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Norm of the pre-Schwarzian derivative, Bloch's constant and coefficient bounds in some classes of harmonic mappings

S. Kanas [a,*], S. Maharana [b], J.K. Prajapat [b]

[a] *Faculty of Mathematic and Natural Sciences, University of Rzeszow, ul. S. Pigonia 1, 35-310 Rzeszow, Poland*

[b] *Department of Mathematics, Central University of Rajasthan, Bandarsindri, Kishangarh-305817, Dist.-Ajmer, Rajasthan, India*

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A B S T R A C T

The aim of the paper is a study of a family of sense-preserving complex valued harmonic functions f that are normalized in the open unit disk, and such that its analytic part is convex in one direction. We prove the univalence and establish estimates on pre-Schwarzian derivative and the Bloch constant for co-analytic part of harmonic mapping. The bounds of the coefficients of co-analytic part are also given.

1. Introduction

A complex valued function f is harmonic in a simply connected domain $\Omega \subset \mathbb{C}$ if both $\Re\{f\}$ and $\Im\{f\}$ are harmonic in Ω. Every such f can be uniquely represented as

$$f = h + \overline{g} \tag{1.1}$$

where h and g are analytic in Ω. The Jacobian of f is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ and the second complex (analytic) dilatation of f is given by $w(z) = g'(z)/h'(z)$. A result of Lewy [19] states that a harmonic mapping f is locally univalent at z if and only if its Jacobian $J_f(z) \neq 0$, and is sense-preserving if the Jacobian is positive. The sense-preserving case implies that $|w(z)| = |g'(z)/h'(z)| < 1$ in Ω.

Let $\mathcal{H}ar(\mathbb{D})$ denote the family of continuous complex valued functions which are harmonic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{H}ol(\mathbb{D})$ denote the class of holomorphic functions f in $\mathbb{D}$ with

* Corresponding author.
E-mail addresses: skanas@ur.edu.pl (S. Kanas), snmmath@gmail.com (S. Maharana), jkprajapat@gmail.com (J.K. Prajapat).

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the normalization $f(0) = f'(0) - 1 = 0$ in $\mathbb{D}$. We note that $\mathcal{H}ol(\mathbb{D}) \subset \mathcal{H}ar(\mathbb{D})$. Further, let $\mathcal{S} := \mathcal{S}_{\mathcal{H}ol}$ be the subclass of $\mathcal{H}ol(\mathbb{D})$ which are additionally univalent in $\mathbb{D}$. Clunie and Sheil-Small in [10] developed the fundamental theory of functions $f \in \mathcal{H}ar(\mathbb{D})$ with the normalization $f(0) = h'(0) - 1 = 0$ in $\mathbb{D}$. Following Clunie and Sheil-Small notation we next denote by $\mathcal{S}_{\mathcal{H}ar}$, the subclass of $\mathcal{H}ar(\mathbb{D})$ consisting of univalent and sense-preserving harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$, where h and g are normalized such that

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1.2}$$

Here, h and g are called the *analytic part* and the *co-analytic part* of f, respectively. Certainly $\mathcal{S}_{\mathcal{H}ol} \subset \mathcal{S}_{\mathcal{H}ar}$. Also, let $\mathcal{S}^0_{\mathcal{H}ar} = \{f \in \mathcal{S}_{\mathcal{H}ar} : g'(0) = f_{\overline{z}}(0) = 0\}$. The family $\mathcal{S}^0_{\mathcal{H}ar}$ is known to be compact and normal, whereas $\mathcal{S}_{\mathcal{H}ar}$ is normal but not compact. For many interesting results and expositions on planar harmonic univalent mappings, we refer to the monograph of Duren [13] and also the expository articles [12,26].

Let

$$\mathcal{G} = \left\{ h \in \mathcal{H}ol(\mathbb{D}) : \ -\frac{1}{2} < \Re\left(1 + \frac{zh''(z)}{h'(z)}\right) < \frac{3}{2}, \quad z \in \mathbb{D} \right\}. \tag{1.3}$$

The functions $h \in \mathcal{G}$ are univalent in $\mathbb{D}$ [30] and map every circle $C_\rho = \{z \in \mathbb{C} : |z| = \rho\}$, $\rho \in (0,1)$, onto a curve bounding a region that is convex in one direction. Recall that, a domain $\Omega \subset \mathbb{C}$ is called convex in the direction φ $(0 \le \varphi < \pi)$ if every line parallel to the line going through 0 and $e^{i\varphi}$ has a connected or empty intersection with Ω. An analytic function h is said to be convex in the direction φ if $h(\mathbb{D})$ is convex in the direction φ [28]. Additionally, one of the conditions $\Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$ or $\Re\left(1 + \frac{zh''(z)}{h'(z)}\right) < \frac{3}{2}$ is a sufficient condition for univalence of h [30]. However, the functions satisfying $\Re\left(1 + \frac{zh''(z)}{h'(z)}\right) < 1 + \frac{\delta}{2}$ are not necessarily univalent in $\mathbb{D}$ if $\delta > 1$ [23]. Recently, the second and third authors have studied the radius of convexity of partial sums of functions in the class $\mathcal{G}$ [21].

