \beta = \frac{1}{2\pi\varepsilon^2}\int_{\mathbb{R}^2} g(x) f(u_\varepsilon)dx. \tag{1.12}$$

Here, N is the total string number defined by (1.2) and a is chosen such that $0 < an_j < 1$ for $1 \leq j \leq b_1$. We also denote

$$f(u) = e^{a(u-e^u)}(1-e^u),$$

$$g(x) = \prod_{j=1}^{b} |x-p_j|^{-2an_j}, \quad g_j(x) = \prod_{k\neq j}^{b} |x-p_k|^{-2an_k},$$

$$\omega_j = \sqrt{g_j(p_j)} \quad \text{for} \quad 1 \le j \le b_1.$$

Generally, at each bubble point $p_j \in \mathcal{P}_1$, we use a different scaling parameter

$$\alpha_j = (1 - an_j)^{-1} \quad \text{for} \quad 1 \le j \le b_1. \tag{1.13}$$

However, due to technical reasons (see the proof of Theorem 2.1), we assume that

$$n_1 = \cdots = n_{b_1} = 1. \tag{1.14}$$

In this case,

$$\alpha_j = \alpha = (1 - a)^{-1} \quad \text{for all} \quad 1 \le j \le b_1.$$

We also need to specify β_j and ε_j. First, thanks to a compatibility condition near blow-up points in $\mathcal{P}_1$, we deduce from (1.9) that for $1 \le j, k \le b_1$,

$$(2 - \beta_j)\alpha = (2 - \beta_k)\alpha = -\beta_0 = \text{constant}. \tag{1.15}$$

Thus, all β_j's are equal and (1.12) implies that

$$\beta_j = \frac{\beta}{b_1} \quad \text{for} \quad 1 \le j \le b_1 \quad \text{and} \quad \beta_0 = \frac{\beta - 2b_1}{b_1(1 - a)}. \tag{1.16}$$

Since $\beta_j > 4/a$ by Theorem A, we get a necessary condition

$$\beta > \frac{4b_1}{a}. \tag{1.17}$$

Furthermore, it is natural to take

$$\varepsilon_j = \omega_j^{-1} = \prod_{k \ne j}^{b} |p_j - p_k|^a, \quad 1 \le j \le b_1,$$

which will be clarified later. We denote $U_j(r) = U_{1, \frac{\beta}{b_1}, \omega_j^{-1}}(r)$ and $\kappa_j = \kappa_{1, \frac{\beta}{b_1}, \omega_j^{-1}}$ for $1 \le j \le b_1$. Then, by Theorem A, as $r \to \infty$,

$$U_j(r) = \Big(2 - \frac{\beta}{b_1}\Big) \ln r + \kappa_j + O\big(r^{2 - \frac{a\beta}{b_1}}\big) = -\frac{\beta_0}{\alpha} \ln r + \kappa_j + O\big(r^{2 - \frac{a\beta}{b_1}}\big). \tag{1.18}$$

By Green's representation formula, (1.11), and (1.15), we see that

$$\begin{aligned} u_\varepsilon(x) - C_\varepsilon &= \sum_{j=1}^{b} 2n_j \ln |x - p_j| - \frac{1}{2\pi\varepsilon^2} \int_{\mathbb{R}^2} g(y) f\big(u_\varepsilon(y)\big) \ln |x - y| \, dy \\ &= H(x) + o(1) \end{aligned}$$

on any $K \Subset \mathbb{R}^2 \backslash \mathcal{P}$ as $\varepsilon \to 0$, where each C_ε is a constant and

$$H(x) = -\sum_{j=1}^{b_1} \frac{\beta_0}{\alpha} \ln |x - p_j| + \sum_{j=b_1+1}^{b} 2n_j \ln |x - p_j|.$$

Comparing this expression with (1.9), we obtain

$$C_\varepsilon = -K_j(p_j) + \beta_0 \ln \varepsilon + \kappa_j,$$

where for $1 \le j \le b_1$

$$K_j(x) = H(x) + \frac{\beta_0}{\alpha} \ln |x - p_j|.$$

Hence, we obtain a constraint on the location of the elements of $\mathcal{P}_1$:

$$K_j(p_j) - \kappa_j = K_k(p_k) - \kappa_k =: D = \text{constant} \quad \text{for} \quad 1 \le j \ne k \le b_1. \tag{1.19}$$

Now, we are in a position to state the first main result of this paper.

Theorem 1.1. *Let $b_1 \in \{1, 2, \cdots, b\}$ be fixed and $\alpha = (1-a)^{-1}$. Assume (1.14), (1.19), and*

$$0 < a < \frac{1}{4}. \tag{1.20}$$

Then, for any

$$\beta > \max\left\{2N, \frac{4b_1}{a}\right\}, \tag{1.21}$$

there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, (1.4) possesses a solution u_ε such that

$$\begin{cases} u_\varepsilon(x) = U_j\left(\dfrac{|x-p_j|}{\varepsilon^\alpha}\right) + O(1) \quad \textit{near } x = p_j \quad \textit{for} \quad 1 \le j \le b_1, \\ \dfrac{1}{2\pi\varepsilon^2} \displaystyle\int_{\mathbb{R}^2} g(x) f(u_\varepsilon) dx = \beta, \\ \dfrac{1}{2\pi\varepsilon^2} g(x) f(u_\varepsilon) \longrightarrow \dfrac{\beta}{b_1} \displaystyle\sum_{j=1}^{b_1} \delta_{p_j} \quad \textit{in the sense of measures.} \end{cases}$$

Moreover, $u_\varepsilon \to -\infty$ in $C_{loc}(\mathbb{R}^2 \backslash \mathcal{P})$ as $\varepsilon \to 0$.

Now, let us consider the problem to find bubbling solutions when $\mathcal{S}$ is compact. If we integrate the first and the second equation of (1.3) and add them, we are led to

$$\int_{\mathcal{S}} K_{h_0} \, dV_{h_0} = 2\pi a N. \tag{1.22}$$

Thus, we infer from the Gauss–Bonnet Theorem that $\chi(\mathcal{S}) = 2 - 2n = aN$ where $\chi(\mathcal{S})$ is the Euler characteristic of $\mathcal{S}$ and n is the genus of $\mathcal{S}$. So, we have two cases: either $n = 1$ and $a = 0$ or $n = 0$ and $a > 0$. In the first case, $\mathcal{S}$ becomes a flat torus such that $K_{h_0} = 0$ and $\eta =$ constant. By letting this constant zero, we obtain (1.5) on a flat torus with $\varepsilon = \tilde{\varepsilon}\tau^{-1}$. By integrating this equation on $\mathcal{S}$, we have

$$0 < \frac{1}{\varepsilon^2} \int_{\mathcal{S}} e^u = \frac{|\mathcal{S}|}{\varepsilon^2} - 4\pi N,$$

where $|\mathcal{S}|$ is the volume of $\mathcal{S}$. So, a necessary condition for the existence is $\varepsilon^2 < |\mathcal{S}|(4\pi N)^{-1}$. It was proved in [16] that this condition is also sufficient.

Next, if $n = 0$ and $a > 0$, then $\mathcal{S}$ is diffeomorphic to a sphere and we get from (1.22) a constraint

$$aN = 2. \tag{1.23}$$

We want to make (1.3) into a single elliptic equation as for the noncompact case. Given a point $p \in \mathcal{S}$, let $G(x,p)$ be the Green function satisfying that

$$\begin{cases} -\Delta_{h_0} G(x,p) = -\dfrac{1}{|\mathcal{S}|_{h_0}} + \delta_p, \\ \displaystyle\int_{\mathcal{S}} G(x,p)\ dV_{h_0}(x) = 0. \end{cases} \tag{1.24}$$

See [2] for the detail. If we set

$$u_0 = -4\pi \sum_{j=1}^{b} n_j G(x, p_j) \tag{1.25}$$

and $\tilde{v} = \tilde{u} - u_0$, then $\tilde{v}$ satisfies that

$$\begin{cases} \Delta_{h_0}\tilde{v} = \dfrac{1}{\tilde{\varepsilon}^2} e^{\eta}(e^{u_0+\tilde{v}} - \tau^2) + \dfrac{4\pi N}{|\mathcal{S}|_{h_0}}, \\ \Delta_{h_0}(-\eta - \dfrac{a}{\tau^2} e^{u_0+\tilde{v}} + a\tilde{v}) = -2K_{h_0} + \dfrac{4\pi a N}{|\mathcal{S}|_{h_0}}. \end{cases} \tag{1.26}$$

By the condition (1.22), the equation

$$-\Delta_{h_0} v_0 = -2K_{h_0} + \frac{4\pi a N}{|\mathcal{S}|_{h_0}} \quad \text{and} \quad \int_{\mathcal{S}} v_0 = 0$$

has a unique solution. So, by the second equation of (1.26),

$$-\eta - \frac{a}{\tau^2} e^{u_0+\tilde{v}} + a\tilde{v} + v_0$$

is a constant. By letting this constant be zero, we have

$$\eta = a\tilde{v} - \frac{a}{\tau^2} e^{u_0+\tilde{v}} + v_0,$$

and insert this in the first equation of (1.26). Then, by setting $v = \tilde{v} - \ln \tau^2$ and $\tilde{\varepsilon}\tau^{-1-a} = \varepsilon$, we obtain the final form on a compact surface $\mathcal{S}$:

$$\Delta_{h_0} v = \frac{1}{\varepsilon^2} e^{v_0} e^{a(v - e^{u_0+v})}(e^{u_0+v} - 1) + \frac{4\pi N}{|\mathcal{S}|_{h_0}}. \tag{1.27}$$

It is not difficult to see that if v is a solution of (1.27), then by the maximum principle

$$e^{u_0+v} < 1 \quad \text{on} \quad \mathcal{S}.$$

By integrating (1.27), we see that any solution v of (1.27) satisfies

$$4\pi N = \frac{1}{\varepsilon^2}\int_{\mathcal{S}} e^{v_0} e^{a(v-e^{u_0+v})}(1-e^{u_0+v}). \tag{1.28}$$

So, as $\varepsilon \to 0$, every solution v satisfies one of the following:

$$\begin{cases} \text{topological solution:} & e^{u_0+v} \to 1, \\ \text{nontopological solution:} & e^{v} \to 0. \end{cases} \tag{1.29}$$

Then, the main research topic regarding (1.27) is to find solutions satisfying each of the asymptotics (1.29).

