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Attractors for the degenerate Kirchhoff wave model with strong damping: Existence and the fractal dimension

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In this paper, we study the long time behavior of the Kirchhoff type wave equation in the space $H_0^1(\Omega) \times L^2(\Omega)$. We prove the existence of the global attractor for the equation which covers the case of possible generation of the stiffness coefficient. We also consider the geometrical property of the global attractor. By means of the Z_2 index, we provide, under suitable assumptions, the fractal dimension of the global attractor is infinite.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following Kirchhoff wave equation:

$$\begin{cases} \partial_{tt}u - \delta\Delta\partial_t u - \phi(\|\nabla u\|^2)\Delta u + f(u) = h(x), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.1}$$

Here $h \in L^2(\Omega)$ is an external force term, $f(u)$ is a given source term and ϕ is a nonlinear scalar function specified later, $\delta > 0$ is a constant.

One dimensional model (1.1) without the damping term was introduced by Kirchhoff [19] to describe small vibrations of an elastic stretched string. The original model is

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$$\rho h \frac{\partial^2 u}{\partial t^2} = \{a_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\} \frac{\partial^2 u}{\partial x^2} + f, \tag{1.2}$$

where $u = u(x,t)$ is the transverse deflection, $0 < x < L$ the space co-ordinate, $t \geq 0$ the time, E the Young's modulus, ρ the mass density, h the cross-section area, L the length, a_0 the initial axial tension, and f the external force. This kind of models have been studied by many authors, see [4,23,28,37] and references therein.

In general, we call the Kirchhoff equation non-degenerate if the stiffness ϕ satisfies the strict hyperbolicity condition

$$\phi(s) \geq \mu_1 > 0, \quad \forall\ s \geq 0.$$

And we call it degenerate if ϕ just satisfies the degenerate hyperbolicity condition

$$\phi(s) \geq 0, \quad \forall\ s \geq 0.$$

In addition, if $\phi(s) \geq 0$ but $\phi(\|\nabla u_0\|^2) > 0$, we call it mildly degenerate. It is easy to see that the degenerate stiffness coefficient $\phi(s)$ in Kirchhoff equation corresponds to the case that the initial axial tension equals zero.

For Kirchhoff type model without damping, well-posedness issues were studied intensively in decades. The reader is referred to [1,2,4] about the local existence results. Global existence has been proved by many authors in heterogeneous sets of initial data, such as analytic data or quasi-analytic data [8,14,33], small data [9,16,27,44], etc. This model with boundary damping is first considered in [42] for $\Omega \subset \mathbb{R}$. Some years later, [29,35] generalised the results in [42] to n-dimensional models. In [21], the authors further studied the n-dimensional model with a nonlinear boundary damping. For the Kirchhoff wave equation with weak interior damping, global existence has been studied both in the strictly hyperbolic case (see [5,39,43]) and in the mildly degenerate case (see [13,34]). However, at the present, the global in time solvability in usual Sobolev spaces is open in the theory of Kirchhoff equations in these cases.

Global existence results of the Kirchhoff equation with strong damping $u_{tt} - m(\|\nabla u\|^2)\Delta u - \Delta u_t = 0$ was first obtained by Nishihara [32], as we know. From the physical point of view, the dissipativeness plays an important spreading role for the energy gather arising from the nonlinearity in real process. At the same time, the strong damping term $-\Delta u_t$ provides an additional a priori estimate as viewed from mathematics. In recent years many mathematicians and physicists paid their attentions in this type of problem under different types of hypotheses. We refer to [6,15,26,30,36,38] and references therein.

Our aim is to show the existence of the global attractor and estimate the fractal dimension of the attractor of the problem (1.1). For the dynamical system generated by (1.1) in the non-degenerate case, the issues on the global attractor have been studied by many authors last years, see [6,12,20,31,45,47], etc. As the first step of investigation, Nakao and Yang [31] obtained the existence of the global attractor in the regular phase space $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ for the problem (1.1) with

$$\phi(s) = 1 + s^{\frac{m}{2}}, \qquad m \geq 2,$$

and the term $f = f(x,u)$ satisfies the condition $|f_u(x,u)| \leq C(|u|^{p-1} + L(x))$, where $1 \leq p < \frac{n+2}{(n-2)^+}$. In general, the exponent $p^* = \frac{n+2}{(n-2)^+}$ is called to be critical when someone study the problem in $H_0^1(\Omega)) \times L^2(\Omega)$. The first result in the supercritical case ($p^* < p < \frac{n+4}{(n-4)^+}$) we are aware of is given by Chueshov [6]. He studied the well-posedness of problem (1.1) with more general stiffness ϕ (cover the case of degenerate) and the supercritical nonlinear term $f(u)$. However, when he proved the existence of a finite-dimensional global attractor for problem (1.1) in the natural energy space $(H_0^1(\Omega) \cap L^{p+1}(\Omega)) \times L^2(\Omega)$ endowed with a partially strong topology (in the sense, if $(u_0^n, u_1^n) \to (u_0, u_1)$ with a partially strong topology, then $(u_0^n, u_1^n) \to (u_0, u_1)$ strongly in $H_0^1 \times L^2$ and $u_0^n \rightharpoonup u_0$ weakly in L^{p+1}), he assumed that

$$\phi(s) > 0, \quad \forall\ s \geq 0, \qquad \phi \in C^1(\mathbb{R}^+).$$

More related work concerning the global attractor of Kirchhoff wave equations can be found in [18,22,24,46], etc. However, all these papers deal with the problem in the non-degenerate case or in the case with $\phi(s) > 0$.

