



Hardy-Sobolev equation on compact Riemannian manifolds involving p -Laplacian



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ABSTRACT

In this paper, we apply the Hardy-Sobolev inequality and minimization method in the spirit of Aubin to obtain the existence theorem for a class of quasilinear elliptic equations with critical Hardy-Sobolev exponent on compact Riemannian manifolds.

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1. Introduction and main results

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ without boundary. Let $1 < p < n$, $s \in (0, p)$ and $a(x), K(x) \in C^0(M)$. For given $x_0 \in M$, we are looking for the distributional solutions $u \in H_1^p(M) \cap L^\infty(M)$ to the following equation

$$\Delta_{p,g} u + a(x)|u|^{p-2}u = K(x) \frac{|u|^{p^*(s)-2}u}{d_g(x, x_0)^s} \quad \text{in } M, \quad (1.1)$$

where $\Delta_{p,g} = -\operatorname{div}_g(|\nabla_g u|_g^{p-2} \nabla_g u)$ is the p -Laplace-Beltrami operator on (M, g) , d_g is the Riemannian distance on (M, g) , and $p^*(s) = \frac{(n-s)p}{n-p}$ is the critical Hardy-Sobolev exponent in the sense of Lemma 3.1 which we will show later. The relevant problem corresponding to (1.1) has been studied extensively in different variants. We quote some of them.

When $s = 0$, problem (1.1) reduces to the so-called generalized scalar curvature equation

$$\Delta_{p,g} u + a(x)u^{p-1} = K(x)u^{p^*-1}, \quad u > 0, \quad \text{in } M. \quad (1.2)$$

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In [15], Druet has proved that under suitable assumptions on M , $a(x)$ and $K(x)$, there exists at least one positive solution for (1.2). In [9], Chen and Liu obtained some existence results for (1.2) with a lower order perturbation term. We point out that in the case of $p = 2$, the problem (1.2) is a natural extension of the classical Yamabe problem. The latter has been completely solved by Yamabe [36], Aubin [1], Trudinger [35] and Schoen [32]. See also [23] for an excellent survey.

When $p = 2$ and $K(x) = 1$, problem (1.1) reduces to the following semilinear problem involving critical Hardy-Sobolev exponent

$$\Delta_g u + a(x)u = \frac{|u|^{2^*(s)-2}u}{d_g(x, x_0)^s} \quad \text{in } M. \quad (1.3)$$

By a minimization method, Jaber (see [18]) obtained the existence of positive distributional solutions when $n \geq 4$ and the potential a is sufficiently below the scalar curvature at x_0 . For $n = 3$, by a global argument, he proved the existence when the mass of the linear operator $\Delta_g + a$ is positive. In [20], Jaber proved the existence of solution for Eq. (1.3) with a perturbation term u^{q-1} for $2 < q < \frac{2n}{n-2}$ by mountain pass theorem.

In the Euclidean setting, Perera and Zou (see [30]) studied the existence of solution for the following p -Laplacian problem involving critical Hardy-Sobolev exponent

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + \frac{|u|^{p^*(s)-2}u}{|x|^s} & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases} \quad (1.4)$$

where Ω is a bounded domain in \mathbb{R}^n containing the origin, and $\lambda > 0$ is a parameter. See also Chen et al. [10] for the fractional p -Laplacian. For other related references on Hardy-Sobolev equations we refer the readers to the works by Ghoussoub and Yuan [16], Kang and Peng [21,22], Peng and Wang [29], Li et al. [25], Musina [28], Li and Gao [24], Zhong and Zou [37], Chen [8]. The list is far to be exhaustive.

Motivated by the above mentioned papers, especially [15] and [18], we study the effect of the geometry on the existence of a nontrivial solution for the problem (1.1). Since the embedding

$$H_1^p(M) \hookrightarrow L^{p^*(s)}(M, d_g(x, x_0)^{-s} dv_g)$$

is not compact (see Lemma 3.1), it is not possible to obtain solutions of problem (1.1) via simple variational arguments. The main ideas of proof are based on the Hardy-Sobolev inequality and minimization method in the spirit of Aubin [1], which is classical because of the lack of compactness for the critical embedding.

Nontrivial weak solutions of (1.1) correspond, modulo nonzero constant multiples, to critical points of the following functional

$$I(u) := \int_M (|\nabla_g u|_g^p + a(x)|u|^p) dv_g$$

on the manifold

$$\Lambda := \left\{ u \in H_1^p(M) \setminus \{0\} : \int_M \frac{K(x)|u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g = 1 \right\}.$$

Let L_g be defined by

$$L_g(u) = \Delta_{p,g} u + a(x)|u|^{p-2}u.$$

It is now well known that the operator L_g plays a fundamental role in the study of (1.1). We limit the study to the case that the operator L_g is coercive. In this case we investigate the existence of solutions to (1.1) by minimization the functional I . We say that a is such that the operator L_g is coercive if there exists $\zeta > 0$ such that

$$\int_M (|\nabla_g u|_g^p + a(x)|u|^p) dv_g \geq \zeta \|u\|^p, \text{ for all } u \in H_1^p(M),$$

where $\|\cdot\|$ is as in (2.1), or equivalently, the operator L_g is coercive if and only if there exists $\zeta > 0$ such that

$$\int_M (|\nabla_g u|_g^p + a(x)|u|^p) dv_g \geq \zeta \int_M |u|^p dv_g, \text{ for all } u \in H_1^p(M).$$

Clearly, this happens if $a > 0$. Moreover, if the operator L_g is coercive, one can check easily that the assumption “ K is strictly positive somewhere” is a necessary condition for (1.1) to admit a positive solution u . In what follows, we say that (H1) and (H2) holds if

$$\max_M K > 0, \tag{H1}$$

and

$$\left(\max_M K\right)^{\frac{p}{p^*(s)}} \inf_{u \in \Lambda} I(u) < S(n, p, s)^{-p}, \tag{H2}$$

where $S(n, p, s)$ is explained later in this section (see (1.7)). Our first result of this paper reads as follows:

Theorem 1.1. *Let (M, g) be a smooth compact Riemannian manifold without boundary of dimension $n \geq 3$ and $p \in (1, n)$. We fix $x_0 \in M$, $s \in (0, p)$, and $a(x), K(x) \in C^0(M)$ such that L_g is coercive and (H1)-(H2) holds. Then problem (1.1) possesses a nontrivial solution $u \in L^\infty(M) \cap C_{loc}^{1,\alpha}(M \setminus \{x_0\})$ for some $\alpha \in (0, 1)$.*

In the proof of Theorem 1.1, we will use the approach developed by Yamabe [36] which include several steps. Moreover, as a straightforward application of Theorem 1.1, notice that

$$u \equiv \left(\int_M \frac{K(x)}{d_g(x, x_0)^s} dv_g \right)^{-\frac{1}{p^*(s)}} \in \Lambda,$$

and then we obtain the following corollary.

Corollary 1.2. *Let (M, g) be a smooth compact Riemannian manifold without boundary of dimension $n \geq 3$ and $1 < p < n$. We fix $x_0 \in M$, $s \in (0, p)$, and $a, K \in C^0(M)$ such that L_g is coercive and $\int_M K(x) d_g(x, x_0)^{-s} dv_g > 0$. If*

$$\left(\frac{\max_M K}{\int_M K(x) d_g(x, x_0)^{-s} dv_g} \right)^{\frac{p}{p^*(s)}} \int_M a(x) dv_g < S(n, p, s)^{-p},$$

then equation (1.1) possesses a nontrivial solution $u \in L^\infty(M) \cap C_{loc}^{1,\alpha}(M \setminus \{x_0\})$ for some $\alpha \in (0, 1)$.

