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Critical exponents for the fast diffusion equation with a nonlinear boundary condition

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ABSTRACT

In this paper we consider the fast diffusion equation $\partial_t u = \Delta(u^m)$ ($x \in \Omega$, $t > 0$) with a nonlinear boundary condition $\partial_\nu u^m = u^p$ ($x \in \partial\Omega$, $t > 0$), where $0 < m < 1$, $p > 0$, $\Omega \subset \mathbf{R}^N$ is a smooth domain and $N \geq 1$. We prove that $p_0 = (m+1)/2$ is the critical global existence exponent for the cases $\Omega = \mathbf{R}^N \setminus \overline{B_1}$ ($N \geq 2$) and $\Omega = B_1 := \{x \in \mathbf{R}^N : |x| < 1\}$ ($N \geq 1$).

1. Introduction

In this paper, we study blow-up of solutions of the fast diffusion equation with a nonlinear boundary condition

$$\begin{cases} \partial_t u = \Delta(u^m), & x \in \Omega, t > 0, \\ \partial_\nu(u^m) = u^p, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}. \end{cases} \tag{1.1}$$

Here $0 < m < 1$, $p > 0$, $\Omega \subset \mathbf{R}^N$ is a smooth domain, $N \geq 1$, $\nu = \nu_x$ is the outer unit normal vector at $x \in \partial\Omega$ and $u_0 \in C^1(\overline{\Omega})$. We always assume that the initial data u_0 is a nonnegative, nontrivial and bounded function satisfying the compatibility condition $\partial_\nu(u_0^m) = u_0^p$ on $\partial\Omega$.

Critical exponents for blow-up of solutions of the problem (1.1) with $m > 0$ have been studied by many authors. We refer to the survey papers [2,4] and references therein. For $m > 1$, Galaktionov and Levine [6] considered the case $N = 1$, $\Omega = \mathbf{R}_+ := \{x \in \mathbf{R} : x > 0\}$ and showed that if

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$$0 < p \leq p_0 := \frac{m+1}{2},$$

then all nonnegative solutions exist globally in time, while if $p_0 < p \leq p_F(\mathbf{R}_+) := m+1$, then all solutions blow up in finite time. Furthermore, if $p > p_F(\mathbf{R}_+)$, then a solution exists globally in time for some initial data. They call p_0 and $p_F(\mathbf{R}_+)$ the critical global existence exponent and the critical exponent of Fujita type, respectively. Note that critical exponents of Fujita type for the problem (1.1) depend on the domain Ω, see [1,7,10] for other domains.

For the fast diffusion case $0 < m < 1$ under the condition that Ω is bounded, Filo [5] considered the case $N = 1$. Wolanski [11] extended the results in [5] to the case $\Omega = B_R := \{x \in \mathbf{R}^N; |x| < R\}$ ($R > 0$, $N \geq 1$) and showed that if $0 < p \leq p_0 = (m+1)/2$, then all solutions exist globally in time. Moreover, if $p > p_0$, then all solutions blow up in finite time. This result says that the critical global existence exponent is p_0 and that the critical exponent of Fujita type $p_F(B_R)$ is infinity.

In the case where $0 < m < 1$ and Ω is unbounded, Ferreira, de Pablo, Quirós and Rossi [3] studied critical exponents and the blow-up profile of solutions of (1.1) with $N = 1$, $\Omega = \mathbf{R}_+$. In particular, they showed that the critical global existence exponent and the critical exponent of Fujita type $p_F(\mathbf{R}_+)$ coincide with the case $m > 1$, respectively. Pang, Wang and Yin [9] considered the case $N \geq 3$, $\Omega = \Omega_e := \mathbf{R}^N \setminus \overline{B_1}$. It was shown that all positive solutions exist globally in time if $0 < p < p_0$. On the other hand, if $p > p_0$, there both exist a finite time blowing up solution and a globally in time solution. This means that, for $\Omega = \Omega_e$, the number p_0 is the critical global existence exponent and it coincides with the critical exponent of Fujita type $p_F(\Omega_e)$, that is, $p_0 = p_F(\Omega_e) = (m+1)/2$. Note that they handled neither the case $N = 2$ nor the critical case $p = p_0$.