In this paper, we consider the problem (1.1) under the condition $\phi(s) \geq 0$, such as $\phi(s) = bs^\gamma$ with $\gamma \geq 1$ in the subcritical case. Based on the result in [6], we first prove the existence of the global attractor in $H_0^1(\Omega)) \times L^2(\Omega)$ if ϕ is really degenerate. However, the method in [6] (Theorem 3.9) can not be applied to the case of $\phi(s) \geq 0$. It is well known that if $\phi(s) > 0$ in $\mathbb{R}^+$ and $\phi \in C^1(\mathbb{R}^+)$, then for every bounded set $B \subset H_0^1(\Omega)$ there exists a positive constant c_0 such that

$$\phi(\|\nabla u\|^2) \geq c_0 > 0,\ \forall\ u \in B. \tag{1.3}$$

Since for every solution u of the problem (1.1) with the initial data $(u_0, u_1) \in B_0$ ($B_0 \subset (H_0^1(\Omega) \cap L^{p+1}(\Omega)) \times L^2(\Omega)$ bounded) the author has proved that there exists $M > 0$ such that $\|\nabla u(t)\| \leq M (t \geq 0)$, it is obvious that

$$\phi(\|\nabla u(t)\|^2) \geq c_0 > 0,\ \forall\ t \geq 0.$$

Then in this case, we can obtain the estimates

$$\phi(\|\nabla u(t)\|^2)(-\Delta u, u) \geq c_0 \|\nabla u\|^2, \tag{1.4}$$

or

$$\phi(\|\nabla u(t)\|^2)(\Delta w, \Delta w) \geq c_0 \|\Delta w\|^2 \tag{1.5}$$

if we decompose the solution semigroup $u = v + w$ for $w \in H_0^1(\Omega) \cap H^2(\Omega)$. It is a critical step when we prove that the semigroup is "asymptotically smooth" (it is one of the necessary conditions for the existence of the global attractor, see Definition 2.4 and Theorem 2.2) by the classical energy method or by the method of semigroup decomposition. The reason is that we need to use Gronwall's lemma to obtain the existence of a bounded set in a more regular space, such as $(H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$. However, the inequality (1.3) doesn't hold under the condition of $\phi(s) \geq 0$ and it leads to the lack of the key estimates (1.4) and (1.5). Then in the degenerate case we can not entirely rely on the method of classical energy estimate or the semigroup decomposition to prove that the semigroup is asymptotically smooth. To overcome the above difficulties caused by the degeneration, we apply a special energy method combining with the criterion via "the measure of noncompactness" (see [25]) to prove that the semigroup is ω-limit compact. The criterion (see Theorem 2.1) is also regarded as one of the methods of proving "asymptotically smooth" via "weak quasi-stability".

Secondly, under some additional assumptions, we show that the fractal dimension of the attractor is infinite. As we know, the global attractor of dynamical systems can be very complicated. And to study the geometry of the attractors, some concepts such as Lyapunov exponents, the Hausdorff dimension and the fractal dimension were proposed. Numerous works have contributed to establish the estimates for the Hausdorff or fractal dimension of the attractors for some physical systems. However, as far as we know, all the present papers that concern the Kirchhoff equation study the finiteness of the fractal dimension. Although the finite dimensionality of the global attractor is essential, we think that it is also interesting to consider the global attractors with infinite dimension. In [11] the authors obtained that the global attractor of porous media equation may be infinite. [10] proved the infinite dimension of the global attractor for parabolic equations involving p-Laplacians. In [48], by using the Z_2 index, the authors provided a new approach to study the geometry of the global attractor. They also proved that the dimension of the attractor is infinite for a class of symmetric p-Laplacian equations. In [49], the authors proved the existence of the multiple equilibrium points in the global attractors for the symmetric dynamical systems by estimating the lower bound of Z_2 index. In this paper, we apply the method providing in [48] to the symmetric Kirchhoff equation and prove that the fractal dimension is infinite for a class of degenerate Kirchhoff equation.

The paper is organized as follows. In section 2, we introduce some notations and preliminaries. In section 3, we discuss the existence of the global attractor for the equation (1.1), see Theorem 3.1. In section 4, we use the means of the Z_2 index to estimate the fractal dimension of the global attractor in Theorem 4.1. As a corollary, we show that the fractal dimension is infinite if the semigroup generated by the system is odd.

2. Preliminaries

In this section, we will give some notations and results. Subsequently, we denote by $\|\cdot\|$ and $(\cdot,\ \cdot)$ the norm and the inner product in $L^2(\Omega)$. Let $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$. We define the norms in $\mathcal{H}$ by

$$\|(u_0, u_1)\|_{\mathcal{H}}^2 = \|\nabla u_0\|^2 + \|u_1\|^2.$$

Let $\lambda_1 > 0$ be the first eigenvalue of $-\Delta$, we know that $\|\nabla w\| \geq \lambda_1^{\frac{1}{2}} \|w\|,\ \forall w \in H_0^1(\Omega)$. Throughout the paper, C stands for a generic positive constant. We also denote the different positive constant by $C_i, i \in \mathbb{N}$, for special differentiation. For the sake of simplicity we assume that $\delta = 1$ in the equation (1.1).

Definition 2.1. A function $u(t)$ is said to be a weak solution to problem (1.1) on an interval $[0, T]$ if u satisfies equation in the sense of distributions and

$$u \in L^\infty(0, T; H_0^1(\Omega)),\ \partial_t u \in L^\infty(0, T; L^2(\Omega)).$$

Now, we recall some definitions and results related to the global attractor, which will be used in the present paper. More details can be found in [3,7,25,41].

Definition 2.2. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a metric space (X, d). A subset $\mathcal{A}$ of X is called a global attractor for the semigroup, if $\mathcal{A}$ is compact and enjoys the following properties:

(1) $\mathcal{A}$ is invariant, that is, $S(t)\mathcal{A} = \mathcal{A}$, for all $t \geq 0$;
(2) $\mathcal{A}$ attracts all bounded sets of X. That is, for any bounded subset B of X,

$$d(S(t)B, \mathcal{A}) \to 0, \quad \text{as } t \to +\infty,$$

where $d(B, A)$ is the Hausdorff semi-distance.

Definition 2.3. A strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ in a Banach space X is said to satisfy the Condition (C) if for any $\varepsilon > 0$ and for any bounded set B of X, there exist $t(B) > 0$ and a finite dimensional subspace $X_1(B)$ of X such that $\{\|PS(t)x\|_X, x \in B, t \geq t(B)\}$ is bounded and

$$\|(I - P)S(t)x\|_X < \varepsilon, \quad t \geq t(B),\ x \in B,$$

where $P : X \to X_1(B)$ is a bounded projector.

