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Sturmian comparison theorems for completely controllable linear Hamiltonian systems in singular case ☆

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ABSTRACT

In this paper we consider two linear Hamiltonian differential systems on an open unbounded interval. We assume that the systems are completely controllable and satisfy the Sturmian majorant condition and the Legendre condition. We derive a singular Sturmian comparison theorem for the case when the minorant system has a solution, which is principal at both endpoints of the considered interval. The main result is new even for the second order differential equations and it generalizes the singular comparison theorem obtained by Aharonov and Elias (2010). We illustrate our new theory by several examples.

1. Introduction

In this paper we investigate Sturmian comparison theorems for two second order linear differential equations, or more generally for two linear Hamiltonian systems (also called canonical systems), which satisfy a certain majorant condition for their coefficients. Such theorems provide a comparison of the numbers of zeros of solutions of these equations. The monographs [8,14,20,21] represent a sample of classical references on this subject.

The motivation of this work comes from a singular Sturmian comparison theorem for the second order linear differential equations, which was proven in [1,2] by Aharonov and Elias. Consider the open unbounded interval (a, ∞) with $a \in \mathbb{R} \cup \{-\infty\}$ and the differential equations

$$\big(r(t)\, x'\big)' + p(t)\, x = 0, \quad t \in (a, \infty), \tag{1.1}$$

$$\big(\hat{r}(t)\, \hat{x}'\big)' + \hat{p}(t)\, \hat{x} = 0, \quad t \in (a, \infty), \tag{1.2}$$

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where $r, p, \hat{r}, \hat{p} : (a, \infty) \to \mathbb{R}$ are continuous functions satisfying the majorant condition

$$p(t) \geq \hat{p}(t), \quad \hat{r}(t) \geq r(t) > 0, \quad t \in (a, \infty). \tag{1.3}$$

The term "singular" refers to the fact that the endpoints of the considered interval (a, ∞) are allowed to be singular points of equations (1.1) and (1.2). The results of [1, Theorem 1] and [2, Theorem 1] read as follows.

Proposition 1.1 *(Singular Sturmian comparison theorem). Assume that* (1.3) *holds with* $r(t) = \hat{r}(t) \equiv 1$ *and* $p(t) \not\equiv \hat{p}(t)$ *on* (a, ∞)*. In addition, suppose that the minorant equation* (1.2) *has a solution* $\hat{x}$*, which is principal at* a *and at the same time at* ∞*. Then every (nontrivial) solution of the majorant equation* (1.1) *has a zero in the interval* (a, ∞)*.*

The result in Proposition 1.1 uses the principal solutions of equation (1.2) at the endpoints ∞ and a. We recall that a solution $\hat{x}$ of (1.2) is principal at ∞ if $\hat{x}(t) \neq 0$ on $[\alpha, \infty)$ for some $\alpha \in (a, \infty)$ and

$$\int_{\alpha}^{\infty} \frac{1}{\hat{r}(t)\, \hat{x}^2(t)}\, \mathrm{d}t = \infty. \tag{1.4}$$

Similarly, a solution $\hat{x}$ of (1.2) is principal at a if $\hat{x}(t) \neq 0$ on $(a, \beta]$ for some $\beta \in (a, \infty)$ and

$$\int_{a}^{\beta} \frac{1}{\hat{r}(t)\, \hat{x}^2(t)}\, \mathrm{d}t = \infty. \tag{1.5}$$

Note that when $a \in \mathbb{R}$ and the functions $\hat{r}$ and $\hat{p}$ are continuous on $[a, \infty)$, then condition (1.5) is equivalent with $\hat{x}(a) = 0$, see e.g. [14, Theorem XI.6.4] with $\omega := a$. The principal solutions of (1.2) at a and at ∞ are unique up to a nonzero multiple and we denote them by $\hat{x}_a$ and $\hat{x}_\infty$. The main assumption of Proposition 1.1 can then be formulated as $\hat{x}_a = \hat{x}_\infty$.

The main purpose of this paper is to establish a generalization of Proposition 1.1 to completely controllable linear Hamiltonian differential systems. That is, we consider the systems

$$y' = \mathcal{J}\mathcal{H}(t)\, y, \quad t \in (a, \infty), \tag{H}$$

$$\hat{y}' = \mathcal{J}\hat{\mathcal{H}}(t)\, \hat{y}, \quad t \in (a, \infty), \tag{$\hat{\mathrm{H}}$}$$

where for a fixed dimension $n \in \mathbb{N}$ the matrix-valued functions $\mathcal{H}, \hat{\mathcal{H}} : (a, \infty) \to \mathbb{R}^{2n \times 2n}$ are symmetric and piecewise continuous, and where $\mathcal{J} \in \mathbb{R}^{2n \times 2n}$ is the canonical skew-symmetric matrix. Furthermore, we assume that $\mathcal{H}(t)$ and $\hat{\mathcal{H}}(t)$ satisfy the Sturmian majorant condition

$$\mathcal{H}(t) \geq \hat{\mathcal{H}}(t), \quad t \in (a, \infty), \tag{1.6}$$

and the Legendre condition

$$B(t) \geq 0, \quad \hat{B}(t) \geq 0, \quad t \in (a, \infty), \tag{1.7}$$

where $B(t)$ and $\hat{B}(t)$ are the lower right $n \times n$ blocks of $\mathcal{H}(t)$ and $\hat{\mathcal{H}}(t)$. In this setting we say that system (H) is a Sturmian majorant of ($\hat{\mathrm{H}}$), or that system ($\hat{\mathrm{H}}$) is a Sturmian minorant of (H). In the special case of the second order differential equations (1.1) and (1.2) satisfying majorant condition (1.3) we have

$$\mathcal{H}(t) = \begin{pmatrix} p(t) & 0 \\ 0 & 1/r(t) \end{pmatrix}, \quad \hat{\mathcal{H}}(t) = \begin{pmatrix} \hat{p}(t) & 0 \\ 0 & 1/\hat{r}(t) \end{pmatrix}, \quad t \in (a, \infty), \tag{1.8}$$

so that systems (H) and ($\hat{\mathrm{H}}$) with (1.8) are completely controllable on (a, ∞) and they satisfy assumptions (1.6) and (1.7).

Our aim is to derive a singular Sturmian comparison theorem for systems (H) and ($\hat{\mathrm{H}}$) in the spirit of Proposition 1.1, that is, for the case when the minorant system ($\hat{\mathrm{H}}$) has a solution $\hat{Y}_a = \hat{Y}_\infty$ which is principal both at a and at ∞. For this purpose we utilize our recent Sturmian comparison theorem for general possibly uncontrollable systems (H) and ($\hat{\mathrm{H}}$) on (a, ∞), which was derived in [27]. Therefore, this work can be regarded as a follow up paper to the latter reference dedicated to completely controllable systems (H) and ($\hat{\mathrm{H}}$) and notably to the second order differential equations (1.1) and (1.2), see [27, Remark 6.9]. The main results of this paper (Theorems 3.1, 3.4, and 3.6) include exact formulas for the numbers of focal points of two conjoined bases Y and $\hat{Y}$ of (H) and ($\hat{\mathrm{H}}$), as well as a detailed analysis of the consequences of the assumption $\hat{Y}_a = \hat{Y}_\infty$ used in Proposition 1.1 for equations (1.1) and (1.2). This analysis connects the majorant condition (1.6), i.e., the coefficients of systems (H) and ($\hat{\mathrm{H}}$), with the distinguished solution of the Riccati matrix differential equation associated with system ($\hat{\mathrm{H}}$). Similarly to [27], the results are formulated in terms of a conjoined basis of an auxiliary transformed linear Hamiltonian system

$$\tilde{y}' = \mathcal{J}\tilde{\mathcal{H}}(t)\,\tilde{y}, \quad t \in (a, \infty), \tag{$\tilde{\mathrm{H}}$}$$

which is obtained from the majorant system (H) by a transformation involving a special symplectic fundamental matrix of the minorant system ($\hat{\mathrm{H}}$). This approach allows to improve the lower bound for the number of focal points of any conjoined basis of (H) in $(-\infty, \infty)$ presented in [11, Theorem 2] by Došlý and Kratz (see Proposition 2.2). Moreover, for systems (H) and ($\hat{\mathrm{H}}$) with (1.8) we then obtain (Theorem 4.1 and Corollaries 4.2 and 4.3) that the above general theory can be applied also in the situation when $p(t) \equiv \hat{p}(t)$ and $r(t) \not\equiv \hat{r}(t)$ on (a, ∞), which is excluded in Proposition 1.1.