The following result is related to the geometry of the manifold at a maximum point of K and the behavior of K up to the second order at this point. More precisely, we have:

Theorem 1.3. Let (M, g) be a smooth compact Riemannian manifold without boundary of dimension $n \geq 3$ and $1 < p^2 < n$. We fix $x_0 \in M$, $s \in (0, p)$, and let $a(x) \in C^0(M)$ be such that L_g is coercive. Furthermore, assume that $K(x) \in C^2(M)$ and $K(x_0) = \max_{x \in M} K(x) > 0$. If we have one of the following cases:

- (i) $p < 2$, $n > 3p - 2$ and $a(x_0) < 0$;
- (ii) $p = 2$ and

$$\frac{4(2n-2-s)}{(n-2)(n-4)}a(x_0) < -\frac{\Delta_g K(x_0)}{K(x_0)} + \frac{6-s}{3(n-4)}\text{Scal}_g(x_0);$$

- (iii) $p > 2$ and

$$\frac{3(n-3p+2)}{3p-s} \frac{\Delta_g K(x_0)}{K(x_0)} < \text{Scal}_g(x_0),$$

then problem (1.1) possesses a nontrivial solution $u \in L^\infty(M) \cap C_{loc}^{1,\alpha}(M \setminus \{x_0\})$ for some $\alpha \in (0, 1)$.

The proof of Theorem 1.3 involves estimates on the growth of some test function localized at a maximum point of K . However, compared with the non-singular cases (i.e., $s = 0$), the terms involved in the expansion of the functional become more complicated so that we need more careful analysis and calculations.

Remark 1.4. (1) In the non-singular case (i.e., the case $s = 0$), some results can be found in Druet [15] (see also [5,6]) for the *generalized scalar curvature equation*.

(2) When $K(x) \equiv 1$, the case (ii) of Theorem 1.3 becomes $p = 2$ and

$$a(x_0) < \frac{(n-2)(6-s)}{12(2n-2-s)}\text{Scal}_g(x_0)$$

and has been investigated by Jaber in [18].

Besides the minimization technique and estimates of the growth of some test functions, the proof of Theorem 1.3 relies on the best constant in the Hardy-Sobolev inequality. It follows from the Hardy-Sobolev embedding (see Lemma 3.1 later) that there exist $A, B > 0$ such that

$$\left(\int_M \frac{|u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} \leq A \int_M |\nabla_g u|_g^p dv_g + B \int_M |u|^p dv_g, \quad (1.5)$$

for all $u \in H_1^p(M)$. We denote $\mathcal{A}_p(M, g, s, x_0)$ to be the best first constant of the Riemannian Hardy-Sobolev inequality, that is

$$\mathcal{A}_p(M, g, s, x_0) := \inf\{A > 0 : \exists B \in \mathbb{R} \text{ such that inequality (1.5) holds}\}. \quad (1.6)$$

Let $S(n, p, s)$ be the optimal constant of the Euclidean Hardy-Sobolev inequality, namely

$$\frac{1}{S(n, p, s)^p} =: \inf_{\varphi \in C_0^\infty(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla \varphi|^p dX}{\left(\int_{\mathbb{R}^n} \frac{|\varphi|^{p^*(s)}}{|X|^s} dX \right)^{p/p^*(s)}}. \quad (1.7)$$

For the existence of minimizers in (1.7) we refer to [16]. Moreover, we have the following result.

Theorem 1.5. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ and $p \in (1, n)$, $s \in (0, p)$, $x_0 \in M$, and $p^*(s) = \frac{(n-s)p}{n-p}$. Then

$$\mathcal{A}_p(M, g, s, x_0) = S(n, p, s)^p.$$

In particular, for any $\epsilon > 0$, there exists a real constant B_ϵ such that

$$\left(\int_M \frac{|u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} \leq (S(n, p, s)^p + \epsilon) \int_M |\nabla_g u|_g^p dv_g + B_\epsilon \int_M |u|^p dv_g, \quad (1.8)$$

for all $u \in H_1^p(M)$.

In fact, Theorem 1.5 is a generalization of Aubin [2] for the case $s = 0$ and Jaber [18] for $p = 2$. Another natural question is the validity of the associated optimal inequalities, that is, if there exists $B > 0$ such that equality (1.5) holds for all $u \in H_1^p(M)$ with $A = S(n, p, s)^p$? This problem is of independent interest. Jaber (see [19]) proved that one can take $A = S(n, 2, s)^2$ in (1.5) for $p = 2$. For the non-singular case in this direction, related results can be found in Aubin and Li [4], Druet [14] and the references therein.

As a byproduct, we prove the invalidity of the associated optimal inequalities on compact Riemannian manifolds in which have positive scalar curvature somewhere. More precisely, we obtain:

Theorem 1.6. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ and $p \in (1, n)$, $s \in (0, p)$, $x_0 \in M$, and $p^*(s) = \frac{(n-s)p}{n-p}$. Assume that $\text{Scal}_g(x_0) > 0$. Then the following optimal inequality

$$\left(\int_M \frac{|u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} \leq S(n, p, s)^p \int_M |\nabla_g u|_g^p dv_g + B \left(\int_M |u|^q dv_g \right)^{\frac{p}{q}}, \quad (1.9)$$

is not valid if $1 < p < \frac{n+2}{3}$ and $q < \frac{np}{n-p+2}$, or if $p = \frac{n+2}{3}$, $n > 4$ and $q \leq \frac{np}{n-p+2}$ for all $u \in H_1^p(M)$.

More general, we have:

Theorem 1.7. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ and $p \in (1, n)$, $s \in (0, p)$, $x_0 \in M$, and $p^*(s) = \frac{(n-s)p}{n-p}$. Assume that $\text{Scal}_g(x_0) > 0$. Then the following optimal inequality

$$\left(\int_M \frac{|u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{r}{p^*(s)}} \leq S(n, p, s)^r \left(\int_M |\nabla_g u|_g^p dv_g \right)^{\frac{r}{p}} + B \left(\int_M |u|^p dv_g \right)^{\frac{r}{p}}, \quad (1.10)$$

is not valid if any of the following situations

- (i) $n = 3$, and $\begin{cases} 1 < p < \frac{5}{3} \text{ and } r > 2, \\ p = \frac{5}{3} \text{ and } r \geq 2; \end{cases}$
- (ii) $n = 4$, $1 < p \leq 2$ and $r > 2$;
- (iii) $n > 4$, and $\begin{cases} 1 < p \leq \sqrt{n} \text{ and } r > 2, \\ \sqrt{n} < p < \frac{n+2}{3} \text{ and } r > \frac{2p(p-1)}{n-p}, \\ p = \frac{n+2}{3} \text{ and } r \geq p, \end{cases}$

happens for all $u \in H_1^p(M)$.

Remark 1.8. In particular, Theorem 1.6 and Theorem 1.7 extend and improve the results obtained by Biezuner [7, Theorem 2.1, Theorem 2.2] and Druet [13, Theorem 1] for the case $s = 0$.

The rest of our paper is organized as follows. In Section 2, we give some preliminary results. In Section 3, we prove Theorem 1.5. In Section 4, we obtain a general existence theorem for equation (1.1). Finally in Section 5, we perform the test-function estimate and give the proofs of Theorems 1.3, 1.6, and 1.7.