The analytic parts of harmonic mappings play a vital role in shaping their geometric properties. For instance if $f = h + \overline{g}$ is a sense preserving harmonic mapping and h is convex univalent, then $f \in \mathcal{S}_{\mathcal{H}ar}$ and maps $\mathbb{D}$ onto a close-to-convex domain [10]. In [15,16] a class of functions $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}ar}$ has been studied, where h and g are given by (1.2), such that $b_1 = \alpha \in (0,1)$, h is convex in $\mathbb{D}$ (or h is a function with bounded boundary rotation) and the dilatation w is of the form $w(z) = (z + \alpha)/(1 + \alpha z)$.

For α $(0 \le \alpha < 1)$ let $\mathcal{G}^\alpha_{\mathcal{H}ar}$ denote the set of all harmonic functions $f = h + \overline{g} \in \mathcal{H}ar(\mathbb{D})$, with $g'(0) = b_1 = \alpha$ and

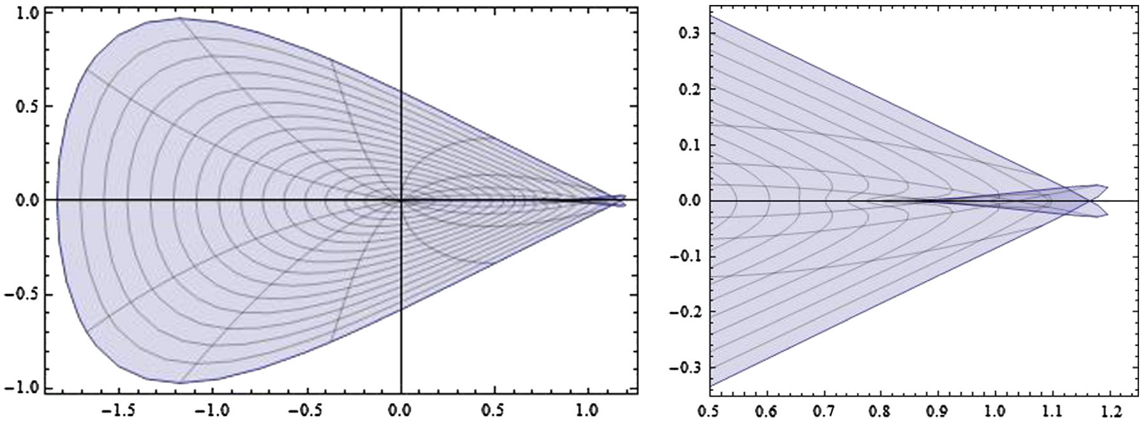
$$g'(z) = w(z)h'(z) \quad \text{and} \quad w \in \mathcal{G} \qquad (z \in \mathbb{D}), \tag{1.4}$$

where w is the Möbius selfmap of $\mathbb{D}$ of the form $w(z) = (z + \alpha)/(1 + \alpha z)$. The function w has the series expansion

$$w(z) = \alpha + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{D},\ c_i \in \mathbb{C},\ i = 1, 2, \ldots). \tag{1.5}$$

Observe that, a harmonic mapping $f = h + \overline{g} \in \mathcal{H}ar(\mathbb{D})$ with $g'(0) = b_1 = \alpha$ $(0 \le \alpha < 1)$ satisfying conditions (1.4) need not necessarily univalent in $\mathbb{D}$. For instance, when $w(z) = \frac{1}{2} + \frac{1}{2}z$, and a harmonic function $f = h + \overline{g}$, where

$$h(z) = z - \frac{1}{2}z^2 \quad \text{and} \quad g(z) = \frac{1}{2}z - \frac{1}{6}z^3,$$



(a) The image of the mapping f (b) An enlarged view of right cusp of image of f

Fig. 1. Graph of the function $f(z) = z - \frac{1}{2}z^2 + \overline{\frac{1}{2}z - \frac{1}{6}z^3}$.

we obtain $g'(z) = w(z)h'(z)$. It follows that

$$\Re\left(1 + \frac{zh''(z)}{h'(z)}\right) < \frac{3}{2} \qquad (z \in \mathbb{D}),$$

but we see that f is not univalent in $\mathbb{D}$ (see the graph of f shown in Fig. 1).

If $f \in \mathcal{G}^{\alpha}_{\mathcal{H}ar}$, then according to the form of $w(z) = (z + \alpha)/(1 + \alpha z)$, and the relation $w = g'/h'$ we have $c_0 = b_1 = \alpha$,

$$\frac{|r - \alpha|}{1 - \alpha r} \leq |w(z)| \leq \frac{r + \alpha}{1 + \alpha r}, \tag{1.6}$$

and

$$|c_n| \leq 1 - |c_0|^2 \quad (n = 1, 2, \ldots), \qquad |w'(z)| \leq \frac{1 - |w(z)|^2}{1 - |z|^2} \quad (z \in \mathbb{D}) \tag{1.7}$$

(see e.g. [2, p. 30, 53]).

