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Area integral means over the annuli ☆

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ABSTRACT

For $0 < p < +\infty$ and an analytic function f in the unit disk $\mathbb{D}$ we show that the L^p integral mean of f on $c \leq |z| \leq r$ or $r \leq |z| \leq c$ with respect to the weighted area measure $|z|^{2\beta}(1-|z|^2)^\alpha\, dA(z)$ is a logarithmically convex function of r if α, β, c satisfy certain conditions.

1. Introduction

Let $H(\mathbb{D})$ denote the space of all analytic functions in the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. For any $f \in H(\mathbb{D})$ and $0 < p < +\infty$ the classical integral means of f are defined by

$$M_p(f,r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta, \qquad 0 \leq r < 1.$$

These integral means play a prominent role in classical analysis, especially in the theory of Hardy spaces. For example, the well-known Hardy convexity theorem asserts that $M_p(f,r)$, as a function of r on $(0,1)$, is logarithmically convex. Logarithmic convexity here means that the function $r \mapsto \log M_p(f,r)$ is convex in $\log r$. See [2] for example.

For any real α, β, we consider the measure

$$dA_{\alpha,\beta}(z) = |z|^{2\beta}(1-|z|^2)^\alpha\, dA(z),$$

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where dA is the area measure on $\mathbb{D}$. When $\beta = 0$, such measures are frequently used in the more recent theory of Bergman spaces. See [3,4]. Let $A(a, b)$ be the annulus $\{z \in \mathbb{C} : a \leq |z| \leq b\}$ if $a < b$ or the annulus $\{z \in \mathbb{C} : b \leq |z| \leq a\}$ if $a > b$. For $f \in H(\mathbb{D})$, $0 < p < +\infty$, $0 < c < 1$, and real α, β, we consider the area integral means

$$M_{p,\alpha,\beta,c}(f, r) = \frac{\displaystyle\int_{A(c,r)} |f(z)|^p |z|^{2\beta} (1 - |z|^2)^\alpha \, dA(z)}{\displaystyle\int_{A(c,r)} |z|^{2\beta} (1 - |z|^2)^\alpha \, dA(z)}.$$

Here we assume that f is analytic in the entire unit disk $\mathbb{D}$, although the integral means are taken over the annuli $A(c, r)$, $r \in (c, 1)$ or $r \in (0, c)$.

In [10] Xiao and Zhu initiated the study of area integral means of analytic functions in the unit disk $\mathbb{D}$ with respect to the weighted area measure $dA_{\alpha,0}(z)$. It was shown in [7,8] that, just like the classical integral means, $M_{p,\alpha,0,0}(f, r)$ is also logarithmically convex on $(0, 1)$ when $-2 \leq \alpha \leq 0$. Furthermore, if $p = 2$, then $M_{2,\alpha,0,0}(f, r)$ is logarithmically convex on $(0, 1)$ when $-3 \leq \alpha \leq 0$, and this range for α is best possible. Cui, Wang and Zhu proved in [1] that the function $M_{2,\alpha,0,c}(f, r)$ is logarithmically convex for $r \in (c, 1)$ if $0 \leq c < 1$ and $-3 \leq \alpha \leq 0$. We will consider the logarithmic convexity problem on $M_{p,\alpha,\beta,c}(f, r)$ for general p and β and give our main results in Section 3. See [5,6,9] for other recent work in the area.

Throughout the paper we use the symbol =: whenever a new notation is being introduced. We will use the notation $A \sim B$ to mean that A and B have the same sign.

2. Preliminaries

In this section we collect several preliminary results that will be needed for the proof of our main results. For any twice differentiable function f on (a, b), we define

$$d_f(x) = x\frac{f'(x)}{f(x)}$$

and

$$D_f(x) = D(f(x)) = \frac{f'(x)}{f(x)} + x\,\frac{f''(x)}{f(x)} - x\left(\frac{f'(x)}{f(x)}\right)^2.$$

It is easy to check

$$(d_f(x))' = D_f(x), \tag{1}$$

and

$$xD_f(x) = d_f(x)(1 + d_{f'}(x) - d_f(x)). \tag{2}$$

Lemmas 1 and 2 were stated and proved in [7,8] for the interval $(0, 1)$. But it is clear that the conclusions still hold if $(0, 1)$ is replaced by any interval (a, b), where $0 \leq a < b < +\infty$. See also [1].