Our main theorem deals with the case $N \geq 2$, $\Omega = \Omega_e$ and shows that the critical case $p = p_0$ belongs to the global existence case.

Theorem 1.1. *Let $N \geq 2$, $0 < m < 1$ and $p > 0$. Assume $\Omega = \Omega_e$. Then the following* (i), (ii) *and* (iii) *hold for the problem* (1.1).

(i) *If $p \leq p_0$, then every solution exists globally in time.*
(ii) *If $p > p_0$, then a solution blows up in finite time for some u_0.*
(iii) *If $p > p_0$, then a solution exists globally in time for some u_0.*

In order to prove this result, we construct comparison functions by modifying the functions employed in de Pablo, Quirós and Rossi [8] and Ferreira, de Pablo, Quirós and Rossi [3]. We note that our method also works for the case $\Omega = B_1$. Indeed, we can give another proof of the next theorem.

Theorem 1.2 *([11]). Let $N \geq 1$, $0 < m < 1$ and $p > 0$. Assume $\Omega = B_1$. Then the following* (i) *and* (ii) *hold for the problem* (1.1).

(i) *If $p \leq p_0$, then every solution exists globally in time.*
(ii) *If $p > p_0$, then every solution blows up in finite time.*

The rest of this paper is devoted to proving the above theorems. In Section 2, we show Theorem 1.1. In Section 3, we give a proof of Theorem 1.2.

2. Proof of Theorem 1.1

In order to prove Theorem 1.1 (i), (ii) and (iii), we modify the functions in [8, Proof of Theorem 1.1 (i)], [3, Proof of Theorem 1.1 (ii)] and [3, Proof of Theorem 1.1 (iii)], respectively.

Proof of Theorem 1.1 (i). We construct a supersolution. For $A > 0$, define $\overline{u}$ by

$$\begin{aligned}\overline{u}(x,t) &:= e^{\alpha t} f(\xi), \quad \xi := (|x|-1)e^{\beta t},\\ f(\xi) &:= (A + e^{-\gamma\xi})^{\frac{1}{m}},\end{aligned} \tag{2.1}$$

where

$$\alpha := (A+1)^2, \quad \beta := \frac{1-m}{2}(A+1)^2, \quad \gamma := (A+1)^{\frac{p}{m}}. \tag{2.2}$$

The constant A will be chosen later.

Let us first consider the first equation in (1.1). By simple computations, we have

$$\partial_t \overline{u} = e^{\alpha t}(\alpha f + \beta\xi f'), \quad \Delta(\overline{u}^m) = e^{(\alpha m + 2\beta)t}\left((f^m)'' + \frac{N-1}{|x|e^{\beta t}}(f^m)'\right).$$

Since $\alpha = \alpha m + 2\beta$ and $(f^m)' \leq 0$, we have

$$\partial_t \overline{u} - \Delta(\overline{u}^m) \geq e^{\alpha t}(\alpha f + \beta \xi f' - (f^m)'').$$

It is easy to see that

$$\alpha f = (A+1)^2(A + e^{-\gamma\xi})^{\frac{1}{m}} \geq A^{2+\frac{1}{m}}.$$

Since $\gamma > 0$, we have $\max_{\xi \geq 0}(\xi e^{-\gamma\xi}) = e^{-1}\gamma^{-1}$, and so

$$\begin{aligned}\beta\xi f' &= -\frac{\beta\gamma}{m}(A+e^{-\gamma\xi})^{\frac{1}{m}-1}\xi e^{-\gamma\xi}\\ &\geq -\frac{\beta}{me}(A+e^{-\gamma\xi})^{\frac{1}{m}-1}\\ &\geq -\frac{1-m}{2me}(A+1)^{1+\frac{1}{m}}.\end{aligned}$$

Recall $p \leq p_0$. Then,

$$(f^m)'' = \gamma^2 e^{-\gamma\xi} \leq (A+1)^{\frac{2p}{m}} \leq (A+1)^{1+\frac{1}{m}}.$$

From the above estimates, it follows that

$$\partial_t \overline{u} - \Delta(\overline{u}^m) \geq A^{2+\frac{1}{m}} - \frac{1-m}{2me}(A+1)^{1+\frac{1}{m}} - (A+1)^{1+\frac{1}{m}}.$$

Hence there exists a positive constant $A_0 = A_0(m)$ such that

$$\partial_t \overline{u} \geq \Delta(\overline{u}^m), \quad x \in \overline{\Omega}, t > 0 \tag{2.3}$$

for any $A \geq A_0$.