The first result on (1.27) was known in [18] where the existence of solution was proved for $0 < \varepsilon \ll 1$. The authors in [10] improved this result by showing that there exists $\varepsilon_c > 0$ such that for all $\varepsilon \in (0, \varepsilon_c)$, (1.27) possesses at least two solutions v_ε^1 and v_ε^2. Moreover, v_ε^1 is a topological solution, that is, $e^{u_0+v_\varepsilon^1} \to 1$ as $\varepsilon \to 0$. One interesting question is whether there exist solutions of (1.27) which bubbles at some string points and have profiles of radially symmetric solutions in Theorem A near blowup points. In other words, can we find bubbling solutions as in Theorem 1.1 when $\mathcal{S}$ is a compact surface? Such solutions on compact surfaces can be found in other self-dual equations, for instance, the self-dual Chern–Simons–Higgs equation [11]. However, the answer is negative for (1.27) due to the obstruction (1.23). We show in this paper that it is not possible to find such bubbling solutions for (1.27). In the following theorem, we prove this by taking p_1 as a bubble point for simplicity. The second main result of this paper is the following.

Theorem 1.2. *Assume* (1.23)*. Given a solution $U_{\lambda,\beta,\varepsilon}$ of* (1.7)*, let $V_{\lambda,\beta,\varepsilon}(r) = U_{\lambda,\beta,\varepsilon} - 2\lambda \ln r$. Then, for any pair $(\lambda, \beta, \xi) \in \mathbb{R}^3_+$,* (1.27) *has no bubbling solutions v_ε satisfying that*

(a) $v_\varepsilon(x) = V_{\lambda,\beta,\xi}\left(\frac{|x-p_1|}{\varepsilon^\alpha}\right) + O(1)$ *as $x \to p_1$ for any small $\varepsilon > 0$,*

(b) $\frac{1}{\varepsilon^2} e^{v_0}\ e^{a(v_\varepsilon - e^{u_0+v_\varepsilon})}(1-e^{u_0+v_\varepsilon}) \to 4\pi N \delta_{p_1}$ *in the sense of measures as $\varepsilon \to 0$.*

This paper is organized as follows. In section 2, we prove Theorem 1.1 by applying the Contraction Mapping Theorem. In Section 3, we prove Theorem 1.2. In Section 4, we prove Theorem 2.1 which deals with the invertibility of the linearized operator and plays a key role in the proof of Theorem 1.1.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We fix $b_1 \in \mathbb{N}$ such that $1 \le b_1 \le b$ and set $\mathcal{P}_1 = \{p_1, \cdots, p_{b_1}\}$. Without loss of generality, we may assume that $|p_j - p_k| \ge 4$ for $1 \le j \ne k \le b$. We set

$$\alpha_* = \frac{2a}{1-a} \quad \text{and} \quad \alpha^* = \frac{1-2a}{1-a}.$$

By the condition (1.20), it holds that $\alpha_* < \alpha^*$. Based on the heuristic argument in the previous section, we define the approximate solution $u_{0,\varepsilon}$ by

$$u_{0,\varepsilon}(x) = \sum_{j=1}^{b_1} \sigma(x-p_j)\cdot\Big(V_{j,\varepsilon}(x) + \eta_{j,\varepsilon}(x)\Big)$$
$$+ \Big[1 - \sum_{j=1}^{b_1}\sigma(x-p_j)\Big]\cdot\big(H(x) + \beta_0 \ln\varepsilon - D\big).$$

Here,

$$\alpha = \frac{1}{1-a}, \quad V_{j,\varepsilon}(x) = U_j\Big(\frac{|x-p_j|}{\varepsilon^\alpha}\Big), \quad \eta_{j,\varepsilon}(x) = K_j(x) - K_j(p_j),$$

and $\sigma(x) = \sigma(|x|) \in C_0^\infty(\mathbb{R}^2)$ is a smooth radially symmetric function such that

$$0 \le \sigma(x) \le 1, \quad \sigma(x) = 0 \quad \text{for } |x| \ge 2 \quad \text{and} \quad \sigma(x) = 1 \quad \text{for } |x| \le 1. \tag{2.1}$$

One may check that

$$u_{0,\varepsilon}(x) = \begin{cases} V_{j,\varepsilon}(x) + \eta_{j,\varepsilon}(x), & x \in B_1(p_j) \text{ with } p_j \in \mathcal{P}_1, \\ \beta_0 \ln\varepsilon + H(x) - D + o(1), & x \in B_2(p_j)\backslash B_1(p_j), \\ H(x) + \beta_0 \ln\varepsilon - D, & x \in B_2(p_j)^c. \end{cases} \tag{2.2}$$

Set

$$u_\varepsilon(x) = u_{0,\varepsilon}(x) + v_\varepsilon(x).$$

Then, (1.4) can be rewritten as

$$\Delta v_\varepsilon + F_\varepsilon(x) v_\varepsilon + G_\varepsilon(x, v_\varepsilon) = 0 \text{ in } \mathbb{R}^2, \tag{2.3}$$

where F_ε is given by

$$F_\varepsilon(x) = \begin{cases} \dfrac{\omega_j^2}{\varepsilon^2}|x-p_j|^{-2a} f'(V_{j,\varepsilon}(x)) \quad \text{on } B_1(p_j) \text{ with } p_j \in \mathcal{P}_1, \\ 0 \quad \text{on } B_1(\mathcal{P}_1)^c, \end{cases} \tag{2.4}$$

and G_ε is defined by, for $t \in \mathbb{R}$,

$$\begin{aligned} &\varepsilon^2 G_\varepsilon(x,t) \\ &= \begin{cases} \varepsilon^2 G_{1,\varepsilon}(x,t) + \varepsilon^2 G_{2,\varepsilon}(x,t) & \text{on } B_1(p_j), \\ g(x) f(u_{0,\varepsilon}(x) + t) + \varepsilon^2 \Delta u_{0,\varepsilon}(x) & \text{on } B_2(p_j)\backslash B_1(p_j), \\ g(x) f(u_{0,\varepsilon}(x) + t) & \text{on } B_2(\mathcal{P}_1)^c, \end{cases} \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \varepsilon^2 G_{1,\varepsilon}(x,t) &= \omega_j^2 |x-p_j|^{-2a} \cdot \Big[f(u_{0,\varepsilon}(x)+t) - f\big(V_{j,\varepsilon}(x)\big) - f'\big(V_{j,\varepsilon}(x)\big)t \Big], \\ \varepsilon^2 G_{2,\varepsilon}(x,t) &= \big(g(x) - \omega_j^2|x-p_j|^{-2a}\big) f(u_{0,\varepsilon}(x)+t). \end{aligned}$$

We will prove Theorem 1.1 by the contraction mapping principle. Define the following two weighted functions ρ and $\hat\rho$ in $\mathbb{R}^2$ by

$$\rho(x) = (1+|x|)^{1+\gamma} \quad \text{and} \quad \hat\rho(x) = \frac{1}{(1+|x|)\cdot\big[\ln(2+|x|)\big]^{1+\gamma}},$$

where $\gamma \in (0, 1/2)$ is a fixed number. Let us consider the following function spaces:

$$Y_{\gamma,\varepsilon} = \big\{ v \in L^2_{loc}(\mathbb{R}^2) : \|v\|_{Y_{\gamma,\varepsilon}} < \infty \big\},$$

$$X_{\gamma,\varepsilon} = \left\{ v \in W^{2,2}_{loc}(\mathbb{R}^2) : \|v\|_{X_{\gamma,\varepsilon}} < \infty \right\}.$$

The norms of $Y_{\gamma,\varepsilon}$ and $X_{\gamma,\varepsilon}$ are defined by

$$\|v\|^2_{Y_{\gamma,\varepsilon}} = \sum_{j=1}^{b_1} \left\| \varepsilon^2 \rho(x) v(p_j + \varepsilon^\alpha x) \right\|^2_{L^2(B_{\varepsilon^{-\alpha}})} + \|\rho v\|^2_{L^2(B_1(\mathcal{P}_1)^c)}$$

and

$$\|v\|^2_{X_{\gamma,\varepsilon}} = \sum_{j=1}^{b_1} \left\| \hat{\rho}(x) v(p_j + \varepsilon^\alpha x) \right\|^2_{L^2(B_{\varepsilon^{-\alpha}})} + \|\hat{\rho} v\|^2_{L^2(B_1(\mathcal{P}_1)^c)} + \|\Delta v\|^2_{Y_{\gamma,\varepsilon}}.$$

Here, we put $B_{\varepsilon^{-\alpha}} = B_{\varepsilon^{-\alpha}}(0)$. We define a linear operator $L_\varepsilon = \Delta + F_\varepsilon(x)$ and rewrite (2.3) as

$$L_\varepsilon v_\varepsilon = -G_\varepsilon(x, v_\varepsilon). \tag{2.6}$$

The following theorem is crucial to prove Theorem 1.1 and will be proved in Section 4.

