



Wolff potentials and regularity of solutions to integral systems on spaces of homogeneous type



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ABSTRACT

Two objectives are accomplished in this paper: First we establish the comparison between Wolff and Riesz potentials on space of homogeneous type in the sense of Coifman and Weiss (Theorem 1.2), followed by a Hardy–Littlewood–Sobolev type inequality for Wolff potentials (Theorem 1.3). Then applying this inequality, we consider a Lane–Emden type integral system and use regularity lifting to derive integrability estimates of positive solutions to the system, by which we also imply L^∞ estimates (see Theorem 1.4 and 1.5). Furthermore, we use a modified regularity lifting method to prove that the positive solutions are also Lipschitz continuous (Theorem 1.6). Our results imply Wolff potentials and regularity of solutions to the Lane–Emden type integral systems on stratified groups and the Carnot–Carathéodory metric spaces defined by a family of vector fields satisfying Hörmander’s finite rank condition as special cases.

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1. Introduction

Wolff potentials on \mathbb{R}^n were originally studied by Hedberg and Wolff [14]: Given $\omega \in \mathbb{M}^+(\mathbb{R}^n)$, the class of all positive locally finite Borel measure on \mathbb{R}^n , the (continuous) Wolff potential $W_{\alpha,p}\omega(x)$ for $\alpha > 0$ and $p > 1$ is defined as

$$W_{\alpha,p}\omega(x) = \int_0^\infty \left[\frac{\omega(B_t(x))}{t^{n-\alpha p}} \right]^{p'-1} \frac{dt}{t}$$

for $x \in \mathbb{R}^n$, where $\omega(B_t(x)) = \int_{B_t(x)} d\omega$ and p' is the conjugate index¹ of p . They also introduced the discrete version of Wolff potentials as

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¹ That is, $pp' = p + p'$.

$$W_{\alpha,p}^D \omega(x) = \sum_{Q \in \mathcal{D}} \left[\frac{\omega(Q)}{|Q|^{1-\frac{\alpha p}{n}}} \right]^{\frac{1}{p-1}} \chi_Q(x),$$

where \mathcal{D} is the set of all the dyadic cubes $Q \subseteq \mathbb{R}^n$ and $|Q|$ denotes its volume. We define the (continuous) Riesz potential of ω for $0 < \lambda < n$ as

$$I_\lambda \omega(x) = c \int_{\mathbb{R}^n} |x-y|^{\lambda-n} d\omega = \int_0^\infty \frac{\omega(B_t(x))}{t^{n-\lambda}} \frac{dt}{t}.$$

It is evident that $I_\lambda \omega = W_{\frac{\lambda}{2},2} \omega$, and the discrete version of Riesz potentials can be similarly established. Wolff's theorem ([14], see also §4.5 in [1]) states:

Theorem (Wolff's theorem). *Let $\alpha > 0$, $1 < p < \infty$, $0 < \alpha p < n$ and $\omega \in \mathbb{M}^+(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} W_{\alpha,p}^D \omega(x) d\omega \simeq \int_{\mathbb{R}^n} (I_\alpha \omega(x))^{p'} dx.$$

Here and hereafter, $A \simeq B$ means two quantities A and B are equivalent, more precisely, there exist $0 < c \leq C < \infty$ such that $cA \leq B \leq CA$.

The brilliant work [14] of Hedberg and Wolff was originally carried out to fill the gap in the study of Sobolev spaces. However, it also has important applications in other areas. Here we mention some interesting examples among them. Note that if $u \geq 0$ is measurable on \mathbb{R}^n , then $d\omega = u dx \in \mathbb{M}^+(\mathbb{R}^n)$.

Example 1.

$$u(x) = W_{\frac{\lambda}{2},2}(u^{\frac{n+\lambda}{n-\lambda}})(x) = I_\lambda(u^{\frac{n+\lambda}{n-\lambda}})(x),$$

and its corresponding semilinear partial differential equation

$$(-\Delta)^\lambda u = u^{\frac{n+\lambda}{n-\lambda}}.$$

This family of equations are closely related to optimizers of sharp Hardy–Littlewood–Sobolev² inequality. For the study of this inequality, we refer the reader to [11,13,16] and the references therein.

Example 2 (*p-Laplacian equations*).

$$u(x) = W_{1,p}(u^q)(x),$$

and its corresponding p -Laplacian equation

$$-\Delta_p u = -\operatorname{div}(\nabla u |\nabla u|^{p-2}) = u^q.$$

Example 3 (*Hessian equations*).

$$u(x) = W_{\frac{2k}{k+1},k+1}(u^q)(x),$$

and its corresponding k -Hessian equation

² We use HLS to denote Hardy–Littlewood–Sobolev in the following content.

$$F_k[-u] = u^q,$$

where $F_k[u]$ is a k -Hessian ($k = 1, 2, \dots, n$) defined by

$$F_k[u] = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the Hessian matrix D^2u , that is, $F_k[u]$ is the sum of $k \times k$ principal minors of D^2u . In particular, $F_1[u] = \Delta u$ when $k = 1$ and $F_n[u] = \det(D^2u)$ when $k = n$.

Based on systematic use of Wolff potentials, Phuc and Verbitsky [21] studied Examples 2 and 3. They gave the existence and pointwise estimate of the positive solutions, in terms of the corresponding Wolff potentials. By using a general Regularity Lifting Theorem, Ma, Chen and Li [17] proved regularity for positive solutions of an integral system associated with Wolff potentials. In this paper, we shall concentrate on some analogous results on spaces of homogeneous type in the sense of Coifman and Weiss.

Recall that a quasi-metric d on a set \mathbb{X} is a mapping $d : \mathbb{X} \times \mathbb{X} \rightarrow [0, \infty)$ satisfying

- (i) $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathbb{X}$;
- (iii) $d(x, z) \leq k_1(d(x, y) + d(y, z))$ for all $x, y, z \in \mathbb{X}$ and some constant $k_1 \in [1, \infty)$ independent of x, y and z .

Such quasi-metric d defines a topology on \mathbb{X} , for which the balls $B_t(x) = \{y \in \mathbb{X} : d(x, y) < t\}$ form a base. To carry over Hardy spaces and Calderón–Zygmund singular integral operators to various settings, in the early 1970's Coifman and Weiss introduced the following spaces of homogeneous type.

Definition (*Spaces of homogeneous type*). A space of homogeneous type (\mathbb{X}, d, μ) is a set \mathbb{X} equipped with a quasi-metric d and a doubling measure μ , that is, μ is a locally finite nonnegative measure on Borel subsets of \mathbb{X} satisfying

- (iv) $\mu(B_{2t}(x)) \leq k_2\mu(B_t(x))$ for all balls $B_t(x) \subseteq \mathbb{X}$ and some constant $k_2 \in (0, \infty)$ independent of x and r .

In [18], Macias and Segovia have proved that one can replace the quasi-metric d by another quasi-metric $d^* \approx d$ such that d^* yields the same topology on \mathbb{X} as d does and, moreover, for some $N > 0$

$$\mu(B_r(x)) \sim r^N \tag{1.1}$$

where $B_r(x) = \{y \in \mathbb{X} : d^*(y, x) < r\}$ and d^* has the following regularity property

$$|d^*(x, y) - d^*(x', y)| \leq C_0[d^*(x, x')]^\theta [d^*(x, y) + d^*(x', y)]^{1-\theta} \tag{1.2}$$

for some regularity exponent $\theta : 0 < \theta < 1, 0 < r < \infty$ and all $x, x', y \in \mathbb{X}$.

Here and hereafter, we drop the $*$ in d^* and simply assume d satisfies (1.1) and (1.2), denote the spaces of homogeneous type (\mathbb{X}, d, μ) defined as above by \mathbb{X} , and call N as the homogeneous dimension of \mathbb{X} . About spaces of homogeneous type \mathbb{X} , as Meyer remarked in his preface to [7], “*The Fourier transform is missing, but a version of harmonic analysis is still present.*” Many classical results of harmonic analysis over Euclidean spaces can be extended to \mathbb{X} . For more background and information on this subject, we refer the reader to [2, 5–7, 10, 12, 15, 18, 19, 23, 24] and the references therein.

Before state our results on \mathbb{X} , we first recall some results on the Euclidean space \mathbb{R}^n .

Proposition 1.1 (Proposition 5.1 in [21]). Let $\alpha > 0$, $1 < p < \infty$, $q > p - 1$, $\omega \in \mathbb{M}^+(\mathbb{R}^n)$ and $0 < r \leq \infty$, then the following quantities are equivalent.

$$\left\| W_{\alpha p, \frac{q}{q-p+1}}^r \omega \right\|_{L^1(d\omega)} = \int_{\mathbb{R}^n} \int_0^r \left[\frac{\omega(B_t(x))}{t^{n-\frac{\alpha p q}{q-p+1}}} \right]^{\frac{q-p+1}{p-1}} \frac{dt}{t} d\omega, \quad (1.3)$$

$$\left\| W_{\alpha, p}^r \omega \right\|_{L^q(dx)}^q = \int_{\mathbb{R}^n} \left\{ \int_0^r \left[\frac{\omega(B_t(x))}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right\}^q dx, \quad (1.4)$$

$$\left\| I_{\alpha p}^r \omega \right\|_{L^{\frac{q}{p-1}}(dx)}^{\frac{q}{p-1}} = \int_{\mathbb{R}^n} \left[\int_0^r \frac{\omega(B_t(x))}{t^{n-\alpha p}} \frac{dt}{t} \right]^{\frac{q}{p-1}} dx. \quad (1.5)$$

Remark. In Proposition 1.1, (1.3) \simeq (1.5) is the truncated version of Wolff's theorem.