2. Preliminaries

In this section, we give some preliminary results. For more details one can see Aubin [3] and Hebey [17] on this subject. Let (M, g) be the Riemannian manifold as mentioned in the beginning of the paper. Let $1 < p < n$. By standard notations, the Sobolev space $H_1^p(M)$ is defined to be the completion of $C^\infty(M)$ with respect to the Sobolev norm

$$\|u\| = \left(\int_M |\nabla_g u|_g^p dv_g + \int_M |u|^p dv_g \right)^{\frac{1}{p}}, \quad (2.1)$$

where ∇_g is the gradient operator and dv_g is the canonical volume element on (M, g) . The operator $\Delta_{p,g} = -\operatorname{div}_g(|\nabla_g u|_g^{p-2} \nabla_g u)$ is the p -Laplace-Beltrami operator associated with g of u . Precisely, in local coordinates (X^i) , we have $|\nabla_g u|^2 = g^{ij} \frac{\partial u}{\partial X^i} \frac{\partial u}{\partial X^j}$, $dv_g = \sqrt{|g|} dX^1 \dots dX^n$ and

$$\Delta_{p,g} u = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial X^i} (\sqrt{|g|} g^{ij} |\nabla_g u|_g^{p-2} \frac{\partial u}{\partial X^j}),$$

where (g_{ij}) is the metric matrix, $(g^{ij}) = (g_{ij})^{-1}$ and $|g| := \det(g_{ij})$ is the determinant of the Riemannian metric g . Here, and in the sequel, the Einstein's summation convention is adopted.

Let $L^p(M, \frac{dv_g}{d_g(x, x_0)^s})$ be the weighted Lebesgue space equipped with the norm

$$\|u\|_{p,s} = \left(\int_M \frac{|u(x)|^p}{d_g(x, x_0)^s} dv_g \right)^{\frac{1}{p}},$$

and $\|u\|_p := \|u\|_{p,0}$ is the norm in $L^p(M)$.

In this paper, we will use the exponential map $\exp : TM \rightarrow M$, defined on the tangent bundle TM of M , which is a C^∞ -map. We may choose normal coordinates or geodesic polar coordinates if necessary. For $x \in M$, the exponential map $\exp_x : T_x(M) \rightarrow M$ is a local diffeomorphism in a neighborhood of the zero tangent vector. Throughout the paper, $B(0, r)$ denotes the ball of radius r centered at 0 in the Euclidean space \mathbb{R}^n and $B_g(x, r)$ denotes the geodesic ball of radius r centered at x in (M, g) . The following lemma is a basic fact in Riemannian geometry.

Lemma 2.1. *In a normal coordinates (X^i) centered at $x_0 \in M$, the expansion of the volume element dv_g in a neighborhood of x_0 is*

$$dv_g = \left(1 - \frac{1}{6} R_{ij} X^i X^j + O(|X|^3) \right) dX, \quad (2.2)$$

where R_{ij} is the Ricci tensor of the metric g at x_0 .

The volume of a bounded open set $S \subset M$ is defined as $\operatorname{Vol}_g(S) := \int_S 1 dv_g$. By Lemma 2.1, the behavior of the volume of small geodesic balls on M can be expressed as follows: for every $x \in M$,

$$\text{Vol}_g(B_g(x, r)) = \alpha_n r^n \left(1 - \frac{\text{Scal}_g(x)}{6(n+2)} r^2 + O(r^3) \right), \text{ as } r \rightarrow 0, \quad (2.3)$$

where α_n is the volume of the unit ball in \mathbb{R}^n and $\text{Scal}_g(x)$ is the scalar curvature at x .

Now we turn to a general boundedness result.

Proposition 2.2. *Let (M, g) be a compact Riemannian n -manifold, $n \geq 2$, and $p \in (1, n)$. We fix $x_0 \in M$, $s \in (0, p)$. Let $u \in H_1^p(M)$ be a weak solution of $-\text{div}_g(|\nabla_g u|_g^{p-2} \nabla_g u) = h(x, u)$ in M for h satisfying*

$$|h(x, r)| \leq C \left(1 + |r|^{p^*-1} + \frac{|r|^{p^*(s)-1}}{d_g(x_0, x)^s} \right), \quad (2.4)$$

for all $(x, r) \in M \times \mathbb{R}$, where C is a positive constant. Then $u \in L^\infty(M)$.

Proof. By [15, Lemma 2.2] it suffices to prove that $h(x, r) \in L^{\bar{q}}(M)$ for some $\bar{q} > \frac{n}{p}$. For simplicity, we assume that $u \geq 0$. Let $\beta > 0$, multiplying the equation with $u^{\beta+1}$ and taking integral, and we have

$$\frac{(\beta+1)p^p}{(\beta+p)^p} \int_M |\nabla_g(u^{\frac{\beta}{p}+1})|_g^p dv_g = \int_M h(x, u) u^{\beta+1} dv_g. \quad (2.5)$$

From (2.4) and (2.5), we get

$$\begin{aligned} \int_M |\nabla_g(u^{\frac{\beta}{p}+1})|_g^p dv_g &= \frac{(\beta+p)^p}{(\beta+1)p^p} \int_M h(x, u) u^{\beta+1} dv_g \\ &\leq C \frac{(\beta+p)^p}{(\beta+1)p^p} \int_M \left(u^{\beta+1} + u^{\beta+p^*} + \frac{u^{\beta+p^*(s)}}{d_g(x_0, x)^s} \right) dv_g. \end{aligned} \quad (2.6)$$

By Hardy-Sobolev inequality (1.5) with weight 1 and $\gamma = 0$, together with weight $d_g(x_0, x)^{-s}$ and $\gamma = s$, we obtain

$$\begin{aligned} \left(\int_M \frac{u^{(\frac{\beta}{p}+1)p^*(\gamma)}}{d_g(x_0, x)^\gamma} dv_g \right)^{\frac{p}{p^*(\gamma)}} &\leq C_\beta \int_M \left(u^{\beta+1} + u^{\beta+p} + u^{\beta+p^*} + \frac{u^{\beta+p^*(s)}}{d_g(x_0, x)^s} \right) dv_g \\ &\leq C_\beta \int_M \left(u^{\beta+p^*} + \frac{u^{\beta+p^*(s)}}{d_g(x_0, x)^s} \right) dv_g + C. \end{aligned} \quad (2.7)$$

For any $K > 1$, we apply Hölder's inequality with $\gamma = 0$ and $\gamma = s$, respectively, and deduce

$$\begin{aligned} \int_M \frac{u^{\beta+p^*(\gamma)}}{d_g(x_0, x)^\gamma} dv_g &= \int_{\{u < K\}} \frac{u^{\beta+p^*(\gamma)}}{d_g(x_0, x)^\gamma} dv_g + \int_{\{u \geq K\}} \frac{u^{\beta+p^*(\gamma)}}{d_g(x_0, x)^\gamma} dv_g \\ &\leq K^\beta \int_M \frac{u^{p^*(\gamma)}}{d_g(x_0, x)^\gamma} dv_g + \left(\int_{\{u \geq K\}} \frac{u^{p^*(\gamma)}}{d_g(x_0, x)^\gamma} dv_g \right)^{\frac{p^*(\gamma)-p}{p^*(\gamma)}} \left(\int_M \frac{u^{(\frac{\beta}{p}+1)p^*(\gamma)}}{d_g(x_0, x)^\gamma} dv_g \right)^{\frac{p}{p^*(\gamma)}}. \end{aligned} \quad (2.8)$$