Theorem 2.1. *There exists $\varepsilon_* \in (0,1)$ such that $L_\varepsilon : X_{\gamma,\varepsilon} \to Y_{\gamma,\varepsilon}$ is an isomorphism for $0 < \varepsilon < \varepsilon_*$. Furthermore, for each $v \in X_{\gamma,\varepsilon}$ and $0 < \varepsilon < \varepsilon_*$,*

$$\|v\|_{X_{\gamma,\varepsilon}} + \|v\|_{L^\infty(\mathbb{R}^2)} \le C_* \left(\ln \frac{1}{\varepsilon} \right) \|L_\varepsilon v\|_{Y_{\gamma,\varepsilon}}, \tag{2.7}$$

where C_* is independent of v and ε.

By virtue of Theorem 2.1, we can define an operator $T_\varepsilon = -L_\varepsilon^{-1} G_\varepsilon(x, \cdot)$ and write (2.6) as

$$v_\varepsilon = T_\varepsilon v_\varepsilon.$$

In the following, we will prove that if ε is small enough, then T_ε is a contraction map on suitable subset A_ε of $X_{\gamma,\varepsilon}$. We define

$$A_\varepsilon = \left\{ v \in X_{\gamma,\varepsilon} : \|v\|_{X_{\gamma,\varepsilon}} + \|v\|_{L^\infty(\mathbb{R}^2)} \le \varepsilon^\theta \right\} \ \text{ for } \ 0 < \varepsilon < \varepsilon_*.$$

Here, θ is a constant such that

$$\alpha_* < \theta < \min\left\{ \frac{1}{2}, a\beta_0 - \gamma - 2, \alpha^* \right\}.$$

We first show that T_ε is well-defined.

Lemma 2.2. *There exists $\varepsilon_1 \in (0, \varepsilon_*)$ such that $T_\varepsilon : A_\varepsilon \to A_\varepsilon$ is well-defined for all $\varepsilon \in (0, \varepsilon_1)$.*

Proof. If $w = T_\varepsilon v$ for $v \in A_\varepsilon$, then by Theorem 2.1

$$\|w\|_{X_{\gamma,\varepsilon}} + \|w\|_{L^\infty(\mathbb{R}^2)} \le C_* \left(\ln \frac{1}{\varepsilon} \right) \|G_\varepsilon(x, v)\|_{Y_{\gamma,\varepsilon}}.$$

We note that

$$f'(u) = e^{a(u-e^u)}\{a(1-e^u)^2 - e^u\},$$
$$f''(u) = e^{a(u-e^u)}\{a^2(1-e^u)^3 - 3a(1-e^u)e^u - e^u\},$$

and hence $|f(u)| + |f'(u)| + |f''(u)| \le ce^{au}$ for all $u \in \mathbb{R}$ and some constant c. We divide the proof into two parts.

Part 1. $x \in B_1(p_j)$ *for some* $p_j \in \mathcal{P}_1$.

By the Mean Value Theorem, there exist $s = s(\varepsilon, x), t = t(\varepsilon, x) \in \mathbb{R}$ with $0 \le s, t \le 1$ such that

$$\varepsilon^2 G_{1,\varepsilon}(x, v) = \omega_j^2 |x - p_j|^{-2a} \cdot \Big[f'\big(V_{j,\varepsilon} + s(\eta_{j,\varepsilon} + v)\big)\eta_{j,\varepsilon} + f''\big(V_{j,\varepsilon} + ts(\eta_{j,\varepsilon} + v)\big) \cdot sv(\eta_{j,\varepsilon} + v)\Big].$$

Hence,

$$\big|\varepsilon^2 G_{1,\varepsilon}(x, v)\big| \le C|x - p_j|^{-2a} e^{a(V_{j,\varepsilon} + |\eta_{j,\varepsilon}|)}\big(v^2 + |\eta_{j,\varepsilon}|\big).$$

Also, we can see that for $x \in B_{\varepsilon^{-\alpha}}$,

$$\begin{aligned}&\Big|\eta_{j,\varepsilon}\big(p_j + \varepsilon^\alpha x\big)\Big| \\ = &\left|\sum_{k=1,k\ne j}^{b_1}\Big(2 - \frac{\beta}{b_1}\Big)\ln\frac{|p_j + \varepsilon^\alpha x - p_k|}{|p_j - p_k|} + \sum_{k=b_1+1}^{b} 2n_k \ln\frac{|p_j + \varepsilon^\alpha x - p_k|}{|p_j - p_k|}\right| \\ \le\ & C\varepsilon^\alpha |x|.\end{aligned}$$

Thus we have for each $j \in \{1, \cdots, b_1\}$,

$$\begin{aligned}&\Big\|\varepsilon^2 \rho(x) \cdot G_{1,\varepsilon}\big(p_j + \varepsilon^\alpha x, v(p_j + \varepsilon^\alpha x)\big)\Big\|^2_{L^2(B_{\varepsilon^{-\alpha}})} \\ \le\ & C\int_{B_{\varepsilon^{-\alpha}}} \rho^2(x) e^{2aU_j(|x|) - 4a\ln|\varepsilon^\alpha x|} \cdot (\varepsilon^{4\theta} + \varepsilon^{2\alpha}|x|^2)dx \\ \le\ & C\Big(\int_{|x|<R} + \int_{|x|\ge R}\Big) \ = \ I_1 + I_2,\end{aligned}$$

where R is chosen from (1.18) such that

$$U_j(|x|) = \Big(2 - \frac{\beta}{b_1}\Big)\ln|x| + O(1)$$

for each $1 \le j \le b_1$ and for all $|x| \ge R$. Here, $O(1)$ denotes a bounded quantity which is independent of $|x|$. So,

$$\begin{aligned}I_2 \le\ & C\int_{|x|\ge R}(1 + |x|)^{2+2\gamma-\frac{2a\beta}{b_1}}\varepsilon^{-4a\alpha}(\varepsilon^{4\theta} + \varepsilon^{2\alpha}|x|^2)\ dx \\ \le\ & C\varepsilon^{2\theta}(\varepsilon^{2\theta-4a\alpha} + \varepsilon^{2\alpha(1-2a)-2\theta})\int_{|x|\ge R}(1 + |x|)^{-2-2\mu}\ dx\ \le\ \frac{1}{4}\varepsilon^{2\theta+\mu},\end{aligned}$$

where the last inequality comes from the choice of θ. Here, μ is chosen such that

$$0<\mu<\mu_1=\min\Big\{\frac{1}{2},\ \theta-\alpha_*,\ \alpha^*-\theta,\ \frac{a\beta}{b_1}-3-\gamma\Big\}.$$

By (1.21) and the choice of θ, μ_1 is well-defined. Then,

$$\begin{aligned}I_1 \le\ & C\int\limits_{|x|<R}(1+|x|)^{2+2\gamma}\varepsilon^{-4a\alpha}(\varepsilon^{4\theta}+\varepsilon^{2\alpha}|x|^2)dx\\ \le\ & CR^{4+2\gamma}\varepsilon^{2\theta}(\varepsilon^{2\theta-4a\alpha}+\varepsilon^{2\alpha(1-2a)-2\theta})\ \le\ \frac{1}{4}\varepsilon^{2\theta+\mu},\end{aligned}$$

where the last inequality holds for all small $\varepsilon>0$. As a consequence,

$$\big\|\varepsilon^2\rho(x)\cdot G_{1,\varepsilon}\big(p_j+\varepsilon^\alpha x,v(p_j+\varepsilon^\alpha x)\big)\big\|^2_{L^2(B_{\varepsilon^{-\alpha}})}\le\frac{1}{2}\varepsilon^{2\theta+\mu}.$$

On the other hand,

$$\begin{aligned}&\big\|\varepsilon^2\rho(x)\cdot G_{2,\varepsilon}\big(p_j+\varepsilon^\alpha x,v(p_j+\varepsilon^\alpha x)\big)\big\|^2_{L^2(B_{\varepsilon^{-\alpha}})}\\ \le\ & C\int\limits_{B_{\varepsilon^{-\alpha}}}\rho^2(x)\big|g_j(p_j+\varepsilon^\alpha x)-g_j(p_j)\big|^2e^{2aU_j(|x|)-4a\ln|\varepsilon^\alpha x|}\ dx\\ \le\ & C\int\limits_{B_{\varepsilon^{-\alpha}}}\rho^2(x)\big|\nabla g_j(p_j+s\varepsilon^\alpha x)\big|^2\cdot|\varepsilon^\alpha x|^2e^{2aU_j(|x|)-4a\ln|\varepsilon^\alpha x|}\ dx\\ \le\ & C\varepsilon^{2\alpha-4a\alpha}\int\limits_{B_{\varepsilon^{-\alpha}}}(1+|x|)^{4+2\gamma}e^{2a(U_j(|x|)-2\ln|x|)}\ dx\\ \le\ & C\varepsilon^{2\alpha^*}\Big(\int\limits_{|x|<R}+\int\limits_{|x|\ge R}\Big)\ \le\ \frac{1}{2}\varepsilon^{2\theta+\mu},\end{aligned}$$

where $s=s(\varepsilon,x)\in[0,1]$ and the last inequality follows from a similar argument as above. In the sequel,

$$\big\|\varepsilon^2\rho(x)\cdot G_\varepsilon\big(p_j+\varepsilon^\alpha x,v(p_j+\varepsilon^\alpha x)\big)\big\|^2_{L^2(B_{\varepsilon^{-\alpha}})}\le\varepsilon^{2\theta+\mu}. \tag{2.8}$$

Part 2. $x\in B_1(\mathcal{P}_1)^c$.