We next examine the second equation in (1.1). Straightforward calculations show that

$$\partial_\nu(\overline{u}^m) = -e^{(\alpha m+\beta)t}(f^m)' = \gamma e^{(\alpha m + \beta)t}e^{-\gamma\xi}, \quad \overline{u}^p = e^{p\alpha t}(A + e^{-\gamma\xi})^{\frac{p}{m}}.$$

Thus, the relation $\xi = 0$ for $|x| = 1$ yields

$$(\partial_\nu(\overline{u}^m) - \overline{u}^p)\big|_{|x|=1} = \gamma e^{(\alpha m+\beta)t} - e^{p\alpha t}(A+1)^{\frac{p}{m}}.$$

Since $p \le p_0$, we have $\alpha m + \beta \ge p\alpha$. This together with $\gamma = (A+1)^{p/m}$ gives

$$\partial_\nu(\overline{u}^m) \ge \overline{u}^p, \quad x \in \partial\Omega, t > 0 \tag{2.4}$$

for any $A > 0$.

Finally, the definition of $\overline{u}$ yields $\overline{u}(x,0) \ge A^{1/m}$. Then, by choosing a positive constant A_1 such that $A_1 \ge \max\{A_0, \sup_{|x|\ge 1} u_0(x)^m\}$, we can see that $\overline{u}$ is a supersolution of (1.1) provided that $A \ge A_1$. Hence, for any u_0, every solution u satisfies $u \le \overline{u}$ in $\overline{\Omega} \times (0,\infty)$ and exists globally in time. The proof is complete. □

Proof of Theorem 1.1 (ii). We construct a subsolution of the form

$$\begin{gathered}\underline{u}(x,t) := (T-t)^{-\alpha} g(\eta), \quad \eta := (|x|-1)(T-t)^{-\beta}, \\ g(\eta) := (A\eta + A^{-c})^{-\gamma}.\end{gathered} \tag{2.5}$$

The constant $A > 1$ will be chosen later. Let $p > p_0$. Set

$$\alpha := \frac{1}{2p-(1+m)}, \quad \beta := \frac{p-m}{2p-(1+m)}, \quad T^\beta = A^{-1} (< 1). \tag{2.6}$$

Fix γ and c such that

$$\gamma > \max\left\{\frac{2}{1-m}, \frac{\sqrt{2\alpha}}{m}, \frac{2(N-1)}{m}\right\} \tag{2.7}$$

$$\frac{1}{(p-m)\gamma - 1} < c \le \frac{2}{(1-m)\gamma - 2}. \tag{2.8}$$

Remark that such c exists if $p > p_0$.

Let us first consider the first equation in (1.1). By $-\alpha - 1 = -\alpha m - 2\beta$ and $g' \le 0$, we have

$$\begin{aligned}\partial_t \underline{u} - \Delta(\underline{u}^m) &= (T-t)^{-\alpha-1}\left(\alpha g + \beta\eta g' - (g^m)'' - \frac{N-1}{|x|}(T-t)^\beta (g^m)'\right) \\ &\le (T-t)^{-\alpha-1}\left(\alpha g - (g^m)'' - \frac{N-1}{|x|}(T-t)^\beta (g^m)'\right).\end{aligned} \tag{2.9}$$

By the definition of g, we have

$$\begin{aligned}-(g^m)'' &= -\gamma m(\gamma m+1)A^2(A\eta + A^{-c})^{-\gamma m-2} \\ &\le -\gamma^2 m^2 A^2 (A\eta + A^{-c})^{-\gamma m - 2}.\end{aligned} \tag{2.10}$$