The interval considered in this paper is of the form (a, ∞) with $a \in \mathbb{R} \cup \{-\infty\}$ being called a *singular endpoint*. If $a \in \mathbb{R}$ and if the coefficients $\mathcal{H}(t)$ and $\hat{\mathcal{H}}(t)$ are defined and satisfy the main assumptions (1.6) and (1.7) on the closed interval $[a, \infty)$, then we call a to be a *regular endpoint*. The main results of this paper are formulated in a way which includes both regular and singular endpoint a.

Remark 1.2. The problem on the singular interval (a, ∞) with $a \in \mathbb{R}$ can be transformed into an equivalent problem on the interval $(-\infty, \infty)$ by a suitable transformation $s = \varphi(t)$ of the independent variable, where φ is an increasing continuously differentiable function mapping (a, ∞) onto $(-\infty, \infty)$. For example, we can use $\varphi(t) = \ln(t - a)$. In a similar way, the singular interval (a, b) with $b \in \mathbb{R}$ used in [1,2] can be transformed into the type (a, ∞) considered in this paper. Therefore, the results from [27, Sections 4–6] regarding the singular interval $(-\infty, \infty)$ can be applied without loss of generality also when the left endpoint of (a, ∞) is singular and $a \in \mathbb{R}$. More details about the transformation theory of differential equations can be found e.g. in the monographs [6,18].

2. Theory of linear Hamiltonian systems

In this section we first review some results from the oscillation theory of completely controllable linear Hamiltonian systems and their solutions. As basic literature we refer to the monographs [8,15,19,20]. In the second part we deal with basic notions regarding a possibly uncontrollable system ($\tilde{\mathrm{H}}$). A $2n \times n$ matrix solution Y of (H) is a *conjoined basis* if

$$Y^T(t)\,\mathcal{J}Y(t) = 0, \quad \operatorname{rank} Y(t) = n, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

for some (and hence for all) $t \in (a,\infty)$. Suppose that systems (H) and ($\hat{\mathrm{H}}$) are completely controllable on (a,∞). This means under condition (1.7), see [15, Theorem 4.1.3], that conjoined bases $Y = (X^T, U^T)^T$ of (H) and $\hat{Y} = (\hat{X}^T, \hat{U}^T)^T$ of ($\hat{\mathrm{H}}$) have their first components $X(t)$ and $\hat{X}(t)$ invertible for all $t \in (a,\infty) \setminus \mathcal{T}$, where $\mathcal{T}$ is finite or countable. Then a point $t_0 \in (a,\infty)$ such that $X(t_0)$ is singular is called a *focal point* of Y and its multiplicity is given by

$$m(t_0) := \operatorname{def}\, X(t_0) = \dim \operatorname{Ker} X(t_0) = n - \operatorname{rank} X(t_0). \tag{2.1}$$

Moreover, we denote by

$$m(\mathcal{I}) := \sum_{t_0 \in \mathcal{I}} m(t_0) \tag{2.2}$$

the total number of focal points of Y in the indicated interval $\mathcal{I} \subseteq (a,\infty)$. Sturmian comparison theorems for completely controllable linear Hamiltonian systems on bounded interval $[a,b]$ are derived in [7, Theorem 4] by Coppel, in [4, pg. 252] by Arnold, and in [15, Section 7.3] by Kratz. In the special case when the two systems (H) and ($\hat{\mathrm{H}}$) coincide we obtain the Sturmian separation theorem, saying that the difference between the numbers of focal points in $\mathcal{I}$ of any two conjoined bases of (H) is at most n, see e.g. [19, Corollary 1, pg. 366] or [15, Corollary 7.3.2]. This justifies the definition of system (H) to be *nonoscillatory* if the quantity $m(a,\infty)$ is finite for all (or equivalently for some) conjoined basis Y of (H). Note that the nonoscillation condition is imposed at both endpoints of the interval (a,∞).

Recently we showed in [26,27] that in the singular Sturmian theory it is essential to include the multiplicities of focal points at ∞ and/or at a. Following [26, Remark 3.7], for a conjoined basis Y of a nonoscillatory and completely controllable system (H) the multiplicity of its focal point at a and at ∞ is defined by the quantity

$$m(t_0) := \operatorname{def}\, W(Y_{t_0}, Y) = n - \operatorname{rank} W(Y_{t_0}, Y), \quad t_0 \in \{a, \infty\}, \tag{2.3}$$

where Y_∞ and Y_a are the principal solutions of (H) at ∞ and at a, respectively, and where $W(Y_{t_0}, Y) := Y_{t_0}^T(t)\mathcal{J}Y(t)$ is the constant Wronskian of Y_{t_0} and Y. Definition (2.3) complies with (2.1), since for $t_0 \in (a,\infty)$ we have $\operatorname{rank} X(t_0) = \operatorname{rank} W(Y_{t_0}, Y)$, where Y_{t_0} is the principal solution of (H) at the point $t_0 \in (a,\infty)$. Recall that the *principal solution* of (H) at the point $s \in (a,\infty)$ is defined as the solution Y satisfying the initial condition $Y(s) = E$, where the matrix $E := (0,\, I)^T$. As in (1.4), the *principal solution* of (H) at ∞ is defined as the conjoined basis $Y = (X^T, U^T)^T$ such that $X(t)$ is invertible on $[\alpha,\infty)$ for some $\alpha \in (a,\infty)$ and

$$\lim_{t\to\infty} \Big(\int_\alpha^t X^{-1}(s)\, B(s)\, X^{T-1}(s)\, \mathrm{d}s \Big)^{-1} = 0. \tag{2.4}$$

And similarly to (1.5), the *principal solution* of (H) at a is defined as the conjoined basis $Y = (X^T, U^T)^T$ such that $X(t)$ is invertible on $(a,\beta]$ for some $\beta \in (a,\infty)$ and

$$\lim_{t\to a^+} \Big(-\int_t^\beta X^{-1}(s)\, B(s)\, X^{T-1}(s)\, \mathrm{d}s \Big)^{-1} = 0. \tag{2.5}$$

The principal solution of (H) at $t_0 \in (a,\infty) \cup \{a,\infty\}$ will be denoted by the common notation Y_{t_0}. This unified notation is justified by the comments in [3, pg. 173] and by the results of [25, Theorem 5.8] and [26, Section 5], where it is shown that the principal solution at a defined according to (2.5) coincides with the principal solution at a defined by the initial condition $Y(a) = E$ when the endpoint a is regular. We note that under the complete controllability assumption the nonoscillation of system (H) is equivalent with the existence (and uniqueness up to a constant nonsingular right multiple) of the principal solutions Y_a and Y_∞.

Given the definition of the multiplicity of a focal point of Y in (2.3), we will use the notation in (2.2) also for unbounded intervals which include the points a or ∞, i.e., for $\mathcal{I} = (a, \infty]$, or $\mathcal{I} = [a, \infty)$. The quantities in (2.1), (2.3), and (2.2) corresponding to the principal solutions Y_a and Y_∞ of (H) will be denoted by $m_a(t_0)$, $m_a(\mathcal{I})$ and $m_\infty(t_0)$, $m_\infty(\mathcal{I})$. For a conjoined basis $\hat{Y}$ of the minorant system ($\hat{\text{H}}$) we will use the notation $\widehat{m}(t_0)$ and $\widehat{m}(\mathcal{I})$ in the corresponding meaning of (2.1), (2.3), and (2.2). Finally, we will denote by $\hat{Y}_s$ the principal solution of system ($\hat{\text{H}}$) at $s \in (a, \infty) \cup \{a, \infty\}$, and use the notation $\widehat{m}_s(t_0)$ and $\widehat{m}_s(\mathcal{I})$ for the corresponding quantities in (2.1), (2.3), and (2.2).