We see from (2.2) that $u_{0,\varepsilon}(x)=\beta_0\ln\varepsilon+O(1)$ if $x\in B_2(p_j)\backslash B_1(p_j)$ for some $j=1,\cdots,b_1$. Hence, it follows from (2.5) that

$$|G_\varepsilon(x,v)|\le C\varepsilon^{a\beta_0-2}\ \text{ for }\ x\in B_2(p_j)\backslash B_1(p_j).$$

On the other hand, if $x\in B_2(\mathcal{P}_1)^c$, then by (2.2)

$$u_{0,\varepsilon}(x)=(2N-\beta)\ln|x|+\beta_0\ln\varepsilon-D+O(1),$$

which implies by (2.5) that

$$|G_\varepsilon(x,v)|=\frac{1}{\varepsilon^2}\big|g(x)\cdot f(u_{0,\varepsilon}+v)\big|\le C\varepsilon^{a\beta_0-2}(1+|x|)^{-a\beta}.$$

By (1.16) and (1.21),

Please cite this article in press as: J. Han, J. Sohn, Bubbling string solutions for the self-dual Einstein–Maxwell–Higgs equation, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.11.011

$$a\beta_0 = \frac{a\beta - 2b_1 a}{b_1(1-a)} > \frac{4-2a}{1-a} > 4.$$

Thus, it holds that

$$\begin{aligned} &\left\|\rho(x)\cdot G_\varepsilon(x,v(x))\right\|^2_{L^2(B_1(\mathcal{P}_1)^c)} \\ \le\ & C\varepsilon^{2a\beta_0-4}\int\limits_{B_1(\mathcal{P}_1)^c}(1+|x|)^{2+2\gamma-2a\beta}dx\ \le\ \varepsilon^{2\theta+\mu}. \end{aligned} \tag{2.9}$$

Now, it follows from (2.8) and (2.9) that for $v\in A_\varepsilon$ and $\varepsilon\ll 1$,

$$C_*\Big(\ln\frac{1}{\varepsilon}\Big)\|G_\varepsilon(x,v)\|_{Y_{\gamma,\varepsilon}}\ \le\ C_*\Big(\ln\frac{1}{\varepsilon}\Big)\cdot C\varepsilon^{\theta+\frac{\mu}{2}} < \varepsilon^\theta.$$

Therefore, T_ε maps A_ε into A_ε. $\square$

Proof of Theorem 1.1. We prove that (2.3) has a solution. By Lemma 2.2, it suffices to show that $T_\varepsilon : A_\varepsilon \to A_\varepsilon$ is a contraction map for any sufficiently small ε. Let $w_1 = T_\varepsilon v_1$ and $w_2 = T_\varepsilon v_2$. By Theorem 2.1, we obtain that

$$\|w_1-w_2\|_{X_{\gamma,\varepsilon}} + \|w_1-w_2\|_{L^\infty(\mathbb{R}^2)} \le C_*\Big(\ln\frac{1}{\varepsilon}\Big)\|G_\varepsilon(x,v_1)-G_\varepsilon(x,v_2)\|_{Y_{\gamma,\varepsilon}}.$$

As in Lemma 2.2, we divide $\mathbb{R}^2$ into two parts.

First, suppose that $x\in B_1(p_j)$ for some $p_j\in\mathcal{P}_1$. By the Mean Value Theorem,

$$\begin{aligned} &\varepsilon^2\big[G_{1,\varepsilon}(x,v_1)-G_{1,\varepsilon}(x,v_2)\big] \\ =\ &\omega_j^2|x-p_j|^{-2a}\cdot\Big[f''\big(V_{j,\varepsilon}+t(\eta_{j,\varepsilon}+sv_1+(1-s)v_2)\big)\ \times \\ &\big(\eta_{j,\varepsilon}+sv_1+(1-s)v_2\big)\Big](v_1-v_2) \end{aligned}$$

for some $s=s(\varepsilon,x)$ and $t=t(\varepsilon,x)$ with $0\le s,t\le 1$. Thus, we can derive that

$$\begin{aligned} &\Big\|\varepsilon^2\rho(x)\Big[G_{1,\varepsilon}\big(p_j+\varepsilon^\alpha x, v_1(p_j+\varepsilon^\alpha x)\big)-G_{1,\varepsilon}\big(p_j+\varepsilon^\alpha x, v_2(p_j+\varepsilon^\alpha x)\big)\Big]\Big\|^2_{L^2(B_{\varepsilon^{-\alpha}})} \\ \le\ & C\int\limits_{B_{\varepsilon^{-\alpha}}}\rho^2(x)e^{2aU_j(|x|)-4a\ln|\varepsilon^\alpha x|}\cdot\varepsilon^{2\theta}(1+\varepsilon^{2\alpha-2\theta}|x|^2)\cdot \\ &\qquad\big|v_1(p_j+\varepsilon^\alpha x)-v_2(p_j+\varepsilon^\alpha x)\big|^2dx \\ \le\ & C\varepsilon^{2\theta-4a\alpha}\int\limits_{B_{\varepsilon^{-\alpha}}}(1+|x|)^{4+2\gamma-\frac{2a\beta}{b_1}}\big|v_1(p_j+\varepsilon^\alpha x)-v_2(p_j+\varepsilon^\alpha x)\big|^2dx \\ \le\ & C\varepsilon^{2(\theta-\alpha_*)}\int\limits_{B_{\varepsilon^{-\alpha}}}\hat\rho^2(x)\big|v_1(p_j+\varepsilon^\alpha x)-v_2(p_j+\varepsilon^\alpha x)\big|^2dx \\ \le\ & C\varepsilon^{2\mu}\|v_1-v_2\|^2_{X_{\gamma,\varepsilon}}. \end{aligned}$$

Meanwhile,

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$$\begin{aligned}&\varepsilon^2\big[G_{2,\varepsilon}(x,v_1)-G_{2,\varepsilon}(x,v_2)\big]\\ =\ &|x-p_j|^{-2a}\big[g_j(x)-g_j(p_j)\big]f'\big(V_{j,\varepsilon}+\eta_{j,\varepsilon}+sv_1+(1-s)v_2\big)(v_1-v_2)\end{aligned}$$

for some $s=s(\varepsilon,x)\in[0,1]$. So,

$$\begin{aligned}&\Big\|\varepsilon^2\rho(x)\Big[G_{2,\varepsilon}\big(p_j+\varepsilon^\alpha x,v_1(p_j+\varepsilon^\alpha x)\big)-G_{2,\varepsilon}\big(p_j+\varepsilon^\alpha x,v_2(p_j+\varepsilon^\alpha x)\big)\Big]\Big\|^2_{L^2(B_{\varepsilon^{-\alpha}})}\\ \le\ &C\int_{B_{\varepsilon^{-\alpha}}}\rho^2(x)e^{2aU_j(|x|)-4a\ln|\varepsilon^\alpha x|}\cdot\big|g_j(p_j+\varepsilon^\alpha x)-g_j(p_j)\big|^2\cdot\\ &\qquad\big|v_1(p_j+\varepsilon^\alpha x)-v_2(p_j+\varepsilon^\alpha x)\big|^2dx\\ \le\ &C\varepsilon^{2\alpha-4a\alpha}\int_{B_{\varepsilon^{-\alpha}}}(1+|x|)^{4+2\gamma-\frac{2a\beta}{b_1}}\big|v_1(p_j+\varepsilon^\alpha x)-v_2(p_j+\varepsilon^\alpha x)\big|^2dx\\ \le\ &C\varepsilon^{2(\theta-\alpha_*)}\int_{B_{\varepsilon^{-\alpha}}}\hat\rho^2(x)\big|v_1(p_j+\varepsilon^\alpha x)-v_2(p_j+\varepsilon^\alpha x)\big|^2dx\\ \le\ &C\varepsilon^{2\mu}\|v_1-v_2\|^2_{X_{\gamma,\varepsilon}}.\end{aligned}$$

Next, we consider the case that $x\in B_1(\mathcal{P}_1)^c$. If $x\in B_2(p_j)\backslash B_1(p_j)$ for some $j=1,\cdots,b_1$, it follows that

$$\begin{aligned}\big|G_\varepsilon(x,v_1)-G_\varepsilon(x,v_2)\big|\ &=\ \frac{1}{\varepsilon^2}\big|g(x)\cdot f'(u_{0,\varepsilon}+sv_1+(1-s)v_2)(v_1-v_2)\big|\\ &\le\ C\varepsilon^{a\beta_0-2}|v_1-v_2|.\end{aligned}$$

On the other hand, for $x\in B_2(\mathcal{P}_1)^c$, we obtain

$$|G_\varepsilon(x,v_1)-G_\varepsilon(x,v_2)|\ \le\ C\varepsilon^{a\beta_0-2}(1+|x|)^{-a\beta}|v_1-v_2|.$$

Thus,

$$\begin{aligned}&\big\|\rho(x)\cdot\big[G_\varepsilon(x,v_1(x))-G_\varepsilon(x,v_2(x))\big]\big\|^2_{L^2(B_1(\mathcal{P}_1)^c)}\\ =\ &\int_{B_1(\mathcal{P}_1)^c}\rho^2(x)\cdot\big|G_\varepsilon(x,v_1(x))-G_\varepsilon(x,v_2(x))\big|^2dx\\ \le\ &C\varepsilon^{2a\beta_0-4}\int_{B_1(\mathcal{P}_1)^c}(1+|x|)^{2+2\gamma-2a\beta}\cdot|v_1-v_2|^2dx\\ \le\ &C\varepsilon^{2\mu}\int_{B_1(\mathcal{P}_1)^c}\hat\rho^2(x)\cdot|v_1-v_2|^2dx.\end{aligned}$$