Again with Hardy-Sobolev inequality, we know that $u^{p^*(\gamma)} \in L^1(M, \frac{dv_g}{d_g(x_0, x)^\gamma})$. We can thus choose K large enough so that

$$C_\beta \left(\int_{\{u \geq K\}} \frac{u^{p^*(\gamma)}}{d_g(x_0, x)^\gamma} dv_g \right)^{\frac{p^*(\gamma)-p}{p^*(\gamma)}} < \frac{1}{6}, \quad \text{for } \gamma = 0 \text{ and } s. \quad (2.9)$$

Hence, bringing (2.8) into (2.7) and using (2.9), we have (summed for $\gamma = 0$ and $\gamma = s$)

$$\begin{aligned} & \left(\int_M u^{(\frac{\beta}{p}+1)p^*} dv_g \right)^{\frac{p}{p^*}} + \left(\int_M \frac{u^{(\frac{\beta}{p}+1)p^*(s)}}{d_g(x_0, x)^s} dv_g \right)^{\frac{p}{p^*(s)}} \\ & \leq C_\beta K^\beta \int_M \left(u^{p^*} + \frac{u^{p^*(s)}}{d_g(x_0, x)^s} \right) dv_g + C \leq C_\beta K^\beta (\|u\|^{p^*} + \|u\|^{p^*(s)}) + C. \end{aligned}$$

This implies that $u \in L^q(M, \frac{dv_g}{d_g(x_0, x)^s})$ for every $q > 1$, and $u^{p^*-1} \in L^q(M)$ for every $q > \frac{n}{p}$. In order to get $h(x, u) \in L^{\bar{q}}(M)$ for some $\bar{q} > \frac{n}{p}$ it remains to show, by (2.4), that

$$v := \frac{u^{p^*(s)-1}}{d_g(x_0, x)^s} \in L^{\bar{q}}(M), \quad \text{for some } \bar{q} > \frac{n}{p}. \quad (2.10)$$

To this end, we choose $\bar{q} \in (\frac{n}{p}, \frac{n}{s})$. Then, Hölder's inequality with exponents q and q' yields

$$\begin{aligned} \int_M v^{\bar{q}} dv_g &= \int_M \frac{u^{\bar{q}(p^*(s)-1)}}{d_g(x_0, x)^{\frac{s}{\bar{q}}}} \cdot \frac{1}{d_g(x_0, x)^{s(\bar{q}-\frac{1}{\bar{q}})}} dv_g \\ &\leq \left(\int_M \frac{u^{\bar{q}(p^*(s)-1)q}}{d_g(x_0, x)^s} dv_g \right)^{\frac{1}{q}} \left(\int_M \frac{1}{d_g(x_0, x)^{s(\bar{q}-\frac{1}{\bar{q}})q'}} dv_g \right)^{\frac{1}{q'}}. \end{aligned}$$

Since $s(\bar{q} - \frac{1}{\bar{q}})q' < n$, we know that the last two integrals in the above inequality are finite. Thus (2.10) and the boundedness of u are proved. \square

3. The best constant in the Hardy-Sobolev inequality

In this section, we will prove Theorem 1.5. We begin with stating the following result which generalizes the well-known Rellich-Kondrachov compactness theorem.

Lemma 3.1. *Let (M, g) be a compact Riemannian n -manifold, $x_0 \in M$. Assume $p \in (1, n)$, $0 \leq s < p$, and $p \leq q \leq p^*(s) := \frac{n-s}{n-p}p$. Then*

(i) *There exists a constant $C > 0$ such that for any $u \in H_1^p(M)$,*

$$\|u\|_{p^*(s), s} \leq C\|u\|;$$

(ii) *The embedding $H_1^p(M) \hookrightarrow L^q(M, \frac{dv_g}{d_g(x, x_0)^s})$ is compact provided $q < p^*(s)$.*

Proof. For $s = 0$, the assertion is just the standard Sobolev inequality and Rellich-Kondrachov theorem on compact Riemannian manifolds. The assertion (i) follows easily from standard compactness arguments since M is compact. We prove assertion (ii). Suppose $0 < s < p$. Let $\{u_j\}$ be a bounded sequence in $H_1^p(M)$. We may assume, up to a subsequence, that

$$\begin{aligned} u_j &\rightharpoonup u \quad \text{weakly in } H_1^p(M), \\ u_j &\rightarrow u \quad \text{strongly in } L^r(M) \text{ for all } r \in [1, p^*), \\ u_j(x) &\rightarrow u(x) \quad \text{almost everywhere on } M. \end{aligned}$$

Then there exists a constant $C > 0$ such that $\|u_j - u\|_{p^*}^{p^*} \leq C$ and $\|u_j - u\|_q^q \leq C$. Let \exp_{x_0} be the exponential chart at x_0 with respect to the metric g . For any $\epsilon > 0$ small enough, \exp_{x_0} gives a homeomorphism from $B(0, \epsilon) \subset \mathbb{R}^n$ onto $B_g(x_0, \epsilon)$. By Hölder's inequality and a change of variable, with the fact that $d_g(x, x_0) = |\exp_{x_0}^{-1} x|$, we then get

$$\begin{aligned} \int_{B_g(x_0, \epsilon)} \frac{|u_j - u|^q}{d_g(x, x_0)^s} dv_g &\leq \left(\int_{B_g(x_0, \epsilon)} |u_j - u|^{p^*} dv_g \right)^{\frac{q}{p^*}} \left(\int_{B_g(x_0, \epsilon)} d_g(x, x_0)^{-\frac{p^* s}{p^* - q}} dv_g \right)^{\frac{p^* - q}{p^*}} \\ &\leq C \left(\int_{B(0, \epsilon)} |X|^{-\frac{p^* s}{p^* - q}} dX \right)^{\frac{p^* - q}{p^*}} = CO(\epsilon^\eta), \end{aligned} \quad (3.1)$$

where $\eta := \frac{p^* - q}{p^*} (-\frac{p^* s}{p^* - q} + n) > 0$, since $q < p^*(s)$. On the other hand, there exists a constant $C_\epsilon > 0$ such that

$$\frac{1}{d_g(x, x_0)^s} \leq C_\epsilon, \text{ for all } x \in M \setminus B_g(x_0, \epsilon).$$

Therefore, we have

$$\int_{M \setminus B_g(x_0, \epsilon)} \frac{|u_j - u|^q}{d_g(x, x_0)^s} dv_g \leq C_\epsilon \int_{M \setminus B_g(x_0, \epsilon)} |u_j - u|^q dv_g = o(1). \quad (3.2)$$

Clearly, (3.1) and (3.2) imply $\lim_{j \rightarrow +\infty} \|u_j - u\|_{q, s}^q = 0$. \square

In order to prove Theorem 1.5 we need the following Aubin type ϵ -level sharp Hardy-Sobolev inequality.

Proposition 3.2 (ϵ -level inequality). *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, $p \in (1, n)$, $s \in (0, p)$ and $x_0 \in M$. Then for any $\epsilon > 0$, there exists B_ϵ such that*

$$\left(\int_M \frac{|u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} \leq (S(n, p, s)^p + \epsilon) \int_M |\nabla_g u|_g^p dv_g + B_\epsilon \int_M |u|^p dv_g, \quad (3.3)$$

for all $u \in H_1^p(M)$.