Now we switch our attention to spaces of homogeneous type \mathbb{X} . We define the continuous truncated version of Wolff potentials on \mathbb{X} for $\omega \in \mathbb{M}^+(\mathbb{X})$ as

$$W_{\alpha, p}^r \omega(x) = \int_0^r \left[\frac{\omega(B_t(x))}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t}.$$

One can similarly define the continuous version $W_{\alpha, p} \omega = W_{\alpha, p}^\infty \omega$ and the discrete version $W_{\alpha, p}^D \omega$, using the dyadic construction on spaces of homogeneous types by Christ [5] and Sawyer and Wheeden [23]. The details of such a construction will be given in Section 2. One of our main theorems is as follows.

Theorem 1.2. Let $\alpha > 0$, $1 < p < \infty$, $q > p - 1$, $\omega \in \mathbb{M}^+(\mathbb{X})$ and $0 < r \leq \infty$, then

$$\left\| W_{\alpha, p}^r \omega \right\|_{L^q(d\mu)}^q = \int_{\mathbb{X}} \left\{ \int_0^r \left[\frac{\omega(B_t(x))}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right\}^q d\mu \quad (1.6)$$

$$\simeq \left\| I_{\alpha p}^r \omega \right\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} = \int_{\mathbb{X}} \left[\int_0^r \frac{\omega(B_t(x))}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \frac{dt}{t} \right]^{\frac{q}{p-1}} d\mu. \quad (1.7)$$

More precisely, there exists constant c and C which depend only on α, p, q and N such that

$$c \left\| W_{\alpha, p}^r \omega \right\|_{L^q(d\mu)}^q \leq \left\| I_{\alpha p}^r \omega \right\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \leq C \left\| W_{\alpha, p}^r \omega \right\|_{L^q(d\mu)}^q.$$

Remark. Wolff's theorem on \mathbb{X} , i.e., the parallel result of (1.3) \simeq (1.5) on \mathbb{X} was proved by Cascante and Ortega (Theorems 2.7 and 3.1 in [3]).

By a HLS inequality proved by Sawyer and Wheeden [23] (see also Sawyer, Wheeden and Zhao [24]) for Riesz potentials on \mathbb{X} (i.e., fractional integrals, and they proved weighted version therein), we derive the following HLS type inequality for Wolff potentials.

Theorem 1.3 (HLS type inequality for Wolff potentials). Let $\alpha > 0$, $1 < p < \infty$, $q > p - 1$ and $\alpha p < N$. If $f \in L^s(d\mu)$ for $s > 1$, then

$$\|W_{\alpha,p}(f)\|_{L^q(d\mu)} \leq C \|f\|_{L^s(d\mu)}^{\frac{1}{p-1}},$$

where $\frac{p-1}{q} = \frac{1}{s} - \frac{\alpha p}{N}$ and constant C depends only on α, p, q and N .

We apply the above inequality to study the Lane–Emden type integral system, that is,

$$\begin{cases} u = W_{\alpha,p}(v^{q_2}), \\ v = W_{\alpha,p}(u^{q_1}), \end{cases} \quad (1.8)$$

under the (critical) condition

$$\frac{p-1}{q_1+p-1} + \frac{p-1}{q_2+p-1} = \frac{N-\alpha p}{N}, \quad (1.9)$$

and when $u = v$ and $q_1 = q_2 = q$, (1.8) is reduced to

$$u = W_{\alpha,p}(u^q),$$

which is the Lane–Emden type integral equation, and deduces Examples 1, 2 and 3 above on \mathbb{X} , given special pairs of α and p . The study to the Lane–Emden integral system continues to attract many researchers. For a detailed history on this subject, we refer readers to the introduction of J. Dou and colleagues' work [8]. Our main regularity theorems state:

Theorem 1.4 (Integrability estimates). *Let $\alpha > 0$, $1 < p \leq 2$, $\alpha p < N$ and $q_1, q_2 > 1$, assume that (u, v) is a pair of positive solutions of (1.8) and (1.9) satisfying $(u, v) \in L^{q_1+p-1}(d\mu) \times L^{q_2+p-1}(d\mu)$, then $(u, v) \in L^{s_1}(d\mu) \times L^{s_2}(d\mu)$ for all s_1 and s_2 such that*

$$\frac{1}{s_1} \in \left(0, \frac{p}{q_1+p-1}\right) \cap \left(-\frac{1}{q_2+p-1} + \frac{1}{q_1+p-1}, \frac{p-1}{q_2+p-1} + \frac{1}{q_1+p-1}\right)$$

and

$$\frac{1}{s_2} \in \left(0, \frac{p}{q_2+p-1}\right) \cap \left(-\frac{1}{q_1+p-1} + \frac{1}{q_2+p-1}, \frac{p-1}{q_1+p-1} + \frac{1}{q_2+p-1}\right).$$

In particular, if $q_1 = q_2 = q$, then

$$\frac{1}{s_1}, \frac{1}{s_2} \in \left(0, \frac{p}{q+p-1}\right).$$

Theorem 1.5 (L^∞ estimates). *Under the same conditions in Theorem 1.4, u and v are both uniformly bounded on \mathbb{X} .*

In order to present Lipschitz continuity estimates, we need the notations of Lipschitz continuous space. Denote by $C^{0,\gamma}(d\mu)$ the class of all bounded functions which are Hölder continuous of order $\gamma \in (0, 1]$, a Banach space under the norm

$$\|f\|_{0,\gamma} = \|f\|_{L^\infty(d\mu)} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\gamma}.$$

Theorem 1.6 (Lipschitz continuity estimates). *Under the same conditions in Theorem 1.4, furthermore assume that $k_1 = 1$, then u and v are both Lipschitz continuous on \mathbb{X} , that is, $u, v \in C^{0,1}(d\mu)$.*

In Section 2, we proved Theorem 1.2 and Theorem 1.3. For the proof of Theorem 1.2, the dyadic decomposition on \mathbb{X} due to Christ [5] and Sawyer and Wheeden [23] plays an important role and the key is Lemma 2.3 which is also proved in Section 2. Theorem 1.3 follows easily from Theorem 1.2 and normed inequalities for fractional integrals in Sawyer and Wheeden [23] (see also [24]).

In Section 3, we prove Theorems 1.4 and 1.5 to verify uniform boundedness of positive solutions of (1.8) and (1.9). Lipschitz continuity estimates in Theorem 1.6 is proved in Section 4. The proofs of Theorems 1.4, 1.5 and 1.6 are more difficult than those on Euclidean spaces \mathbb{R}^n because of the complicated structures of spaces of homogeneous type \mathbb{X} .

In addition, we would like to point out that our results on \mathbb{X} imply the corresponding ones on Wolff potentials and regularity of solutions to the Lane–Emden type integral systems on stratified groups and Carnot–Carathéodory metric spaces defined by a family of Hörmander’s vector fields, which are both special examples of \mathbb{X} (see Folland–Stein [10], Nagel [19], Rothschild–Stein [22], Fefferman–Phong [9], Nagel–Stein–Wainger [20]).

We shall mention that avoiding misunderstanding of notations is a quite difficult job here, since a lot of indices are needed and they are used differently in the literature (with some very basic convention). We properly design the assignments: μ denotes the Borel measure (with doubling property) in the definition of space of homogeneous type \mathbb{X} , while $\omega \in \mathbb{M}^+(\mathbb{X})$ is a positive Borel measure on \mathbb{X} . α and p are assigned to represent the indices of Wolff potential, and we use c or C to represent positive constants depending only on α , p , q_1 , q_2 and N .

2. Proof of discrete version of Theorem 1.2

To introduce the discrete Wolff potentials on spaces of homogeneous type \mathbb{X} , we need the following construction given by Christ (Theorem 1.1 in [5]), which provides an analogue of the grid of Euclidean dyadic cubes on \mathbb{X} . A similar construction was given by Sawyer and Wheeden independently [23].

Lemma 2.1 (Dyadic cubes on \mathbb{X}). *For every integer $k \in \mathbb{Z}_+$, there exists a collection of open subsets $\{Q_\tau^k \subseteq \mathbb{X} : \tau \in I_k\}$, where I_k denotes some index set depending on k , and $c_1, c_2 > 0$ such that*

- (i) $\mu(\{X \setminus \cup Q_\tau^k\}) = 0$;
- (ii) If $l \geq k$, then for all $\tau' \in I_l$ and $\tau \in I_k$ either $Q_{\tau'}^l \subseteq Q_\tau^k$ or $Q_{\tau'}^l \cap Q_\tau^k = \emptyset$;
- (iii) If $l < k$, for each $\tau \in I_k$, there is a unique $\tau' \in I_l$ such that $Q_\tau^k \subseteq Q_{\tau'}^l$, $\text{diam}(Q_\tau^k) \leq c_1 2^{-k}$, and each Q_τ^k contains some ball $B(z_\tau^k, c_2 2^{-k})$.