Let f be a locally univalent function in $\mathbb{D}$, then the pre-Schwarzian derivative T_f of f is defined by

$$T_f(z) = \frac{f''(z)}{f'(z)}.$$

The pre-Schwarzian derivative T_f is a basic part of the Schwarzian derivative S_f that equals $S_f = (T_f)' - (T_f)^2/2$. We also define the norm of T_f by

$$\|T_f\| = \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right) |T_f|\,. \tag{1.8}$$

It is well known that $\|T_f\| < \infty$ if and only if f is uniformly locally univalent. Here, an analytic function f on $\mathbb{D}$ is said to be uniformly locally univalent if f is univalent on each hyperbolic disk in $\mathbb{D}$ with fixed radius [31,32]. If f is univalent in $\mathbb{D}$, then $\|T_f\| \leq 6$ and the bound 6 is sharp. Conversely if $\|T_f\| \leq 1$ then f is univalent in $\mathbb{D}$ [3]. The pre-Schwarzian derivative has been studied in detail by many authors because of its connection with univalence, convexity, quasiconformality, Teichmüller spaces, etc. (see, for example [1,18,14,22]), where several applications were indicated, for a survey we refer to [25]. Therefore, some relevant estimates of its norm in classical subclasses of univalent functions were discussed in [6,17], and also [24,27,29].

An important problem in the theory of harmonic mappings turned out to be finding a suitable definition of the Schwarzian derivative (the pre-Schwarzian derivative, respectively). Chuaqui et al. [7,8] introduced the harmonic Schwarzian using the differential geometry of the associated minimal surface, this for functions that are not necessarily locally univalent, and then derived the necessary condition for univalence [9]. Following the definition in the analytic case the harmonic pre-Schwarzian has been proposed in [15], and described by the formula

$$T_f = \frac{2\,\partial(\log\lambda)}{\partial z}, \tag{1.9}$$

where $\lambda = |h'| + |g'|$. In the case, when f is analytic, $\lambda = |f'|$, so that $\log\lambda = \log f'/2 + \log \overline{f'}/2$, therefore (1.9) in the analytic case agrees with the classical formula f''/f'. We observe that if $g' = qh'$, and $q = w^2$, then [13, p. 191]

$$T_f = \frac{2\,\partial(\log\lambda)}{\partial z} = \frac{h''}{h'} + \frac{2w'\overline{w}}{1+|w|^2} = T_h + \frac{2w'\overline{w}}{1+|w|^2} \tag{1.10}$$

(here local univalence of a function h is required).

The classical Bloch theorem asserts the existence of a positive constant b such that for any holomorphic mapping f of the unit disk $\mathbb{D}$, with the normalization $f'(0) = 1$, the image $f(\mathbb{D})$ contains a Schlicht disk of radius b. By Schlicht disk, we mean a disk which is the univalent image of some region in $\mathbb{D}$. The Bloch constant is defined as the "best" such constant, that is supremum of such constants b. Chen et al. [4] estimated Bloch constant for harmonic mappings.

A function $f \in \mathcal{H}ar(\mathbb{D})$ is called a *harmonic Bloch mapping* if and only if

$$\mathcal{B}_f = \sup_{z,w\in\mathbb{D}, z\neq w} \frac{|f(z) - f(w)|}{\mathcal{Q}(z,w)} < \infty, \tag{1.11}$$

where

$$\mathcal{Q}(z,w) = \frac{1}{2}\log\left(\frac{1+\left|\frac{z-w}{1-\overline{z}w}\right|}{1-\left|\frac{z-w}{1-\overline{z}w}\right|}\right) = \operatorname{arctanh}\left|\frac{z-w}{1-\overline{z}w}\right|$$

denotes the hyperbolic distance between z and w in $\mathbb{D}$, and $\mathcal{B}_f$ is called the *Bloch's constant* of f. In [11] Colonna proved that

$$\mathcal{B}_f = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)\Lambda_f \tag{1.12}$$

where

$$\begin{aligned}\Lambda_f = \Lambda_f(z) &= \max_{0\le\theta\le 2\pi} \left|f_z(z) - e^{-2i\theta} f_{\overline{z}}(z)\right| = |f_z(z)| + |f_{\overline{z}}(z)| \\ &= |h'(z)| + |g'(z)| = |h'(z)|(1+|w(z)|).\end{aligned}$$

Moreover, the set of all harmonic Bloch mappings forms a complex Banach space with the norm $\|\cdot\|$ given by

$$\|f\| = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)\Lambda_f(z).$$

This definition agrees with the notion of the Bloch's constant for analytic functions. Recently, many authors have studied Bloch's constant for harmonic mappings (see [4,5,16,20]).

To investigate our main results, we shall use the following Lemmas.

Lemma 1.1. *[21] Let $h \in \mathcal{G}$ be of the form (1.2). Then for $|z| = r < 1$, the following statements are true*

(a) $\left|\dfrac{zh''(z)}{h'(z)}\right| \le \dfrac{r}{1-r}$,
(b) $1 - r \le |h'(z)| \le 1 + r$.

Both inequalities (a) and (b) are sharp, with equality for $h(z) = z - z^2/2$.