Proof. We follow basically from the lines of Hebey [17] and Jaber [18]. By standard approximation, we just need to prove the proposition for smooth functions.

Step 1: Covering of M by geodesic balls. For any $x \in M$, let \exp_x be the exponential map at x with respect to the metric g . For any $x \in M$ and any $\rho > 0$, there exist $r = r(x, \rho) \in (0, i_g(M))$ with $\lim_{\rho \rightarrow 0} r(x, \rho) = 0$ (here $i_g(M)$ denotes the injectivity radius of (M, g)) such that the exponential chart $(B_g(x, 2r), \exp_x^{-1})$ satisfies the following properties, namely, on geodesic ball $B_g(x, 2r)$,

- (i) $(1 - \rho)e \leq g \leq (1 + \rho)e$,
- (ii) $(1 - \rho)^{\frac{n}{2}} dX \leq dv_g \leq (1 + \rho)^{\frac{n}{2}} dX$,
- (iii) $D_\rho^{-1} |T|_e \leq |T|_g \leq D_\rho |T|_e$, for all $T \in \chi(T^*M)$.

Here $\lim_{\rho \rightarrow 0} D_\rho = 1$, $\chi(T^*M)$ denotes the space of 1-covariant tensor fields on M , dX is the volume element in \mathbb{R}^n , e is the Euclidean metric on \mathbb{R}^n . Then we have assimilated g to the local metric $(\exp_x)^* g$ on \mathbb{R}^n via the exponential map.

It follows from the compactness of M that there exists $N_\rho \in \mathbb{N}$ and $x_1, \dots, x_{N_\rho} \in M \setminus B_g(x_0, \frac{r_0}{2})$ such that

$$M \setminus B_g(x_0, \frac{r_0}{2}) \subset \cup_{m=1}^{N_\rho} B_g(x_m, r_m),$$

where $r_0 = r(x_0, \rho)$ and $r_m = r(x_m, \rho) \leq r_0^2$, for $m = 1, \dots, N_\rho$.

Step 2: We claim that for every $\epsilon > 0$, there exist $\rho_0 = \rho_0(\epsilon) > 0$ such that $\lim_{\epsilon \rightarrow 0} \rho_0(\epsilon) = 0$ and for all $\rho \in (0, \rho_0)$, all $m \in \{0, 1, \dots, N_\rho\}$ and all $u \in C_0^\infty(B_g(x_m, r_m))$, we have that

$$\left(\int_M \frac{|u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} \leq \left(S(n, p, s)^p + \frac{\epsilon}{2} \right) \int_M |\nabla_g u|_g^p dv_g. \quad (3.4)$$

Indeed, it follows from (1.7) that for all $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} \frac{|\varphi|^{p^*(s)}}{|X|_e^s} dX \right)^{\frac{p}{p^*(s)}} \leq S(n, p, s)^p \int_{\mathbb{R}^n} |\nabla \varphi|_e^p dX. \quad (3.5)$$

We now consider $\rho > 0$, $m \in \{0, 1, \dots, N_\rho\}$ and $u \in C_0^\infty(B_g(x_m, r_m))$ such that

$$(B_g(x_m, r_m), \exp_{x_m}^{-1})$$

is an exponential chart as in **Step 1**.

We distinguish two cases:

Case 1: if $m = 0$, then by **Step 1** and Hardy-Sobolev inequality (3.5), we write

$$\begin{aligned} \left(\int_M \frac{|u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} &\leq (1 + \rho)^{\frac{np}{2p^*(s)}} S(n, p, s)^p \int_{\mathbb{R}^n} |\nabla(u \circ \exp_{x_0})|_e^p dX \\ &\leq (D_\rho)^p (1 + \rho)^{\frac{np}{2p^*(s)}} (1 - \rho)^{-\frac{n}{2}} S(n, p, s)^p \int_M |\nabla_g u|_g^p dv_g. \end{aligned}$$

Therefore we get (3.4) as $\rho \rightarrow 0$. This proves (3.4) in the case of $m = 0$.

Case 2: If $m \in \{1, \dots, N_\rho\}$, then for all $x \in B_g(x_m, r_m)$, we have

$$d_g(x, x_0) \geq d_g(x_0, x_m) - d_g(x, x_m) > \frac{r_0}{2} - r_m.$$

Denote $\lambda_0 := \frac{r_0}{2} - r_m > 0$. Applying Hölder's inequality and Sobolev's inequality (i.e., $s = 0$ in (3.4)), we have that for any $u \in C_0^\infty(B_g(x_m, r_m))$,

$$\begin{aligned} &\left(\int_M \frac{|u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} \\ &\leq \lambda_0^{-\frac{ps}{p^*(s)}} \left(\int_{B_g(x_m, r_m)} |u|^{p^*(s)} dv_g \right)^{\frac{p}{p^*(s)}} \\ &\leq \lambda_0^{-\frac{ps}{p^*(s)}} (\text{Vol}_g(B_g(x_m, r_m)))^{p(\frac{1}{p^*(s)} - \frac{1}{p^*})} \left(\int_{B_g(x_m, r_m)} |u|^{p^*} dv_g \right)^{\frac{p}{p^*}} \\ &\leq C \lambda_0^{-\frac{ps}{p^*(s)}} (\text{Vol}_g(B_g(x_m, r_m)))^{p(\frac{1}{p^*(s)} - \frac{1}{p^*})} \int_{B_g(x_m, r_m)} |\nabla_g u|_g^{p^*} dv_g, \end{aligned} \quad (3.6)$$

where $C > 0$ is a constant and $p^* = \frac{np}{n-p}$ is the critical Sobolev exponent. On the other hand, by (2.3), we have

$$\text{Vol}_g(B_g(x, r)) = \alpha_n r^n (1 + o(r)), \text{ as } r \rightarrow 0, \quad (3.7)$$

where α_n is the volume of the unit ball in \mathbb{R}^n . Combining now (3.7) and $r_m \leq r_0^2$, we get

$$\lim_{\rho \rightarrow 0} \frac{(\text{Vol}_g(B_g(x_m, r_m)))^{p(\frac{1}{p^*(s)} - \frac{1}{p^*})}}{\lambda_0^{\frac{ps}{p^*(s)}}} = \lim_{\rho \rightarrow 0} \left(\frac{r_m}{\frac{r_0}{2} - r_m} \right)^{\frac{(n-p)s}{n-s}} = 0$$

Taking $\rho \rightarrow 0$ in (3.6), we get (3.4) for all $u \in C_0^\infty(B_g(x_m, r_m))$ when $m \geq 1$. This ends **Step 2**.