With the settings defined above, we say that a cube $Q \subseteq \mathbb{X}$ is a dyadic cube if $Q = Q_\tau^k$ for some $k \in \mathbb{Z}_+$, $\tau \in I_k$ and $\text{diam}(Q) \sim 2^{-k}$. Denote by $Q_\tau^{k, \nu}$, $\nu = 1, 2, \dots, N(k, \tau)$, the set of all cubes $Q_\tau^{k+j} \subseteq Q_\tau^k$, where j is a fixed large positive integer. Then for $\alpha > 0$, $1 < p < \infty$ and $\omega \in M^+(\mathbb{X})$, we define the discrete Wolff potentials on \mathbb{X} by

$$W_{\alpha, p}^D \omega(x) = \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \left[\frac{\omega(Q)}{\mu(Q)^{1 - \frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \chi_Q(x),$$

and when $\alpha = \lambda/2$ and $p = 2$, the discrete Riesz follows as

$$I_\lambda^D \omega(x) = \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1 - \frac{\lambda}{N}}} \chi_Q(x).$$

In this section we shall prove the discrete version of Theorem 1.2, that is

Theorem 2.2 (Discrete version of Theorem 1.2). Let $\alpha > 0$, $1 < p < \infty$, $q > p - 1$ and $\omega \in \mathbb{M}^+(\mathbb{X})$, then

$$\|W_{\alpha,p}^D \omega\|_{L^q(d\mu)}^q = \int_{\mathbb{X}} \left\{ \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \left[\frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \chi_Q(x) \right\}^q d\mu \quad (2.1)$$

$$\simeq \|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} = \int_{\mathbb{X}} \left[\sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \chi_Q(x) \right]^{\frac{q}{p-1}} d\mu. \quad (2.2)$$

Theorem 1.2 follows evidently from its discrete counterpart, and a short proof of the HLS type inequality for Wolff potentials in Theorem 1.3 is as follows.

Proof of Theorem 1.3. From [23], one have for Riesz potentials

$$\|I_\lambda(f)\|_{L^q(d\mu)} \leq C \|f\|_{L^s(d\mu)},$$

where $1 < s \leq q < \infty$, $0 < \lambda < N$, $\frac{1}{q} = \frac{1}{s} - \frac{\lambda}{N}$ and $f \in L^s(d\mu)$. Thus by taking $\alpha > 0$, $1 < p < \infty$ and $\lambda = \alpha p$, we have

$$\|I_{\alpha p}(f)\|_{L^{\frac{q}{p-1}}(d\mu)} \leq C \|f\|_{L^s(d\mu)},$$

where

$$\frac{p-1}{q} = \frac{1}{s} - \frac{\alpha p}{N}.$$

Then by comparison of Wolff and Riesz potentials in Theorem 1.2, we arrive at

$$\|W_{\alpha,p}(f)\|_{L^q(d\mu)} \leq C \|I_{\alpha p}(f)\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{1}{p-1}} \leq C \|f\|_{L^s(d\mu)}^{\frac{1}{p-1}},$$

and Theorem 1.3 is verified. \square

In order to prove Theorem 2.2, discrete version of Theorem 1.2, we need the following equivalent recording of discrete Riesz potentials.

Lemma 2.3. Assume the same conditions in Theorem 2.2, define

$$\Lambda(\omega, \mu) := \int_{\mathbb{X}} \left[\sup_{k \in \mathbb{Z}_+, \text{diam}(Q) \sim 2^{-k}, x \in Q} \frac{1}{\mu(Q)^{1-\frac{\alpha p}{N}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{\frac{q}{p-1}} d\mu.$$

Then we have

$$\Lambda(\omega, \mu) \simeq \|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}}. \quad (2.3)$$

Proof of Lemma 2.3.

- $\Lambda(\omega, \mu) \lesssim \|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}}$

We need dyadic Hardy–Littlewood maximal function M^{dy} on \mathbb{X} , which is defined for all $\nu \in \mathbb{M}^+(\mathbb{X})$ by

$$M^{dy}(\nu)(x) = \sup_{x \in Q} \frac{\nu(Q)}{\mu(Q)}.$$

Since for $d\nu = |f|d\mu$, the operator M^{dy} is bounded on $L^{\frac{q}{p-1}}(\mu)$ for $q > p-1$, (See, e.g. Theorem 3.1(c) in [2].) we have

$$\begin{aligned} \Lambda(\omega, \mu) &= \int_{\mathbb{X}} \left[\sup_{k \in \mathbb{Z}_+, \text{diam}(Q) \sim 2^{-k}, x \in Q} \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q)^{1-\frac{\alpha p}{N}}} \right]^{\frac{q}{p-1}} d\mu \\ &\leq \int_{\mathbb{X}} M^{dy} \left[\sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \right]^{\frac{q}{p-1}} d\mu \\ &\leq C \|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}}, \end{aligned}$$

which finishes the proof $\Lambda(\omega, \mu) \lesssim \|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}}$.

- $\Lambda(\omega, \mu) \gtrsim \|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}}$

First we show that for all $x \in \mathbb{X}$,

$$\begin{aligned} &\left[\sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \omega(Q) \chi_Q(x) \right]^{\frac{q}{p-1}} \\ &\leq \frac{q}{p-1} \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \omega(Q) \chi_Q(x) \left[\sum_{Q' \subseteq Q} \omega(Q') \chi_{Q'}(x) \right]^{\frac{q}{p-1}-1} \end{aligned} \quad (2.4)$$

in three cases.

Case I: If

$$\sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \omega(Q) \chi_Q(x) \leq \infty.$$

Note that for a fixed $x \in \mathbb{X}$, the dyadic cubes containing x form a nested family of cubes. Hence using the elementary $b^t - a^t \leq t(b-a)b^{t-1}$ for $0 \leq a \leq b$ and $1 \leq t < \infty$, we have

$$\begin{aligned} &\left[\sum_{Q' \subseteq Q} \omega(Q') \chi_{Q'}(x) \right]^{\frac{q}{p-1}} - \left[\sum_{Q' \subsetneq Q} \omega(Q') \chi_{Q'}(x) \right]^{\frac{q}{p-1}} \\ &\leq \frac{q}{p-1} \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \omega(Q) \chi_Q(x) \left[\sum_{Q' \subseteq Q} \omega(Q') \chi_{Q'}(x) \right]^{\frac{q}{p-1}-1}. \end{aligned}$$

From this (2.4) follows by a telescoping sum argument, taking the sums of both sides over all dyadic cubes Q that contain x .

Case II: If

$$\sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \omega(Q) \chi_Q(x) = \infty,$$

but

$$\sum_{Q \subseteq Q_0} \omega(Q) \chi_Q(x) \leq \infty$$

for some (and hence every) dyadic cube Q_0 which contains x , then (2.4) follows by the same argument as in Case I taking the sums over all $Q \subseteq Q_0$ and then letting $\mu(Q_0) \rightarrow \infty$.

Case III: If

$$\sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \omega(Q) \chi_Q(x) = \infty,$$

but

$$\sum_{Q \subseteq Q_0} \omega(Q) \chi_Q(x) = \infty$$

for some Q_0 , then both side of (2.3) are obviously infinite. This completes the proof of (2.4).

Next we use induction on $\frac{q}{p-1} > 1$ to prove

$$\begin{aligned} & \|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \\ & \leq C \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \left[\frac{1}{\mu(Q)^{1-\frac{\alpha p}{N}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{\frac{q}{p-1}-1}, \end{aligned} \quad (2.5)$$

where C only depends on \mathbb{X} , p and q .

Step 1: To verify (2.5) is true if $1 < \frac{q}{p-1} \leq 2$. By (2.4),

$$\begin{aligned} & \|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \\ & = \int_{\mathbb{X}} \left[\sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \chi_Q(x) \right]^{\frac{q}{p-1}} d\mu \\ & \leq \frac{q}{p-1} \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \int_Q \left[\sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \right]^{\frac{q}{p-1}-1} d\mu, \end{aligned}$$

then by Hölder's inequality with exponents $\frac{p-1}{q-p+1}$ and $\frac{p-1}{2p-2-q}$, we have

$$\begin{aligned} & \int_Q \left[\sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \right]^{\frac{q}{p-1}-1} d\mu \\ & \leq \left[\int_Q \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) d\mu \right]^{\frac{q-p+1}{p-1}} \mu(Q)^{\frac{2p-2-q}{p-1}} \end{aligned}$$

$$\begin{aligned}
&\leq \left[\int_Q \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) d\mu \right]^{\frac{q}{p-1}-1} \left[\frac{1}{\mu(Q)} \right]^{\frac{q}{p-1}-1} \mu(Q) \\
&= \mu(Q) \left[\frac{1}{\mu(Q)} \int_Q \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) d\mu \right]^{\frac{q}{p-1}-1} \\
&\leq \mu(Q) \left[\frac{1}{\mu(Q)} \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \right]^{\frac{q}{p-1}-1} \\
&\leq \mu(Q) \left[\frac{1}{\mu(Q)^{1-\frac{\alpha p}{N}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{\frac{q}{p-1}-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \\
&\leq \frac{q}{p-1} \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \int_Q \left[\sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \right]^{\frac{q}{p-1}-1} d\mu \\
&\leq \frac{q}{p-1} \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \left[\frac{1}{\mu(Q)^{1-\frac{\alpha p}{N}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{\frac{q}{p-1}-1},
\end{aligned}$$

which means (2.5) holds for $1 < \frac{q}{p-1} \leq 2$.