From the estimate

$$\frac{(T-t)^\beta}{|x|} = \frac{1}{(T-t)^{-\beta} + \eta} \le \frac{1}{T^{-\beta} + \eta} = \frac{1}{A+\eta} \quad \text{for } |x| > 1,$$

it follows that

$$
\begin{aligned}
-\frac{N-1}{|x|}(T-t)^{\beta}(g^m)' &= \gamma m(N-1)\frac{(T-t)^{\beta}}{|x|}A(A\eta+A^{-c})^{-\gamma m-1}\\
&\le \gamma m(N-1)\frac{A}{A+\eta}(A\eta+A^{-c})^{-\gamma m-1}.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
(T-t)^{\alpha+1}(\partial_t\underline{u}-\Delta(\underline{u}^m)) &\le \alpha(A\eta+A^{-c})^{-\gamma}-\gamma^2m^2A^2(A\eta+A^{-c})^{-\gamma m-2}\\
&\quad+\gamma m(N-1)\frac{A}{A+\eta}(A\eta+A^{-c})^{-\gamma m-1}.
\end{aligned}
\tag{2.11}
$$

We claim that the following estimates hold.

$$
\alpha(A\eta+A^{-c})^{-\gamma} \le \frac{1}{2}\gamma^2m^2A^2(A\eta+A^{-c})^{-\gamma m-2}, \tag{2.12}
$$

$$
\gamma m(N-1)\frac{A}{A+\eta}(A\eta+A^{-c})^{-\gamma m-1} \le \frac{1}{2}\gamma^2m^2A^2(A\eta+A^{-c})^{-\gamma m-2}. \tag{2.13}
$$

Note that once we prove these inequalities, we immediately see that the right-hand side of (2.11) is smaller than or equal to 0. (2.12) and (2.13) will be proved once we prove the following equivalent inequalities.

$$
\begin{aligned}
&\frac{m^2A^2}{2\alpha}\gamma^2(A\eta+A^{-c})^{\gamma(1-m)-2}\ge 1,\\
&\left(\frac{1}{2}\gamma m-(N-1)\right)A\eta+\frac{1}{2}\gamma mA^2-(N-1)A^{-c}\ge 0.
\end{aligned}
$$

First, by $A>1$, (2.7) and (2.8), we have

$$
\frac{m^2A^2}{2\alpha}\gamma^2(A\eta+A^{-c})^{\gamma(1-m)-2} \ge \frac{m^2}{2\alpha}\gamma^2A^{2-c(\gamma(1-m)-2)} \ge \frac{m^2}{2\alpha}\gamma^2 \ge 1.
$$

Next, from $A>1$, $c>0$ and (2.7), it follows that

$$
\left(\frac{1}{2}\gamma m-(N-1)\right)A\eta+\frac{1}{2}\gamma mA^2-(N-1)A^{-c} \ge \frac{1}{2}\gamma m-(N-1)\ge 0,
$$

and the claim follows. As noted above, the inequality

$$
\partial_t\underline{u}\le\Delta(\underline{u}^m), \quad x\in\overline{\Omega}, t\in(0,T)
$$

holds for any $A>1$.

We next consider the boundary condition. From the facts that $-\alpha m-\beta=-\alpha p$ and $\eta=0$ for $|x|=1$, it follows that

$$
\begin{aligned}
&(\partial_\nu(\underline{u}^m)-\underline{u}^p)\big|_{|x|=1}\\
&=(T-t)^{-\alpha p}\left(m\gamma A(A\eta+A^{-c})^{-m\gamma-1}-(A\eta+A^{-c})^{-p\gamma}\right)\Big|_{\eta=0}\\
&=(T-t)^{-\alpha p}A^{1+c(m\gamma+1)}\left(m\gamma-A^{c((p-m)\gamma-1)-1}\right).
\end{aligned}
\tag{2.14}
$$