Step 3: We fix $\epsilon > 0$, $\rho \in (0, \rho_0(\epsilon))$ and $x_0, x_1, \dots, x_{N_\rho}$ as in **Step 1** and **Step 2**. Let $\{\alpha_m\}_{m=0}^{N_\rho}$ be a smooth partition of unity subordinate to the covering $\{B_g(x_m, r_m)\}_{m=0}^{N_\rho}$ of M and define, for all $m = 0, 1, \dots, N_\rho$, a function

$$\eta_m = \frac{\alpha_i^{[p]+1}}{\sum_{i=0}^{N_\rho} \alpha_i^{[p]+1}},$$

where $[p]$ is the greatest integer not exceeding p . Clearly, $\{\eta_m\}_{m=0}^{N_\rho}$ is a smooth partition of unity subordinate to the covering $\{B_g(x_m, r_m)\}_{m=0}^{N_\rho}$ of M such that $\eta_m^{\frac{1}{p}} \in C^1(M)$ for any m . We choose $H > 0$ such that for any m ,

$$|\nabla_g \eta_m^{\frac{1}{p}}|_g \leq H. \quad (3.8)$$

Step 4: In this step, we will prove the Hardy-Sobolev inequality on $C_0^\infty(M)$. Indeed, we let $\epsilon > 0$ and $\{\eta_m\}_{m=0}^{N_\rho}$ is a smooth partition of unity as in **Step 3**. We consider $u \in C_0^\infty(M)$. Since $\frac{p^*(s)}{p} > 1$, we deduce that

$$\begin{aligned} \left(\int_M \frac{|u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} &\leq \left(\int_M \frac{|\sum_{m=0}^{N_\rho} \eta_m |u|^p|^{\frac{p^*(s)}{p}}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} \\ &\leq \left\| \sum_{m=0}^{N_\rho} \eta_m |u|^p \right\|_{\frac{p^*(s)}{p}, s} \leq \sum_{m=0}^{N_\rho} \left\| \eta_m |u|^p \right\|_{\frac{p^*(s)}{p}, s} \\ &= \sum_{m=0}^{N_\rho} \left(\int_M \frac{|\eta_m^{\frac{1}{p}} u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}}. \end{aligned} \quad (3.9)$$

Coming back to (3.4) in **Step 2** and by density (since $\eta_m^{\frac{1}{p}} u \in C^1(M)$), we get

$$\left(\int_M \frac{|\eta_m^{\frac{1}{p}} u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} \leq (S(n, p, s)^p + \frac{\epsilon}{2}) \int_M |\nabla_g(\eta_m^{\frac{1}{p}} u)|_g^p dv_g. \quad (3.10)$$

Applying Hölder's inequality and the elementary inequality

$$(1+t)^p \leq 1 + \mu t + \nu t^p, \quad \text{for any } t \geq 0,$$

(e.g., one can take $\mu = p \max\{1, 2^{p-2}\}$ and $\nu = \max\{1, 2^{p-2}\}$), together with (3.8), (3.9) and (3.10) we obtain

$$\begin{aligned}
& \left(\int_M \frac{|u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} \\
& \leq (S(n, p, s)^p + \frac{\epsilon}{2}) \sum_{m=0}^{N_p} \int_M (\eta_m^{\frac{1}{p}} |\nabla_g u|_g + |u| |\nabla_g \eta_m^{\frac{1}{p}}|_g)^p dv_g \\
& \leq (S(n, p, s)^p + \frac{\epsilon}{2}) \sum_{m=0}^{N_p} \int_M (|\nabla_g u|_g^p \eta_m + \mu |\nabla_g u|_g^{p-1} \eta_m^{\frac{p-1}{p}} |u| |\nabla_g \eta_m^{\frac{1}{p}}|_g \\
& \quad + \nu |u|^p |\nabla_g \eta_m^{\frac{1}{p}}|_g^p) dv_g \\
& \leq (S(n, p, s)^p + \frac{\epsilon}{2}) (\|\nabla_g u\|_p + \mu n H \|\nabla_g u\|_p^{p-1} \|u\|_p + \nu n H^p \|u\|_p^p).
\end{aligned} \tag{3.11}$$

We choose now $\epsilon_0 > 0$ such that

$$(S(n, p, s)^p + \frac{\epsilon}{2})(1 + \epsilon_0) \leq S(n, p, s)^p + \epsilon. \tag{3.12}$$

For any positive real numbers x , y , and λ , Young's inequality gives

$$px^{p-1}y \leq \lambda(p-1)x^p + \lambda^{1-p}y^p. \tag{3.13}$$

By taking $x = \|\nabla_g u\|_p$, $y = \|u\|_p$, and $\lambda = \frac{p\epsilon_0}{\mu(p-1)nH}$ in (3.13), we then get that for any $u \in C_0^\infty(M)$,

$$\mu n H \|\nabla_g u\|_p^{p-1} \|u\|_p \leq \epsilon_0 \|\nabla_g u\|_p^p + C \|u\|_p^p, \tag{3.14}$$

where

$$C = \frac{\mu n H}{p} \left(\frac{p\epsilon_0}{\mu(p-1)nH} \right)^{1-p}.$$

Hence, combining (3.11) with (3.12) and (3.14), for any $u \in C_0^\infty(M)$, we get

$$\begin{aligned}
\left(\int_M \frac{|u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} & \leq (S(n, p, s)^p + \frac{\epsilon}{2})(1 + \epsilon_0) \int_M |\nabla_g u|_g^p dv_g + B_\epsilon \int_M |u|^p dv_g \\
& \leq (S(n, p, s)^p + \epsilon) \int_M |\nabla_g u|_g^p dv_g + B_\epsilon \int_M |u|^p dv_g,
\end{aligned}$$

where $B_\epsilon = (S(n, p, s)^p + \frac{\epsilon}{2})(c + \nu n H^p)$. This proves (3.3) for any $u \in C_0^\infty(M)$. \square

Remark 3.3. Proposition 3.2 has been proved by Aubin [2] for the case $s = 0$ (see also [17] for an exposition in book form) and by Jaber [18] for the case $p = 2$ (see also Thiam [33] for a similar result with addition of an extra remainder term).

Remark 3.4. Since $(x + y)^{\frac{1}{p}} \leq x^{\frac{1}{p}} + y^{\frac{1}{p}}$, for all $x, y \geq 0$ and $p > 1$, it then follows from (3.3) that for all $\epsilon > 0$, there exists $B_\epsilon > 0$ such that for any $u \in C_0^\infty(M)$,

$$\left(\int_M \frac{|u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{1}{p^*(s)}} \leq (S(n, p, s) + \epsilon^{\frac{1}{p}}) \left(\int_M |\nabla_g u|_g^p dv_g \right)^{\frac{1}{p}} + B_\epsilon^{\frac{1}{p}} \left(\int_M |u|^p dv_g \right)^{\frac{1}{p}}.$$

We are now in a position to prove Theorem 1.5.

Proof of Theorem 1.5. We let $A \in \mathbb{R}$ and $B > 0$ such that the inequality (1.5) holds for all $u \in H_1^p(M)$. Therefore, we have

$$\left(\int_M \frac{|u|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} \leq A \int_M |\nabla_g u|_g^p dv_g + B \int_M |u|^p dv_g \quad (3.15)$$

for all $u \in H_1^p(M)$. We consider $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp}(\phi) \subset B(0, R)$, $R > 0$ and $(B_g(x_0, \rho_0), \exp_{x_0}^{-1})$ is an exponential chart centered at x_0 with $\rho_0 \in (0, i_g(M))$. For all $0 < \mu \leq \frac{\rho_0}{R}$ sufficiently small, we choose $\phi_\mu \in C^\infty(B_g(x_0, \rho_0))$ such that

$$\phi_\mu = \phi(\mu^{-1} \exp_{x_0}^{-1}(x))$$

for all $x \in B_g(x_0, \rho_0)$. Applying (3.15) to ϕ_μ , we have

$$\left(\int_{B_g(x_0, \mu R)} \frac{|\phi_\mu|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} \leq A \int_{B_g(x_0, \mu R)} |\nabla_g \phi_\mu|_g^p dv_g + B \int_{B_g(x_0, \mu R)} |\phi_\mu|^p dv_g. \quad (3.16)$$