Step 2: Given an integer $m \geq 2$, we assume that (2.5) holds for any $\frac{q}{p-1} \leq m$, then and we show that it also holds for $\frac{q}{p-1} \leq m+1$.

By (2.4) and the induction hypothesis, we have

$$\begin{aligned}
&\|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \\
&\leq \frac{q}{p-1} \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \int_Q \left[\sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \right]^{\frac{q}{p-1}-1} d\mu \\
&\leq C \frac{q}{p-1} \left(\frac{q}{p-1} - 1 \right) \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \left[\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right]^{\frac{q}{p-1}-2} \\
&= C \frac{q(q-p+1)}{(p-1)^2} \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \left[\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right]^{\frac{q}{p-1}-2} \sum_{Q' \subseteq Q} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \\
&\leq C \frac{q(q-p+1)}{(p-1)^2} \int_{\mathbb{X}} \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \left[\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right]^{\frac{q}{p-1}-2} \\
&\quad \times \left[\sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \chi_Q(x) \right] d\mu.
\end{aligned}$$

Note that $\frac{q}{p-1} - 1 > m - 1 \geq 2$, by Hölder's inequality with exponents $\frac{q}{p-1} - 1 = \frac{q-p+1}{p-1}$ and $\frac{q-p+1}{q-2p+2}$, we have

$$\begin{aligned}
 & \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \left[\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right]^{\frac{q}{p-1}-2} \\
 &= \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \left[\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \right]^{\frac{p-1}{q-p+1}} \left[\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \right]^{\frac{q-2p+2}{q-p+1}} \\
 & \quad \times \left[\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right]^{\frac{q}{p-1}-2} \\
 &\leq \left[\sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \left(\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right)^{\frac{q}{p-1}-1} \right]^{\frac{q-2p+2}{q-p+1}} \\
 & \quad \times \left[\sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \right]^{\frac{p-1}{q-p+1}}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \\
 &\leq C \frac{q(q-p+1)}{(p-1)^2} \int_{\mathbb{X}} \left[\sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \chi_Q(x) \right] \left[\sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \right]^{\frac{p-1}{q-p+1}} \\
 & \quad \times \left[\sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \left(\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right)^{\frac{q}{p-1}-1} \right]^{\frac{q-2p+2}{q-p+1}} d\mu \\
 &\leq C \frac{q(q-p+1)}{(p-1)^2} \int_{\mathbb{X}} \left[\sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \right]^{\frac{q}{q-p+1}} \\
 & \quad \times \left[\sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \left(\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right)^{\frac{q}{p-1}-1} \right]^{\frac{q-2p+2}{q-p+1}} d\mu.
 \end{aligned}$$

By using Hölder's inequality with exponents $\frac{q-p+1}{p-1}$ and $\frac{q-p+1}{q-2p+2}$ again, we have

$$\begin{aligned}
 & \|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \\
 &\leq C \frac{q(q-p+1)}{(p-1)^2} \left[\int_{\mathbb{X}} \left(\sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \right)^{\frac{q}{p-1}} d\mu \right]^{\frac{p-1}{q-p+1}}
 \end{aligned}$$

$$\begin{aligned}
& \times \left[\int_{\mathbb{X}} \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \left(\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right)^{\frac{q}{p-1}-1} d\mu \right]^{\frac{q-2p+2}{q-p+1}} \\
& = C \frac{q(q-p+1)}{(p-1)^2} \left[\|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \right]^{\frac{p-1}{q-p+1}} \\
& \times \left[\int_{\mathbb{X}} \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \left(\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right)^{\frac{q}{p-1}-1} d\mu \right]^{\frac{q-2p+2}{q-p+1}} \\
& \leq C \frac{q(q-p+1)}{(p-1)^2} \left[\|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \right]^{\frac{p-1}{q-p+1}} \\
& \times \left[\sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \left(\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right)^{\frac{q}{p-1}-1} \right]^{\frac{q-2p+2}{q-p+1}}.
\end{aligned}$$

From the above inequality it follows that (2.5) holds for $m < \frac{q}{p-1} \leq m+1$, where C only depends on \mathbb{X} , p and q , and then (2.5) is verified for every $1 < \frac{q}{p-1} < \infty$.

With the help of (2.5), we compute

$$\begin{aligned}
& \|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \\
& \leq C \int_{\mathbb{X}} \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \chi_Q(x) \left[\frac{1}{\mu(Q)^{1-\frac{\alpha p}{N}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{\frac{q}{p-1}-1} d\mu \\
& \leq C \int_{\mathbb{X}} \left[\sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \chi_Q(x) \right] \left[\sup_{k, \text{diam}(Q) \sim 2^{-k}, x \in Q} \frac{1}{\mu(Q)^{1-\frac{\alpha p}{N}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{\frac{q}{p-1}-1} d\mu \\
& \leq C \left(\|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \right)^{\frac{p-1}{q}} [\Lambda(\omega, \mu)]^{1-\frac{p-1}{q}},
\end{aligned}$$

where the last estimate we have used Hölder's inequality with exponents $\frac{q}{p-1}$ and $\frac{q}{q-p+1}$. Thus

$$\|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \leq C^{\frac{q}{q-p+1}} \Lambda(\omega, \mu) \lesssim \Lambda(\omega, \mu),$$

and the proof of Lemma 2.3 is completed. \square

Theorem 2.2 can be proved easily with the assistance of Lemma 2.3.

Proof of Theorem 2.2.

$$\bullet \quad \|W_{\alpha, p}^D \omega\|_{L^q(d\mu)}^q \gtrsim \|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}}$$

It becomes obvious once one notices that

$$\|I_{\alpha p}^D \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \simeq \Lambda(\omega, \mu)$$

$$\begin{aligned}
&= \int_{\mathbb{X}} \left[\sup_{k \in \mathbb{Z}_+, \text{diam}(Q) \sim 2^{-k}, x \in Q} \frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{\frac{q}{p-1}} d\mu \\
&\leq \int_{\mathbb{X}} \left\{ \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \left[\frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \chi_Q(x) \right\}^q d\mu \\
&= \|W_{\alpha,p}^D \omega\|_{L^q(d\mu)}^q.
\end{aligned}$$

We omit the proof of the opposite direction since it is quite similar to the case on \mathbb{R}^n that was given in [21]. Thus we have completed the proof of Theorem 2.2. \square

3. Integrability and L^∞ estimates

In this section, we prove regularity estimates in Theorems 1.4 and 1.5. Let us begin with setting the frame, that is, suppose V is a topological vector space with two extended norms,

$$\|\cdot\|_X, \|\cdot\|_Y : V \rightarrow [0, \infty],$$

let $X := \{v \in V : \|v\|_X < \infty\}$ and $Y := \{v \in V : \|v\|_Y < \infty\}$. The operator $T : X \rightarrow Y$ is said to be contracting if

$$\|Tf - Th\|_Y \leq \eta \|f - h\|_X,$$

$\forall f, h \in X$ and some $0 < \eta < 1$. And T is said to be shrinking if

$$\|Tf\|_Y \leq \theta \|f\|_X,$$

$\forall f \in X$ and some $0 < \theta < 1$.

Remark. It is obvious that for a linear operator T , these two conditions above are equivalent. Thus the following theorem is also true for linear shrinking operators.

To prove Theorem 1.4, we need the following regularity lifting lemma (an slight different earlier version was introduced in [4]).

Lemma 3.1 (Regularity lifting by contracting operators [17]). *Let T be a contracting operator from X to itself and from Y to itself, and assume that X, Y are both complete. If $f \in X$, and there exists $g \in Z := X \cap Y$ such that $f = Tf + g$ in X , then $f \in Z$.*

From now, without causing any confusion, we simply denote $\|\cdot\|_{L^q(d\mu)}$ by $\|\cdot\|_q$, and $L^q(d\mu)$ by L^q .