By (2.8), we have $c((p-m)\gamma-1)-1>0$. Therefore, there exists a constant $A_0:=A_0(m,N,p,\gamma)>1$ such that

$$\partial_\nu(\underline{u}^m) \le \underline{u}^p, \quad x \in \partial\Omega, t \in (0,T) \tag{2.15}$$

for any $A \ge A_0$. Now we fix $A \ge A_0$ and choose an initial data u_0 such that $u_0 \ge \underline{u}(\cdot,0)$ on $\mathbf{R}^N \setminus B_1$. Then $\underline{u}$ is a subsolution of the problem (1.1), and so every solution u with $u(\cdot,0) = u_0 \ge \underline{u}(\cdot,0)$ blows up at some $t = T'$ with $T' \le T < +\infty$. This proves Theorem 1.1 (ii). $\square$

Proof of Theorem 1.1 (iii). We construct a supersolution of the form

$$\overline{u}(x,t) := (b+t)^{-\alpha} h(\zeta), \quad \zeta := (|x|-1)(b+t)^{-\beta},$$
$$h(\zeta) := k(A + a(\zeta+1)^2)^{-\frac{1}{1-m}},$$

where $\alpha = \alpha(m,p)$ and $\beta = \beta(m,p)$ are given by (2.6) and

$$a := \frac{1-m}{m}\alpha, \quad b^{-\beta} = \frac{4}{1-m}.$$

The constants A and $0 < k < 1$ are chosen later. By similar computations to (2.9), $h', (h^m)' \le 0$ and the estimate

$$\frac{(b+t)^\beta}{|x|} = \frac{1}{(b+t)^{-\beta} + \zeta} \ge \frac{1}{b^{-\beta} + \zeta} \quad \text{for } |x| > 1,$$

we have

$$\begin{aligned} \partial_t \overline{u} - \Delta(\overline{u}^m) &= (b+t)^{-\alpha-1}\left(-\alpha h - \beta\zeta h' - (h^m)'' - \frac{N-1}{|x|}(b+t)^\beta (h^m)'\right) \\ &\ge (b+t)^{-\alpha-1}\left(-\alpha h - (h^m)'' - \frac{N-1}{b^{-\beta}+\zeta}(h^m)'\right). \end{aligned} \tag{2.16}$$

The definition of h shows that

$$\begin{aligned} &-\alpha h - (h^m)'' - \frac{N-1}{b^{-\beta}+\zeta}(h^m)' \\ &= (A + a(\zeta+1)^2)^{-\frac{1}{1-m}}\left(-\alpha k - \frac{4m}{(1-m)^2}a^2 k^m (\zeta+1)(A + a(\zeta+1)^2)^{-1}\right. \\ &\quad \left. + \frac{2m}{1-m}ak^m + \frac{N-1}{b^{-\beta}+\zeta} \times \frac{2m}{1-m}ak^m(\zeta+1)\right). \end{aligned}$$

By the choice of a, we have

$$\frac{m}{1-m}ak^m \ge \alpha k \quad \text{for any } 0 < k < 1,$$

and so

$$-\alpha h - (h^m)'' - \frac{N-1}{b^{-\beta}+\zeta}(h^m)' \ge (A + a(\zeta+1)^2)^{-\frac{1}{1-m}} \frac{m}{1-m} ak^m H(\zeta), \tag{2.17}$$

where

$$
\begin{aligned}
H(\zeta) &:= 1 + \frac{N-1}{b^{-\beta}+\zeta} 2(\zeta+1) - \frac{4}{1-m} a(\zeta+1)(A+a(\zeta+1)^2)^{-1} \\
&= \frac{(A+a(\zeta+1)^2)^{-1}}{b^{-\beta}+\zeta} \Big((b^{-\beta}+\zeta)(A+a(\zeta+1)^2) \\
&\qquad + 2(N-1)(\zeta+1)(A+a(\zeta+1)^2) - \frac{4}{1-m} a(\zeta+1)(b^{-\beta}+\zeta) \Big).
\end{aligned}
$$