For all $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$(1 - \epsilon)e \leq g \leq (1 + \epsilon)e$$

in $B_g(x_0, R_\epsilon)$. Then, for all $\mu > 0$ sufficiently small such that $\mu R < R_\epsilon$, we get successively that

$$\int_{B_g(x_0, \mu R)} \frac{|\phi_\mu|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \geq (1 - \epsilon)^{\frac{n}{2}} \mu^{n-s} \int_{B(0, R)} \frac{|\phi|^{p^*(s)}(X)}{|X|_e^s} dX, \quad (3.17)$$

$$\int_{B_g(x_0, \mu R)} |\nabla_g \phi_\mu|_g^p dv_g \leq (1 + \epsilon)^{\frac{n}{2}} (1 - \epsilon)^{-\frac{n}{2}} \mu^{n-p} \int_{B(0, R)} |\nabla \phi|_e^p dX \quad (3.18)$$

and

$$\int_{B_g(x_0, \mu R)} |\phi_\mu|^p dv_g \leq (1 + \epsilon)^{\frac{n}{2}} \mu^n \int_{B(0, R)} |\phi|^p dX. \quad (3.19)$$

Plugging the estimates (3.17)-(3.19) into (3.16), taking $\mu \rightarrow 0$ and then $\epsilon \rightarrow 0$, we get

$$\left(\int_{B(0, R)} \frac{|\phi|^{p^*(s)}(X)}{|X|_e^s} dX \right)^{\frac{p}{p^*(s)}} \leq A \int_{B(0, R)} |\nabla \phi|_e^p dX.$$

It then follows from the definition of $S(n, p, s)$ that $A \geq S(n, p, s)^p$. Moreover, it follows from the definition of $\mathcal{A}_p(M, g, s, x_0)$ that $\mathcal{A}_p(M, g, s, x_0) \geq S(n, p, s)^p$. On the other hand, by Proposition 3.2, we have that $\mathcal{A}_p(M, g, s, x_0) \leq S(n, p, s)^p$. Therefore, $\mathcal{A}_p(M, g, s, x_0) = S(n, p, s)^p$. This finishes the proof. \square

4. A general existence theorem

In this section, we will prove Theorem 1.1. First let us introduce some notions. For any $q \in [p, p^*(s)]$, we define

$$J_q(u) := I(u) \cdot \left(\int_M \frac{K(x)|u|^q}{d_g(x, x_0)^s} dv_g \right)^{-\frac{p}{q}}, \quad u \in H_1^p(M) \setminus \{0\},$$

and

$$\Lambda_q := \left\{ u \in H_1^p(M) : \int_M \frac{K(x)|u|^q}{d_g(x, x_0)^s} dv_g = 1 \right\}.$$

Finally, we set

$$\lambda_q := \inf_{u \in H_1^p(M) \setminus \{0\}} J_q(u) = \inf_{u \in \Lambda_q} I(u).$$

Now we need the following important lemma in order to prove Theorem 1.1. The similar argument can be found in [15] and [17] for generalized scalar curvature equation.

Lemma 4.1 (*Subcritical solutions*). *Let (M, g) be a compact smooth Riemannian manifold of dimension $n \geq 2$ and $p \in (1, n)$. Given $x_0 \in M$, $s \in (0, p)$, and $a(x), K(x) \in C^0(M)$ such that L_g is coercive and (H1) holds. Then for any $q \in (p, p^*(s))$, there exists $u_q \in H_1^p(M) \cap \Lambda_q$, $u_q \geq 0$ in M , such that*

$$\Delta_{p,g} u + a(x)u^{p-1} = \lambda_q K(x) \frac{u^{q-1}}{d_g(x, x_0)^s}, \quad (4.1)$$

where Λ_q and λ_q are defined as above. In addition, $\lambda_q = I(u_q)$.

We now prove Theorem 1.1 via the subcritical approach. For simplicity, we denote $\lambda := \lambda_{p^*(s)}$ and $J(u) := J_{p^*(s)}(u)$.

Proof of Theorem 1.1. The proof is divided into several steps.

Step 1: We claim that

$$\limsup_{q \rightarrow p^*(s)} \lambda_q \leq \lambda. \quad (4.2)$$

Indeed, we follow the ideas from [17]. By Lebesgue's dominated convergence theorem, we have for all $u \in H_1^p(M) \setminus \{0\}$ that

$$J_q(u) = I(u) \cdot \left(\int_M \frac{K(x)|u|^q}{d_g(x, x_0)^s} dv_g \right)^{-\frac{p}{q}} \rightarrow J(u) \quad (4.3)$$

as $q \rightarrow p^*(s)$. Given $\epsilon > 0$, according to the definition of λ , there exists $u \in H_1^p(M) \setminus \{0\}$ such that

$$J(u) \leq \lambda + \epsilon.$$

Hence, from (4.3), we have

$$\lim_{q \rightarrow p^*(s)} J_q(u) \leq \lambda + \epsilon.$$

This implies that there exists $q_0 \in (p, p^*(s)]$ such that for all $q > q_0$, we have

$$\lambda_q \leq J_q(u) \leq \lambda + 2\epsilon.$$

As a result, we get $\limsup_{q \rightarrow p^*(s)} \lambda_q \leq \lambda + 2\epsilon$, and this proves the claim. In what follows, up to a subsequence, we assume that

$$\bar{\lambda} = \lim_{q \rightarrow p^*(s)} \lambda_q. \quad (4.4)$$

Step 2: Let $\{u_q\}$ be as in Lemma 4.1 with the additional property (4.4). As one checks, the sequence $\{u_q\}$ is bounded in $H_1^p(M)$ independently of q . Therefore, up to a subsequence, and as $q \rightarrow p^*(s)$, we can assume u_q converge to u weakly in $H_1^p(M)$, and strongly in $L^p(M)$, and pointwise a.e. in M . Clearly, $u \geq 0$.

We claim that $u \in H_1^p(M)$ is a weak solution to

$$\Delta_{p,g} u + a(x)u^{p-1} = \bar{\lambda}K(x) \frac{u^{p^*(s)-1}}{d_g(x, x_0)^s}. \quad (4.5)$$

Indeed, we denote by Σ_q and Π the vector fields $|\nabla_g u_q|_g^{p-2} \nabla_g u_q$ and $|\nabla_g u|_g^{p-2} \nabla_g u$ respectively. Then $\{\Sigma_q\}$ is bounded in $L^{\frac{p}{p-1}}(M)$ and thus, we can assume that Σ_q converges weakly in $L^{\frac{p}{p-1}}(M)$ to some vector field $\Sigma \in L^{\frac{p}{p-1}}(M)$ as $q \rightarrow p^*(s)$. Similarly, we can suppose that

$$u_q^{q-1} \rightarrow u^{p^*(s)-1} \quad \text{weakly in } L^{\frac{p^*(s)}{p^*(s)-1}}\left(M, \frac{dv_g}{d_g(x, x_0)^s}\right),$$

since $\{u_q^{q-1}\}$ is bounded in $L^{\frac{p^*(s)}{q-1}}\left(M, \frac{dv_g}{d_g(x, x_0)^s}\right) \subset L^{\frac{p^*(s)}{p^*(s)-1}}\left(M, \frac{dv_g}{d_g(x, x_0)^s}\right)$ (due to Lemma 3.1). Then, passing to the limit in Eq. (4.1) one gets that

$$-\operatorname{div}_g(\Sigma) + a(x)u^{p-1} = \bar{\lambda}K(x) \frac{u^{p^*(s)-1}}{d_g(x, x_0)^s}.$$