Proof of Theorem 1.4. For a fixed real number $a > 0$, define

$$v_a(u) = \begin{cases} v(x) & \text{if } |v(x)| > a, \text{ or } |x| > a, \\ 0 & \text{otherwise.} \end{cases}$$

Let $v_b(u) = v(u) - v_a(u)$, and similarly we define u_a and u_b , then v_b and u_b are uniformly bounded by a in $B_a(0)$ obviously. It is evident that $v_a \cdot v_b = 0$ and $v^r = (v_a + v_b)^r = v_a^r + v_b^r$ for all $r > 0$. Define the linear operator T_1 ,

$$T_1 h(x) = \int_0^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{2-p}{p-1}} \left[\frac{\int_{B_t(x)} v_a^{q_2-1} h d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \frac{dt}{t}.$$

Since u satisfies (1.6), $u = W_{\alpha,p}(v^{q_2})$, we have

$$\begin{aligned} u(x) &= W_{\alpha,p}(v^{q_2})(x) \\ &= \int_0^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{2-p}{p-1}} \left[\frac{\int_{B_t(x)} (v_a^{q_2} + v_b^{q_2}) d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \frac{dt}{t} \\ &= T_1 v(x) + \int_0^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{2-p}{p-1}} \left[\frac{\int_{B_t(x)} v_b^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \frac{dt}{t} \\ &:= T_1 v(x) + F(x), \end{aligned}$$

and thus $u = T_1 v + F$, in which

$$F(x) = \int_0^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{2-p}{p-1}} \left[\frac{\int_{B_t(x)} v_b^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \frac{dt}{t}.$$

Similarly, we define

$$T_2 h(x) = \int_0^\infty \left[\frac{\int_{B_t(x)} u^{q_1} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{2-p}{p-1}} \left[\frac{\int_{B_t(x)} u_a^{q_1-1} h d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \frac{dt}{t}$$

and

$$G(x) = \int_0^\infty \left[\frac{\int_{B_t(x)} u^{q_1} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{2-p}{p-1}} \left[\frac{\int_{B_t(x)} u_b^{q_1} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \frac{dt}{t}.$$

Then we have $v = T_2 u + G$. Define the operator $T(f, g) = (T_1 g, T_2 f)$, equip the product space $L^{q_1+p-1} \times L^{q_2+p-1}$ with norm $\|(f, g)\|_{q_1+p-1, q_2+p-1} = \|f\|_{q_1+p-1} + \|g\|_{q_2+p-1}$, and $L^{s_1} \times L^{s_2}$ with norm $\|(f, g)\|_{s_1, s_2} = \|f\|_{s_1} + \|g\|_{s_2}$. It is easy to see they are both complete under these norms respectively.

Thus we immediately observe that (u, v) solves the equation $(f, g) = T(f, g) + (F, G)$. In order to apply regularity lifting by contracting operators (Lemma 3.1), we fix the indices s_1 and s_2 satisfying

$$\frac{1}{s_1} - \frac{1}{s_2} = \frac{1}{q_1 + p - 1} - \frac{1}{q_2 + p - 1}. \quad (3.1)$$

Note that the interval conditions in Theorem 1.4 guarantee the existence of such pairs (s_1, s_2) and T is linear. To arrive at the conclusion that $(f, g) \in L^{s_1} \times L^{s_2}$, we only need to verify the following conditions, for sufficiently large a .

- (1) T is shrinking from $L^{q_1+p-1} \times L^{q_2+p-1}$ to itself.
- (2) T is shrinking from $L^{s_1} \times L^{s_2}$ to itself.
- (3) $(F, G) \in L^{q_1+p-1} \times L^{q_2+p-1} \cap L^{s_1} \times L^{s_2}$, i.e., $F \in L^{q_1+p-1} \cap L^{s_1}$ and $G \in L^{q_2+p-1} \cap L^{s_2}$.

(1). T is shrinking from $L^{q_1+p-1} \times L^{q_2+p-1}$ to itself.

First, we show that $\|T_1 h\|_{q_1+p-1} \leq \frac{1}{2}\|h\|_{q_2+p-1}$ for all $h \in L^{q_2+p-1}$. By choosing $\frac{1}{2-p}$ and $\frac{1}{p-1}$ as two conjugate indices in Hölder's inequality, we have

$$\begin{aligned} |T_1 h(x)| &= \left| \int_0^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{2-p}{p-1}} \left[\frac{\int_{B_t(x)} v_a^{q_2-1} h d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \frac{dt}{t} \right| \\ &\leq \left\{ \int_0^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right\}^{2-p} \left\{ \int_0^\infty \left[\frac{\int_{B_t(x)} v_a^{q_2-1} |h| d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right\}^{p-1} \\ &= [W_{\alpha,p}(v^{q_2})(x)]^{2-p} [W_{\alpha,p}(v_a^{q_2-1}|h|)(x)]^{p-1} \\ &= u^{2-p}(x) [W_{\alpha,p}(v_a^{q_2-1}|h|)(x)]^{p-1}. \end{aligned}$$

Thus, applying Hölder's inequality again,

$$\begin{aligned} \|T_1 h\|_{q_1+p-1} &\leq \|u^{2-p}\|_{\frac{q_1+p-1}{2-p}} \| [W_{\alpha,p}(v_a^{q_2-1}|h|)]^{p-1} \|_s \\ &= \|u\|_{q_1+p-1}^{2-p} \|W_{\alpha,p}(v_a^{q_2-1}|h|)\|_{s(p-1)}^{p-1} \\ &\leq C \|u\|_{q_1+p-1}^{2-p} \|v_a^{q_2-1}|h|\|_{\frac{q_2+p-1}{q_2}} \\ &\leq C \|u\|_{q_1+p-1}^{2-p} \|v_a^{q_2-1}\|_{\frac{q_2+p-1}{q_2-1}} \|h\|_{q_2+p-1} \\ &= C \|u\|_{q_1+p-1}^{2-p} \|v_a\|_{q_2+p-1}^{q_2-1} \|h\|_{q_2+p-1}, \end{aligned}$$

in which we used HLS type inequality for Wolff potentials in Theorem 1.3 and have

$$\frac{1}{q_1+p-1} = \frac{2-p}{q_1+p-1} + \frac{1}{s}$$

and

$$\frac{1}{s} = \frac{q_2}{q_2+p-1} - \frac{\alpha p}{N},$$

which is ensured by the condition (1.9). Thus we choose a sufficiently large that

$$C \|u\|_{q_1+p-1}^{2-p} \|v_a\|_{q_2+p-1}^{q_2-1} \leq \frac{1}{2},$$

since $u \in L^{q_1+p-1}$ and $v \in L^{q_2+p-1}$. Then $\|T_1 h\|_{q_1+p-1} \leq \frac{1}{2}\|h\|_{q_2+p-1}$ is verified. Similarly we can prove that $\|T_2 h\|_{q_2+p-1} \leq \frac{1}{2}\|h\|_{q_1+p-1}$ for all $h \in L^{q_1+p-1}$ by choosing a large enough. Combining them together, we have no difficulty to get

$$\begin{aligned} &\|T(f, g)\|_{q_1+p-1, q_2+p-1} \\ &= \|T_1 g\|_{q_1+p-1} + \|T_2 f\|_{q_2+p-1} \\ &\leq \frac{1}{2}(\|g\|_{q_2+p-1} + \|f\|_{q_1+p-1}) \\ &= \frac{1}{2}\|(f, g)\|_{q_1+p-1, q_2+p-1}, \end{aligned}$$

and this shows that T is shrinking from $L^{q_1+p-1} \times L^{q_2+p-1}$ to itself.

(2). T is shrinking from $L^{s_1} \times L^{s_2}$ to itself.

We use the same tool as we did in (1), that is, HLS type inequality for Wolff potentials in Theorem 1.3 with assistance of Hölder's inequality, by properly choosing the indices. Here, we prove that $\|T_2 h\|_{s_2} \leq \frac{1}{2} \|h\|_{s_1}$ first,

$$\begin{aligned} & \|T_2 h\|_{s_2} \\ & \leq \|v^{2-p}\|_{\frac{q_2+p-1}{2-p}} \left\| \left[W_{\alpha,p}(u_a^{q_1-1}|h|) \right]^{p-1} \right\|_{t_1} \\ & = \|v\|_{q_2+p-1}^{2-p} \|W_{\alpha,p}(u_a^{q_1-1}|h|)\|_{t_1(p-1)}^{p-1} \\ & \leq C \|v\|_{q_2+p-1}^{2-p} \|u_a^{q_1-1}|h|\|_{t_2} \\ & \leq C \|v\|_{q_2+p-1}^{2-p} \|u_a^{q_1-1}\|_{\frac{q_1+p-1}{q_1-1}} \|h\|_{s_1} \\ & = C \|v\|_{q_2+p-1}^{2-p} \|u_a\|_{q_1+p-1}^{q_1-1} \|h\|_{s_1}, \end{aligned}$$

in which we choose a sufficiently large such that

$$C \|v\|_{q_2+p-1}^{2-p} \|u_a\|_{q_1+p-1}^{q_1-1} \leq \frac{1}{2},$$

since $v \in L^{q_2+p-1}$ and $u \in L^{q_1+p-1}$. Thus, $\|T_2 h\|_{s_2} \leq \frac{1}{2} \|h\|_{s_1}$ for all $h \in L^{s_1}$. The indices s_1 , s_2 , t_1 and t_2 above satisfy

$$\begin{aligned} \frac{1}{s_2} &= \frac{2-p}{q_2+p-1} + \frac{1}{t_1}, \\ \frac{1}{t_2} &= \frac{q_1-1}{q_1+p-1} + \frac{1}{s_1} \end{aligned}$$