Lemma 1.2. *[23] Let $h \in \mathcal{G}$ be of the form (1.2). Then*

(a) $|a_n| \le [n(n-1)]^{-1}$ *for all $n \ge 2$. Equality holds for f_n such, that $f_n'(z) = (1 - z^{n-1})^{1/(n-1)}$, $n \ge 2$.*
(b) $\left|a_3 - \lambda a_2^2\right| \le \begin{cases} \lambda/4, & \text{for} \quad |\lambda| \ge 2/3, \\ 1/6, & \text{for} \quad |\lambda| < 2/3. \end{cases}$

The equality in the Fekete–Szegö functional $\left|a_3 - \lambda a_2^2\right|$ is attained in each case.

2. Main results

2.1. The harmonic pre-Schwarzian derivative

In this section we shall find bounds on the norm of the pre-Schwarzian derivative for functions in the class $\mathcal{G}^{\alpha}_{\mathcal{H}ar}$.

Theorem 2.1. *Let $\alpha \in (0,1)$ and $f \in \mathcal{G}^{\alpha}_{\mathcal{H}ar}$. The norm of the pre-Schwarzian derivative of f is bounded by*

$$\|T_f\| \le 1 + r_0 + \frac{2(1-\alpha^2)(1-r_0^2)(r_0+\alpha)}{(1+\alpha r_0)[(1+r_0^2)(1+\alpha^2) - 4\alpha\, r_0]}, \tag{2.1}$$

where $r_0 \in (0,1)$ is the unique root of the equation

$$\begin{aligned} &\alpha(1+\alpha^2)^2 r^5 + (-2\alpha^6 - 13\alpha^4 + 4\alpha^2 - 1)r^4 + \alpha(10\alpha^4 - 4\alpha^2 + 2)r^3 \\ &+ (8\alpha^6 - 14\alpha^4 + 20\alpha^2 - 6)r^2 + (-7\alpha^5 + 10\alpha^3 - 15\alpha)r + (2\alpha^6 - 9\alpha^4 + 8\alpha^2 + 3) = 0. \end{aligned}$$

Proof. If $f = h + \overline{g} \in \mathcal{G}^{\alpha}_{\mathcal{H}ar}$, then $h \in \mathcal{G}$. It is then clear from Lemma 1.1 that

$$|T_h| = \left|\frac{h''}{h'}\right| \le \frac{1}{1-r}, \quad |z| = r < 1.$$

From (1.10) and the above, we have

$$|T_f| = \left|\frac{h''}{h'} + \frac{2w'\overline{w}}{1+|w|^2}\right| \le |T_h| + \frac{2|w'||\overline{w}|}{1+|w|^2} \le \frac{1}{1-r} + \frac{2|w'||w|}{1+|w|^2}. \tag{2.2}$$

Using that, (1.6) and (1.7), we obtain

$$\begin{aligned}\|T_f\| &\le \sup_{0<r<1}(1-r^2)\left[\frac{1}{1-r}+\frac{2(1-\alpha^2)(1-r^2)(r+\alpha)}{(1-r^2)(1+\alpha r)[(1+r^2)(1+\alpha^2)-4\alpha r]}\right]\\ &= 1+r+\frac{2(1-\alpha^2)(1-r^2)(r+\alpha)}{(1+\alpha r)[(1+r^2)(1+\alpha^2)-4\alpha r]} =: F(r).\end{aligned}$$

We note that $F'(r)=0$ if and only if $G(r)=0$, where G is given as

$$\begin{aligned}G(r) &= \alpha(1+\alpha^2)^2r^5+(-2\alpha^6-13\alpha^4+4\alpha^2-1)r^4+2\alpha(5\alpha^4-2\alpha^2+1)r^3\\ &+ (8\alpha^6-14\alpha^4+20\alpha^2-6)r^2+(-7\alpha^5+10\alpha^3-15\alpha)r+(2\alpha^6-9\alpha^4+8\alpha^2+3).\end{aligned}$$

We have

$$G(0)=2\alpha^6-9\alpha^4+8\alpha^2+3=2\alpha^6+8\alpha^2(1-\alpha^2)+(1-\alpha^4)+2>0,$$

and

$$G(1)=-4(1-\alpha)^3(1+6\alpha+7\alpha^2+2\alpha^3)<0.$$

Then, there exists at least one $r_0\in(0,1)$ such, that $G(r_0)=0$. We will prove that the zero of G in $(0,1)$ is unique. In order to do this we will prove that G is decreasing in $(0,1)$, equivalently that $G'(r)<0$ in $(0,1)$. We have

$$\begin{aligned}G'(r) &= 5\alpha(1+\alpha^2)^2r^4+4(-2\alpha^6-13\alpha^4+4\alpha^2-1)r^3+3\alpha(10\alpha^4-4\alpha^2+2)r^2\\ &+ 2(8\alpha^6-14\alpha^4+20\alpha^2-6)r+(-7\alpha^5+10\alpha^3-15\alpha),\end{aligned}$$

and

$$G'(0)=(-7\alpha^5+10\alpha^3-15\alpha)<0,\quad G'(1)=-4(4+13\alpha+13\alpha^2+2\alpha^3)(1-\alpha)^3<0.$$