Now, we deduce that if ε is small enough, then

$$\begin{aligned}\|T_\varepsilon v_1-T_\varepsilon v_2\|_{X_{\gamma,\varepsilon}}\ &\le\ C_*\Big(\ln\frac{1}{\varepsilon}\Big)\|G_\varepsilon(x,v_1)-G_\varepsilon(x,v_2)\|_{Y_{\gamma,\varepsilon}}\\ &\le\ C_*\varepsilon^\mu\Big(\ln\frac{1}{\varepsilon}\Big)\|v_1-v_2\|_{X_{\gamma,\varepsilon}}\ <\ \frac{1}{2}\|v_1-v_2\|_{X_{\gamma,\varepsilon}}.\qquad\square\end{aligned}$$

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. We explain how (1.23) gets into problems about the construction of bubbling solutions of (1.27) on a compact surface. Without loss of generality, we consider the case where solutions v_ε blow up at one point, say p_1. Let $\mathcal{W}$ be an isothermal coordinate of p_1. Without loss of generality, we may assume that $p_j \notin \mathcal{W}$ for $j = 2, \cdots, b$ and $B_2(p_1) \subset \mathcal{W}$. On $\mathcal{W}$, we may write the Green function $G(x, p_1)$ in (1.24) as

$$G(x, p_1) = \gamma(x, p_1) - \frac{1}{2\pi} \ln |x - p_1|,$$

where $\gamma(x, p_1)$ is a smooth function defined on $\mathcal{W}$. We also note that $\Delta\gamma(x, p_1) = 1/|\mathcal{S}|_{h_0}$, where Δ is the usual Laplacian on the Euclidean space.

Given a solution $U_{\lambda,\beta,\varepsilon}$ of (1.7), let $V_{\lambda,\beta,\varepsilon}(r) = U_{\lambda,\beta,\varepsilon}(r) - 2\lambda \ln r$. Then, $V = V_{\lambda,\beta,\varepsilon}$ satisfies that

$$\begin{cases} V'' + \dfrac{1}{r}V' + \dfrac{1}{\varepsilon^2} e^{a(V - r^{2\lambda} e^V)}(1 - r^{2\lambda} e^V) = 0, \quad r > 0, \\ V(r) = -\beta \ln r + \kappa_{\lambda,\beta,\varepsilon} + O(r^{2-a\beta}) \quad \text{as} \quad r \to \infty. \end{cases}$$

We expect that the local profile of v_ε near p_1 looks like

$$V_\varepsilon(x) := V_{n_1,\beta,\xi}\left(\frac{|x - p_1|}{\varepsilon^\alpha}\right)$$

up to a bounded function for some $\beta > 0$, $\xi > 0$ and $\alpha > 0$. Namely, we want to find a solution v_ε of (1.27) satisfying that

$$v_\varepsilon(x) = V_\varepsilon(x) + o(1) \quad \text{as } x \to p_1, \tag{3.1}$$

and

$$\frac{1}{\varepsilon^2} e^{v_0} e^{a(v_\varepsilon - e^{u_0 + v_\varepsilon})}(1 - e^{u_0 + v_\varepsilon}) \ \to \ 4\pi N \delta_{p_1} \tag{3.2}$$

in the sense of measures as $\varepsilon \to 0$.

By (3.1), it is reasonable to set

$$v_\varepsilon(x) = V_\varepsilon(x) + E_\varepsilon(x) \ \text{ near } \ p_1. \tag{3.3}$$

Then,

$$\Delta v_\varepsilon = \frac{1}{\varepsilon^{2\alpha}\xi^2} e^{-aE_\varepsilon} e^{a(v_\varepsilon - e^{v_\varepsilon + 2n_1 \ln \frac{|x-p_1|}{\varepsilon^\alpha} - E_\varepsilon})} \left(e^{v_\varepsilon + 2n_1 \ln \frac{|x-p_1|}{\varepsilon^\alpha} - E_\varepsilon} - 1\right) + \Delta E_\varepsilon.$$

So, we put $E_\varepsilon(x) = 4\pi N \gamma(x, p_1) + A_\varepsilon$ for some constant A_ε such that

$$\Delta E_\varepsilon = \frac{4\pi N}{|\mathcal{S}|_{h_0}}.$$

We note that

$$\begin{aligned} & v_\varepsilon + 2n_1 \ln \frac{|x-p_1|}{\varepsilon^\alpha} - E_\varepsilon \\ = \;& u_0 + v_\varepsilon + 4\pi(n_1 - N)\gamma(x,p_1) + 4\pi \sum_{j=2}^{b} n_j G(x,p_j) - 2n_1\alpha \ln\varepsilon - A_\varepsilon. \end{aligned}$$

By letting $x \to p_1$, we obtain

$$A_\varepsilon = 4\pi(n_1 - N)\gamma(p_1,p_1) + 4\pi \sum_{j=2}^{b} n_j G(p_1,p_j) - 2n_1\alpha \ln\varepsilon.$$

With this value A_ε, we obtain

$$\begin{aligned} & \frac{1}{\varepsilon^{2\alpha}\xi^2} e^{-aE_\varepsilon} \\ = \;& \frac{1}{\varepsilon^{2\alpha(1-n_1 a)}} \cdot \frac{1}{\xi^2} \cdot e^{-a\left(4\pi N[\gamma(x,p_1)-\gamma(p_1,p_1)] + 4\pi n_1 \gamma(p_1,p_1) + 4\pi \sum_{j=2}^{b} n_j G(p_1,p_j)\right)}. \end{aligned}$$

By letting $x \to p_1$ and recalling the equation (1.27), we obtain

$$\begin{cases} \alpha = (1-n_1 a)^{-1}, \\ \xi^2 = e^{-a[4\pi n_1 \gamma(p_1,p_1) + 4\pi \sum_{j=2}^{b} n_j G(p_1,p_j)]} \cdot e^{-v_0(p_1)}. \end{cases} \tag{3.4}$$

For x apart from p_1, since $\Delta v_\varepsilon \to 4\pi N/|\mathcal{S}|_{h_0}$ by (3.3) as $\varepsilon \to 0$, we expect that

$$v_\varepsilon(x) \;=\; 4\pi N G(x,p_1) + B_\varepsilon \tag{3.5}$$

for some constant B_ε which will be given later. In view of (3.3) and (3.5), we define an approximate solution $v_{0,\varepsilon}$ of (1.27) by

$$\begin{aligned} v_{0,\varepsilon}(x) &= \sigma(x-p_1)\{V_\varepsilon(x) + E_\varepsilon(x)\} + \big(1 - \sigma(x-p_1)\big) \cdot \Big(4\pi N G(x,p_1) + B_\varepsilon\Big) \\ &= \sigma(x-p_1)\Big\{V_\varepsilon(x) + 2N \ln|x-p_1| + A_\varepsilon - B_\varepsilon\Big\} + 4\pi N G(x,p_1) + B_\varepsilon. \end{aligned}$$

Here, σ is a function defined by (2.1). For this function, we want to check that

$$\Delta v_{0,\varepsilon} - \frac{4\pi N}{|\mathcal{S}|_{h_0}} =: Z_\varepsilon \to 0 \quad \text{on} \quad B_2(p_1)\backslash B_1(p_1) \quad \text{as} \quad \varepsilon \to 0.$$

Indeed, we deduce that for $B_2(p_1)\backslash B_1(p_1)$,

$$\begin{aligned} Z_\varepsilon(x) \;=\;& \Delta\sigma(x-p_1)\Big\{(2N-\beta)\ln|x-p_1| + \alpha\beta\ln\varepsilon + \kappa_{n_1,\beta,\xi} \\ & + O\big(\varepsilon^{\alpha(a\beta-2)}\big) + A_\varepsilon - B_\varepsilon\Big\} \\ & + \; 2\sigma'(|x-p_1|)\Big\{\frac{1}{\varepsilon^\alpha} V'_{n_1,\beta,\xi}\Big(\frac{|x-p_1|}{\varepsilon^\alpha}\Big) + \frac{2N}{|x-p_1|}\Big\} \\ & + \; \sigma(x-p_1)\Delta V_\varepsilon(x) \\ =\;& (i) + (ii) + (iii). \end{aligned}$$

It is necessary to put

$$\beta = 2N, \quad B_\varepsilon = 2N\alpha \ln \varepsilon + \kappa_{n_1,\beta,\xi} + A_\varepsilon. \tag{3.6}$$

Then, since $aN = 2$ by (1.23), we deduce that $|(i)| = O(\varepsilon^{\alpha(2aN-2)}) = O(\varepsilon^{2\alpha})$. We also note from (3.6) that

$$(ii) = 2\sigma'(|x-p_1|) \cdot \left(\frac{2N-\beta}{|x-p_1|} + o(1) \right) = o(1).$$

Moreover, we observe that

$$|(iii)| \le \frac{C}{\varepsilon^{2\alpha}} e^{aV_\varepsilon} = O(\varepsilon^{2aN\alpha - 2\alpha}) = O(\varepsilon^{2\alpha}).$$