Lemma 1. *Suppose that f is positive and twice differentiable on (a, b). Then*

(i) *$f(x)$ is convex in $\log x$ if and only if $f(x^2)$ is convex in $\log x$.*
(ii) *$\log f(x)$ is convex in $\log x$ if and only if $D_f(x) \geq 0$ for all $x \in (a, b)$.*

Lemma 2. *Suppose that $f = f_1/f_2$ is a quotient of two positive and twice differentiable functions on (a, b). Then*

$$D_f(x) = D_{f_1}(x) - D_{f_2}(x)$$

for $x \in (a, b)$. Consequently, $\log f(x)$ is convex in $\log x$ if and only if

$$D_{f_1}(x) - D_{f_2}(x) \geq 0$$

on (a, b).

To simplify notation, we let

$$\varphi = \varphi(x) = \int_{x_0}^{x} \varphi'(t)\,dt, \qquad \varphi' = \varphi'(x) = x^{\beta}(1-x)^{\alpha}, \tag{3}$$

where x_0, x are in $(0, 1)$ and α, β are real numbers. We also write

$$d_0 = d_\varphi = d_\varphi(x), \quad D_0 = D_\varphi = D_\varphi(x).$$

It is easy to check that

$$d_1 = d_{\varphi'} = d_{\varphi'}(x) = \beta + \frac{-\alpha x}{1-x}, \tag{4}$$

$$D_1 = D_{\varphi'} = D_{\varphi'}(x) = \frac{-\alpha}{(1-x)^2}, \tag{5}$$

and

$$(xD_0)' = (1 + d_1)D_0 + d_0(D_1 - 2D_0). \tag{6}$$

Lemma 3. *Suppose $-\infty < \beta < +\infty$ and $-2 \leq \alpha \leq 0$. Then $D_1 - 2D_0 \geq 0$ holds for $x \in (x_0, 1)$ if one of the following conditions is satisfied:*

(i) $\beta \leq 0$.
(ii) $\beta > 0$ *and* $x_0 \geq \frac{\beta}{\alpha+\beta+2}$.
(iii) $\beta > 0$, $x_0 < \frac{\beta}{\alpha+\beta+2}$ *and* $(D_1 - 2D_0)\big|_{x=\frac{\beta}{\alpha+\beta+2}} \geq 0$.

Proof. It is easy to see that $D_1 - 2D_0 \geq 0$ is equivalent to $g_1(x) \geq 0$, where

$$g_1(x) = \frac{\varphi^2}{\varphi'}(D_1 - 2D_0) = \frac{\varphi^2}{\varphi'}D_1 - 2\varphi\left(1 + x\frac{\varphi''}{\varphi'}\right) + 2x\varphi'.$$

Since D_1 and φ' are both positive for $\alpha \leq 0$, we have

$$\begin{aligned} g_1'(x) &= D_1'\frac{\varphi^2}{\varphi'} - D_1\frac{\varphi^2\varphi''}{(\varphi')^2} \\ &\sim \frac{D_1'}{D_1} - \frac{\varphi''}{\varphi'} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha+2}{1-x} - \frac{\beta}{x} \\
&\sim (\alpha+\beta+2)x - \beta.
\end{aligned}
$$

In case (i) and (ii), we see that $g_1'(x) \geq 0$ and hence $g_1(x) \geq g_1(x_0) \geq 0$. In case (iii), $g_1'(x)$ has a unique zero $x_1 = \frac{\beta}{\alpha+\beta+2}$ at which $g_1(x)$ attains its minimum in $(x_0, 1)$ and hence $g_1(x) \geq g_1(x_1) \geq 0$. $\square$

It should be pointed out that for $-2 \leq \alpha \leq 0$ and general $\beta > 0, x_0$, $D_1 - 2D_0$ is not always positive. Let $\alpha = -1$, $\beta = 3$, $x_0 = \frac{1}{4}$ and $x = \frac{3}{4}$. It follows from direct computations that

$$
\varphi\left(\frac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} (1-t)^{-1} t^3\, dt = \log 3 - \frac{85}{96},
$$

and

$$
d_0 = \frac{81}{64\left(\log 3 - \frac{85}{96}\right)} = 5.936\cdots, d_1 = 6, D_1 = 16.
$$

Thus we have

$$
xD_1 - 2xD_0 = 2(6 - 7d_0 + d_0^2) = 2(1-d_0)(6-d_0) < 0.
$$

Lemma 4. *Suppose $\alpha \leq 0$ and $\beta \geq -1$. Then $(1+d_1)D_0 - d_0 D_1 \leq 0$ holds for $x \in (0, x_0)$ if one of the following conditions is satisfied:*