For our reference we state the following comparison theorem for nonoscillatory and completely controllable linear Hamiltonian systems, see [8, Proposition 2.10] and [15, Proposition 5.1.5(v)].

Proposition 2.1. *Assume that* (1.6) *and* (1.7) *hold. If system* (H) *is nonoscillatory, then system* ($\hat{\text{H}}$) *is also nonoscillatory. If system* ($\hat{\text{H}}$) *is completely controllable, then system* (H) *is also completely controllable.*

The following comparison result involving any conjoined basis of (H) and the principal solution $\hat{Y}_\infty$ of ($\hat{\text{H}}$) at ∞ is proven in [11, Theorem 2].

Proposition 2.2. *Assume that* (1.6) *and* (1.7) *hold and that systems* (H) *and* ($\hat{\text{H}}$) *are nonoscillatory and completely controllable. Then for any conjoined basis Y of* (H) *we have the estimate $m[t, \infty) \geq \widehat{m}_\infty[t, \infty)$ for all $t \in (a, \infty)$, so that*

$$m(a, \infty) \geq \widehat{m}_\infty(a, \infty). \tag{2.6}$$

The formulation of the main results employs system ($\tilde{\text{H}}$), which is connected with the majorant system (H) by the transformation $\tilde{y} = \hat{Z}^{-1}(t)\, y$, where $\hat{Z}$ is a symplectic fundamental matrix of the minorant system ($\hat{\text{H}}$). In this paper we will utilize a special symplectic fundamental matrix $\hat{Z}_\infty$ of ($\hat{\text{H}}$), which contains the principal solution $\hat{Y}_\infty$ of ($\hat{\text{H}}$) as its second block column, i.e., $\hat{Y}_\infty(t) = \hat{Z}_\infty(t)\, E$ on (a, ∞). Then, as in [10], we can verify that

$$\tilde{\mathcal{H}}(t) = \hat{Z}_\infty^T(t)\, [\mathcal{H}(t) - \hat{\mathcal{H}}(t)]\, \hat{Z}_\infty(t), \quad t \in (a, \infty). \tag{2.7}$$

It follows that $\tilde{\mathcal{H}}(t) \geq 0$ on (a, ∞) under the majorant condition (1.6), and in particular the lower right $n \times n$ block $\tilde{B}(t)$ of $\tilde{\mathcal{H}}(t)$ satisfies the Legendre condition

$$\tilde{B}(t) \geq 0, \quad t \in (a, \infty). \tag{2.8}$$

It is obvious that we cannot expect that system ($\tilde{\text{H}}$) is in general completely controllable. For instance, if $\mathcal{H}(t) \equiv \hat{\mathcal{H}}(t)$ on (a, ∞), then $\tilde{\mathcal{H}}(t) \equiv 0$ on (a, ∞) by (2.7) and system ($\tilde{\text{H}}$) is not controllable. Therefore, regarding system ($\tilde{\text{H}}$) we need to recall the concept of left and right focal points, introduced in [29, Definition 1.1] by Wahrheit and in [17, pg. 513] by Kratz and the second author. The point $t_0 \in (a, \infty)$ is a *left focal point*, resp. a *right focal point*, of a conjoined basis $\tilde{Y}$ of ($\tilde{\text{H}}$) if $\operatorname{Ker} \tilde{X}(t_0^-) \subsetneqq \operatorname{Ker} \tilde{X}(t_0)$, resp. $\operatorname{Ker} \tilde{X}(t_0^+) \subsetneqq \operatorname{Ker} \tilde{X}(t_0)$, and then

$$\widetilde{m}_L(t_0) := \operatorname{rank} \tilde{X}(t_0^-) - \operatorname{rank} \tilde{X}(t_0), \quad \widetilde{m}_R(t_0) := \operatorname{rank} \tilde{X}(t_0^+) - \operatorname{rank} \tilde{X}(t_0) \tag{2.9}$$

are the corresponding multiplicities. The notations $\operatorname{Ker} \tilde{X}(t_0^\pm)$ and $\operatorname{rank} \tilde{X}(t_0^\pm)$ represent the one-sided limits at t_0 of the quantities $\operatorname{Ker} \tilde{X}(t)$ and $\operatorname{rank} \tilde{X}(t)$, which are under condition (2.8) piecewise constant on (a, ∞)

by [16, Theorem 3]. The multiplicities in (2.9) then coincide with the number $\widetilde{m}(t_0)$ defined by (2.1) when the system ($\tilde{\mathrm{H}}$) is completely controllable. The nonoscillation of a possibly uncontrollable system ($\tilde{\mathrm{H}}$) means that the kernel of $\tilde{X}(t)$ is constant in some neighborhoods of a and ∞ for some (and hence for any) conjoined basis $\tilde{Y} = (\tilde{X}^T, \tilde{U}^T)^T$ of ($\tilde{\mathrm{H}}$), see [28, Theorem 2.2]. The following result is based on a generalized reciprocity principle in [12, Theorem 2.2].

Proposition 2.3. *Assume that* (1.6) *and* (1.7) *hold on* (a, ∞) *and that system* (H) *is nonoscillatory. Then system* ($\tilde{\mathrm{H}}$) *is also nonoscillatory, where the coefficient matrix* $\tilde{\mathcal{H}}(t)$ *is given in* (2.7) *with the symplectic fundamental matrix* $\hat{Z}_\infty$ *of* ($\hat{\mathrm{H}}$) *satisfying* $\hat{Z}_\infty E = \hat{Y}_\infty$.

The quantities in (2.3) are extended to a possibly uncontrollable system ($\tilde{\mathrm{H}}$) as

$$\left.\begin{aligned} \widetilde{m}_R(a) &:= \big(\lim_{t\to a^+} \operatorname{rank} \tilde{X}(t)\big) - \operatorname{rank} W(\tilde{Y}_a, \tilde{Y}), \\ \widetilde{m}_L(\infty) &:= \big(\lim_{t\to \infty} \operatorname{rank} \tilde{X}(t)\big) - \operatorname{rank} W(\tilde{Y}_\infty, \tilde{Y}), \end{aligned}\right\} \tag{2.10}$$

see [26, Section 3]. In (2.10) the conjoined bases $\tilde{Y}_a$ and $\tilde{Y}_\infty$ are the *minimal principal solutions* of ($\tilde{\mathrm{H}}$) at a and at ∞, which are defined as in (2.4) and (2.5) with the replacement of all inverses by the Moore–Penrose pseudoinverses and imposing the smallest possible rank on the first blocks $\tilde{X}_a(t)$ and $\tilde{X}_\infty(t)$ of $\tilde{Y}_a$ and $\tilde{Y}_\infty$ near the corresponding singular endpoint, see [25, Definition 5.1] and [22–24] for more details. The notation from (2.2) then extends naturally to the notation

$$\widetilde{m}_L(\mathcal{I}) := \sum_{t_0\in\mathcal{I}} \widetilde{m}_L(t_0), \quad \widetilde{m}_R(\mathcal{I}) := \sum_{t_0\in\mathcal{I}} \widetilde{m}_R(t_0),$$

where $\mathcal{I} \subseteq (a, \infty) \cup \{a, \infty\}$ is an interval which may include one of the singular endpoints. When dealing with the principal solution $\tilde{Y}_s$ of ($\tilde{\mathrm{H}}$) at the point $s \in (a, \infty)$ and with the minimal principal solution $\tilde{Y}_s$ of ($\tilde{\mathrm{H}}$) at $s \in \{a, \infty\}$, we will use the notation $\widetilde{m}_{Ls}(t_0)$, $\widetilde{m}_{Ls}(\mathcal{I})$ and $\widetilde{m}_{Rs}(t_0)$, $\widetilde{m}_{Rs}(\mathcal{I})$ in the corresponding meaning.

In the following statement we complement the nonoscillation of system ($\tilde{\mathrm{H}}$) in Proposition 2.3 by the information about the transformation of the minimal principal solutions of systems (H) and ($\tilde{\mathrm{H}}$) at a and at ∞.