Theorem 2.1. *(Theorem 3.11, [25]) Let X be a uniformly convex Banach space and $\{S(t)\}_{t\geq 0}$ be a strongly continuous semigroup in X. Then $\{S(t)\}$ has a global attractor $\mathcal{A}$ in X if and only if*

(1) *$\{S(t)\}$ satisfies the Condition (C);*
(2) *there is a bounded absorbing set $B \subset X$.*

Now it is known that the Condition (C) is a convenient criteria for asymptotic smoothness/compactness of evolution operators and dynamical systems.

Definition 2.4. An evolution semigroup $\{S(t)\}_{t\geq 0}$ in a complete metric space X is said to be asymptotically smooth if the following condition is valid: for every bounded set B such that $S(t)B \subset B$ for $t > 0$ there exists a compact set K in the closure $\overline{B}$ of B, such that $S(t)B$ converges uniformly to K in the sense that

$$\lim_{t\to+\infty} d(S(t)B, K) = 0.$$

Proposition 2.1. *(Proposition 2.29, [7]) Let $\{S(t)\}_{t\geq 0}$ be an evolution semigroup on a reflexive Banach space X. Assume that the Condition (C) holds. Then the evolution semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically smooth.*

Theorem 2.2. *(Theorem 2.3.5, [7]) Let $\{S(t)\}_{t\geq 0}$ be an evolution semigroup on a complete metric space X. Then $\{S(t)\}$ has a global attractor $\mathcal{A}$ in X if*

(1) *$\{S(t)\}$ is asymptotically smooth;*
(2) *there is a bounded absorbing set $B \subset X$.*

In the following, we recall some basic results on the Z_2 index. More details can be found in [40]. The concept of an index theory is most easily explained for an even functional E on some Banach space X, with symmetry group $G = Z_2 = \{id, -id\}$. Define $\Sigma = \{A \subset X | A \text{ closed}, A = -A\}$ to be the class of closed symmetric subsets of X.

Definition 2.5. For $A \in \Sigma$, $A \neq \emptyset$, let

$$\gamma(A) = \begin{cases} \inf\{m : \ \exists\ g \in C^0(A; \mathbb{R}^m \backslash \{0\}),\ g(-u) = -g(u)\}, \\ \infty,\ \text{if } \{\cdots\} = \emptyset, \text{in particular},\ \text{if } 0 \in A, \end{cases}$$

and define $\gamma(\emptyset) = 0$. $\gamma(A)$ is called the Z_2 index or Z_2 genus.

The Z_2 index has properties as follows.

Lemma 2.1. *A Z_2 index defined on Σ satisfies:*

(1) $\gamma(A) = 0 \Leftrightarrow A = \emptyset$;
(2) *if $A \subset B$, then $\gamma(A) \leq \gamma(B)$, for any $A,\ B \subset \Sigma$;*
(3) *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$ for any $A,\ B \subset \Sigma$;*
(4) *if $A \in \Sigma$ is a compact set, then $\exists \delta > 0$ such that $\gamma(\overline{N_\delta(A)}) = \gamma(A)$, where $N_\delta(A)$ is a symmetric δ-neighborhood of A;*
(5) *$\gamma(A) \leq \gamma(\overline{h(A)})$, $\forall A \in \Sigma$, and $h : X \to X$ is odd and continuous.*

For the proof of the above lemma, we refer readers to [40]. In order to use the Z_2 index to prove the infinite dimensional of the global attractor, we need the following lemma (see [48], Lemma 2.3).

Lemma 2.2. *Assume that $S(t)$ is an odd and continuous semigroup on X, and $\mathcal{A}$ is a symmetric global attractor. If there exists a bounded symmetric set B such that $\gamma(B) \geq m$, $m < \infty$, and $\omega(B) \subset \mathcal{A}\backslash\{0\}$, then $\gamma(\mathcal{A}\backslash N(0)) \geq m$ for some neighborhood $N(0)$ of 0, where $\omega(B) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)B}$.*

By the definition of the Z_2 index, we know that a set B with Z_2 index $\gamma(B) \geq m$ can not be mapped into $\mathbb{R}^{m-1}\backslash\{0\}$ by an odd continuous map. On the other hand, any compact set B with fractal dimension $dim_F = m$ can be mapped into $\mathbb{R}^{2m-1}$ by a linear (odd) Hölder continuous one-to-one projector from the Mane projection theorem, see [17]. It means that if the Z_2 index of a set B is larger than $2m$, then the fractal dimension of the set B must be larger than m. Thus, by estimation of the lower bound of the Z_2 index of the global attractor, we can obtain the corresponding lower bound of the fractal dimension of the global attractor. In particular we have the following corollary.

Corollary 2.1. *Let $\mathcal{A}$ be an symmetric compact set. If for any $m \in \mathbb{N}$ the inequality $\gamma(\mathcal{A}) \geq m$ holds, then the fractal dimension of $\mathcal{A}$ is infinite.*

3. Existence of the global attractor

In this section we prove the existence of the global attractor when $\phi(s)$ is really degenerate and $f(u)$ is subcritical. We assume that f and ϕ satisfy the following conditions.

Assumptions 3.1. $f(u)$ is a C^1 function, $f(0) = 0$ (without loss of generality),

$$\mu_f = \liminf_{|s|\to+\infty} \frac{f(s)}{s} > -\infty, \tag{3.1}$$

and the following properties hold:

(a) if $n = 1$, then f is arbitrary;
(b) if $n = 2$, then $|f'(u)| \leq C(1 + |u|^{p-1})$ for some $p \geq 1$;
(c) if $n \geq 3$, then

$$|f'(u)| \leq C(1 + |u|^{p-1}) \ \text{ with } 1 \leq p < p_* = \frac{n+2}{n-2}, \tag{3.2}$$

where c_i are positive constants.

Assumptions 3.2. The function $\phi \in C^1(\mathbb{R}^+)$ possess the following properties:

(i) $\phi(s) \geq \min\{L_1 s^\alpha, L_2\}$, where $\alpha \geq 0$, $L_1, L_2 > 0$ are constants;
(ii) $\hat{\mu}_\phi \lambda_1 + \mu_f > 0$, where λ_1 is the first eigenvalue of $-\Delta$, $\hat{\mu}_\phi = \liminf_{s\to+\infty} \phi(s) > 0$.