and by (3.1) and (1.9),

$$\begin{aligned} \frac{1}{t_1} &= \frac{1}{s_2} - \frac{2-p}{q_2+p-1} \\ &= \frac{1}{s_1} - \frac{1}{q_1+p-1} + \frac{1}{q_2+p-1} - \frac{2-p}{q_2+p-1} \\ &= \frac{1}{s_1} - \frac{1}{q_1+p-1} + \frac{p-1}{q_2+p-1} \\ &= \frac{1}{s_1} - \frac{1}{q_1+p-1} + \frac{N-\alpha p}{N} - \frac{p-1}{q_1+p-1} \\ &= \frac{1}{s_1} + \frac{q_1-1}{q_1+p-1} - \frac{\alpha p}{N} \\ &= \frac{1}{t_2} - \frac{\alpha p}{N}, \end{aligned}$$

which ensures us to use HLS type inequality for Wolff potentials in Theorem 1.3, and we need

$$\frac{1}{t_2} = \frac{q_1-1}{q_1+p-1} + \frac{1}{s_1} < 1,$$

that is

$$\frac{1}{s_1} < \frac{p}{q_1+p-1}.$$

Similarly we estimate T_1 for $h \in L^{s_2}$ if

$$\frac{1}{s_2} < \frac{p}{q_2 + p - 1},$$

and easily pass the results to $L^{s_1} \times L^{s_2}$, i.e.,

$$\|T(f, g)\|_{s_1, s_2} \leq \frac{1}{2} \|(f, g)\|_{s_1, s_2},$$

which shows that T is shrinking from $L^{s_1} \times L^{s_2}$ to itself.

(3). $F \in L^{q_1+p-1} \cap L^{s_1}$ and $G \in L^{q_2+p-1} \cap L^{s_2}$.

We only estimate F , one notices that v_b is uniformly bounded by a in $B_a(0)$, thus $v_b \in L^{q_2+p-1} \cap L^{s_2}$. Because T_1 is bounded from L^{q_2+p-1} to L^{q_1+p-1} by (1), then $F = T_1 v_b \in L^{q_1+p-1}$. Because T_1 is bounded from L^{s_2} to L^{s_1} by (2), then $F = T_1 v_b \in L^{s_1}$, and we conclude $F \in L^{q_1+p-1} \cap L^{s_1}$.

Applying regularity lifting Lemma 3.1 we finish the proof of Theorem 1.4. \square

Now we are able to prove L^∞ estimate.

Proof of Theorem 1.5. It is sufficient to show for u , then the estimate of v can be proved similarly. For any $x \in \mathbb{X}$, we divide

$$\begin{aligned} u(x) &= W_{\alpha, p}(v^{q_2})(x) \\ &= \int_0^1 \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} + \int_1^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \\ &:= I_1(x) + I_2(x), \end{aligned}$$

in which the first integral

$$\begin{aligned} I_1(x) &= \int_0^1 \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \int_0^1 \left[\frac{\left(\int_{B_t(x)} 1^{s'} d\mu \right)^{\frac{1}{s'}} \left(\int_{B_t(x)} v^{q_2 s} d\mu \right)^{\frac{1}{s}}}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \|v\|_{q_2 s}^{\frac{q_2}{p-1}} \int_0^1 [\mu(B_t(x))]^{\frac{1}{p-1}(\frac{1}{s'}-1+\frac{\alpha p}{N})} \frac{dt}{t} \\ &\lesssim \|v\|_{q_2 s}^{\frac{q_2}{p-1}} \int_0^1 t^{\frac{N}{p-1}(\frac{1}{s'}-\frac{N-\alpha p}{N})-1} dt \\ &\leq C_1, \end{aligned}$$

as we choose s such that $\|v\|_{q_2 s} < \infty$ and $\frac{1}{s'} - \frac{N-\alpha p}{N} > 0$, that is, $\frac{1}{q_2 s} < \frac{\alpha p}{q_2 N}$. By integrability estimate of v in Theorem 1.4, we only need to check

$$\frac{\alpha p}{q_2 N} > -\frac{1}{q_1 + p - 1} + \frac{1}{q_2 + p - 1},$$

this is plain by a simple computation.

We notice that C_1 is independent of x . To estimate the second integral I_2 , given $\delta > 0$, for all $y \in \mathbb{X}$ such that $d(x, y) \leq \delta$, thus we have $d(z, y) \leq k_1(d(z, x) + d(x, y)) \leq k_1(t + \delta)$ for all $z \in B_t(x)$. (Recall the definition of quasi-metric on spaces of homogeneous types.) We compute

$$\begin{aligned}
 I_2(x) &= \int_1^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \\
 &\leq \int_1^\infty \left[\frac{\int_{B_{k_1(t+\delta)}(y)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \\
 &\leq \int_1^\infty \left[\frac{\int_{B_{k_1(t+\delta)}(y)} v^{q_2} d\mu}{\mu(B_{k_1(t+\delta)}(y))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \left[\frac{\mu(B_{k_1(t+\delta)}(y))}{\mu(B_t(x))} \right]^{\frac{1}{p-1}(1-\frac{\alpha p}{N})} \frac{dt}{t} \\
 &\leq \int_1^\infty \left[\frac{\int_{B_{k_1(t+\delta)}(y)} v^{q_2} d\mu}{\mu(B_{k_1(t+\delta)}(y))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \left[\frac{k_1(t+\delta)}{t} \right]^{\frac{N-\alpha p}{p-1}+1} \frac{dt}{k_1(t+\delta)} \\
 &\leq k_1^{\frac{N-\alpha p}{p-1}+1} (1+\delta)^{\frac{N-\alpha p}{p-1}+1} \int_{k_1(1+\delta)}^\infty \left[\frac{\int_{B_{k_1(t+\delta)}(y)} v^{q_2} d\mu}{\mu(B_{k_1(t+\delta)}(y))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{k_1(t+\delta)} \\
 &\leq k_1^{\frac{N-\alpha p}{p-1}} (1+\delta)^{\frac{N-\alpha p}{p-1}+1} \int_{k_1(1+\delta)}^\infty \left[\frac{\int_{B_t(y)} v^{q_2} d\mu}{\mu(B_t(y))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \\
 &\leq C_2 W_{\alpha,p}(v^{q_2})(y) \\
 &= C_2 u(y),
 \end{aligned}$$

in which C_2 is independent of x and y . Thus, combining I_1 and I_2 , we have

$$u(x) \leq C_1 + C_2 u(y),$$

for any x and y such that $d(x, y) \leq \delta$. s -th powering and integrating both sides,

$$\int_{B_\delta(x)} u^s(x) d\mu \leq \int_{B_\delta(x)} (C_1 + C_2 u(y))^s d\mu \lesssim C \|u\|_s^s$$

by choosing $s > 1$ in the integrability interval such that $\|u\|_s < \infty$. Then we finish L^∞ estimate by noticing that C is independent of x . \square

4. Lipschitz continuity estimates

In this section, we prove Lipschitz continuity estimates in Theorem 1.6, and a modified regularity lifting method is needed, that is, regularity lifting by combinations of contracting and shrinking operators. We adopt all the settings in Section 3, and introduce the following “XY-pair” that appears in the modified version of regularity lifting.

Definition (“XY-pair”). Two normed subspaces X and Y are called an “XY-pair”, if whenever the sequence $\{u_n\} \subseteq X$ with $u_n \rightarrow u$ in X and $\|u_n\|_Y \leq C$ will imply $u \in Y$.

Remark. There are some “XY-pairs” of important spaces, and the pair we use here is L^∞ and $C^{0,1}$.

Lemma 4.1 (Regularity lifting by combinations of contracting and shrinking operators). *Let X and Y be an “XY-pair”, and assume that X, Y are both complete. Let \mathcal{X} and \mathcal{Y} be closed subsets of X and Y respectively, and T be an operator, which is contracting from \mathcal{X} to X and shrinking from \mathcal{Y} to Y .*

Define $Sw = Tw + g$ for some $g \in \mathcal{X} \cap \mathcal{Y}$, and assume that $S : \mathcal{X} \cap \mathcal{Y} \rightarrow \mathcal{X} \cap \mathcal{Y}$. Then there exists a unique solution u of the equation $w = Tw + g$ in \mathcal{X} , and $u \in Y$.

One can find the detailed proof of Lemma 4.1 in [17]. Next we apply the above regularity lifting to prove Lipschitz continuity estimates in Theorem 1.6. We simply denote $C^{0,1}(d\mu)$ by $C^{0,1}$ without causing any confusion. We further assume $k_1 = 1$, that is, d is a metric on \mathbb{X} , and the reasons why we need this condition are embedded in the following proof.