In order to determine whether $G'(r)<0$ in $(0,1)$ we define a function

$$L(x)=(1+x)^4G'\left(\frac{x}{1+x}\right)=d_0+d_1x+d_2x^2+d_3x^3+d_4x^4,$$

where the coefficients $d_0,d_1,...,d_4$ are negative, since

$$\begin{aligned}d_0 &= -7\alpha^5+10\alpha^3-15\alpha=-\alpha(7\alpha^4+10(1-\alpha^2)+5)<0,\\ d_1 &= 4\alpha^6-7\alpha^5-7\alpha^4+10\alpha^2-15\alpha-3\\ &= -4\alpha(1-\alpha^5)-7\alpha^5-7\alpha^4-10\alpha(1-\alpha)-\alpha-3<0,\\ d_2 &= 4\alpha^6-\alpha^5-7\alpha^4+4\alpha^3+10\alpha^2-7\alpha-3\\ &= -(1-\alpha)(4\alpha^5+3\alpha^4+4\alpha(1-\alpha^2)+6\alpha+1)<0,\\ d_3 &= 5\alpha^6+4\alpha^5-17\alpha^4+2\alpha^3+17\alpha^2-6\alpha-5\\ &= -(1-\alpha)(5\alpha^5+9\alpha^4+4\alpha^3+6\alpha^2+11\alpha+5)<0,\\ d_4 &= 2\alpha^6+7\alpha^5-20\alpha^4+2\alpha^3+14\alpha^2-\alpha-4\\ &= -(1-\alpha)^3(11\alpha^3+13\alpha^2+13\alpha+4)<0.\end{aligned}$$

Thus, there are no sign variation of L in the coefficients of L on $(0,1)$. Hence $G'\left(\frac{x}{1+x}\right)<0$ on $(0,\infty)$ and so $G'(r)<0$ on $(0,1)$ as $\phi(x)=x/(1+x)$ maps $(0,\infty)$ onto $(0,1)$. Thus G is decreasing for $0<\alpha<1$ and $r\in(0,1)$, which means that there exists the unique point $r_0\in(0,1)$ such that $G(r_0)=0$. The value $F(r_0)$ is the maximum point of F and the assertion is proved. $\square$

2.2. Estimate of Bloch constant

In this section, we shall find bounds on the Bloch constant for co-analytic part of functions in the class $\mathcal{G}^{\alpha}_{\mathcal{H}ar}$.

Theorem 2.2. *Let $\alpha \in (0,1)$, and let $f \in \mathcal{G}^{\alpha}_{\mathcal{H}ar}$. The Bloch constant $\mathcal{B}_f$ of f is bounded by*

$$\mathcal{B}_f \le \frac{(1+\alpha)(1-r_0^2)(1+r_0)^2}{1+\alpha r_0}, \tag{2.3}$$

where r_0 is the unique root of the equation $2-\alpha-6r^2-4(1+\alpha)r^3-3\alpha r^4=0$ in the interval $(0,1)$.

Proof. Let $f = h + \overline{g} \in \mathcal{G}^{\alpha}_{\mathcal{H}ar}$ and $h \in \mathcal{G}$. Using Lemma 1.1 along with (1.12) and (1.6), we obtain

$$\begin{aligned}\mathcal{B}_f &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)|h'(z)|\left(1+|w(z)|\right)\\ &\le \sup_{0<r<1} \left(1-r^2\right)(1+r)\left(1+\frac{r+\alpha}{1+\alpha r}\right) = (1+\alpha)\sup_{0<r<1} P(r),\end{aligned}$$

where

$$P(r) := \frac{(1-r^2)(1+r)^2}{1+\alpha r}.$$

The derivative of $P(r)$ is equal to zero if $Q(r)=0$ for $r \in (0,1)$, where

$$Q(r) = 2-\alpha-6r^2-4(1+\alpha)r^3-3\alpha r^4.$$

We note that $Q(0)=2-\alpha>0$ and $Q(1)=-8-8\alpha<0$ so that there exist a root $r_0 \in (0,1)$ such that $Q(r_0)=0$. Now it suffices to prove that r_0 is unique. It is enough to prove that the derivative $Q'(r)<0$ for $r \in (0,1)$ and $\alpha \in (0,1)$. This holds by virtue of the inequality $Q'(r) = -12r(1+(1+\alpha)r+\alpha r^2)<0$. Hence

$$\sup_{0<r<1} P(r) = \frac{(1-r_0^2)(1+r_0)^2}{1+\alpha r_0},$$

where r_0 is unique root of $Q(r)=0$ for $r \in (0,1)$. This proves the result. $\square$

2.3. Coefficient bounds

In this section we shall find bounds on coefficients and the Fekete–Szegö functional for the co-analytic parts of functions in the class $\mathcal{G}^{\alpha}_{\mathcal{H}ar}$.

Theorem 2.3. *If $f = h + \overline{g} \in \mathcal{G}^{\alpha}_{\mathcal{H}ar}$, where h and g are given by (1.2), then*

$$|b_2| \le \frac{1+\alpha-\alpha^2}{2}. \tag{2.4}$$

This estimate is sharp and the extremal function is

$$f(z) = \begin{cases} z-\frac{1}{2}z^2+\overline{(-1-\frac{1}{\alpha}+\frac{1}{\alpha^2})z+\frac{1}{2\alpha}z^2+(1-\frac{1}{\alpha}-\frac{1}{\alpha^2}+\frac{1}{\alpha^3})\ln(1-\alpha z)}, & \alpha \ne 0,\\ z-\frac{1}{2}z^2+\overline{\frac{1}{2}z^2-\frac{1}{3}z^3}, & \alpha = 0.\end{cases}$$

Moreover,

$$|b_n| \le \frac{1-\alpha^2}{n}\big(1+H_{n-2}\big) + \frac{\alpha}{n(n-1)} \quad (n=3,4,\ldots), \tag{2.5}$$

where H_n denotes n-th harmonic number $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$.