The condition (ii) in Assumptions 3.2 is from Assumptions 3.11 (ii) in [6].

Remark 3.1. (1) $\phi(s) = L_1 s^\alpha$ $(\alpha \geq 1)$ satisfies Assumptions 3.2. It indicates that we include into the consideration the case of possibly degenerate ϕ since $\phi(0) = 0$. Moreover, because $\hat{\mu}_\phi = +\infty$ in this case, we need no additional assumptions concerning f except Assumptions 3.1.

(2) If $\phi(s) = \begin{cases} L_1 s^\alpha, & 0 \le s \le 1 \\ L_2, & s > 1 \end{cases}$, then $\hat{\mu}_\phi = L_2$ and Assumptions 3.2 (ii) will be $L_2\lambda_1 + \mu_f > 0$.

(3) If $\alpha = 0$, then $\phi(s) \equiv \min\{L_1, L_2\}$ and the equation (1.1) is the nonlinear wave equation with strong damping.

Now, we state some results in [6], which will still be used in the present paper.

Proposition 3.1. *([6], Theorem 2.2) Let Assumptions 3.1 and 3.2 be in force and $(u_0, u_1) \in \mathcal{H}$. Then for every $T > 0$ problem (1.1) has a unique weak solution $u(t)$ with $(u, u_t) \in C([0, T]; \mathcal{H})$. This solution possesses the following properties:*

(1) *For every $t \in [0, T]$, $\|(u_0, u_1)\|_{\mathcal{H}} \le R$, there exist $C_{R,T} > 0$ such that*

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \int_0^t \|\nabla u_t(\tau)\|^2 d\tau \le C_{R,T}, \tag{3.3}$$

and

$$E(u(t), u_t(t)) + \int_s^t \|\nabla u(\tau)\|^2 d\tau = E(u(s), u_t(s)),\ \ t > s \ge 0, \tag{3.4}$$

where $E(u_0, u_1) = \frac{1}{2}[\|u_1\|^2 + \Phi(\|\nabla u_0\|^2)] + \int_\Omega F(u_0)dx - \int_\Omega h u_0 dx$, $\Phi(s) = \int_0^s \phi(\tau)d\tau$, $F(s) = \int_0^s f(\tau)d\tau$.

(2) *For every $0 < a < T$,*

$$u_t \in L^\infty(a, T; H_0^1(\Omega)),\ \ u_{tt} \in L^\infty(a, T; H^{-1}(\Omega)) \cap L^2(a, T; L^2(\Omega)),$$

and there exist $\beta > 0$ and $C_{R,T} > 0$ such that

$$\| u_{tt}(t) \|_{-1}^2 + \| \nabla u_t(t) \|^2 + \int_t^{t+1} \| u_{tt}(\tau) \|^2 \, d\tau \le \frac{C_{R,T}}{t^\beta} \tag{3.5}$$

for every $t \in (0, T]$ and $\| (u_0, u_1) \|_{\mathcal{H}} \le R$.

Define the operator $S(t) : \mathcal{H} \to \mathcal{H}$, $S(t)(u_0, u_1) = (u(t), u_t(t))$, where $u(t)$ is the solution of problem (1.1).

Proposition 3.2. *([6], Proposition 3.2, 3.5) Let Assumptions 3.1 and 3.2 be in force. Then the evolution semigroup $S(t)$ possesses the following properties.*

(1) *It is a continuous mapping in $\mathcal{H}$ with respect to the $\mathcal{H}$-norm.*
(2) *It has a bounded absorbing set $B(0, R_0)$, i.e., for every $R > 0$ and $(u_0, u_1) \in B(0, R)$, there exists $t_R \ge 0$, such that*

$$\|(u(t), u_t(t)\|_{\mathcal{H}} \le R_0,\ \ \forall\ t \ge t_R.$$

According to Proposition 3.2, we know that

$$\beta^* = \{(u_0, u_1) \in \mathcal{H} :\ \|(u_0, u_1)\|_{\mathcal{H}} \le R_0\}$$

is a bounded absorbing set. Let $\beta_0 = \overline{\bigcup_{t\geq 1+t^*} S(t)\beta^*}$, where $t^* \geq 0$ is chosen such that $S(t)\beta^* \subset \beta^*$ for $t \geq t^*$. It is obvious that β_0 is the close bounded absorbing set for $S(t)$ with $S(t)\beta_0 \subset \beta_0 \subset \beta^*$. Then for every $(u_0, u_1) \in \beta_0$, $(u(t), u_t(t)) = S(t)(u_0, u_1)$, one can get

$$\|(u(t), u_t(t)\|_{\mathcal{H}} \leq R_0, \ \forall\, t \geq 0. \tag{3.6}$$

Now, let $(u_0, u_1) \in \bigcup_{t\geq 1+t^*} S(t)\beta^*$. Then there exists $t_0 \geq 1 + t^*$ and $(u_0^*, u_1^*) \in \beta^*$, such that $(u_0, u_1) = s(t_0)(u_0^*, u_1^*)$. For every $t \geq 0$, it gives that

$$(u(t), u_t(t)) = S(t)(u_0, u_1) = S(t + t_0)(u_0^*, u_1^*) = S(1)S(t + t_0 - 1)(u_0^*, u_1^*).$$

Choosing $T = 1$ and $R = R_0$ in (3.5) and noticing $\|S(t + t_0 - 1)(u_0^*, u_1^*)\|_{\mathcal{H}} \leq R_0$, we find that

$$\|u_{tt}(t)\|_{-1}^2 + \|\nabla u_t(t)\|^2 + \int_t^{t+1} \|u_{tt}(\tau)\|^2 d\tau \leq \frac{C_{R_0,1}}{1} \triangleq C_{R_0}, \tag{3.7}$$

where C_{R_0} is independent of t. It means that β_0 is bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$. Moreover, by taking $s = 0$ in the energy relation (3.4), we get that

$$\sup_{t\in\mathbb{R}^+} E(u(t), u_t(t)) + \int_0^{+\infty} \|\nabla u_t(\tau)\|^2 d\tau \leq C_{R_0}. \tag{3.8}$$

Therefore, by using (3.6)-(3.8), we have that

$$\|u_{tt}(t)\|_{-1}^2 + \|\nabla u(t)\|^2 + \|\nabla u_t(t)\|^2 + \int_0^t \|\nabla u_t(\tau)\|^2 d\tau + \int_t^{t+1} \|u_{tt}(\tau)\|^2 d\tau \leq C_{\beta_0}^2. \tag{3.9}$$

For simplicity, we assume that $\alpha > 0$, $L_1 = L_2 = 1$ in the following.