Proof of Theorem 1.6. By the L^∞ estimates in Theorem 1.5, we have $u, v \in L^\infty$. Write

$$M = 2\|u\|_\infty + 2\|v\|_\infty,$$

and let

$$\mathcal{X} = \{w \in L^\infty \mid \|w\|_\infty \leq M\} \text{ and } \mathcal{Y} = \{w \in C^{0,1} \mid \|w\|_\infty \leq M\}.$$

Note that $\mathcal{X} \subseteq L^\infty$ and $\mathcal{Y} \subseteq C^{0,1}$ are both closed, and

$$\begin{aligned} u(x) &= W_{\alpha,p}(v^{q_2})(x) \\ &= \int_0^m \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} + \int_m^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t}, \end{aligned}$$

for a fixed real number $m > 0$. Define

$$T_1 w(x) = \int_0^m \left[\frac{\int_{B_t(x)} w^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t}$$

and

$$g_1(x) = \int_m^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t}.$$

Then obviously, $u = T_1 v + g_1$. Similarly, we define,

$$T_2 w(x) = \int_0^m \left[\frac{\int_{B_t(x)} w^{q_1} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t}$$

and

$$g_2(x) = \int_m^\infty \left[\frac{\int_{B_t(x)} u^{q_1} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t}.$$

Then we have $v = T_2u + g_2$. Define the operator $T(w_1, w_2) = (T_1w_2, T_2w_1)$. Equip the product space $L^\infty \times L^\infty$ (and its subspace $\mathcal{X} \times \mathcal{X}$) with norm $\|(w_1, w_2)\|_\infty = \|w_1\|_\infty + \|w_2\|_\infty$, and similarly equip the product space $C^{0,1} \times C^{0,1}$ (and its subspace $\mathcal{Y} \times \mathcal{Y}$) with norm $\|(w_1, w_2)\|_{0,1} = \|w_1\|_{0,1} + \|w_2\|_{0,1}$. It is easy to see that $L^\infty \times L^\infty$ and $C^{0,1} \times C^{0,1}$ are an “XY-pair”, and $\mathcal{X} \times \mathcal{X}$ and $\mathcal{Y} \times \mathcal{Y}$ are both closed.

Thus we can immediately observe that (u, v) solves the equation $(w_1, w_2) = T(w_1, w_2) + (g_1, g_2)$. Let $S(w_1, w_2) = T(w_1, w_2) + (g_1, g_2)$, to apply Lemma 4.1, and then arrive at the conclusion that $(u, v) \in C^{0,1} \times C^{0,1}$, we need to verify the following conditions, for sufficiently small m .

- (1) T is contracting from $\mathcal{X} \times \mathcal{X}$ to $L^\infty \times L^\infty$.
- (2) T is shrinking from $\mathcal{Y} \times \mathcal{Y}$ to $C^{0,1} \times C^{0,1}$.
- (3) $(g_1, g_2) \in (\mathcal{X} \times \mathcal{X}) \cap (\mathcal{Y} \times \mathcal{Y})$, i.e., $g_1, g_2 \in \mathcal{X} \cap \mathcal{Y}$.
- (4) $S : (\mathcal{X} \times \mathcal{X}) \cap (\mathcal{Y} \times \mathcal{Y}) \rightarrow (\mathcal{X} \times \mathcal{X}) \cap (\mathcal{Y} \times \mathcal{Y})$.

(1). To show T is contracting from $\mathcal{X} \times \mathcal{X}$ to $L^\infty \times L^\infty$. Notice that $\forall (f_1, f_2), (h_1, h_2) \in \mathcal{X} \times \mathcal{X}$,

$$\begin{aligned} T(f_1, f_2) - T(h_1, h_2) &= (T_1f_2, T_2f_1) - (T_1h_2, T_2h_1) \\ &= (T_1f_2 - T_1h_2, T_2f_1 - T_2h_1). \end{aligned} \quad (4.1)$$

For all $x \in \mathbb{X}$,

$$\begin{aligned} &|T_1f_2(x) - T_1h_2(x)| \\ &= \left| \int_0^m \left[\frac{\int_{B_t(x)} f_2^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} - \int_0^m \left[\frac{\int_{B_t(x)} h_2^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right| \\ &\leq \int_0^m \left[\frac{\int_{B_t(x)} |f_2^{q_2} - h_2^{q_2}| d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \cdot \left[\frac{\int_{B_t(x)} (f_2^{q_2} + h_2^{q_2}) d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{2-p}{p-1}} \frac{dt}{t} \\ &\leq \int_0^m \left[\frac{\int_{B_t(x)} q_2 |\xi^{q_2-1}(y)| |f_2(y) - h_2(y)| d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \cdot \left[\frac{\int_{B_t(x)} (f_2^{q_2} + h_2^{q_2}) d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{2-p}{p-1}} \frac{dt}{t} \\ &\leq q_2 2^{q_2} M^{\frac{q_2}{p-1}-1} \|f_2 - h_2\|_\infty \int_0^m \mu(B_t(x))^{\frac{\alpha p}{N(p-1)}} \frac{dt}{t} \\ &\leq CM^{\frac{q_2}{p-1}-1} m^{\frac{\alpha p}{p-1}} \|f_2 - h_2\|_\infty \\ &\leq \frac{1}{4} \|f_2 - h_2\|_\infty, \end{aligned}$$

by choosing m small enough. Here we applied the Mean Value Theorem for x^p , with $\xi(y)$ between $f_2(y)$ and $h_2(y)$, and $\mu(B_t(x)) = t^N$ by the definition of homogeneous dimension in Section 1.

Similarly, choose m small such that

$$|T_2f_1(x) - T_2h_1(x)| \leq \frac{1}{4} \|f_1 - h_1\|_\infty.$$

Therefore

$$\begin{aligned}
|T(f_1, f_2)(x) - T(h_1, h_2)(x)| &= |(T_1 f_2(x) - T_1 h_2(x), T_2 f_1(x) - T_2 h_1(x))| \\
&\leq \frac{1}{4}(\|f_2 - h_2\|_\infty + \|f_1 - h_1\|_\infty) \\
&\leq \frac{1}{4}\|(f_1, f_2) - (h_1, h_2)\|_\infty,
\end{aligned}$$

for all $(f_1, f_2), (h_1, h_2) \in \mathcal{X} \times \mathcal{X}$, and $x \in \mathbb{X}$, thus T is contracting from $\mathcal{X} \times \mathcal{X}$ to $L^\infty \times L^\infty$.

(2). To show T is shrinking from $\mathcal{Y} \times \mathcal{Y}$ to $C^{0,1} \times C^{0,1}$. $\forall (f_1, f_2) \in \mathcal{Y} \times \mathcal{Y}$ and $x, y \in \mathbb{X}$, by (4.1), we shall consider T_1 first.

$$\begin{aligned}
&|T_1 f_2(x) - T_1 f_2(z)| \tag{4.2} \\
&= \left| \int_0^m \left[\frac{\int_{B_t(x)} f_2^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} - \int_0^m \left[\frac{\int_{B_t(y)} f_2^{q_2} d\mu}{\mu(B_t(y))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right| \\
&\leq \int_0^m \left[\frac{|\int_{B_t(x)} f_2^{q_2} d\mu - \int_{B_t(y)} f_2^{q_2} d\mu|}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \cdot \left[\frac{\int_{B_t(x)} f_2^{q_2} d\mu + \int_{B_t(y)} f_2^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{2-p}{p-1}} \frac{dt}{t}.
\end{aligned}$$

To estimate

$$\left| \int_{B_t(x)} f_2^{q_2} d\mu - \int_{B_t(y)} f_2^{q_2} d\mu \right|,$$

we write $\delta = d(x, y)$, for $k > 1$ and $k\delta \leq t \leq m$. Therefore, $\delta = \frac{1}{k}t < t$ and $x \in B_t(y)$. Thus, for all $z \in B_t(x)$ such that $d(z, x) < t - \delta$, we have $d(z, y) \leq d(z, x) + d(x, y) = t$ and $B_{t-\delta}(x) \subseteq B_t(y) \cap B_t(x)$. Then

$$\begin{aligned}
&\mu(B_t(y) \setminus B_t(x)) \\
&\leq \mu(B_t(y)) - \mu(B_{t-\delta}(x)) \\
&= t^N - (t - \delta)^N \\
&\leq Ct^{N-1}\delta,
\end{aligned}$$

for sufficiently large k such that $t \geq k\delta \gg \delta$. We denote $B_t(x) \oplus B_t(y) = (B_t(x) \setminus B_t(y)) \cup (B_t(y) \setminus B_t(x))$, therefore,

$$\mu(B_t(x) \oplus B_t(y)) \lesssim t^{N-1}\delta. \tag{4.3}$$

Furthermore,

$$\begin{aligned}
&\left| \int_{B_t(x)} f_2^{q_2} d\mu - \int_{B_t(y)} f_2^{q_2} d\mu \right| \\
&\leq M^{q_2-1} \|f_2\|_{0,1} \mu(B_t(x) \oplus B_t(y)) \\
&\leq M^{q_2-1} t^{N-1} \|f_2\|_{0,1} \delta.
\end{aligned}$$