Proposition 2.4. *Assume that* (1.6) *and* (1.7) *hold and that systems* (H) *and* ($\hat{\mathrm{H}}$) *are nonoscillatory. In addition, assume that* $\hat{Y}_a = \hat{Y}_\infty$ *holds. Let* $\hat{Z}_\infty$ *be the symplectic fundamental matrix of* ($\hat{\mathrm{H}}$) *satisfying* $\hat{Z}_\infty E = \hat{Y}_\infty$. *Then the minimal principal solutions* Y_a, Y_∞ *and* $\tilde{Y}_a$, $\tilde{Y}_\infty$ *of* (H) *and* ($\tilde{\mathrm{H}}$) *satisfy* $\tilde{Y}_a = \hat{Z}_\infty^{-1} Y_a$ *and* $\tilde{Y}_\infty = \hat{Z}_\infty^{-1} Y_\infty$.

Proof. This result is proven in [27, Theorem 4.4] for the case when $a \in \mathbb{R}$ is a regular point. When a is a singular point (including $a = -\infty$), the result follows from [27, Theorem 6.3 and Remark 6.7], taking Remark 1.2 into account. We note that under the assumption $\hat{Y}_a = \hat{Y}_\infty$ we have $\hat{Z}_\infty = \hat{Z}_a$, where $\hat{Z}_a$ is the symplectic fundamental matrix of ($\hat{\mathrm{H}}$) satisfying $\hat{Z}_a E = \hat{Y}_a$. $\square$

We conclude this section by the following auxiliary statement from matrix analysis, see [15, Lemma 3.1.10] or [5, Proposition 8.2.3]. For this purpose we split the matrix $\tilde{\mathcal{H}}(t)$ in (2.7) into $n \times n$ blocks as

$$\tilde{\mathcal{H}}(t) = \begin{pmatrix} -\tilde{C}(t) & \tilde{A}^T(t) \\ \tilde{A}(t) & \tilde{B}(t) \end{pmatrix}, \quad t \in (a, \infty). \tag{2.11}$$

Proposition 2.5. *Assume that* (1.6) *holds on* (a,∞)*. Then the matrices* $\tilde{A}(t)$*,* $\tilde{B}(t)$*, and* $\tilde{C}(t)$ *defined by* (2.11) *satisfy*

$$\tilde{A}(t) = \tilde{B}(t)\,\tilde{D}(t), \quad \tilde{B}(t) \geq 0, \quad \tilde{D}^T(t)\,\tilde{B}(t)\,\tilde{D}(t) + \tilde{C}(t) \leq 0, \quad t \in (a,\infty), \tag{2.12}$$

for some $n \times n$ *matrix function* $\tilde{D}(t)$ *on* (a,∞)*.*

Proof. The result follows from [15, Lemma 3.1.10] or [5, Proposition 8.2.3], since we have $\tilde{\mathcal{H}}(t) \geq 0$ on (a,∞) under (1.6). Note that we may take $\tilde{D}(t) = \tilde{B}^\dagger(t)\,\tilde{A}(t)$ for $t \in (a,\infty)$, where the dagger refers to the Moore–Penrose pseudoinverse of the given matrix. □

3. Singular Sturmian comparison theorems

In this section we present the main results of this paper regarding the singular Sturmian comparison theorems for completely controllable and nonoscillatory systems (H) and (Ĥ) on the interval (a,∞). In the first result we present formulas for the numbers of focal points of conjoined bases of systems (H) and (Ĥ) under the assumption $\hat{Y}_a = \hat{Y}_\infty$ for the principal solutions of (Ĥ) at a and ∞. We recall from Section 1 that we allow a to be a singular point, especially we allow $a = -\infty$.

Theorem 3.1 *(Singular Sturmian comparison theorem). Assume that* (1.6) *and* (1.7) *hold on* (a,∞) *and that systems* (H) *and* (Ĥ) *are nonoscillatory and completely controllable. In addition, assume that* $\hat{Y}_a = \hat{Y}_\infty$ *holds. Then*

$$m_a(a,\infty] = \widehat{m}_a(a,\infty] + \widetilde{m}_{La}(a,\infty], \quad m_a[a,\infty) = \widehat{m}_a[a,\infty) + \widetilde{m}_{Ra}[a,\infty), \tag{3.1}$$

and for any conjoined basis Y *of* (H) *we have the equality*

$$m(a,\infty) = \operatorname{rank} W(Y_a, Y) + \widehat{m}_a(a,\infty) + \widetilde{m}_R[a,\infty), \tag{3.2}$$

where the quantity $\widetilde{m}_R[a,\infty)$ *refers to the conjoined basis* $\tilde{Y} := \hat{Z}_\infty^{-1} Y$ *of* (H̃)*. In particular,*

$$m(a,\infty) \geq m_a(a,\infty) = \widehat{m}_a(a,\infty) + \widetilde{m}_{Ra}[a,\infty). \tag{3.3}$$

Proof. We split the proof into two parts. Assume first that the endpoint a is regular. The formulas in (3.1) then follow from [27, Corollary 5.4] by using the fact that $\hat{Y}_\infty(a) = \hat{Y}_a(a) = E$. Similarly, for any conjoined basis Y of (H) we obtain from [27, Theorem 5.3] by using the complete controllability assumption that

$$m[a,\infty) = \widehat{m}_\infty[a,\infty) + \widetilde{m}_R[a,\infty) = \widehat{m}_a[a,\infty) + \widetilde{m}_R[a,\infty). \tag{3.4}$$

According to (2.1) we have $m(a) = n - \operatorname{rank} X(a)$ with $\operatorname{rank} X(a) = \operatorname{rank} W(Y_a, Y)$, as well as $\widehat{m}_a(a) = \operatorname{def} \hat{X}_a(a) = n$. Then formula (3.2) follows from (3.4). Finally, the inequality $m(a,\infty) \geq m_a(a,\infty)$ holds by the Sturmian separation theorem [26, Theorem 5.10], while the equality $m_a(a,\infty) = \widehat{m}_a(a,\infty) + \widetilde{m}_{Ra}[a,\infty)$ in (3.3) follows from (3.1) or from (3.2) with $Y := Y_a$ (note that in this case $\tilde{Y} = \hat{Z}_\infty^{-1} Y_a = \tilde{Y}_a$ by Proposition 2.4). Next we assume that the point a is singular and in this case we may take without loss of generality $a = -\infty$ by Remark 1.2. The equalities in (3.1) are then proven in [27, Theorem 6.8]. By the same reference (with $\hat{Y} := \hat{Y}_{-\infty}$ and under the complete controllability assumption) we also obtain that for every conjoined basis Y of (H) the equality

$$m[-\infty,\infty) = \widehat{m}_{-\infty}[-\infty,\infty) + \widetilde{m}_R[-\infty,\infty) \tag{3.5}$$

holds. Since by (2.3) we have $m(-\infty) = n - \operatorname{rank} W(Y_{-\infty}, Y)$ and $\widehat{m}_{-\infty}(-\infty) = n$, separating these multiplicities in formula (3.5) yields

$$m(-\infty,\infty) = \operatorname{rank} W(Y_{-\infty}, Y) + \widehat{m}_{-\infty}(-\infty,\infty) + \widetilde{m}_R[-\infty,\infty), \tag{3.6}$$

which is (3.2) with $a = -\infty$. Next, by the separation theorem in [26, Eq. (8.7)] and by equation (3.6) with $Y := Y_{-\infty}$ (note that again $\tilde{Y} = \hat{Z}_\infty^{-1} Y_{-\infty} = \tilde{Y}_{-\infty}$ by Proposition 2.4) we get

$$m(-\infty,\infty) \geq m_{-\infty}(-\infty,\infty) = \widehat{m}_{-\infty}(-\infty,\infty) + \widetilde{m}_{R-\infty}[-\infty,\infty),$$

which is formula (3.3) with $a = -\infty$. The proof is complete. □

As a consequence of Theorem 3.1 we obtain that the conjoined bases of the majorant system (H), except of the principal solution Y_a of (H) at a (up to a right nonsingular multiple), have at least one focal point in the open interval (a,∞).