Theorem 3.1. *Let Assumptions 3.1 and 3.2 hold. Then the problem (1.1) has a global attractor in $\mathcal{H}$.*

Proof. According to Proposition 3.2, one can see that $S(t) : [0, +\infty) \times \mathcal{H} \to \mathcal{H}$ is continuous and β_0 is a bounded absorbing set in $\mathcal{H}$. Therefore, to prove the existence of the global attractor it is sufficient to prove that the solution semigroup $\{S(t)\}$ satisfies the condition (C) in $\mathcal{H}$ for any bounded set $B \subset \mathcal{H}$ by Theorem 2.1. Without loss of generality, we assume $(u(t), u_t(t)) \in \beta_0$ for $t \geq 0$, since β_0 is an absorbing set in $\mathcal{H}$. Then we need to verify that for every $\varepsilon > 0$, there exist $t_0 > 0$ and a finite dimensional subspace Y of $\mathcal{H}$, such that for any $(u_0, u_1) \in \beta_0$ and $t \geq t_0$,

$$\|(I - P)S(t)(u_0, u_1)\|_{\mathcal{H}} < \varepsilon,$$

where $P : \mathcal{H} \to Y$ is an orthogonal projector.

Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of $L^2(\Omega)$ which consists of eigenvectors of $-\Delta$. It is well known that $\{e_j\}_{j=1}^{\infty}$ is also an orthogonal basis of $H_0^1(\Omega)$. The corresponding eigenvalues are denoted by $\{\lambda_j\}_{j=1}^{\infty}$:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \to \infty,$$

i.e., $-\Delta e_j = \lambda_j e_j, j \in \mathbb{N}$. Let $V_m = span\{e_1, \cdots, e_m\}$ in $H_0^1(\Omega)$ and let $P_m : H_0^1(\Omega) \to V_m$ be an orthogonal projector. Denote $Q_m = I - P_m$ and

$$u = P_m u + Q_m u \triangleq u^1 + u^2. \tag{3.10}$$

It is easy to see that $(u^1, u^2) = 0$ by the orthogonality of the projectors. Using the multiplier u^2 in equation (1.1), we obtain that

$$\begin{aligned} &\frac{d}{dt}(\|\nabla u^2\|^2) + 2\phi(\|\nabla u\|^2)\|\nabla u^2\|^2 \\ &= -2(u_{tt}^2, u^2) - 2(f(u), u^2) + 2(h^2, u^2) \\ &\leq 2\|u^2\| \cdot \|u_{tt}^2\| + \|f(u)\|_{L^{\frac{p+1}{p}}} \cdot \|u^2\|_{L^{p+1}} + \|u^2\| \cdot \|h^2\|. \end{aligned} \tag{3.11}$$

Since $2 \leq p+1 < p^* = \dfrac{2n}{n-2}$, there exists $\theta \in (0, 1]$ such that $\dfrac{1}{p+1} = \dfrac{\theta}{2} + \dfrac{1-\theta}{p^*}$. Then the interpolation inequality implies that

$$\|u^2\|_{L^{p+1}} \leq \|u^2\|^\theta \cdot \|u^2\|_{L^{p^*}}^{1-\theta} \leq C\|u^2\|^\theta \cdot \|\nabla u^2\|^{1-\theta} \leq C \cdot C_{\beta_0}^{1-\theta}\|u^2\|^\theta =: C_1\|u^2\|^\theta. \tag{3.12}$$

At the same time, noticing that the nonlinear term $f(u)$ is of subcritical growth, there exists $M > 0$ such that $\|f(u)\|_{L^{\frac{p+1}{p}}} \leq M$ holds for every $(u(t), u_t(t)) \in \beta_0$.

Since β_0 is bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ compactly, for any $\varepsilon > 0$, there exists $m_1 \in \mathbb{N}$, such that for $m \geq m_1$, $P_m : L^2(\Omega) \to V_m$,

$$\|(u^2, u_t^2)\|_{L^2(\Omega)\times L^2(\Omega)} < \varepsilon, \quad \forall\ (u, u_t) \in \beta_0, \tag{3.13}$$

where $u^2 = Q_m u$ and $u_t^2 = Q_m u_t$. Choosing appropriate ε in (3.13), by (3.12), we can find V_{m_2} $(m_2 \geq m_1)$, such that

$$\|u^2\|_{L^{p+1}} < \frac{\varepsilon_1^2}{M} \text{ for } u^2 = Q_{m_2} u, \tag{3.14}$$

where $\varepsilon_1 = \dfrac{\varepsilon^{2+2\alpha}}{2C_{\beta_0}^2 + 2}$ and $C_{\beta_0}^2$ is a constant as shown in (3.9). Similarly, because of $h \in L^2(\Omega)$, without loss of generality, we assume that $h = h^1 + h^2$, $h^1 \in V_{m_2}$, $h^2 = Q_{m_2} h$, and

$$\|h_2\| < \varepsilon_1. \tag{3.15}$$

In the following, we will verify the existence of $P_{m_3} : H_0^1(\Omega) \to V_{m_3} \subset H_0^1(\Omega)$ and $t_0 \geq 0$, such that for any $(u(t), u_t(t)) \in \beta_0$ and $t \geq t_0$,

$$\|\nabla u^2(t)\| < \varepsilon \quad \text{for} \quad u^2 = Q_{m_3} u. \tag{3.16}$$

Thus by taking $m_0 = \max\{m_2, m_3\}$, $P_{m_0} : H_0^1(\Omega) \to V_{m_0} \subset H_0^1(\Omega)$, $u^2 = Q_{m_0} u$ and $u_t^2 = Q_{m_0} u_t$, we can conclude from (3.13) and (3.16) that

$$\|(u^2, u_t^2)\|_{\mathcal{H}} < \varepsilon, \quad \forall\ (u, u_t) \in \beta_0.$$

Now, we will consider two situations to prove (3.16). Without loss of generality, we assume $0 < \varepsilon < \frac{1}{3}$.