Now we fix k , if $t \leq k\delta$, then for all $z_1 \in B_t(x)$ and $z_2 \in B_t(y)$,

$$d(z_1, z_2) \leq d(z_1, x) + d(x, y) + d(y, z_2) \leq 2t + \delta \leq (k+2)\delta$$

and

$$|f_2^{q_2}(z_1) - f_2^{q_2}(z_2)| \leq \|f_2\|_\infty^{q_2-1} \|f_2\|_{0,1} (k+2)\delta \leq (k+2)M^{q_2-1} \|f_2\|_{0,1}\delta.$$

Thus,

$$\begin{aligned} & \left| \int_{B_t(x)} f_2^{q_2} d\mu - \int_{B_t(y)} f_2^{q_2} d\mu \right| \\ & \leq \mu(B_t(x)) \max_{z_1 \in B_t(x), z_2 \in B_t(y)} |f_2^{q_2}(z_1) - f_2^{q_2}(z_2)| \\ & \leq (k+2)M^{q_2-1} t^N \|f_2\|_{0,1}\delta. \end{aligned}$$

Combining two cases for $k\delta \leq t \leq m$ and $t \leq k\delta$, we have for all $t \leq m$,

$$\left| \int_{B_t(x)} f_2^{q_2} d\mu - \int_{B_t(y)} f_2^{q_2} d\mu \right| \leq CM^{q_2-1} t^N \|f_2\|_{0,1}\delta.$$

Returning to (4.2),

$$\begin{aligned} & |T_1 f_2(x) - T_1 f_2(z)| \\ & \leq \int_0^m \left[\frac{|\int_{B_t(x)} f_2^{q_2} d\mu - \int_{B_t(y)} f_2^{q_2} d\mu|}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \cdot \left[\frac{\int_{B_t(x)} f_2^{q_2} d\mu + \int_{B_t(y)} f_2^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{2-p}{p-1}} \frac{dt}{t} \\ & \leq CM^{\frac{q_2}{p-1}-1} m^{\frac{\alpha p}{p-1}} \|f_2\|_{0,1}\delta \\ & \leq \frac{1}{4}d(x, y), \end{aligned}$$

by choosing m small. Thus we can derive that for sufficiently small m ,

$$\sup_{x \neq y} \frac{|T_1 f_2(x) - T_1 f_2(y)|}{d(x, y)} \leq \frac{1}{4} \|f_2\|_{0,1}.$$

From the proof of condition (1), by letting $h_2 = 0$, we have

$$\|T_1 f_2(x)\|_\infty \leq \frac{1}{4} \|f_2\|_\infty \leq \frac{1}{4} \|f_2\|_{0,1}. \quad (4.4)$$

Then,

$$\|T_1 f_2\|_{0,1} = \|T_1 f_2(x)\|_\infty + \sup_{x \neq y} \frac{|T_1 f_2(x) - T_1 f_2(z)|}{d(x, y)} \leq \frac{1}{2} \|f_2\|_{0,1}.$$

Similarly, choose m small such that

$$\|T_2 f_1\|_{0,1} \leq \frac{1}{2} \|f_1\|_{0,1},$$

and consequently

$$\begin{aligned}
\|T(f_1, f_2)\|_{0,1} &= \|(T_1 f_2, T_2 f_1)\|_{0,1} \\
&= \|T_1 f_2\|_{0,1} + \|T_2 f_1\|_{0,1} \\
&\leq \frac{1}{2} \|f_2\|_{0,1} + \frac{1}{2} \|f_1\|_{0,1} \\
&\leq \frac{1}{2} \|(f_1, f_2)\|_{0,1},
\end{aligned}$$

which means T is shrinking from $\mathcal{Y} \times \mathcal{Y}$ to $C^{0,1} \times C^{0,1}$. In (3) and (4) following, we shall fix $0 < m \leq 1$ which makes conditions (1) and (2) satisfied.

(3). To show $g_1, g_2 \in \mathcal{X} \cap \mathcal{Y}$, it suffices to prove for g_1 , and the case for g_2 follows similarly. Observe that

$$\|g_1\|_\infty = \sup_{x \in \mathbb{X}} \int_m^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \leq \|u\|_\infty \leq M. \quad (4.5)$$

Thus $g_1 \in \mathcal{X}$. For the Lipschitz continuity estimate of g_1 , we divide,

$$g_1(x) = \int_m^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} = \int_m^1 + \int_1^\infty = J_1(x) + J_2(x).$$

For J_2 , we apply the results in estimating I_2 in Section 3. However notice that since we assume here $k_1 = 1$, then given $\delta > 0$, for all $x, y \in \mathbb{X}$ such that $d(x, y) = \delta$, we have $d(z, y) \leq d(z, x) + d(x, y) \leq t + \delta$ for all $z \in B_t(x)$. We compute

$$\begin{aligned}
&J_2(x) - J_2(y) \\
&\leq (1 + \delta)^{\frac{N-\alpha p}{p-1}+1} \int_{1+\delta}^\infty \left[\frac{\int_{B_t(y)} v^{q_2} d\mu}{\mu(B_t(y))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} - \int_1^\infty \left[\frac{\int_{B_t(y)} v^{q_2} d\mu}{\mu(B_t(y))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \\
&\leq \frac{N-\alpha p}{p-1} \delta \int_1^\infty \left[\frac{\int_{B_t(y)} v^{q_2} d\mu}{\mu(B_t(y))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \\
&\leq CM\delta.
\end{aligned}$$

Also $J_2(x) - J_2(y) \leq CM\delta$, and thus J_2 is Lipschitz continuous. To estimate J_1 , we have

$$\begin{aligned}
&|J_1(x) - J_1(y)| \\
&= \left| \int_m^1 \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} - \int_m^1 \left[\frac{\int_{B_t(y)} v^{q_2} d\mu}{\mu(B_t(y))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right| \\
&\leq \int_m^1 \left[\frac{\int_{B_t(x)} v^{q_2} d\mu - \int_{B_t(y)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \cdot \left[\frac{\int_{B_t(x)} v^{q_2} d\mu + \int_{B_t(y)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{2-p}{p-1}} \frac{dt}{t} \\
&\leq CM^{\frac{q_2}{p-1}} \mu(B_m(x) \oplus B_m(y)) \int_m^1 [\mu(B_t(x))]^{\frac{\alpha p}{N(p-1)} - \frac{1}{N}} \frac{dt}{t} \\
&\leq CM^{\frac{q_2}{p-1}} (1 - m^{\frac{\alpha p}{p-1}-1}) \mu(B_m(x) \oplus B_m(y)) \\
&\leq C\delta.
\end{aligned}$$

Here, we applied the fact from (4.3) that

$$\mu(B_m(x) \oplus B_m(y)) \lesssim m^{N-1} \delta.$$

Thus J_1 is also Lipschitz continuous and $g_1 \in C^{0,1}$, and therefore $g_1 \in \mathcal{X} \cap \mathcal{Y}$.

(4). To show $S : (\mathcal{X} \times \mathcal{X}) \cap (\mathcal{Y} \times \mathcal{Y}) \rightarrow (\mathcal{X} \times \mathcal{X}) \cap (\mathcal{Y} \times \mathcal{Y})$. Given $(w_1, w_2) \in (\mathcal{X} \times \mathcal{X}) \cap (\mathcal{Y} \times \mathcal{Y})$, we have $w_1 \in \mathcal{X} \cap \mathcal{Y}$ and $w_2 \in \mathcal{X} \cap \mathcal{Y}$, thus $\|w_1\|_\infty \leq M$ and $\|w_2\|_\infty \leq M$.

It is evident that $T(w_1, w_2) \in (L^\infty \times L^\infty) \cap (C^{0,1} \times C^{0,1})$, since T is contracting from $\mathcal{X} \times \mathcal{X}$ to $L^\infty \times L^\infty$ by (1), and is shrinking from $\mathcal{Y} \times \mathcal{Y}$ to $C^{0,1} \times C^{0,1}$ by (2).

Then it follows that $S(w_1, w_2) = T(w_1, w_2) + (g_1, g_2) \in (L^\infty \times L^\infty) \cap (C^{0,1} \times C^{0,1})$, since $(g_1, g_2) \in (\mathcal{X} \times \mathcal{X}) \cap (\mathcal{Y} \times \mathcal{Y}) \subseteq (L^\infty \times L^\infty) \cap (C^{0,1} \times C^{0,1})$ by (3). Furthermore, by (4.4) and (4.5),

$$\|T_1 w_2 + g_1\|_\infty \leq \|T_1 w_2\|_\infty + \|g_1\|_\infty \leq \frac{1}{4} \|w_2\|_\infty + \|g_1\|_\infty \leq \frac{3}{2} \|u\|_\infty + \|v\|_\infty \leq M,$$

and similarly,

$$\|T_2 w_1 + g_2\|_\infty \leq \|u\|_\infty + \frac{3}{2} \|v\|_\infty \leq M.$$

Therefore, $S(w_1, w_2) = (T_1 w_2 + g_1, T_2 w_1 + g_2) \in (\mathcal{X} \times \mathcal{X}) \cap (\mathcal{Y} \times \mathcal{Y})$. This finishes the proof and we can conclude that $(u, v) \in C^{0,1} \times C^{0,1}$, i.e., u and v are Lipschitz continuous on \mathbb{X} . \square

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