Consequently, $|Z_\varepsilon(x)| = o(1)$ as $\varepsilon \to 0$ and we have the final form of an approximate solution $v_{0,\varepsilon}$ as follows:

$$\begin{aligned} v_{0,\varepsilon}(x) &= \sigma(x-p_1)\Big\{ V_\varepsilon(x) + 2N \ln \frac{|x-p_1|}{\varepsilon^\alpha} - \kappa_{n_1,2N,\xi} \Big\} \\ &\quad + 4\pi N G(x,p_1) + \kappa_{n_1,2N,\xi} + 4\pi \sum_{j=2}^{b} n_j G(p_1,p_j) \\ &\quad + 2(n_1 - N) \ln \frac{1}{\varepsilon^\alpha} + 4\pi(n_1 - N)\gamma(p_1,p_1). \end{aligned} \tag{3.7}$$

We recall the constants α, β and ξ are determined by (3.4) and (3.6). There is another constraint on the range of β according to Theorem A. In fact, by Theorem A, only when $0 < an_1 < 1$ or $1 < an_1 < 2$, we can consider the radial profile $V_{n_1,\beta,\xi}$ as a nontopological type I solution. If $0 < an_1 < 1$, then Theorem A (i) implies that

$$2N = \beta > \frac{4}{a}, \quad \text{i.e.,} \quad aN > 2.$$

This contradicts to the condition (1.23). On the other hand, if $1 < an_1 < 2$, then

$$2n_1 < \beta = 2N < 4/a, \quad \text{i.e.,} \quad aN < 2,$$

which is also absurd. As a consequence, we cannot take any radial solutions $V_{n_1,\beta,\xi}$ as a local profile of bubbling solutions. This completes the proof of Theorem 1.2. □

4. Proof of Theorem 2.1

In this section, we prove Theorem 2.1. The proof is similar to that of Theorem 2.1 of [7] or Theorem B.1 of [11]. We modify arguments in [7,11] to show the invertibility of L_ε and the estimate (2.7). We provide an outline of the proof and the detailed proof is given only when our situation is different from [7,11].

We start with the study of the linearization of the radial equation (1.7). Suppose that $a > 0$ and $\lambda > 0$ such that $0 < a\lambda < 1$. By Theorem A, there exists a type I solution $U_{\lambda,\beta,\varepsilon}$ of (1.7) satisfying (1.8) such that

$$U_{\lambda,\beta,\varepsilon}(r) = \begin{cases} 2\lambda \ln r + O(1) & \text{as} \quad r \to 0, \\ (2\lambda - \beta)\ln r + O(1) & \text{as} \quad r \to \infty. \end{cases}$$

For $k \in \mathbb{N} \cup \{0\}$, consider the following ODE:

$$\varphi'' + \frac{1}{r}\varphi' - \frac{k^2}{r^2}\varphi + \frac{1}{\varepsilon^2} r^{-2a\lambda} f'\big(U_{\lambda,\beta,\varepsilon}(r)\big)\varphi = 0. \tag{4.1}$$

Since $f'(u) = e^{a(u-e^u)}\{a(1-e^u)^2 - e^u\}$, we deduce that as $r \to 0$,

$$r^{-2a\lambda} f'\big(U_{\lambda,\beta,\varepsilon}(r)\big) = r^{-2a\lambda} e^{aU_{\lambda,\beta,\varepsilon}(r)}\big[a + o(1)\big] = a + o(1).$$

Thus, $r = 0$ is a regular singular point of (4.1) for $k \geq 1$ such that as $r \to 0$, either $\varphi(r) = r^k\big[A + o(1)\big]$ or $\varphi(r) = r^{-k}\big[A + o(1)\big]$ for some $A \neq 0$. One may refer to Theorem 5.7 of [3]. If $\psi(r) = \varphi(r^{-1})$, then ψ satisfies

$$\psi'' + \frac{1}{r}\psi' - \frac{k^2}{r^2}\psi + \frac{1}{\varepsilon^2} r^{2a\lambda-4} f'\big(U_{\lambda,\beta,\varepsilon}(r^{-1})\big)\psi = 0. \tag{4.2}$$

Since $a\beta > 4$ by Theorem A, it holds that as $r \to 0$,

$$\begin{aligned} r^{2a\lambda-4} f'\big(U_{\lambda,\beta,\varepsilon}(r^{-1})\big) &= r^{2a\lambda-4} e^{aU_{\lambda,\beta,\varepsilon}(r^{-1})}\big[a + o(1)\big] \\ &= r^{a\beta-4}\big[a + o(1)\big] = o(1). \end{aligned}$$

Hence, $r = 0$ is a regular singular point of (4.2) for $k \geq 1$ such that either $\psi(r) = r^k\big[B + o(1)\big]$ or $\psi(r) = r^{-k}\big[B + o(1)\big]$ for some $B \neq 0$. In the sequel, if φ is a solution of (4.1) for $k \geq 1$, then either $\varphi(r) = r^k\big[B + o(1)\big]$ or $\varphi(r) = r^{-k}\big[B + o(1)\big]$ for some $B \neq 0$ as $r \to \infty$.

Now let $\varphi_k = \varphi_{\lambda,\beta,\varepsilon,k}(r)$ with $r = |x|$ be the unique solution of the linearized equation at $U_{\lambda,\beta,\varepsilon}$:

$$\begin{cases} \varphi_k'' + \frac{1}{r}\varphi_k' - \frac{k^2}{r^2}\varphi_k + \frac{1}{\varepsilon^2} r^{-2a\lambda} f'\big(U_{\lambda,\beta,\varepsilon}(r)\big)\varphi_k = 0, \\ \varphi_k(r) = r^k\big[1 + o(1)\big] \quad \text{near } r = 0. \end{cases}$$

If $k \geq 1$, then the above argument shows that $\varphi_k(r) = O(r^k)$ or $\varphi_k(r) = O(r^{-k})$ as $r \to \infty$. We also set $w_{\lambda,\beta,\varepsilon}(r) = U'_{\lambda,\beta,\varepsilon}(r)$. Then $w = w_{\lambda,\beta,\varepsilon}$ satisfies that

$$\begin{cases} w'' + \frac{1}{r}w' - \frac{1}{r^2}w + \frac{1}{\varepsilon^2} r^{-2a\lambda} f'\big(U_{\lambda,\beta,\varepsilon}(r)\big)w = \frac{2a\lambda}{\varepsilon^2} r^{-2a\lambda-1} f\big(U_{\lambda,\beta,\varepsilon}(r)\big), \\ w(r) = \frac{2\lambda}{r} + o(1) \quad \text{near } r = 0. \end{cases} \tag{4.3}$$

We note that

$$w(r) = (2\lambda - \beta)/r + O(1) \text{ and } w'(r) = O(r^{-2}) \text{ as } r \to \infty. \tag{4.4}$$

It is known (see Lemma 2.2 of [9]) that $U_{\lambda,\beta,\varepsilon}$ has a unique maximum point $z_0 = z_0(\lambda,\beta,\varepsilon)$. Thus, $w(z_0) = 0$, $w > 0$ for $0 < r < z_0$, and $w < 0$ for $r > z_0$.

Proposition 4.1. *Suppose that* $0 < a\lambda < 1$. *Let* $U_{\lambda,\beta,\varepsilon}(r)$ *be a type I nontopological solution of* (1.7) *satisfying* (1.8).

(i) φ_0 *has exactly one zero in* $(0,\infty)$ *and* $\lim_{r\to\infty}\varphi_0(r) = -\infty$. *Moreover,*

$$\varphi_0(r) = -c_0 \ln r + d_0 + o(1) \quad \text{as } r \to \infty,$$

where $c_0 = c_0(\lambda,\beta,\varepsilon) > 0$ *and* $d_0 = d_0(\lambda,\beta,\varepsilon) \in \mathbb{R}$ *are constants.*

(ii) *Either* φ_1 *has only one zero and* $\varphi_1(r) = -\big[c_1 + o(1)\big]r$ *as* $r \to \infty$, *or* $\varphi_1 > 0$ *on* $(0,\infty)$ *and* $\varphi_1(r) = \big[\tilde{c}_1 + o(1)\big]r$ *as* $r \to \infty$. *Here,* c_1 *and* $\tilde{c}_1$ *are positive constants.*

(iii) *If* $0 < a < 1$ *and* $0 < \lambda \le 1$, *then* $\varphi_k > 0$ *on* $(0, \infty)$ *for* $k \ge 2$. *In addition,* $\varphi_k(r) = \big[c_k + o(1)\big] r^k$ *for some* $c_k = c_k(\lambda, \beta, \varepsilon) > 0$ *as* $r \to \infty$.

Proof. Let us fix λ, β, ε and set $U = U_{\lambda,\beta,\varepsilon}$ and $w = w_{\lambda,\beta,\varepsilon}$ for brevity.

(i) See Lemma 2.10 of [9].

(ii) Let r_0 be the unique zero of φ_0. If φ_1 has the first zero at r_1, then $r_0 < r_1$. Otherwise, we obtain a contradiction:

$$\begin{aligned} 0 < \int_0^{r_1} \frac{1}{r}\varphi_1(r)\varphi_0(r)dr &= \int_0^{r_1} \big[(r\varphi_1')'\varphi_0 - (r\varphi_0')'\varphi_1\big] dr \\ &= r_1\varphi_1'(r_1)\varphi_0(r_1) < 0. \end{aligned}$$

If φ_1 has the second zero at r_2, then we also have a contradiction:

$$0 < \int_{r_1}^{r_2} \frac{1}{r}\varphi_1(r)\varphi_0(r)dr = r_2\varphi_1'(r_2)\varphi_0(r_2) - r_1\varphi_1'(r_1)\varphi_0(r_1) < 0.$$

This means that either φ_1 has only one zero or φ_1 has no zeros. In each case, we will show that $\varphi_1(r) = O(r)$ at infinity.