Proof. Applying the relations $g'(z) = w(z)h'(z)$ and the series expansions of h, g and w given by (1.2) and (1.5), we obtain

$$nb_n = \sum_{p=0}^{n-1}(p+1)a_{p+1}\,c_{n-p-1} \quad (n=2,3,\cdots), \tag{2.6}$$

where $c_n \in \mathbb{C}$, $\;n = 0,1,2,\ldots$, and $|c_0| = |w(0)| = |g'(0)| = |b_1| = \alpha$. For $n = 2$, we have

$$2b_2 = a_1c_1 + 2a_2c_0.$$

By (1.7) and Lemma 1.2, we see that

$$2|b_2| \le |a_1|\,|c_1| + 2|a_2|\,|c_0| \le (1-|c_0|^2) + |c_0| \le 1-\alpha^2+\alpha.$$

Now, we will show that the estimate is sharp. To this end let for $0 \le \alpha < 1$, consider a function $f(z) = h(z) + \overline{g(z)}$, such that $h(z) = z - \frac{1}{2}z^2 \in \mathcal{G}$ and the dilation of f is of the form

$$w(z) = \frac{z-\alpha}{1-\alpha z}, \quad z \in \mathbb{D}.$$

Under the condition $g'(z) = w(z)h'(z)$, we obtain for $z \in \mathbb{D}$

$$\begin{aligned} g'(z) &= \frac{-\alpha + (1+\alpha)z - z^2}{1-\alpha z} = -\frac{\alpha - (1+\alpha)z + z^2}{1-\alpha z} \\ &= -\alpha + (1+\alpha-\alpha^2)z + (-1+\alpha+\alpha^2-\alpha^3)z^2 + \ldots, \end{aligned}$$

that implies that the estimate (2.4) is sharp. Since $g(0) = 0$, by integration, we uniquely deduce for $z \in \mathbb{D}$ that

$$g(z) = \begin{cases} (-1 - \frac{1}{\alpha} + \frac{1}{\alpha^2})z + \frac{1}{2\alpha}z^2 + (1 - \frac{1}{\alpha} - \frac{1}{\alpha^2} + \frac{1}{\alpha^3})\ln(1-\alpha z), & \alpha \ne 0, \\ \frac{1}{2}z^2 - \frac{1}{3}z^3, & \alpha = 0. \end{cases}$$

Further, from (2.6), we obtain for $n = 3,4,\ldots$ that

$$\begin{aligned} |b_n| &\le \frac{1}{n}\sum_{p=0}^{n-1}(p+1)|a_{p+1}|\,|c_{n-p-1}| \\ &= \frac{1}{n}\left[|a_1|\,|c_{n-1}| + \sum_{p=1}^{n-2}(p+1)|a_{p+1}|\,|c_{n-p-1}| + n|a_n|\,|c_0|\right]. \end{aligned} \tag{2.7}$$

Applying (1.7), we obtain $|c_{n-p-1}| \leq 1 - |c_0|^2 = 1 - |b_1|^2 = 1 - \alpha^2$, hence

$$\begin{aligned}
|b_n| &\leq \frac{1}{n}|c_{n-1}| + \frac{1}{n}\sum_{p=1}^{n-2}(p+1)|a_{p+1}|\,|c_{n-p-1}| + |a_n|\,|c_0| \\
&\leq \frac{1-\alpha^2}{n} + \frac{1-\alpha^2}{n}\sum_{p=1}^{n-2}\frac{1}{p} + \frac{\alpha}{n(n-1)} \\
&< \frac{1-\alpha^2}{n} + \frac{1-\alpha^2}{n}H_{n-2} + \frac{\alpha}{n(n-1)} \\
&= \frac{1-\alpha^2}{n}[1 + H_{n-2}] + \frac{\alpha}{n(n-1)}.
\end{aligned}$$

This completes the proof of theorem. □

Theorem 2.4. *Let $f = h + \overline{g} \in \mathcal{G}^{\alpha}_{\mathcal{H}ar}$, where h and g are given by (1.2). Then for $\mu \in \mathbb{R}$, we have*

$$|b_3 - \mu b_2^2| \leq \frac{1-\alpha^2}{12}\left[4 + 3|\mu|(1-\alpha^2) + 2|2 - 3\mu b_1|\right] + \alpha \max\left\{\frac{|\mu b_1|}{4}, \frac{1}{6}\right\},$$

and

$$|b_{n+1} - b_n| \leq \frac{2\alpha}{n^2-1} + \frac{2(1-\alpha^2)(2n+1)}{n(n+1)} + (1-\alpha^2)\frac{n\log(n-1) + (n+1)\log(n-2)}{n(n+1)}.$$