(i) $-2 - 2\beta \leq \alpha \leq 0$.
(ii) $\alpha < -2 - 2\beta$ *and* $x_0 \leq -\frac{\beta+1}{\alpha+\beta+1}$.
(iii) $\alpha < -2 - 2\beta$, $x_0 > -\frac{\beta+1}{\alpha+\beta+1}$ *and*

$$
\left((1+d_1)D_0 - d_0 D_1\right)\Big|_{x=-\frac{\beta+1}{\alpha+\beta+1}} \leq 0.
$$

Proof. Note that $\varphi < 0$ and $\varphi' \geq 0$. It is easy to see that $(1+d_1)D_0 - d_0 D_1 \leq 0$ is equivalent to $g_2(x) \leq 0$, where

$$
g_2(x) = -\left(\frac{xD_1}{(1+d_1)^2} - 1\right)\varphi - \frac{x\varphi'}{1+d_1}.
$$

Since $\varphi < 0$, $D_1 > 0$ and $1 + d_1 \geq 0$, we have

$$
\begin{aligned}
g_2'(x) &= -\left(\frac{D_1 + xD_1'}{(1+d_1)^2} - \frac{2xD_1^2}{(1+d_1)^3}\right)\varphi \\
&\sim (1+d_1)\left(1 + x\frac{D_1'}{D_1}\right) - 2xD_1 \\
&= \frac{\beta+1+(\alpha+\beta+1)x}{1-x}.
\end{aligned}
$$

In case (i) and (ii), we see that $g_2'(x) \geq 0$ and hence $g_2(x) \leq g_2(x_0) \leq 0$. In case (iii), $g_2'(x)$ has a unique zero $x_2 = -\frac{\beta+1}{\alpha+\beta+1}$ at which $g_2(x)$ attains its maximum in $(0, x_0)$ and hence $g_2(x) \leq g_2(x_2) \leq 0$. $\square$

3. Main results

For $M(x) = M_p(f, \sqrt{x})$, we let

$$h = h(x) = \int_{x_0}^{x} M(t)\varphi'(t)dt, \qquad x, x_0 \in (0,1), \tag{7}$$

where x_0 is fixed. Noting that $d_M \geq 0, D_M \geq 0$, we easily obtain

$$d_{h'} = d_M + d_{\varphi'} = d_M + d_1 \geq d_1, \tag{8}$$

and

$$D_{h'} = D_M + D_{\varphi'} = D_M + D_1 \geq D_1. \tag{9}$$

If $x > x_0$, since

$$h(x) = \int_{x_0}^{x} M(t)\varphi'(t)dt \leq M(x)\int_{x_0}^{x} \varphi'(t)dt = M(x)\varphi(x),$$

we have $d_h \geq d_0$. If $x < x_0$, since

$$h(x) = -\int_{x}^{x_0} M(t)\varphi'(t)dt \leq -M(x)\int_{x}^{x_0} \varphi'(t)dt = M(x)\varphi(x),$$

we also have $d_h \geq d_0$.

If $x > x_0$, noting that $d_0 \geq 0, d_M \geq 0$ and $d_{h'} = d_M + d_1$, we have

$$\begin{aligned}
&(1 + d_{h'})^2 - 4xD_0 \\
&= (1 + d_1 - 2d_0)^2 + 2(1 + d_1)d_M + d_M^2 \\
&= (1 + d_1 - 2d_0 + d_M)^2 + 4d_0 d_M \\
&\geq (1 + d_1 - 2d_0 + d_M)^2.
\end{aligned} \tag{10}$$

If $D_0 \leq 0$ whenever $x > x_0$ or $x < x_0$, we have

$$\begin{aligned}
&(1 + d_{h'})^2 - 4xD_0 \\
&= (1 + d_1 - 2d_0)^2 + 2(1 + d_1)d_M + d_M^2 \\
&= \left(1 + d_1 - 2d_0 + \frac{1 + d_1}{1 + d_1 - 2d_0}d_M\right)^2 + \frac{-4xD_0}{(1 + d_1 - 2d_0)^2}d_M^2 \\
&\geq \left(1 + d_1 - 2d_0 + \frac{1 + d_1}{1 + d_1 - 2d_0}d_M\right)^2.
\end{aligned} \tag{11}$$

We can now prove the main results of the paper.