Corollary 3.2. *Assume that (1.6) and (1.7) hold on (a,∞) and that systems (H) and ($\hat{\mathrm{H}}$) are nonoscillatory and completely controllable. If $\hat{Y}_a = \hat{Y}_\infty$ and Y is a conjoined basis of (H) such that $Y \neq Y_a$, then $m(a,\infty) \geq 1$.*

Proof. The result follows from (3.2), since under the given assumptions we have $\operatorname{rank} W(Y_a, Y) \geq 1$. □

Remark 3.3. The equality in (3.3) yields the traditional comparison theorem for principal solutions Y_a and $\hat{Y}_a$. Namely, if Y_a has no focal points in (a,∞), then also $\hat{Y}_a$ has no focal points in (a,∞). Moreover, compared to inequality (2.6) in Proposition 2.2, the estimate in (3.3) provides an improved lower bound for the number of focal points in (a,∞) of any conjoined basis Y of (H) for the case when $\hat{Y}_a = \hat{Y}_\infty$.

In the following result we derive the properties of the coefficients of the transformed system ($\tilde{\mathrm{H}}$) for the case when the principal solution Y_a of (H) has no focal points in (a,∞). We also show that the number of focal points in (a,∞) of any conjoined basis Y of (H) is given by the rank of the Wronskian of Y with Y_a. This result will be utilized in the subsequent Sturmian comparison theorem (Theorem 3.6).

Theorem 3.4. *Assume that (1.6) and (1.7) hold on (a,∞) and that systems (H) and ($\hat{\mathrm{H}}$) are nonoscillatory and completely controllable. In addition, assume that $\hat{Y}_a = \hat{Y}_\infty$ and that Y_a has no focal points in (a,∞). Then $Y_a = Y_\infty$ and the coefficient matrices of the transformed system ($\tilde{\mathrm{H}}$) in (2.11) satisfy*

$$\tilde{A}(t) \equiv 0, \quad \tilde{B}(t) \equiv 0, \quad \tilde{C}(t) \leq 0, \quad t \in (a,\infty). \tag{3.7}$$

Moreover, for every conjoined basis Y of (H) we have the equalities

$$m(a,\infty) = \operatorname{rank} W(Y_a, Y), \quad m(a) = \operatorname{def} W(Y_a, Y) = m(\infty). \tag{3.8}$$

Proof. The assumptions imply that $m_a(a,\infty) = 0$. Then from (3.3) and (3.1) in Theorem 3.1 and from $\widehat{m}_a(\infty) = n$ we obtain

$$\widehat{m}_a(a,\infty) = 0, \quad \widetilde{m}_{Ra}[a,\infty) = 0, \quad m_a(\infty) = n + \widetilde{m}_{La}(a,\infty]. \tag{3.9}$$

Since the multiplicity of a focal point at ∞ is always at most n, the third condition in (3.9) implies that $m_a(\infty) = n$ and the kernel of $\tilde{X}_a(t)$ is nonincreasing on (a,∞). By (2.3) (with $t_0 := \infty$ and $Y := Y_a$) it follows that $\operatorname{rank} W(Y_\infty, Y_a) = 0$, so that $Y_a = Y_\infty$. Next, the second condition in (3.9) implies that

$\widetilde{m}_{Ra}(a) = 0$ and the kernel of $\tilde{X}_a(t)$ is nondecreasing on (a, ∞). This means that the kernel of $\tilde{X}_a(t)$ is constant on (a, ∞). From the definition of the multiplicity of the right focal point of $\tilde{Y}_a$ at a in (2.10) we obtain that $\tilde{X}_a(t) \equiv 0$ on some right neighborhood of a, and hence by the above mentioned property of the kernel of $\tilde{X}_a(t)$ it follows that $\tilde{X}_a(t) \equiv 0$ on the entire interval (a, ∞). Consequently, $\tilde{U}_a(t)$ is invertible on (a, ∞). Using the first equation of system ($\tilde{\mathrm{H}}$), i.e., the equation $\tilde{X}_a'(t) = \tilde{A}(t)\, \tilde{X}_a(t) + \tilde{B}(t)\, \tilde{U}_a(t)$ on (a, ∞), we derive that $\tilde{B}(t) \equiv 0$ on (a, ∞). The properties in (3.7) then follow from (2.12) in Proposition 2.5. For (3.8) we note that, under $\tilde{B}(t) \equiv 0$ on (a, ∞), for every conjoined basis $\tilde{Y}$ of ($\tilde{\mathrm{H}}$) the matrix $\tilde{X}(t)$ has constant kernel on (a, ∞). Therefore,

$$\widetilde{m}_R[a, \infty) = \widetilde{m}_R(a) \overset{(2.10)}{=} \operatorname{rank} \tilde{X}(a^+) - \operatorname{rank} W(\tilde{Y}_a, \tilde{Y}),$$
$$\operatorname{rank} W(\tilde{Y}_a, \tilde{Y}) \equiv \operatorname{rank} \tilde{U}_a^T(t)\, \tilde{X}(t) = \operatorname{rank} \tilde{X}(t), \quad t \in (a, \infty).$$

This implies that $\widetilde{m}_R[a, \infty) = 0$ for every conjoined basis $\tilde{Y}$ of ($\tilde{\mathrm{H}}$). Therefore, given a conjoined basis Y of (H), equality (3.2) then implies that $m(a, \infty) = \operatorname{rank} W(Y_a, Y)$. Finally, the second condition in (3.8) follows from the definition of $m(a)$ and $m(\infty)$ in (2.3) and from the already proven property $Y_a = Y_\infty$. The proof is complete. □

Remark 3.5. Under the assumptions of Theorem 3.4, the equalities in (3.8) can be combined to obtain $m[a, \infty) = n = m(a, \infty]$ for every conjoined basis Y of (H). We note that the same conclusion $\widehat{m}[a, \infty) = n = \widehat{m}(a, \infty]$ for every conjoined basis $\hat{Y}$ of ($\hat{\mathrm{H}}$) follows from the singular Sturmian separation theorem for system ($\hat{\mathrm{H}}$) in [26, Theorem 5.7 and Eq. (8.7)] under the assumption $\hat{Y}_a = \hat{Y}_\infty$. Also, given that $\hat{Y}_a = \hat{Y}_\infty$ and $Y_a = Y_\infty$, it follows by the transformation result in Proposition 2.4 that the statement of Theorem 3.4 can be supplemented by the additional conclusion $\tilde{Y}_a = \tilde{Y}_\infty$. Note that when the point a is regular, then in (3.8) we have $m(a, \infty) = \operatorname{rank} X(a)$ and $m(a) = \operatorname{def} X(a)$, since in this case $W(Y_a, Y) = -X(a)$.

In the second main result of this paper we complement the result in Theorem 3.1 (resp. in Remark 3.3) by providing an exact relationship between the considered properties of the principal solutions Y_a and $\hat{Y}_a$. This relationship is expressed in terms of the coefficients of systems (H) and ($\hat{\mathrm{H}}$) via the Riccati quotient associated with the principal solution $\hat{Y}_a$ of ($\hat{\mathrm{H}}$).

Theorem 3.6 *(Singular Sturmian comparison theorem). Assume that (1.6) and (1.7) hold on (a, ∞) and that systems (H) and ($\hat{\mathrm{H}}$) are nonoscillatory and completely controllable. In addition, assume that $\hat{Y}_a = \hat{Y}_\infty$ holds. Then the following statements are equivalent.*

(i) *The principal solution Y_a of (H) has no focal points in (a, ∞).*
(ii) *The principal solution $\hat{Y}_a$ of ($\hat{\mathrm{H}}$) has no focal points in (a, ∞) and*

$$\mathcal{H}(t) - \hat{\mathcal{H}}(t) = \big(-\hat{Q}_a(t),\ I \big)^T [B(t) - \hat{B}(t)] \big(-\hat{Q}_a(t),\ I \big), \quad t \in (a, \infty), \tag{3.10}$$

where $\hat{Q}_a(t) := \hat{U}_a(t)\, \hat{X}_a^{-1}(t)$ is the symmetric Riccati quotient which corresponds to the principal solution $\hat{Y}_a$ of ($\hat{\mathrm{H}}$) on (a, ∞).