We claim that there exists a constant $A_0 = A_0(m, N, p) > 0$ independent of k such that for any $A \geq A_0$, we have

$$
H(\zeta) \geq 0 \text{ for any } \zeta \geq 0. \tag{2.18}
$$

Direct expansion gives

$$
\begin{aligned}
&(b^{-\beta}+\zeta)(A+a(\zeta+1)^2) + 2(N-1)(\zeta+1)(A+a(\zeta+1)^2) - \frac{4}{1-m} a(\zeta+1)(b^{-\beta}+\zeta) \\
&= (2N-1)a\zeta^3 + \left(b^{-\beta} + 6N - 4 - \frac{4}{1-m} \right) a\zeta^2 \\
&\quad + \left(2b^{-\beta}a + (2N-1)A + (6N-5)a - \frac{4}{1-m} a(b^{-\beta}+1) \right) \zeta^1 \\
&\quad + b^{-\beta}(A+a) + 2(N-1)(A+a) - \frac{4}{1-m} ab^{-\beta}.
\end{aligned}
$$

By $b^{-\beta} = 4/(1-m)$, we have

$$
\begin{aligned}
&(b^{-\beta}+\zeta)(A+a(\zeta+1)^2) + 2(N-1)(\zeta+1)(A+a(\zeta+1)^2) - \frac{4}{1-m} a(\zeta+1)(b^{-\beta}+\zeta) \\
&= (2N-1)a\zeta^3 + (6N-4)a\zeta^2 \\
&\quad + \left((2N-1)A + \left(\frac{4}{1-m} + 6N - 5 - \frac{16}{(1-m)^2} \right) a \right) \zeta^1 \\
&\quad + \left(\left(\frac{4}{1-m} + 2(N-1) \right) A + \left(\frac{4}{1-m} + 2(N-1) - \frac{16}{(1-m)^2} \right) a \right) \zeta^0.
\end{aligned}
$$

We see that each of the coefficients of η^3, η^2, η^1 and η^0 is positive if $A > 1$ is large. Hence the claim follows. By (2.16), (2.17) and (2.18), we obtain

$$
\partial_t \overline{u} \geq \Delta(\overline{u}^m), \quad x \in \overline{\Omega}, t > 0
$$

for any $A \geq A_0$ and $0 < k < 1$. Fix $A \geq A_0$.

Next, we consider the boundary condition. Since $-\alpha m - \beta = -\alpha p$ and $\zeta = 0$ for $|x| = 1$, we have

$$
\begin{aligned}
&(\partial_\nu(\overline{u}^m) - \overline{u}^p)|_{|x|=1} \\
&= (b+t)^{-\alpha p} \Big(\frac{2m}{1-m} ak^m (A+a(\zeta+1)^2)^{-\frac{1}{1-m}} (\zeta+1) \\
&\qquad - k^p (A+a(\zeta+1)^2)^{-\frac{p}{1-m}} \Big) \Big|_{\zeta=0} \\
&= (b+t)^{-\alpha p} \Big(2\frac{m}{1-m} ak^m (A+a)^{-\frac{1}{1-m}} - k^p (A+a)^{-\frac{p}{1-m}} \Big).
\end{aligned}
$$

Since m, a, p, A are fixed and $m < p$, there exists a positive constant $k_0 = k_0(A, m, p) < 1$ such that

$$2\frac{m}{1-m}ak^m(A+a)^{-\frac{1}{1-m}} - k^p(A+a)^{-\frac{p}{1-m}} \geq 0$$

for any $0 < k \leq k_0$, and so

$$\partial_\nu(\overline{u}^m) \geq \overline{u}^p, \quad x \in \partial\Omega, t > 0$$

for any $0 < k \leq k_0$. We fix $0 < k \leq k_0$ and choose u_0 such that $u_0 \leq \overline{u}(\cdot, 0)$ on $\overline{\Omega}$. Then $\overline{u}$ is a supersolution of (1.1), and every solution u with $u(\cdot, 0) = u_0 \leq \overline{u}(\cdot, 0)$ exists globally in time. The proof is complete. $\square$

3. Proof of Theorem 1.2

We construct comparison functions by modifying (2.1) and (2.5) and show Theorem 1.2.