Proof. From (2.6), we have

$$b_2 = \frac{1}{2}c_1 + a_2c_0 \quad \text{and} \quad b_3 = \frac{1}{3}c_2 + \frac{2}{3}a_2c_1 + a_3c_0,$$

since $a_1 = 1$. Therefore

$$\begin{aligned}
\left|b_3 - \mu b_2^2\right| &= \left|\frac{1}{3}c_2 + \frac{2}{3}a_2c_1 + a_3c_0 - \mu\left(\frac{1}{2}c_1 + a_2c_0\right)^2\right| \qquad (2.8)\\
&= \left|\frac{1}{3}c_2 - \frac{1}{4}\mu c_1^2 + a_2c_1\left(\frac{2}{3} - \mu c_0\right) + c_0\left(a_3 - \mu c_0 a_2^2\right)\right| \\
&\leq \frac{1}{3}\left|c_2 - \frac{3}{4}\mu c_1^2\right| + |a_2|\,|c_1|\,\left|\frac{2}{3} - \mu c_0\right| + |c_0|\,\left|a_3 - \mu c_0 a_2^2\right|.
\end{aligned}$$

By using the relation (1.7) along with Lemma 1.2, we obtain that

$$\begin{aligned}
\left|b_3 - \mu b_2^2\right| &\leq \frac{1}{3}\left[|c_2| + \frac{3}{4}|\mu c_1^2|\right] + |a_2|\,|c_1|\,\left|\frac{2}{3} - \mu c_0\right| + \alpha\left|a_3 - \mu b_1 a_2^2\right| \\
&\leq \frac{1}{3}\left[(1-\alpha^2) + \frac{3}{4}|\mu|(1-\alpha^2)^2\right] + \frac{(1-\alpha^2)}{2}\left|\frac{2}{3} - \mu b_1\right| + \alpha\max\left\{\frac{|\mu b_1|}{4}, \frac{1}{6}\right\} \\
&= \frac{1-\alpha^2}{12}\left[4 + 3|\mu|(1-\alpha^2) + 2|2 - 3\mu b_1|\right] + \alpha\max\left\{\frac{|\mu b_1|}{4}, \frac{1}{6}\right\}.
\end{aligned}$$

Further, using (2.6), we have

$$\begin{aligned}
|b_{n+1}-b_n| &= \left|\frac{1}{n+1}\sum_{p=1}^{n+1} pa_pc_{n-p+1} - \frac{1}{n}\sum_{p=1}^{n} pa_pc_{n-p}\right| \\
&= \left|\frac{1}{n+1}\sum_{p=1}^{n} pa_pc_{n-p+1} + a_{n+1}c_0 - \frac{1}{n}\sum_{p=1}^{n-1} pa_pc_{n-p} - a_nc_0\right| \\
&\leq |a_{n+1}|\,|c_0| + |a_n|\,|c_0| + \frac{1}{n+1}\sum_{p=1}^{n} p|a_p|\,|c_{n-p+1}| + \frac{1}{n}\sum_{p=1}^{n-1} p|a_p|\,|c_{n-p}| \\
&\leq \frac{\alpha}{n(n+1)} + \frac{\alpha}{n(n-1)} + \frac{|a_1|\,|c_n|}{n+1} + \frac{1}{n+1}\sum_{p=2}^{n} p|a_p|\,|c_{n-p+1}| \\
&\qquad + \frac{|a_1|\,|c_{n-1}|}{n} + \frac{1}{n}\sum_{p=2}^{n-1} p|a_p|\,|c_{n-p}| \\
&\leq \frac{2\alpha}{n^2-1} + \frac{1-\alpha^2}{n+1} + \frac{1-\alpha^2}{n+1}\sum_{p=2}^{n}\frac{1}{p-1} + \frac{1-\alpha^2}{n} + \frac{1-\alpha^2}{n}\sum_{p=2}^{n-1}\frac{1}{p-1} \\
&\leq \frac{2\alpha}{n^2-1} + \frac{1-\alpha^2}{n+1} + \frac{1-\alpha^2}{n+1}\sum_{p=1}^{n-1}\frac{1}{p} + \frac{1-\alpha^2}{n} + \frac{1-\alpha^2}{n}\sum_{p=1}^{n-2}\frac{1}{p} \\
&\leq \frac{2\alpha}{n^2-1} + \frac{1-\alpha^2}{n+1} + \frac{1-\alpha^2}{n+1}[1+\log(n-1)] + \frac{1-\alpha^2}{n} + \frac{1-\alpha^2}{n}[1+\log(n-2)] \\
&= \frac{2\alpha}{n^2-1} + \frac{1-\alpha^2}{n+1}[2+\log(n-1)] + \frac{1-\alpha^2}{n}[2+\log(n-2)],
\end{aligned}$$

where in the last but one inequality we use known inequality $H_n \leq 1 + \log n$ for harmonic number H_n (it can be also easily show by induction). This finishes the proof. □