Case I. There exists $t_1 \geq 0$, such that $\|\nabla u(t_1)\| < \varepsilon$.

In this case, we claim that the following inequality is true, i.e., for every $t \geq t_1$,

$$\|\nabla u^2(t)\| \leq 2\varepsilon, \quad \text{for} \quad u^2 = Q_{m_2}u. \tag{3.17}$$

In fact, if there exists $t_2 > t_1$, such that

$$2\varepsilon < \|\nabla u^2(t_2)\| < 3\varepsilon, \tag{3.18}$$

then we know from $u \in C([0, +\infty), H_0^1(\Omega))$ that there exists $t_3 : t_1 < t_3 < t_2$ satisfying

$$\frac{3}{2}\varepsilon \leq \|\nabla u^2(t)\| \leq 3\varepsilon, \quad \forall t \in [t_3, t_2], \tag{3.19}$$

$$\|\nabla u^2(t_3)\| = \frac{3}{2}\varepsilon. \tag{3.20}$$

Notice $\|\nabla u\| \geq \|\nabla u^2\|$ and $\|\nabla u^2(t)\|^2 \leq (3\varepsilon)^2 \leq 1$ for $t \in [t_3, t_2]$, we have that

$$\phi(\|\nabla u(t)\|^2) \geq \min\{\|\nabla u(t)\|^{2\alpha}, 1\} \geq \min\{\|\nabla u^2(t)\|^{2\alpha}, 1\} = \|\nabla u^2(t)\|^{2\alpha}. \tag{3.21}$$

Then integrating (3.11) in dt on $(t_3,\ t_2)$ and combining the above inequality and (3.14) yield that

$$\begin{aligned}
&\|\nabla u^2(t_2)\|^2 + 2\int_{t_3}^{t_2} \|\nabla u^2(s)\|^{2\alpha+2} ds \\
&\leq \|\nabla u^2(t_3)\|^2 + 2\varepsilon_1 \int_{t_3}^{t_2} \|u_{tt}(s)\| ds + 4\varepsilon_1^2(t_2 - t_3) \\
&\leq (\frac{3}{2}\varepsilon)^2 + 2\varepsilon_1 (\int_{t_3}^{t_2} \|u_{tt}(s)\|^2 ds)^{\frac{1}{2}} \cdot \sqrt{t_2 - t_3} + 4\varepsilon_1^2(t_2 - t_3).
\end{aligned} \tag{3.22}$$

If $0 < t_2 - t_3 \leq 1$, according to (3.9), (3.19) and (3.20), we can get

$$\begin{aligned}
&\|\nabla u^2(t_2)\|^2 + 2(\frac{3}{2}\varepsilon)^{2\alpha+2}(t_2 - t_3) \\
&\leq (\frac{3}{2}\varepsilon)^2 + 2\varepsilon_1 C_{\beta_0} \cdot \sqrt{t_2 - t_3} + 2\varepsilon_1^2(t_2 - t_3) \\
&\leq (\frac{3}{2}\varepsilon)^2 + \frac{\varepsilon_1}{2} + (2\varepsilon_1 C_{\beta_0}^2 + 2\varepsilon_1)(t_2 - t_3) \\
&\leq (\frac{3}{2}\varepsilon)^2 + \frac{\varepsilon_1}{2} + \varepsilon^{2+2\alpha}(t_2 - t_3).
\end{aligned} \tag{3.23}$$

Thus

$$\|\nabla u^2(t_2)\|^2 \leq (\frac{3}{2}\varepsilon)^2 + \frac{\varepsilon_1}{2} \leq (\frac{3}{2}\varepsilon)^2 + \frac{\varepsilon^{2+2\alpha}}{2} < 4\varepsilon^2, \tag{3.24}$$

which is in contradiction with (3.18).

On the other hand, if $t_2 - t_3 > 1$, then it can be written as $n_0 + \tau$ with $0 < \tau \leq 1$. Using (3.23) on the interval $[t_3, t_3 + 1]$, one can easily get

$$\|\nabla u^2(t_3+1)\|^2 + 2(\frac{3}{2}\varepsilon)^{2\alpha+2} \le (\frac{3}{2}\varepsilon)^2 + \frac{\varepsilon_1}{2} + 2\varepsilon_1 C_{\beta_0}^2 + 2\varepsilon_1 \le (\frac{3}{2}\varepsilon)^2 + \frac{\varepsilon^{2+2\alpha}}{2} + \varepsilon^{2+2\alpha}, \tag{3.25}$$

which indicates $\|\nabla u^2(t_3+1)\| \le \frac{3}{2}\varepsilon$. Similarly, we obtain that $\|\nabla u^2(t_3+n_0)\| \le \frac{3}{2}\varepsilon$. Then repeat the previous steps as for $0 < t_2 - t_3 \le 1$, we obtain that $\|\nabla u^2(t_2)\| \le 2\varepsilon$, which is in contradiction with (3.18). It shows that the claim (3.17) is correct and Condition (C) holds for every $t \ge t_1$.

Case II. For every $t \ge 0$, the inequality

$$\|\nabla u(t)\| \ge \varepsilon \tag{3.26}$$

always holds.