In this case, in addition to (3.7) we have

$$\tilde{C}(t) = -\hat{X}_a^{-1}(t)\, [B(t) - \hat{B}(t)]\, \hat{X}_a^{T-1}(t), \quad t \in (a, \infty), \tag{3.11}$$

and the symmetric Riccati quotient $Q_a(t) := U_a(t)\, X_a^{-1}(t)$ corresponding to the principal solution Y_a of (H) satisfies $Q_a(t) = \hat{Q}_a(t)$ on (a, ∞).

Proof. Assume that (i) holds, i.e., $m_a(a,\infty)=0$. Then by Remark 3.3 we have $\widehat{m}_a(a,\infty)=0$. From (3.7) in Theorem 3.4 in combination with (2.7) and $\hat{Y}_\infty=\hat{Y}_a$ we obtain that

$$\begin{aligned}\mathcal{H}(t)-\hat{\mathcal{H}}(t) &\overset{(2.7)}{=} \hat{Z}_\infty^{T-1}(t)\,\tilde{\mathcal{H}}(t)\,\hat{Z}_\infty^{-1}(t) \overset{(3.7)}{=} \mathcal{J}\hat{Z}_a(t)\,\mathcal{J}\begin{pmatrix}-\tilde{C}(t) & 0\\ 0 & 0\end{pmatrix}\mathcal{J}\hat{Z}_a^T(t)\,\mathcal{J}\\ &= \mathcal{J}\hat{Z}_a(t)\begin{pmatrix}0 & 0\\ 0 & \tilde{C}(t)\end{pmatrix}\hat{Z}_a^T(t)\,\mathcal{J} = \mathcal{J}\hat{Y}_a(t)\,\tilde{C}(t)\,\hat{Y}_a^T(t)\,\mathcal{J}\\ &= -\big(-\hat{Q}_a(t),\ I\big)^T\,\hat{X}_a(t)\,\tilde{C}(t)\,\hat{X}_a^T(t)\,\big(-\hat{Q}_a(t),\ I\big)\end{aligned} \tag{3.12}$$

on (a,∞). Comparing the right lower blocks of the matrices $\mathcal{H}(t)-\hat{\mathcal{H}}(t)$ and (3.12) we get $B(t)-\hat{B}(t)=-\hat{X}_a(t)\,\tilde{C}(t)\,\hat{X}_a^T(t)$, which yields equality (3.11). Upon inserting (3.11) back to (3.12) we obtain formula (3.10). Conversely, assume that (ii) holds. Then $\widehat{m}_a(a,\infty)=0$ and

$$\tilde{\mathcal{H}}(t) \overset{(2.7)}{=} \hat{Z}_a^T(t)\,[\mathcal{H}(t)-\hat{\mathcal{H}}(t)]\,\hat{Z}_a(t) \overset{(3.10)}{=} \begin{pmatrix}\hat{X}_a^{-1}(t)\,[B(t)-\hat{B}(t)]\,\hat{X}_a^{T-1}(t) & 0\\ 0 & 0\end{pmatrix}$$

on (a,∞), where we also used that $\hat{Z}_a=(\hat{\bar{Y}}_a,\,\hat{Y}_a)$ with $W(\hat{\bar{Y}}_a,\hat{Y}_a)=I$. Therefore, $\tilde{A}(t)\equiv 0$, $\tilde{B}(t)\equiv 0$, and (3.11) holds. It follows that every solution $\tilde{Y}$ of system ($\tilde{\text{H}}$) has its first component $\tilde{X}(t)$ with constant kernel on (a,∞). In particular, the principal solution $\tilde{Y}_a$ has $\tilde{X}_a(t)\equiv 0$ and $\tilde{U}_a(t)$ invertible on (a,∞), since $\tilde{Y}_a$ is the minimal principal solution of ($\tilde{\text{H}}$) at a (note that $\tilde{Y}_a$ is a conjoined basis of ($\tilde{\text{H}}$) with the smallest possible rank of $\tilde{X}(t)$ near a and that system ($\tilde{\text{H}}$) has the maximal possible order of abnormality equal to n when $\tilde{B}(t)\equiv 0$), see [25, Definition 5.1]. This yields that $\widetilde{m}_{Ra}[a,\infty)=0$. The equality in (3.3) in Theorem 3.1 then implies that $m_a(a,\infty)=0$, which proves part (i). Finally, since $Y_a=\hat{Z}_a\tilde{Y}_a$ by Proposition 2.4, it follows that $Y_a(t)=\hat{Y}_a(t)\,\tilde{U}_a(t)$ on (a,∞), where the matrix $\tilde{U}_a(t)$ is invertible. Therefore, the equality $Q_a(t)=\hat{Q}_a(t)$ holds on (a,∞) and the proof is complete. □

Remark 3.7. If we split the symmetric matrices $\mathcal{H}(t)$ and $\hat{\mathcal{H}}(t)$ into $n\times n$ blocks as

$$\mathcal{H}(t)=\begin{pmatrix}-C(t) & A^T(t)\\ A(t) & B(t)\end{pmatrix},\quad \hat{\mathcal{H}}(t)=\begin{pmatrix}-\hat{C}(t) & \hat{A}^T(t)\\ \hat{A}(t) & \hat{B}(t)\end{pmatrix},\quad t\in(a,\infty),$$

see also (2.11), then condition (3.10) has the form

$$\left.\begin{aligned}A(t)&=\hat{A}(t)-[B(t)-\hat{B}(t)]\,\hat{Q}_a(t),\\ C(t)&=\hat{C}(t)-\hat{Q}_a(t)\,[B(t)-\hat{B}(t)]\,\hat{Q}_a(t),\end{aligned}\right\}\quad t\in(a,\infty).$$

The advantage of Theorem 3.6 lies in the fact that knowing the relationship between the coefficients $\mathcal{H}(t)$ and $\hat{\mathcal{H}}(t)$ we can obtain the information about the number of focal points of the principal solution Y_a of (H) from the information about the principal solution $\hat{Y}_a$ of ($\hat{\text{H}}$) alone. Hence, via the formulas in (3.8) in Theorem 3.1 we can obtain the information about the number of focal points of any conjoined basis Y of (H).

Corollary 3.8. *Assume that (1.6) and (1.7) hold on (a,∞) and that systems (H) and ($\hat{\text{H}}$) are nonoscillatory and completely controllable. In addition, assume that $\hat{Y}_a=\hat{Y}_\infty$ holds and condition (ii) of Theorem 3.6 is satisfied. Then for every conjoined basis Y of (H) the equalities in (3.8) hold.*

4. Second order differential equations

In this section we apply Theorem 3.6 to systems (H) and ($\hat{\text{H}}$) representing the Sturm–Liouville differential equations (1.1) and (1.2). Even in this special situation we obtain a new result, which extends Proposition 1.1 to the case of variable coefficients $r(t)$ and/or $\hat{r}(t)$. We note that the property in (2.3) now translates as

$$t_0 \in \{a, \infty\}, \quad m(t_0) = \begin{cases} 1, & \Leftrightarrow \ x \text{ is principal at } t_0, \\ 0, & \Leftrightarrow \ x \text{ is nonprincipal at } t_0, \end{cases}$$

that is, the principal solution of (1.1) at ∞ (or at a) has by definition a zero at ∞ (or at a). With this extended terminology the condition $\hat{x}_a = \hat{x}_\infty$ means that the principal solution $\hat{x}_\infty$ of (1.2) has zeros at a and at ∞, so that the statement of Proposition 1.1 agrees with the classical Sturmian comparison theorem on a bounded interval $[a, b]$.