Theorem 2.5. *Let $f = h + \overline{g} \in \mathcal{G}^{\alpha}_{\mathcal{H}ar}$, where h and g are given by (1.2). Then g is univalent, and for $|z| = r < 1$, the following statements hold true;*

$$\left|\frac{r\alpha(-2+(2-r)\alpha+2\alpha^2) - 2(1-\alpha^2)(1-\alpha)\log(1-r\alpha)}{2\alpha^3}\right| \leq |g(z)| \tag{2.9}$$
$$\leq \frac{2\alpha^3r^2 + 2\alpha(\alpha^2+\alpha-1)r + 2(1-\alpha^2)(1-\alpha)\log(1+r\alpha)}{2\alpha^3},$$

$$\frac{r}{1+r} - \frac{r(1-\alpha^2)}{(1-\alpha r)|r-\alpha|} \leq \left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{r}{1-r} + \frac{r(1-\alpha^2)}{(1-\alpha r)|r-\alpha|} \tag{2.10}$$

and

$$\Re\left(1 + \frac{zg''(z)}{g'(z)}\right) > \frac{r(\alpha^2-1)}{(1-\alpha r)|r-\alpha|} + \frac{1}{1+r}. \tag{2.11}$$

Proof. Repeated application of Lemma 1.1 along with the relation $g'(z) = w(z)h'(z)$ and (1.6) enables us to write

$$\frac{|\alpha - r|(1-r)}{1-\alpha r} \le |g'(z)| \le \frac{(\alpha+r)(1+r)}{1+\alpha r} \qquad (|z| = r < 1). \tag{2.12}$$

Integrating along a radial line $\zeta = te^{i\theta}$ the right hand side of (2.9) is obtained immediately.

In order to prove the left hand side of (2.9), we first note that $g(z)$ is univalent. Indeed, since $g'(z) = w(z)h'(z)$ then computing the logarithmic derivative, we get

$$\frac{zg''(z)}{g'(z)} = \frac{zw'(z)}{w(z)} + \frac{zh''(z)}{h'(z)}, \tag{2.13}$$

hence

$$1 + \frac{zg''(z)}{g'(z)} = \frac{zw'(z)}{w(z)} + 1 + \frac{zh''(z)}{h'(z)}. \tag{2.14}$$

Since we have

$$\Re\frac{zw'(z)}{w(z)} > \frac{1-\alpha}{1+\alpha}, \quad \text{and} \quad \Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2},$$

then

$$\Re\left(1 + \frac{zg''(z)}{g'(z)}\right) = \Re\left(1 + \frac{zh''(z)}{h'(z)}\right) + \Re\frac{zw'(z)}{w(z)} > -\frac{1}{2} + \frac{1-\alpha}{1+\alpha} > -\frac{1}{2},$$

that is sufficient for univalence of the function g [30].

Let $\Gamma = g(\{z : |z| = r\})$ and let $\xi_1 \in \Gamma$ be the nearest point to the origin. By a rotation we may assume that $\xi_1 > 0$. Let γ be the line segment $0 \le \xi \le \xi_1$ and suppose that $z_1 = g^{-1}(\xi_1)$ and $L = g^{-1}(\gamma)$ with ζ as the variable of integration on L, we have that $d\xi = g'(\zeta)d\zeta > 0$ on L. Hence

$$\begin{aligned}\xi_1 &= \int_0^{\xi_1} d\xi = \int_0^{z_1} g'(\zeta)d\zeta = \int_0^{z_1} |g'(\zeta)|\,|d\zeta| \ge \int_0^{r} |g'(te^{i\theta})|dt \\ &\ge \int_0^{r} \frac{|\alpha - r|(1-r)}{(1-\alpha r)}dr = \left|\frac{r\alpha(-2 + (2-r)\alpha + 2\alpha^2) - 2(1-\alpha^2)(1-\alpha)\log(1-r\alpha)}{2\alpha^3}\right|.\end{aligned}$$

From (1.6) and (1.7), we conclude that

$$|w'(z)| \le \frac{1-\alpha^2}{(1-\alpha r)^2}. \tag{2.15}$$

The relation (2.13) together with Lemma 1.1 (2.15) gives for $|z| = r < 1$

$$\left|\frac{zg''(z)}{g'(z)}\right| \le \left|\frac{zh''(z)}{h'(z)}\right| + \left|\frac{zw'(z)}{w(z)}\right| \le \frac{(1-\alpha^2)r}{(1-\alpha r)|r-\alpha|} + \frac{r}{1-r}.$$

In view of [23, Corollary 1], we have

$$\left|\frac{zg''(z)}{g'(z)}\right| \ge \left|\frac{zh''(z)}{h'(z)}\right| - \left|\frac{zw'(z)}{w(z)}\right| \ge \frac{r}{1+r} - \frac{r(1-\alpha^2)}{(1-\alpha r)|r-\alpha|}.$$

By the properties of $h \in \mathcal{G}$, and (2.14), we have

$$\Re\left(1+\frac{zg''(z)}{g'(z)}\right)=\Re\left(\frac{zw'(z)}{w(z)}\right)+\Re\left(1+\frac{zh''(z)}{h'(z)}\right)>\Re\left(\frac{zw'(z)}{w(z)}\right)+\frac{1}{1+r}.$$

The relation (1.6) and (1.7) therefore yield

$$\Re\left(1+\frac{zg''(z)}{g'(z)}\right)>\frac{r(\alpha^2-1)}{(1-\alpha r)|\alpha-r|}+\frac{1}{1+r},$$

and the assertion follows. □

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