Theorem 5. *Suppose* $0 < p < +\infty$, $0 \leq c = \sqrt{x_0} < 1$, $-2 \leq \alpha \leq 0$, *and* $f \in H(\mathbb{D})$. *Then the function* $M_{p,\alpha,\beta,c}(f, r)$ *is logarithmically convex for* $r \in (c, 1)$ *if one of the following conditions is satisfied:*

(i) $-1 \le \beta \le 0$.
(ii) $\beta > 0$ *and* $x_0 \ge \frac{\beta}{\alpha+\beta+2}$.
(iii) $\beta > 0$, $x_0 < \frac{\beta}{\alpha+\beta+2}$ *and* $(D_1 - 2D_0)\big|_{x=\frac{\beta}{\alpha+\beta+2}} \ge 0$.
(iv) $\beta < -1$ *and* $x_0 \ge \frac{\beta+1}{\alpha+\beta+1}$.

Proof. Let $x = r^2$ and $x_0 = c^2$. By polar coordinates and Lemma 1, the logarithmic convexity of $M_{p,\alpha,\beta,c}(f,r)$ for $r \in (c,1)$ is equivalent to the logarithmic convexity of $h(x)/\varphi(x)$ for $x \in (x_0,1)$, where φ and h are defined by (3) and (7) respectively. According to Lemma 2, it is enough for us to show that $\Delta(x) \ge 0$ for $x \in (x_0,1)$, where

$$\begin{aligned}\Delta(x) &= xD_h(x) - xD_0(x)\\ &= -d_h^2 + (1+d_{h'})d_h - xD_0\\ &= \left(d_h - \frac{1+d_{h'} - \sqrt{(1+d_{h'})^2 - 4xD_0}}{2}\right)\\ &\quad \cdot \left(\frac{1+d_{h'} + \sqrt{(1+d_{h'})^2 - 4xD_0}}{2} - d_h\right).\end{aligned}$$

Noting that $d_h \ge d_0$, it follows from (8) and (10) that

$$\begin{aligned}&d_h - \frac{1}{2}\left(1+d_{h'} - \sqrt{(1+d_{h'})^2 - 4xD_0}\right)\\ &\ge d_0 - \frac{1}{2}\left(1+d_1+d_M - |1+d_1-2d_0+d_M|\right)\\ &= \frac{1}{2}\left(|1+d_1-2d_0+d_M| - (1+d_1-2d_0+d_M)\right)\\ &\ge 0.\end{aligned}$$

Since $1+d_{h'} \ge 1+d_1 \ge 0$ when $\beta \ge -1$ or in case (iv), we have

$$1+d_{h'} + \sqrt{(1+d_{h'})^2 - 4xD_0} \ge 0.$$

Thus

$$\begin{aligned}\Delta(x) &\sim \frac{1+d_{h'} + \sqrt{(1+d_{h'})^2 - 4xD_0}}{2} - d_h\\ &\sim h - \frac{2xh'}{1+d_{h'} + \sqrt{(1+d_{h'})^2 - 4xD_0}} =: \delta(x).\end{aligned}$$

It is easy to check that

$$\lim_{x\to x_0} D_0 = \lim_{x\to x_0} \frac{\varphi(\varphi' + x\varphi'') - x(\varphi')^2}{\varphi^2} = -\infty$$

when $0 < x_0 < 1$ and

$$\lim_{x\to x_0} xD_0 = \lim_{x\to x_0} d_0(1+d_1-d_0) = 0$$

for $x_0 = 0$. Hence for all $0 \le x_0 < 1$, $\delta(x_0) = 0$.

It follows from direct computations that

$$\delta' = h' - \frac{2h' + 2xh''}{1 + d_{h'} + \sqrt{(1 + d_{h'})^2 - 4xD_0}} + \frac{2xh'}{\left(1 + d_{h'} + \sqrt{(1 + d_{h'})^2 - 4xD_0}\right)^2} \left(D_{h'} + \frac{(1 + d_{h'})D_{h'} - 2(xD_0)'}{\sqrt{(1 + d_{h'})^2 - 4xD_0}} \right).$$