Theorem 4.1 *(Singular Sturmian comparison theorem). Assume that (1.3) holds on (a, ∞) and that equations (1.1) and (1.2) are nonoscillatory. In addition, assume that the principal solution $\hat{x}_\infty$ of (1.2) has a zero at a, i.e., $\hat{x}_\infty = \hat{x}_a$. Then the following statements are equivalent.*

(i) *The principal solution x_a of (1.1) satisfies $x_a(t) \neq 0$ on (a, ∞).*
(ii) *The principal solution $\hat{x}_a$ of (1.2) satisfies $\hat{x}_a(t) \neq 0$ on (a, ∞) and*

$$[r(t) - \hat{r}(t)]\, \hat{x}_a'(t) = 0, \quad p(t) = \hat{p}(t), \quad t \in (a, \infty). \tag{4.1}$$

In this case $x_\infty = x_a$ and every nonprincipal solution of (1.1) at ∞ (or at a) has exactly one zero in the interval (a, ∞).

Proof. The equivalence of (i) and (ii) above follows from the equivalence of (i) and (ii) in Theorem 3.6 when we use the form of $\mathcal{H}(t)$ and $\hat{\mathcal{H}}(t)$ in (1.8). In particular, condition (3.10) reduces to (4.1). Moreover, if condition (i) holds, then Theorem 3.4 applies to this case and consequently, we have $x_\infty = x_a$ and $m(a, \infty) = 1$ for every nonprincipal solution x of (1.1) at ∞ (or at a) by (3.8). □

We note that the assumption $\hat{x}_\infty = \hat{x}_a$ in Theorem 4.1 is naturally satisfied when the point a is chosen to be a zero of the principal solution $\hat{x}_\infty$ of (1.2). The result in Theorem 4.1 also shows that, under the given assumptions, the property $m(a, \infty) \geq 1$ for a solution x of (1.1) in Proposition 1.1 always holds for nonprincipal solutions of (1.1) at ∞, and it also holds for the principal solution x_∞ of (1.1) exactly when condition (ii) in Theorem 4.1 is violated.

Corollary 4.2 *(Singular Sturmian comparison theorem). Assume that (1.3) holds on (a, ∞) and that equations (1.1) and (1.2) are nonoscillatory. In addition, assume that the principal solution $\hat{x}_\infty$ of (1.2) has a zero at a, i.e., $\hat{x}_\infty = \hat{x}_a$. Then every nonprincipal solution of (1.1) at ∞ has at least one zero in (a, ∞), and the principal solution x_∞ of (1.1) has at least one zero in (a, ∞) if and only if condition (ii) in Theorem 4.1 is violated.*

Proof. Let x be a nonprincipal solution of (1.1) at ∞. The number of zeros $m(a, \infty)$ of x in the interval (a, ∞) depends on the number of zeros of the principal solution x_a of (1.1) in this interval. If $m_a(a, \infty) = 0$, then $m(a, \infty) = 1$ by Theorem 4.1. If $m_a(a, \infty) \geq 1$, then $m(a, \infty) \geq m_a(a, \infty) \geq 1$ by the singular Sturmian separation theorem, i.e., by (3.3) in Theorem 3.1. Consider now the principal solution x_∞ of (1.1) and its number of zeros in (a, ∞). By [11, Corollary 1] we know that $m_\infty(a, \infty) = m_a(a, \infty)$. Hence, applying Theorem 4.1 we obtain that $m_\infty(a, \infty) \geq 1$ if and only if condition (ii) in Theorem 4.1 is violated. □

The statement in Corollary 4.2 provides a way to explicitly describe sufficient conditions in terms of the coefficients of equations (1.1) and (1.2), which guarantee that every (nontrivial) solution of (1.1) has at least one zero in the interval (a, ∞). These sufficient conditions are formulated as a violation of condition (4.1). One such a condition, namely $r(t) = \hat{r}(t) \equiv 1$ and $p(t) \not\equiv \hat{p}(t)$ on (a, ∞), was discovered by Aharonov and Elias in [1, Theorem 1] and [2, Theorem 1], see also Proposition 1.1. Observe that the assumption of the nonoscillation of the majorant equation (1.1) can now be dropped.

Corollary 4.3 *(Singular Sturmian comparison theorem). Assume that* (1.3) *holds on* (a, ∞) *and that the minorant equation* (1.2) *is nonoscillatory. In addition, assume that the principal solution* $\hat{x}_\infty$ *of* (1.2) *has a zero at* a*, i.e.,* $\hat{x}_\infty = \hat{x}_a$*, and that condition* (4.1) *does not hold. Then every (nontrivial) solution of* (1.1) *has at least one zero in the interval* (a, ∞)*.*

Proof. If equation (1.1) is oscillatory, then the result holds trivially. If equation (1.1) is nonoscillatory, then the result follows directly from Corollary 4.2. □

Remark 4.4. Condition (4.1) does not hold for example when a is a regular point and $r(t) < \hat{r}(t)$ on $[a, a+\varepsilon)$ for some $\varepsilon > 0$, because in this case $\hat{x}_a(a) = 0$, $\hat{x}_a'(a) = 1/\hat{r}(a)$, and thus $\hat{x}_a'(t) \neq 0$ for t close to a. Further sufficient conditions of a similar type can now be formulated based on this argument.

In the remaining part of this paper we present several examples, which illustrate our new singular Sturmian comparison theorems. These examples are based on a specific choice of the reference equation (1.2) for which $\hat{x}_a = \hat{x}_\infty$. First we consider a regular point a.

Example 4.5. Consider equations (1.1) and (1.2) on $[0, \infty)$ with the coefficients

$$\hat{r}(t) = \begin{cases} \mathrm{e}^{2t}, & \text{for } t \in [0,1], \\ \mathrm{e}^{2}, & \text{for } t \in (1,\infty), \end{cases} \qquad p(t) = \hat{p}(t) = \begin{cases} \dfrac{6\mathrm{e}^{2t}(1-t)}{t^2 - 3t + 3}, & \text{for } t \in [0,1], \\ 0, & \text{for } t \in (1,\infty), \end{cases}$$

where $r(t)$ will be specified later. Then the function $\hat{x}_\infty(t) = \frac{1}{3}(t-1)^3 + \frac{1}{3}$ for $t \in [0,1]$ and $\hat{x}_\infty(t) \equiv \frac{1}{3}$ for $t \in (1, \infty)$ is the principal solution of (1.2) at ∞ satisfying $\hat{x}_\infty(0) = 0$, i.e., $\hat{x}_\infty = \hat{x}_0$. Also, we have $\hat{x}_0(t) \neq 0$ on $(0, \infty)$, $\hat{x}_0'(t) = (t-1)^2$ on $[0,1]$, and $\hat{x}_0'(t) \equiv 0$ on $[1, \infty)$. Let $r(t)$ be any positive continuous function on $[0, \infty)$ such that $r(t) = \mathrm{e}^{2t}$ on $[0,1]$ and $r(t) \leq \mathrm{e}^2$ on $(1, \infty)$. Then condition (4.1) is satisfied, equation (1.1) is nonoscillatory, and hence by Theorem 4.1 the principal solution x_0 of (1.1) satisfies $x_0(t) \neq 0$ on $(0, \infty)$ and $x_\infty = x_0$. Moreover, every nonprincipal solution x of (1.1) at 0 or at ∞ has exactly one zero in the interval $(0, \infty)$.

Another example is motivated by [9, Example 1].

Example 4.6. Consider equations (1.1) and (1.2) on $[0, \infty)$ with the coefficients

$$\hat{r}(t) = \begin{cases} \dfrac{1 + t - 2\mathrm{e}^{t-1}}{1-t}, & \text{for } t \neq 1, \\ 1, & \text{for } t = 1, \end{cases} \qquad p(t) = \hat{p}(t) \equiv 1, \quad t \in [0, \infty),$$

where $r(t)$ will be specified later. Then the function $\hat{x}_\infty(t) = t\mathrm{e}^{-t}$ for $t \in [0, \infty)$ is the principal solution of (1.2) at ∞ satisfying $\hat{x}_\infty(0) = 0$, i.e., $\hat{x}_\infty = \hat{x}_0$. Also, we have $\hat{x}_0(t) \neq 0$ on $(0, \infty)$ and $\hat{x}_0'(t) \neq 0$ for all $t \neq 1$. Let $r(t)$ be any positive continuous function on $[0, \infty)$ such that $r(t) \leq \hat{r}(t)$ and $r(t) \not\equiv \hat{r}(t)$ on $[0, \infty)$. Then by Corollary 4.3 every (nontrivial) solution of (1.1) has at least one zero in the interval $(0, \infty)$.