First, assume that φ_1 has a unique zero at r_1 such that $\varphi_1(r) < 0$ on (r_1, ∞). We recall that $r_0 < r_1$ where r_0 is the unique zero of φ_0. On the other hand, it follows (see Lemma 2.5 of [9]) that $z_0 < r_0$. So, $z_0 < r_1$ such that $w(r_1) < 0$. If φ_1 is bounded such that $\varphi_1(r) = O(r^{-1})$ as $r \to \infty$, then by (4.4)

$$0 > \frac{2a\lambda}{\varepsilon^2} \int_{r_1}^{\infty} r^{-2a\lambda} f(U(r))\varphi_1(r)dr = r_1\varphi_1'(r_1)w(r_1) > 0,$$

a contradiction. Hence, φ_1 is unbounded and $\varphi_1(r) = -\big[c_1 + o(1)\big]r$ for some $c_1 > 0$ as $r \to \infty$.

Next, we consider the case that $\varphi_1(r) > 0$ on $(0, \infty)$. Since

$$(r\varphi_1')' = \Big(\frac{1}{r} - \frac{1}{\varepsilon^2} r^{1-2a\lambda} f'(U)\Big)\varphi_1 > 0 \quad \text{as } r \to \infty,$$

φ_1 does not oscillate for sufficiently large r. Suppose that φ_1 is bounded. Then,

$$\begin{cases} \varphi_1(r) = \big[1 + o(1)\big]r \text{ as } r \to 0, \\ \varphi_1(r) = \big[\bar{c}_1 + o(1)\big]r^{-1} \text{ for some } \bar{c}_1 > 0 \text{ as } r \to \infty. \end{cases} \tag{4.5}$$

Let $\{\varphi_1, \hat{\varphi}_1\}$ form a fundamental set of solutions for (4.1) with $k = 1$. Since $\varphi_1(r) = O(r)$ near $r = 0$ and $\varphi_1(r) = O(r^{-1})$ as $r \to \infty$, we deduce that $\hat{\varphi}_1(r) = O(r^{-1})$ near $r = 0$ and $\hat{\varphi}_1(r) = O(r)$ as $r \to \infty$. By the above discussion, we may assume that

$$\begin{cases} \hat{\varphi}_1(r) = \big[1 + o(1)\big]r^{-1} \text{ as } r \to 0, \\ \hat{\varphi}_1(r) = \big[\hat{c}_1 + o(1)\big]r \text{ for some } \hat{c}_1 \ne 0 \text{ as } r \to \infty. \end{cases} \tag{4.6}$$

Let $W(r) = \varphi_1'(r)\hat{\varphi}_1(r) - \varphi_1(r)\hat{\varphi}_1'(r)$ be the Wronskian of φ_1 and $\hat{\varphi}_1$. Since φ_1 and $\hat{\varphi}_1$ are solutions of (4.1) with $k = 1$, it holds that

$$\big[r\varphi_1'(r)\big]'\hat{\varphi}_1(r) - \big[r\hat{\varphi}_1'(r)\big]'\varphi_1(r) = 0 \quad \text{for} \quad r > 0.$$

Integrating this equation on $(0, r)$, we see from (4.5) and (4.6) that $W(r) = 2r^{-1}$. Then, w can be written as

$$w(r) = \nu_1\varphi_1(r) + \hat{\nu}_1\hat{\varphi}_1(r) + w_p(r) \quad \text{for} \quad r > 0.$$

Here ν_1 and $\hat{\nu}_1$ are constants and w_p is a particular solution given by $w_p(r) = -v_1(r)\varphi_1(r) + \hat{v}_1(r)\hat{\varphi}_1(r)$, where

$$v_1(r) = \frac{2a\lambda}{\varepsilon^2}\int_1^r \frac{\hat{\varphi}_1(s)s^{-2a\lambda-1}f\big(U(s)\big)}{2s^{-1}}\,ds,$$

$$\hat{v}_1(r) = \frac{2a\lambda}{\varepsilon^2}\int_1^r \frac{\varphi_1(s)s^{-2a\lambda-1}f\big(U(s)\big)}{2s^{-1}}\,ds.$$

Since $f\big(U(r)\big) = O(e^{au}) = O(r^{2a\lambda})$ as $r \to 0$, we are led from (4.6) that $v_1(r) = O(\ln r)$ as $r \to 0$. Thus,

$$w(r) = \big[\hat{\nu}_1 + \hat{v}_1(0)\big]\hat{\varphi}_1(r) + o(1) = \big[\hat{\nu}_1 + \hat{v}_1(0)\big]\cdot\frac{1}{r} + o(1) \quad \text{as} \quad r \to 0.$$

In view of the initial condition of w in (4.3), we get

$$\hat{\nu}_1 + \hat{v}_1(0) = 2\lambda. \tag{4.7}$$

Meanwhile, since $f\big(U(r)\big) = O(e^{au}) = O(r^{a(2\lambda-\beta)})$ as $r \to \infty$ and $a\beta > 4$ by Theorem A, it follows that

$$v_1(\infty) := \lim_{r\to\infty} v_1(r) \quad \text{and} \quad \hat{v}_1(\infty) := \lim_{r\to\infty}\hat{v}_1(r)$$

exist. So, as $r \to \infty$,

$$w(r) = \big[\nu_1 - v_1(\infty) + o(1)\big]\varphi_1(r) + \big[\hat{\nu}_1 + \hat{v}_1(\infty) + o(1)\big]\hat{\varphi}_1(r),$$

which implies by (4.4) that

$$\bar{c}_1\big[\nu_1 - v_1(\infty)\big] = 2\lambda - \beta \quad \text{and} \quad \hat{c}_1\big[\hat{\nu}_1 + \hat{v}_1(\infty)\big] = 0. \tag{4.8}$$

Since $\hat{c}_1 \neq 0$, (4.7) and (4.8) tell us that

$$2\lambda = \hat{v}_1(0) - \hat{v}_1(\infty) = -\frac{a\lambda}{\varepsilon^2}\int_0^\infty \varphi_1(r)r^{-2a\lambda}f\big(U(r)\big)\,dr = -2\lambda.$$

Hence, we get $\lambda = 0$, a contradiction. Here, the last equality is derived by the equations of φ_1 and w:

$$\frac{2a\lambda}{\varepsilon^2}\int_0^\infty \varphi_1(r)r^{-2a\lambda}f\big(U(r)\big)\,dr = \Big[rw'(r)\varphi_1(r) - r\varphi_1'(r)w(r)\Big]_0^\infty = 4\lambda.$$

In the sequel, φ_1 is unbounded and $\varphi_1(r) = \big[\tilde{c}_1 + o(1)\big]r$ for some $\tilde{c}_1 > 0$ as $r \to \infty$.

(iii) Let $k \geq 2$ be fixed and set

$$t = r^{\lambda+1}, \quad \tilde{u}(t) = U(r) - 2\lambda\ln r \quad \text{and} \quad \tilde{\varphi}_k(t) = \varphi_k(r).$$

It is not difficult to show that

$$\begin{aligned}\frac{1}{t}(t\tilde{\varphi}_k')' &= \frac{k^2}{(\lambda+1)^2t^2}\tilde{\varphi}_k - \frac{1}{\varepsilon^2(\lambda+1)^2}t^{\frac{-2a\lambda-2\lambda}{\lambda+1}}f'(U)\tilde{\varphi}_k,\\ \frac{1}{t}(t\tilde{u}')' &= -\frac{1}{\varepsilon^2(\lambda+1)^2}t^{\frac{-2a\lambda-2\lambda}{\lambda+1}}f(U).\end{aligned}$$

Set $\tilde{v}(t) = \tilde{u}'(t)$. Then we get

$$\begin{aligned}\tilde{v}(t) &= \frac{1}{(\lambda+1)t}(rU'(r) - 2\lambda) < 0 \quad \text{for all } t > 0,\\ \frac{1}{t}(t\tilde{v}')' &= \frac{1}{t^2}\tilde{v} - \frac{1}{\varepsilon^2(\lambda+1)^2}t^{\frac{-2a\lambda-2\lambda}{\lambda+1}}f'(U)\tilde{v} + \frac{2a\lambda}{\varepsilon^2(\lambda+1)^3}t^{\frac{2-2a\lambda}{\lambda+1}-3}f(U)\\ &\quad + \frac{2\lambda}{\varepsilon^2(\lambda+1)^3}t^{\frac{2-2a\lambda}{\lambda+1}-3}[f(U) - f'(U)].\end{aligned}$$

Assume that $\tilde{\varphi}_k$ has a zero on $(0,\infty)$. If t_1 is the first zero of $\tilde{\varphi}_k$, then $\tilde{\varphi}_k > 0$ on $(0,t_1)$. It follows from the comparison argument that

$$\begin{aligned}0 > -t_1\tilde{\varphi}_k'(t_1)\tilde{v}(t_1) &= t\Big[\tilde{v}'\tilde{\varphi}_k - \tilde{\varphi}_k'\tilde{v}\Big]_{t=0}^{t_1}\\ &= \int_0^{t_1}\Big(1 - \frac{k^2}{(\lambda+1)^2}\Big)\frac{1}{t}\tilde{v}\tilde{\varphi}_k dt \;+\; \frac{2a\lambda}{\varepsilon^2(\lambda+1)^3}\int_0^{t_1} t^{\frac{-2a\lambda-2\lambda}{\lambda+1}}f(U)\tilde{\varphi}_k dt\\ &\quad + \frac{2\lambda}{\varepsilon^2(\lambda+1)^3}\int_0^{t_1} t^{\frac{-2a\lambda-2\lambda}{\lambda+1}}\big[f(U) - f'(U)\big]\tilde{\varphi}_k dt \;>\; 0,\end{aligned}$$

a contradiction. Here, we use the fact that $k^2 \geq (\lambda+1)^2$ for $k \geq 2$ and $0 < \lambda \leq 1$, and

$$f(U) - f'(U) = e^{a(U-e^U)}\big[(1-e^U)(1-a+ae^U) + e^U\big] > 0 \text{ for } 0 < a < 1.$$

Thus, $\tilde{\varphi}_k > 0$ on $(0,\infty)$ and consequently, $\varphi_k > 0$ on $(0,\infty)$.