Noticing that $h' > 0$, we have

$$\begin{aligned}\delta' &\sim -\frac{1 + d_{h'} - \sqrt{(1 + d_{h'})^2 - 4xD_0}}{1 + d_{h'} + \sqrt{(1 + d_{h'})^2 - 4xD_0}} \cdot \frac{\left(1 + d_{h'} + \sqrt{(1 + d_{h'})^2 - 4xD_0}\right)^2}{2x} \\ &\quad + \left(D_{h'} + \frac{(1 + d_{h'})D_{h'} - 2(xD_0)'}{\sqrt{(1 + d_{h'})^2 - 4xD_0}} \right) \\ &= -2D_0 + D_{h'} + \frac{(1 + d_{h'})D_{h'} - 2(xD_0)'}{\sqrt{(1 + d_{h'})^2 - 4xD_0}} \\ &\sim -2D_0\sqrt{(1 + d_{h'})^2 - 4xD_0} - 2(xD_0)' \\ &\quad + D_{h'}\left(1 + d_{h'} + \sqrt{(1 + d_{h'})^2 - 4xD_0}\right) =: \delta_1(x).\end{aligned}$$

In case (i), (ii), (iii) and (iv), we always have $1 + d_1 \geq 0$ and, by Lemma 3, $D_1 - 2D_0 \geq 0$. Combining this with (10) and $D_{h'} \geq D_1$, we obtain

$$\begin{aligned}\delta_1(x) &\geq -2D_0\sqrt{(1 + d_{h'})^2 - 4xD_0} - 2(xD_0)' \\ &\quad + D_1\left(1 + d_{h'} + \sqrt{(1 + d_{h'})^2 - 4xD_0}\right) \\ &= (D_1 - 2D_0)\left(\sqrt{(1 + d_{h'})^2 - 4xD_0} + 1 + d_1 - 2d_0\right) + d_M D_1 \\ &\geq (D_1 - 2D_0)\left(|1 + d_1 - 2d_0| + 1 + d_1 - 2d_0\right) \\ &\geq 0.\end{aligned}$$

Hence $\delta'(x) \geq 0$. This implies $\delta(x) \geq \delta(x_0) = 0$, and $\Delta(x) \geq 0$ on $(x_0, 1)$. □

If $x_0 = c^2 = 0$ and $\beta = 0$, we obtain the main result in [7] as a special case; If $p = 2$ and $\beta = 0$, our approach here yields a proof for the main theorem in [1] for $-2 \leq \alpha \leq 0$, unfortunately, not for $-3 \leq \alpha \leq 0$.

If the annuli $c \leq |z| \leq r$ are replaced by $r \leq |z| \leq c$, we have the following result.

Theorem 6. *Suppose $0 < p < +\infty$, $0 < c = \sqrt{x_0} < 1$, $\beta \geq -1$, and $f \in H(\mathbb{D})$. Then the function $r \mapsto M_{p,\alpha,\beta,c}(f, r)$ is logarithmically convex for $r \in (0, c)$ if one of the following conditions is satisfied:*

(i) $-2 - 2\beta \leq \alpha \leq 0$.
(ii) $\alpha < -2 - 2\beta$ *and* $x_0 \leq -\frac{\beta+1}{\alpha+\beta+1}$.
(iii) $\alpha < -2 - 2\beta$, $x_0 > -\frac{\beta+1}{\alpha+\beta+1}$ *and*

$$\left((1 + d_1)D_0 - d_0 D_1\right)\Big|_{x=-\frac{\beta+1}{\alpha+\beta+1}} \leq 0.$$

Proof. As in the proof of Theorem 5, we need only to consider the function

$$\frac{\int_x^{x_0} M(t)t^\beta(1-t)^\alpha dt}{\int_x^{x_0} t^\beta(1-t)^\alpha dt} = \frac{h(x)}{\varphi(x)}$$

and show that $\Delta(x) = xD_h(x) - xD_0(x) \geq 0$, where $h(x), \varphi(x), \Delta(x)$ are defined as before but for $x \in (0, x_0)$.

Since $d_h \leq 0, D_0 \leq 0$ for $\alpha \leq 0, \beta \geq -1$, we easily obtain

$$\frac{1 + d_{h'} + \sqrt{(1+d_{h'})^2 - 4xD_0}}{2} - d_h \geq 0.$$

Thus

$$\begin{aligned}\Delta(x) &\sim d_h - \frac{1 + d_{h'} - \sqrt{(1+d_{h'})^2 - 4xD_0}}{2} \\ &\sim -h - \frac{2xh'}{\sqrt{(1+d_{h'})^2 - 4xD_0} - 1 - d_{h'}} =: \delta(x).\end{aligned}$$

It is easy to check that

$$\lim_{x\to x_0} D_0 = \lim_{x\to x_0} \frac{\varphi(\varphi' + x\varphi'') - x(\varphi')^2}{\varphi^2} = -\infty$$

when $0 < x_0 < 1$. Hence $\delta(x_0) = 0$.