Next we consider the case of a singular point $a = -\infty$.

Example 4.7. Consider equation (1.2) of the form $\hat{x}'' = 0$ on $(-\infty, \infty)$, i.e., $\hat{r}(t) \equiv 1$ and $\hat{p}(t) \equiv 0$. Then $\hat{x}_\infty(t) \equiv 1$ is the principal solution of (1.2) at $\pm\infty$, i.e., $\hat{x}_\infty = \hat{x}_{-\infty}$. Also, we have $\hat{x}_\infty(t) \neq 0$ on $(-\infty, \infty)$ and $\hat{x}'_\infty(t) \equiv 0$. Let $r(t)$ and $p(t)$ be continuous functions with $0 < r(t) \leq 1$ and $p(t) \geq 0$ on $(-\infty, \infty)$ such that $p(t) \not\equiv 0$ on $(-\infty, \infty)$. Then by Corollary 4.3 every (nontrivial) solution of (1.1) has at least one zero in the interval $(-\infty, \infty)$.

In the last example we consider a singular point $a = 0$.

Example 4.8. Consider equation (1.2) to be the Euler differential equation $\hat{x}'' + \frac{1}{4t^2}\,\hat{x} = 0$ on $(0, \infty)$, i.e., $\hat{r}(t) \equiv 1$ and $\hat{p}(t) = 1/(4t^2)$ on $(0, \infty)$. Then $\hat{x}_\infty(t) = \sqrt{t}$ is the principal solution of (1.2) at ∞ and at 0, i.e., $\hat{x}_\infty = \hat{x}_0$. Let $r(t)$ and $p(t)$ be continuous functions with $0 < r(t) \leq 1$ and $p(t) \geq 1/(4t^2)$ on $(0, \infty)$ such that $r(t) \not\equiv 1$ or $p(t_0) > 1/(4t_0^2)$ for some $t_0 > 0$. Then by Corollary 4.3 every (nontrivial) solution of (1.1) has at least one zero in the interval $(0, \infty)$.

Further examples can be constructed e.g. by considering [1, Example, pg. 762] as the reference equation (1.2) and incorporating the leading coefficient $r(t)$. We refer to [13] for a possible source of such singular Sturm–Liouville equations depending on a parameter $\lambda \in \mathbb{R}$.

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References

[1] D. Aharonov, U. Elias, Singular Sturm comparison theorems, J. Math. Anal. Appl. 371 (2) (2010) 759–763.
[2] D. Aharonov, U. Elias, Sturm comparison theorem in singular situations, Funct. Differ. Equ. 18 (3–4) (2011) 171–175.
[3] C.D. Ahlbrandt, Principal and antiprincipal solutions of self-adjoint differential systems and their reciprocals, Rocky Mt. J. Math. 2 (1972) 169–182.
[4] V.I. Arnold, Sturm theorems and symplectic geometry, Funkc. Anal. Prilozh. 19 (4) (1985) 1–10; translated in Funct. Anal. Appl. 19 (4) (1985) 251–259.
[5] D.S. Bernstein, Matrix Mathematics. Theory, Facts, and Formulas with Application to Linear Systems Theory, Princeton University Press, Princeton, 2005.
[6] O. Borůvka, Linear Differential Transformations of the Second Order, Translated from the German by F.M. Arscott, The English Universities Press, Ltd., London, 1971.
[7] W.A. Coppel, Comparison theorems for canonical systems of differential equations, J. Math. Anal. Appl. 12 (2) (1965) 306–315.
[8] W.A. Coppel, Disconjugacy, Lecture Notes in Mathematics, vol. 220, Springer-Verlag, Berlin-New York, 1971.
[9] Z. Došlá, M. Marini, S. Matucci, A Dirichlet problem on the half-line for nonlinear equations with indefinite weight, Ann. Mat. Pura Appl. (4) 196 (1) (2017) 51–64.
[10] O. Došlý, Relative oscillation of linear Hamiltonian differential systems, Math. Nachr. 290 (14–15) (2017) 2234–2246.
[11] O. Došlý, W. Kratz, Singular Sturmian theory for linear Hamiltonian differential systems, Appl. Math. Lett. 26 (12) (2013) 1187–1191.
[12] J.V. Elyseeva, On symplectic transformations of linear Hamiltonian differential systems without normality, Appl. Math. Lett. 68 (2017) 33–39.
[13] W.N. Everitt, A catalogue of Sturm–Liouville differential equations, in: W.O. Amrein, A.M. Hinz, D.B. Pearson (Eds.), Sturm–Liouville Theory. Past and Present, Proceedings of the International Colloquium, Geneva, 2003, Birkhäuser, Basel, 2005, pp. 271–331.
[14] P. Hartman, Ordinary Differential Equations, John Wiley, New York, 1964.
[15] W. Kratz, Quadratic Functionals in Variational Analysis and Control Theory, Akademie Verlag, Berlin, 1995.
[16] W. Kratz, Definiteness of quadratic functionals, Analysis 23 (2) (2003) 163–183.
[17] W. Kratz, R. Šimon Hilscher, Rayleigh principle for linear Hamiltonian systems without controllability, ESAIM Control Optim. Calc. Var. 18 (2) (2012) 501–519.
[18] F. Neuman, Global Properties of Linear Ordinary Differential Equations, Mathematics and Its Applications (East European Series), vol. 52, Kluwer Academic Publishers Group, Dordrecht, 1991.

[19] W.T. Reid, Ordinary Differential Equations, John Wiley & Sons, Inc., New York – London – Sydney, 1971.
[20] W.T. Reid, Sturmian Theory for Ordinary Differential Equations, Springer-Verlag, New York – Berlin – Heidelberg, 1980.
[21] C.A. Swanson, Comparison and Oscillation Theory of Linear Differential Equations, Academic Press, New York, 1968.
[22] P. Šepitka, R. Šimon Hilscher, Minimal principal solution at infinity for nonoscillatory linear Hamiltonian systems, J. Dyn. Differ. Equ. 26 (1) (2014) 57–91.
[23] P. Šepitka, R. Šimon Hilscher, Principal solutions at infinity of given ranks for nonoscillatory linear Hamiltonian systems, J. Dyn. Differ. Equ. 27 (1) (2015) 137–175.
[24] P. Šepitka, R. Šimon Hilscher, Principal and antiprincipal solutions at infinity of linear Hamiltonian systems, J. Differ. Equ. 259 (9) (2015) 4651–4682.
[25] P. Šepitka, R. Šimon Hilscher, Focal points and principal solutions of linear Hamiltonian systems revisited, J. Differ. Equ. 264 (9) (2018) 5541–5576.
[26] P. Šepitka, R. Šimon Hilscher, Singular Sturmian separation theorems on unbounded intervals for linear Hamiltonian systems, J. Differ. Equ. 266 (11) (2019) 7481–7524.
[27] P. Šepitka, R. Šimon Hilscher, Singular Sturmian comparison theorems for linear Hamiltonian systems, J. Differ. Equ. (2020), https://doi.org/10.1016/j.jde.2020.02.016, in press.
[28] R. Šimon Hilscher, On general Sturmian theory for abnormal linear Hamiltonian systems, in: W. Feng, Z. Feng, M. Grasselli, A. Ibragimov, X. Lu, S. Siegmund, J. Voigt (Eds.), Dynamical Systems, Differential Equations and Applications, Proceedings of the 8th AIMS Conference on Dynamical Systems, Differential Equations and Applications, Dresden, 2010, in: Discrete Contin. Dynam. Systems, Suppl. 2011, American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2011, pp. 684–691.
[29] M. Wahrheit, Eigenvalue problems and oscillation of linear Hamiltonian systems, Int. J. Difference Equ. 2 (2) (2007) 221–244.