Proof of Theorem 1.2. Let us first prove (i). Define $\overline{u}$ by

$$\overline{u}(x,t) := e^{\alpha t}F(\xi), \quad \xi := (1-|x|^2)e^{\beta t},$$
$$F(\xi) := (A + e^{-\gamma\xi})^{\frac{1}{m}}$$

for $A > 0$, where α, β and γ are given by (2.2). Simple calculations yield

$$\partial_t\overline{u} - \Delta(\overline{u}^m) = e^{\alpha t}\left(\alpha F + \beta\xi F' - 4|x|^2(F^m)'' + \frac{2N}{e^{\beta t}}(F^m)'\right).$$

Similar computations to the proof of Theorem 1.1 (i) show that

$$\alpha F \geq A^{2+\frac{1}{m}}, \quad \beta\xi F' \geq -\frac{1-m}{2me}(A+1)^{1+\frac{1}{m}},$$
$$-4|x|^2(F^m)'' = -4|x|^2\gamma^2 e^{-\gamma\xi} \geq -4(A+1)^{\frac{2p}{m}} \geq -4(A+1)^{1+\frac{1}{m}},$$
$$\frac{2N}{e^{\beta t}}(F^m)' = -2N\gamma e^{-\beta t - \gamma\xi} \geq -2N(A+1)^{\frac{p}{m}} \geq -2N(A+1)^{1+\frac{1}{m}}.$$

Then there exists a positive constant $A_0 = A_0(m, N)$ such that (2.3) holds for any $A \geq A_0$. Furthermore, since $\nu = x/|x|$, we have

$$(\partial_\nu\overline{u} - u^p)\big|_{|x|=1} = 2\gamma e^{(\alpha m+\beta)t} - e^{-\alpha pt}(A+1)^{\frac{p}{m}} \geq 0.$$

Hence (2.4) holds. The rest of the proof of Theorem 1.2 (i) is the same as that of Theorem 1.1 (i).

We next show (ii). Let $T > 0$. By modifying (2.5), we define

$$\underline{u}(x,t) := (T-t)^{-\alpha}G(\eta), \quad \eta := (1-|x|)(T-t)^{-\beta},$$
$$G(\eta) := (A\eta + A^{-c})^{-\gamma}$$

for $A > 1$, where α, β, γ and c are chosen so that (2.6), (2.7) and (2.8) hold. Similar calculations to (2.9) and (2.10) show that

$$\begin{aligned}(T-t)^{\alpha+1}(\partial_t\underline{u} - \Delta(\underline{u}^m)) &= \alpha G + \beta\eta G' - (G^m)'' + \frac{N-1}{|x|}(T-t)^\beta(G^m)' \\ &\leq \alpha G - (G^m)'' \\ &\leq \alpha(A\eta + A^{-c})^{-\gamma} - \gamma^2 m^2 A^2(A\eta + A^{-c})^{-\gamma m - 2}.\end{aligned}$$

The right-hand side is nonnegative, since (2.12) holds for $A > 1$. On the other hand, by $\nu = x/|x|$, (2.14) holds. Then there exists a constant $A_0 := A_0(m, N, p) > 1$ such that (2.15) holds for $A \geq A_0$. Fix $A \geq A_0$. Note that A is independent of T and that

$$\underline{u}(x,0) = T^{-\alpha}\left(A\frac{1-|x|}{T^\beta} + A^{-c}\right)^{-\gamma} \leq T^{-\alpha}A^{c\gamma}. \tag{3.1}$$

Let u be a solution of (1.1) with $u(\cdot,0) = u_0$. There exists $t_0 > 0$ such that $u(x,t_0) > 0$ on $\overline{B_1}$. Furthermore, since $u(\cdot,t_0) \in C(\overline{B_1})$, we obtain $\min_{\overline{\Omega}} u(\cdot,t_0) > 0$. By (3.1), we can choose $T > 0$ so large that

$$\underline{u}(x,0) \leq \min_{\overline{\Omega}} u(\cdot,t_0) \leq u(x,t_0), \quad x \in \overline{B_1}.$$

Fix such a T. Then $\underline{u}(x,t)$ is a subsolution of $u(x,t_0+t)$, and so u blows up in finite time. Thus (ii) follows, and the proof is complete. □

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