It follows from direct computations that

$$\begin{aligned}\delta' = -h' &- \frac{2h' + 2xh''}{\sqrt{(1+d_{h'})^2 - 4xD_0} - 1 - d_{h'}} \\ &+ \frac{2xh'}{\left(\sqrt{(1+d_{h'})^2 - 4xD_0} - 1 - d_{h'}\right)^2}\left(\frac{(1+d_{h'})D_{h'} - 2(xD_0)'}{\sqrt{(1+d_{h'})^2 - 4xD_0}} - D_{h'}\right).\end{aligned}$$

Noticing that $h' > 0, D_0 \leq 0$ and $D_{h'} \geq D_1 > 0$, we have

$$\begin{aligned}\delta' &\sim -\frac{\sqrt{(1+d_{h'})^2 - 4xD_0} + 1 + d_{h'}}{\sqrt{(1+d_{h'})^2 - 4xD_0} - 1 - d_{h'}} \cdot \frac{\left(\sqrt{(1+d_{h'})^2 - 4xD_0} - 1 - d_{h'}\right)^2}{2x} \\ &\quad + \left(\frac{(1+d_{h'})D_{h'} - 2(xD_0)'}{\sqrt{(1+d_{h'})^2 - 4xD_0}} - D_{h'}\right) \\ &= 2D_0 + \frac{(1+d_{h'})D_{h'} - 2(xD_0)'}{\sqrt{(1+d_{h'})^2 - 4xD_0}} - D_{h'} \\ &\sim 2D_0\sqrt{(1+d_{h'})^2 - 4xD_0} + D_{h'}\left(1 + d_{h'} - \sqrt{(1+d_{h'})^2 - 4xD_0}\right) - 2(xD_0)' \\ &\leq 2D_0\sqrt{(1+d_{h'})^2 - 4xD_0} + D_1\left(1 + d_{h'} - \sqrt{(1+d_{h'})^2 - 4xD_0}\right) - 2(xD_0)' \\ &= (2D_0 - D_1)\sqrt{(1+d_{h'})^2 - 4xD_0} + d_M D_1 + (D_1 - 2D_0)(1 + d_1 - 2d_0) \\ &=: \delta_2(x).\end{aligned}$$

Note that $D_0 \leq 0$, $D_1 \geq 0$, $1 + d_1 \geq 0$ and $d_0 < 0$ for $\alpha \leq 0$ and $\beta \geq -1$. By (11) and Lemma 4,

$$\begin{aligned}\delta_2(x) &\leq (2D_0 - D_1)\left|1 + d_1 - 2d_0 + \frac{1 + d_1}{1 + d_1 - 2d_0} d_M\right| \\ &\quad + d_M D_1 + (D_1 - 2D_0)(1 + d_1 - 2d_0) \\ &= 2d_M \frac{(1 + d_1)D_0 - d_0 D_1}{1 + d_1 - 2d_0} \\ &\leq 0.\end{aligned}$$

Hence $\delta'(x) \leq 0$. This implies $\delta(x) \geq \delta(x_0) = 0$, and $\Delta(x) \geq 0$ on $(0, x_0)$. □

4. A remark

Note that Theorems 5 and 6 are proved for $\beta \geq -1$. It is natural to ask if the results remain true for $\beta < -1$. We show by an example that the answer is negative even for $\alpha = 0$.

Consider the case where $p = 2$, $\alpha = 0$, $\beta = -3$, $c = \frac{1}{\sqrt{2}}$ and $f(z) = z$. It follows from a direct computation that

$$M_{2,0,-3,\frac{1}{\sqrt{2}}}(z, r) = \frac{2r^2}{1 + 2r^2}.$$

As before, we let $x_0 = c^2 = \frac{1}{2}$ and $x = r^2$. Then by Lemma 2, we just need to verify if the function

$$H(x) = \frac{2x}{1 + 2x}$$

is logarithmically convex on x. By direct computations,

$$D_H(x) = -\frac{2}{(1 + 2x)^2} < 0.$$

This implies that $\log H(x)$ is concave in $\log x$ for $x \in (x_0, 1)$ and $x \in (0, x_0)$. And hence $M_{2,0,-3,\frac{1}{\sqrt{2}}}(z, r)$ is logarithmically concave whenever $r \in (c, 1)$ or $r \in (0, c)$.

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