In this case, it is clear that $\phi(\|\nabla u\|^2) \ge \min\{\|\nabla u\|^{2\alpha}, 1\} \ge \min\{\varepsilon^{2\alpha}, 1\} = \varepsilon^{2\alpha}$. Combining with (3.13) and (3.14), one can find $m_3 \ge m_2$, such that $\|u_t^2(t)\| + \|u^2(t)\| < \varepsilon_1$ for $u_t^2 = Q_{m_3}u_t$, $u^2 = Q_{m_3}u$. Moreover, because of $m_3 \ge m_2$, it is clear that (3.13)-(3.15) hold. Multiplying (1.1) by u^2, we have

$$\begin{aligned} &\frac{d}{dt}\left((u_t^2, u^2) + \frac{1}{2}\|\nabla u^2\|^2\right) + \varepsilon^{2\alpha}\|\nabla u^2\|^2 \\ &\le \|u_t^2\|^2 + (f(u), u^2) + (h^2, u^2). \end{aligned} \tag{3.27}$$

Denote $Y(t) = (u_t^2, u^2) + \dfrac{1}{2}\|\nabla u^2\|^2$. By (3.27), we can find that

$$\begin{aligned} &\frac{d}{dt}Y(t) + 2\varepsilon^{2\alpha}Y(t) \\ &\le 2\varepsilon^{2\alpha}(u_t^2, u^2) + \|u_t^2\|^2 + (f(u), u^2) + (h^2, u^2) \\ &\le 2\varepsilon^{2\alpha} \cdot \|u_t^2\| \cdot \|u^2\| + \|u_t^2\|^2 + \|f(u)\|_{L^{\frac{p+1}{p}}} \cdot \|u^2\|_{L^{p+1}} + \|h^2\| \cdot \|u^2\| \\ &\le 2\varepsilon^{2\alpha}\varepsilon_1^2 + 3\varepsilon_1^2 \\ &\le 4\varepsilon_1^2 = \frac{4\varepsilon^{4+4\alpha}}{2C_{\beta_0}^2 + 2} \end{aligned}$$

By Gronwall's inequality, we obtain that

$$Y(t) \le Y(0)e^{-2\varepsilon^{2\alpha}t} + \frac{\varepsilon^{4+2\alpha}}{C_{\beta_0}^2 + 1}(1 - e^{-2\varepsilon^{2\alpha}t}) \le Y(0)e^{-2\varepsilon^{2\alpha}t} + \varepsilon^{4+2\alpha}. \tag{3.28}$$

Since $Y(0)$ is bounded, there exists $t_0 \ge 0$, such that $Y(0)e^{-2\varepsilon^{2\alpha}t} \le \dfrac{\varepsilon^2}{4}$ for $t \ge t_0$. Then

$$Y(t) \le \frac{\varepsilon^2}{2} + \varepsilon^{4+2\alpha},\ \forall t \ge t_0.$$

And because of $Y(t) \ge \frac{1}{2}\|\nabla u^2\|^2 - \varepsilon_1^2$, we get that

$$\|\nabla u^2\|^2 \le 2(\varepsilon_1^2 + \frac{\varepsilon^2}{4} + \varepsilon^{4+2\alpha}) \le 4\varepsilon^2,$$

i.e., $\|\nabla u^2(t)\| \le 2\varepsilon$ for every $t \ge t_0$. Thus the proof of Theorem 3.1 is complete. $\square$

4. Fractal dimension of the global attractor

In this section, we will estimate the fractal dimension of the global attractor for problem (1.1) by the Z_2 index. However, the Z_2 index is defined for the class of symmetric subsets, so we consider the following problem, which is the special case of (1.1):

$$\begin{cases} \partial_{tt}u - \eta\Delta\partial_t u - \phi(\|\nabla u\|^2)\Delta u + f(u) = 0, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{4.1}$$

Here $\eta > 0$, $\phi(s) = L_1 s^\alpha($ $\forall s \geq 0$, without loss of generality, we assume $L_1 = 1)$ with $\alpha \geq 1$.

Because of the limit of Z_2 index, we assume that the dynamical system is odd. It means that $f(u)$ should be an odd function. Moreover, we need $f(u)$ to satisfy some specific structure (there should be a negative term in f) to guarantee the infinite of dimension. Concretely, we assume that f satisfies the following conditions.

Assumptions 4.1. $f(u) = |u|^{r-2}u - b|u|^{s-2}u + g(u)$, where g is a C^1 function such that $g(-u) = -g(u)$, $q,\ r,\ s \geq 2$, $b > 0$,

$$\mu_g = \liminf_{|s|\to+\infty} \frac{g(s)}{s} > -\infty, \tag{4.2}$$

$$\lim_{|s|\to 0} \frac{|g(s)s|}{|s|^q} = a \geq 0, \tag{4.3}$$

and the following properties hold:

(a) if $n = 1$, then g is arbitrary;
(b) if $n = 2$, then $|g'(u)| \leq C(1 + |u|^{p-1})$ for some $p \geq 1$;
(c) if $n \geq 3$, then

$$|g'(u)| \leq C(1 + |u|^{p-1}) \ \text{ with } 1 \leq p < p_* = \frac{n+2}{n-2},\quad 2 \leq r < \frac{2n}{n-2}, \tag{4.4}$$

where C is a positive constant. We also assume that $2 \leq s < \min\{p+1,\ r,\ q,\ 2\alpha+2\}$.

By the above assumptions, we know that

$$|G(u)| \leq C_1|u|^q + C_2|u|^{p+1} \tag{4.5}$$

for every $u \in \mathbb{R}$ with $G(u) = \int_0^u g(s)ds$. Moreover, under the assumptions given in this section, Assumptions 3.1 and 3.2 hold naturally, so the existence of the global attractor for problem (4.1) follows directly from Theorem 3.1. And because the system that we consider is odd, we can prove that the global attractor $\mathcal{A}$ is symmetric.

Lemma 4.1. *Under the assumptions 4.1, the semigroup $S(t)$ generated by problem (4.1) is odd, and the global attractor $\mathcal{A}$ is symmetric.*

Proof. We prove the result using the same method as in [48].

Let $(u(t), u_t(t)) = S(t)(u_0, u_1)$. Since f is an odd function, $-u(t)$ is the unique solution of problem (1.1) corresponding to initial data $(-u_0, -u_1)$. Thus

$$S(t)(-u_0,-u_1)=(-u(t),-u_t(t))=-S(t)(u_0,u_1),$$

which means that $S(t)$ is an odd semigroup.