Next, we claim that $\varphi_k(r) \to \infty$ as $r \to \infty$. Since $\tilde{u}(t) = -\beta(\lambda+1)^{-1}\ln t + O(1)$ as $t \to \infty$, we have $\tilde{v}(t) = O(t^{-1})$ and $\tilde{v}'(t) = O(t^{-2})$ as $t \to \infty$. If $\tilde{\varphi}_k$ is bounded, then $\tilde{\varphi}_k(t) = O(t^{-k/(\lambda+1)})$, which yields a contradiction:

$$\begin{aligned}0 &= \lim_{t\to\infty} t(\tilde{v}'\tilde{\varphi}_k - \tilde{\varphi}_k'\tilde{v}) - \lim_{t\to 0} t(\tilde{v}'\tilde{\varphi}_k - \tilde{\varphi}_k'\tilde{v})\\ &= \int_0^{\infty}\Big(1 - \frac{k^2}{(\lambda+1)^2}\Big)\frac{1}{t}\tilde{v}\tilde{\varphi}_k dt + \frac{2\lambda}{\varepsilon^2(\lambda+1)^3}\int_0^{\infty} t^{\frac{-2a\lambda-2\lambda}{\lambda+1}}[f(U) - f'(U)]\tilde{\varphi}_k dt\\ &\quad + \frac{2a\lambda}{\varepsilon^2(\lambda+1)^3}\int_0^{t_1} t^{\frac{-2a\lambda-2\lambda}{\lambda+1}}f(U)\tilde{\varphi}_k dt > 0.\end{aligned}$$

Therefore, $\tilde{\varphi}_k$ is unbounded and $\varphi_k(r) = [c_k + o(1)]r^k$ as $r \to \infty$ for some $c_k > 0$. □

For $1 \leq j \leq b_1$, we define the linearized operator $L_j : X_\gamma \to Y_\gamma$ by

$$L_j \;=\; \Delta + \omega_j^2|x|^{-2a}f'(U_j(|x|)).$$

Here, the function spaces X_γ, Y_γ are defined as follows:

$$\begin{aligned} X_\gamma &= \big\{v \in W^{2,2}_{loc}(\mathbb{R}^2) : \|v\|^2_{X_\gamma} = \|\rho\Delta v\|^2_{L^2(\mathbb{R}^2)} + \|\hat{\rho} v\|^2_{L^2(\mathbb{R}^2)} < \infty\big\}, \\ Y_\gamma &= \big\{v \in L^2_{loc}(\mathbb{R}^2) : \|v\|^2_{Y_\gamma} = \|\rho v\|^2_{L^2(\mathbb{R}^2)} < \infty\big\}. \end{aligned}$$

Corollary 4.2. *If $L_j v = 0$ and v is bounded in $\mathbb{R}^2$, then $v(x) \equiv 0$.*

Proof. By the Fourier expansion, one can express v by

$$v(r,\theta) = \sum_{k=0}^{\infty} a_k \varphi_k(r) \cos k\theta + \sum_{k=1}^{\infty} b_k \varphi_k(r) \sin k\theta$$

where $a_k, b_k \in \mathbb{R}$. By Proposition 4.1, the functions φ_k are unbounded for all $k \geq 0$. This contradicts to the boundedness of v unless v vanishes identically. See Lemma 3.2 of [7] for details. $\square$

Theorem 4.3. *For each $1 \leq j \leq b_1$, the linearized operator $L_j : X_\gamma \to Y_\gamma$ is an isomorphism such that*

$$\|v\|_{X_\gamma} + \|v\|_{L^\infty(\mathbb{R}^2)} \;\leq\; C\|L_j v\|_{Y_\gamma} \quad \text{for all } v \in X_\gamma. \tag{4.9}$$

Proof. The proof is based on the property that L_j is a compact perturbation of $\Delta : X_\gamma \to Y_\gamma$. It is known that $\Delta : X_\gamma \to Y_\gamma$ is a Fredholm operator with Ker $\Delta = \{c \in \mathbb{R}\}$ and Range $(\Delta : X_\gamma \to Y_\gamma) = Y_\gamma \cap Z$, where $Z = \big\{u \in L^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} u(x)dx = 0\big\}$. See Lemma 3.3 of [7]. By Corollary 4.2, one can show that $\mathrm{Ker}L_j = 0$ in X_γ and the estimate (4.9) is valid. Then, since L_j is a Fredholm operator of index zero, it is onto and the proof is complete. See Theorem 4.1 of [5] and Theorem 3.4 of [7] for details. $\square$

Proposition 4.4. *Let $\tilde{X}_{\gamma,\varepsilon} = \{v \in X_{\gamma,\varepsilon} : L_\varepsilon v \equiv 0 \;\; \text{on} \;\; B_1(\mathcal{P}_1)\}$. Then, there exists $0 < \varepsilon_* < 1$ such that for any $\varepsilon \in (0, \varepsilon_*)$ and $v \in \tilde{X}_{\gamma,\varepsilon}$,*

$$\|v\|_{X_{\gamma,\varepsilon}} + \|v\|_{L^\infty(\mathbb{R}^2)} \;\leq\; C\Big(\ln\frac{1}{\varepsilon}\Big)\|L_\varepsilon v\|_{Y_{\gamma,\varepsilon}}.$$

Proof. The proof is parallel to that of Theorem 3.6 of [7] and we omit the details. One thing to emphasize is that Corollary 4.2 is used in the proof. Refer to the proof of Lemma 3.5 of [7] to see how Corollary 4.2 is used. $\square$

Proof of Theorem 2.1. It suffices to show the inequality (2.7). Indeed, since L_ε is a Fredholm operator of index zero and L_ε is one-to-one by (2.7), L_ε is also onto. The proof of (2.7) is almost the same as the proof of Theorem 2.1 of [7] and we give only brief outline of it. Define

$$\tilde{h}_j(x) = \begin{cases} h(x), & x \in B_1(p_j), \quad p_j \in \mathcal{P}_1, \\ 0, & \text{otherwise}. \end{cases}$$

Since $\varepsilon^{2\alpha}\tilde{h}_j(p_j + \varepsilon^\alpha y) \in Y_\gamma$, it follows from Theorem 4.3 that there exists a unique $u_j \in X_\gamma$ such that for each $1 \leq j \leq b_1$,

$$\begin{aligned} &L_j u_j(y) = \varepsilon^{2\alpha}\tilde{h}_j(p_j + \varepsilon^\alpha y), \\ &\|u_j\|_{X_\gamma} + \|u_j\|_{L^\infty(\mathbb{R}^2)} \leq C\|\varepsilon^{2\alpha}\tilde{h}_j(p_j + \varepsilon^\alpha y)\|_{Y_\gamma} \leq C\|h\|_{Y_{\gamma,\varepsilon}}. \end{aligned}$$

If we set

$$\hat{w}(x) = v(x) - \sum_{j=1}^{b_1} \sigma(x-p_j)\tilde{v}_j(x),$$

then $L_\varepsilon \hat{w} = h^*$ where

$$\begin{cases} \tilde{v}_j(x) = u_j\Big(\dfrac{x-p_j}{\varepsilon^\alpha}\Big), \\ h^*(x) = h(x) - \displaystyle\sum_{j=1}^{b_1} \sigma(x-p_j)\tilde{h}_j(x) - \tilde{h}(x), \\ \tilde{h}(x) = \displaystyle\sum_{j=1}^{b_1} \big[\Delta\sigma(x-p_j)\tilde{v}_j(x) + 2\nabla\sigma(x-p_j)\cdot\nabla\tilde{v}_j(x)\big]. \end{cases}$$

Since $h^* \equiv 0$ on $B_1(\mathcal{P}_1)$ and $\|h^*\|_{Y_{\gamma,\varepsilon}} \le C\|h\|_{Y_{\gamma,\varepsilon}}$, we deduce from Proposition 4.4 that for small $0 < \varepsilon < \varepsilon_*$,

$$\|\hat{w}\|_{X_{\gamma,\varepsilon}} + \|\hat{w}\|_{L^\infty(\mathbb{R}^2)} \le C\Big(\ln\frac{1}{\varepsilon}\Big)\|h\|_{Y_{\gamma,\varepsilon}}.$$

Then, recovering v from this inequality, we arrive at the estimate (2.7). □

Remark 4.5. The authors thank the anonymous referee for letting them know the result of Ref. [13]. In [13], Lin and Yan improved their results of [5,11] by removing the constraint on the location of blow-up vortex points in the construction of nontopological solutions to the self-dual Chern–Simons–Higgs equation. The referee mentioned that the constraint (1.19) for our solutions to (1.4) may not be necessary in the spirit of [13]. We will treat this issue in the forthcoming paper.

Acknowledgments

Jongmin Han was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2015R1D1A1A01057499).

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