It is well known that

$$\mathcal{A}=\omega(\beta^*)=\bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}S(t)\beta^*},$$

where β^* is the absorbing set generated by Proposition 3.2 and $\omega(\beta^*)$ is its ω-limit set. Clearly, β^* is a symmetric set. Assume $(x,y)\in\mathcal{A}$. Then there exists a sequence $\{(x_n,y_n)\}\subset\beta^*$ and $t_n\to\infty$ (as $n\to\infty$), such that

$$S(t_n)(x_n,y_n)\to(x,y),\ \text{in } \mathcal{H}.$$

Because the semigroup $S(t)$ is odd, we have

$$S(t_n)(-x_n,-y_n)=-S(t_n)(x_n,y_n)\to(-x,-y),\ \text{in } \mathcal{H}.$$

Hence, $\mathcal{A}$ is symmetric, and the proof is completed. □

Our main results in this section is as follows:

Theorem 4.1. *Let Assumptions 4.1 hold. Then for any $m\in\mathbb{N}$, there exists a neighborhood $N(0)$ of 0 such that the Z_2 index of the set $\mathcal{A}\backslash N(0)$ satisfies $\gamma(\mathcal{A}\backslash N(0))\geq m$.*

As indicated in the last paragraph in Section 2, we conclude from Theorem 4.1 that the following result.

Corollary 4.1. *Let Assumptions 4.1 hold. Then the fractal dimension of the global attractor $\mathcal{A}$ is infinite.*

Proof of Theorem 4.1. Following the ideas of Lemma 2.2, we only need to verify that for any $m\in\mathbb{N}$, there exists a symmetric set $B_m\in\mathcal{H}$ with $\gamma(B_m)\geq m$, $\omega(B_m)\subset\mathcal{A}\backslash\{0\}$.

Let $u(t)$ be the solution of problem (4.1). Multiplying (4.1) by u_t and integrating by parts, we obtain that

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2+\frac{1}{2\alpha+2}\|\nabla u\|^{2\alpha+2}+\frac{1}{r}\|u\|_{L^r}^r-\frac{b}{s}\|u\|_{L^s}^s+\int_\Omega G(u)dx\right)+\eta\|\nabla u_t\|^2=0. \tag{4.6}$$

Denote

$$E(u_0,u_1)=\frac{1}{2}\|u_1\|^2+\frac{1}{2\alpha+2}\|\nabla u_0\|^{2\alpha+2}+\frac{1}{r}\|u_0\|_{L^r}^r-\frac{b}{s}\|u_0\|_{L^s}^s+\int_\Omega G(u_0)dx,$$

we can get from (4.6) that

$$E(u(t),u_t(t))+\eta\int_{t_0}^{t}\|\nabla u_t\|^2d\tau=E(u(t_0),u_t(t_0)) \tag{4.7}$$

holds for every $t>t_0\geq 0$. Hence $t\to E(u(t),u_t(t))$ is nonincreasing.

Now, let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of $L^2(\Omega)$ which consists of eigenvectors of $-\Delta$. The corresponding eigenvalues are denoted by $\{\lambda_j\}_{j=1}^{\infty}$:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \to \infty.$$

For any integer $m > 0$, let $V_m = span\{e_1, \cdots, e_m\}$ in $H_0^1(\Omega)$ and let $V'_m = span\{e_1, \cdots, e_m\}$ in $L^2(\Omega)$. Then $\gamma(V_m \times V'_m) \geq m$. Setting $A_m = \{(u_0,\ u_1) \in V_m \times V'_m :\ \|\nabla u_0\|^2 + \|u_1\|^2 = 1\}$, then A_m is compact in $\mathcal{H}$, $\gamma(A_m) = \gamma(V_m \times V'_m) \geq m$, and there exists $\delta_1 > 0$ such that

$$\inf_{(u_0,u_1)\in A_m} \|u\|_{L^s}^s = \delta_1. \tag{4.8}$$

Setting $A_m^\epsilon = \{(\epsilon u_0,\ \epsilon^{s+1} u_1) :\ (u_0, u_1) \in A_m\}$, we get that $\gamma(A_m^\epsilon) = \gamma(A_m) \geq m$ by Lemma 2.1. For $(v_0,\ v_1) = (\epsilon u_0,\ \epsilon^{s+1} u_1)\ \in A_m^\epsilon$, according to the Sobolev embedding inequality and (4.5) we have that

$$\begin{aligned} E(v_0,\ v_1) &= \frac{\epsilon^{2s+2}}{2}\|u_1\|^2 + \frac{\epsilon^{2\alpha+2}}{2\alpha+2}\|\nabla u_0\|^{2\alpha+2} + \frac{\epsilon^r}{r}\|u_0\|_{L^r}^r - \frac{b\epsilon^s}{s}\|u_0\|_{L^s}^s \\ &\quad + \int_\Omega (C_1\epsilon^q|u_0|^q + C_2\epsilon^{p+1}|u_0|^{p+1})dx \\ &\leq \frac{\epsilon^{2s+2}}{2} + \frac{\epsilon^{2\alpha+2}}{2\alpha+2} + C_0\frac{\epsilon^r}{r} - \frac{b\epsilon^s}{s}\delta_1 + C_1'\epsilon^q + C_2'\epsilon^{p+1}. \end{aligned} \tag{4.9}$$

Since $2 \leq s < \min\{p+1,\ r,\ q,\ 2\alpha+2\}$, for ϵ small enough, we have $E(v_0,\ v_1) < 0$ for any $(v_0,\ v_1) \in A_m^\epsilon$. Because $E(0,0) = 0$ and the function $t \to E(v(t),\ v_t(t))$ in nonincreasing for $(v(t),\ v_t(t)) = S(t)(v_0,\ v_1)$, we have $\omega(A_m^\epsilon) \subset \mathcal{A}\backslash\{0\}$. By Lemma 2.2 and the properties of Z_2 index, we get that

$$\gamma(\mathcal{A}\backslash N(0)) \geq m, \tag{4.10}$$

for some neighborhood $N(0)$ of 0. This completes the proof. $\square$

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