We need to prove

$$\Sigma = \Pi. \quad (4.6)$$

We adapt the ideas from [31] and [12]. By Egoroff's theorem, for any given $\delta > 0$, there exists $E_\delta \subset M$ such that $\operatorname{Vol}_g(M \setminus E_\delta) < \delta$, while $\{u_q\}$ converges uniformly to u in E_δ . As a consequence, for a given $\epsilon > 0$, we can take q sufficiently close to $p^*(s)$ to get that $|u_q(x) - u(x)| < \epsilon$ for all $x \in E_\delta$. We define the cut-off function $T_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$ as

$$T_\epsilon(s) = \begin{cases} s, & \text{if } |s| \leq \epsilon, \\ \epsilon \cdot \operatorname{sign}(s), & \text{if } |s| > \epsilon. \end{cases}$$

By Lindqvist's formula [26, Page 162] we know that for any $X, Y \in \mathbb{R}^n$, there exists a constant $c_p > 0$ such that

$$\langle |X|^{p-2}X - |Y|^{p-2}Y, X - Y \rangle \geq \begin{cases} c_p |X - Y|^p, & \text{if } p \geq 2, \\ c_p \frac{|X - Y|^2}{(|X| + |Y|)^{2-p}}, & \text{if } 1 < p \leq 2, \end{cases} \quad (4.7)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in \mathbb{R}^n . One then gets from this elegant inequality that

$$\langle \Sigma_q - \Pi, \nabla_g(T_\epsilon \circ (u_q - u)) \rangle_g \geq 0$$

holds almost everywhere in M . Thus, for q sufficiently close to $p^*(s)$, we have

$$\int_{E_\delta} \langle \Sigma_q - \Pi, \nabla_g(u_q - u) \rangle_g dv_g \leq \int_M \langle \Sigma_q - \Pi, \nabla_g(T_\epsilon \circ (u_q - u)) \rangle_g dv_g, \quad (4.8)$$

since $\nabla_g(T_\epsilon \circ (u_q - u)) = T'_\epsilon(u_q - u)\nabla_g(u_q - u)$ and $T'_\epsilon \geq 0$. Now we note that $T_\epsilon \circ (u_q - u)$ converges weakly to 0 in $H_1^p(M)$ and then

$$\int_M \langle \Pi, \nabla_g(T_\epsilon \circ (u_q - u)) \rangle_g dv_g \rightarrow 0.$$

On another hand, since $T_\epsilon \circ (u_q - u) \in H_1^p(M)$, multiplying (4.1) by $T_\epsilon \circ (u_q - u)$ and integrating by parts one gets that

$$\begin{aligned} & \left| \int_M \langle \Sigma_q, \nabla_g(T_\epsilon \circ (u_q - u)) \rangle_g dv_g \right| \\ & \leq \left| \int_M \lambda_q K(x) \frac{u_q^{q-1}}{d_g(x, x_0)^s} (T_\epsilon \circ (u_q - u)) dv_g \right| + \left| \int_M a(x) u_q^{p-1} (T_\epsilon \circ (u_q - u)) dv_g \right| \\ & \leq C\epsilon \end{aligned}$$

for $|T_\epsilon \circ (u_q - u)| \leq \epsilon$. Consequently, we get from (4.8) that

$$\limsup_{q \rightarrow p^*(s)} \int_{E_\delta} \langle \Sigma_q - \Pi, \nabla_g(u_q - u) \rangle_g dv_g \leq C\epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $\langle \Sigma_q - \Pi, \nabla_g(u_q - u) \rangle_g$ converges to 0 in $L^1(E_\delta)$ and thus, up to a subsequence, also holds almost everywhere in E_δ . Inequality (4.7) implies that $\nabla_g u_q \rightarrow \nabla_g u$ holds almost everywhere in E_δ . Since $\delta > 0$ is arbitrary, it follows that $\nabla_g u_q$ converges to $\nabla_g u$ holds almost everywhere in M and thus, $|\nabla_g u_q|_g^{p-2} \nabla_g u_q \rightarrow \Pi$ holds almost everywhere in M . Since $\{|\nabla_g u_q|_g^{p-2} \nabla_g u_q\}$ is bounded in $L^{\frac{p}{p-1}}(M)$, we get that $|\nabla_g u_q|_g^{p-2} \nabla_g u_q$ converges weakly to Π in $L^{\frac{p}{p-1}}(M)$ and thus that $\Sigma = \Pi$. This proves (4.6). Hence, u is a solution of (4.5).

Step 3: We claim that $\bar{\lambda} = \lambda$ and $u \in \Lambda$, and so that u is a minimizer of I on Λ . Indeed, multiplying (4.5) by u and integrating by parts one gets

$$\begin{aligned} \bar{\lambda} \int_M K(x) \frac{u^{p^*(s)}}{d_g(x_0, x)^s} dv_g &= \int_M (|\nabla_g u|_g^p + a(x) u^p) dv_g \\ &\leq \liminf_{q \rightarrow p^*(s)} \int_M (|\nabla_g u_q|_g^p + a(x) u_q^p) dv_g = \liminf_{q \rightarrow p^*(s)} \lambda_q. \end{aligned}$$

Hence, $\int_M K(x) \frac{u^{p^*(s)}}{d_g(x_0, x)^s} dv_g \leq 1$. Let $v = u \left(\int_M K(x) \frac{u^{p^*(s)}}{d_g(x_0, x)^s} dv_g \right)^{-\frac{1}{p^*(s)}}$, and then $v \in \Lambda$. According to (4.2), we have

$$\bar{\lambda} \leq I(v) = \bar{\lambda} \left(\int_M K(x) \frac{u^{p^*(s)}}{d_g(x_0, x)^s} dv_g \right)^{1 - \frac{p}{p^*(s)}}.$$

Hence we have $\int_M K(x) \frac{u^{p^*(s)}}{d_g(x_0, x)^s} dv_g \geq 1$, which implies $\int_M K(x) \frac{u^{p^*(s)}}{d_g(x_0, x)^s} dv_g = 1$. Therefore, $u \in \Lambda$ and $\bar{\lambda}$ is the infimum of I on Λ , i.e. $\bar{\lambda} = \lambda$. This proves the claim.

Step 4: We claim that $u \not\equiv 0$. Indeed, by the energy assumption, there exists $\epsilon_0 > 0$ such that

$$(S(n, p, s)^p + \epsilon_0)\lambda < \frac{1}{(\max_M K)^{\frac{p}{p^*(s)}}}. \quad (4.9)$$

Now from Proposition 3.2, we know that there exists B_{ϵ_0} such that for any $u \in H_1^p(M)$ and $q \in (p, p^*(s))$, we have

$$\left(\int_M \frac{|u_q|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} \leq (S(n, p, s)^p + \epsilon_0) \int_M |\nabla_g u_q|^p dv_g + B_{\epsilon_0} \int_M |u_q|^p dv_g. \quad (4.10)$$

On the other hand, one has that

$$\begin{aligned} \frac{1}{\max_M K} &= \frac{1}{\max_M K} \int_M \frac{K(x) |u_q|^q}{d_g(x, x_0)^s} dv_g \leq \int_M \frac{|u_q|^q}{d_g(x, x_0)^s} dv_g \\ &\leq \left(\int_M \frac{|u_q|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{q}{p^*(s)}} \left(\int_M \frac{1}{d_g(x, x_0)^s} dv_g \right)^{1 - \frac{q}{p^*(s)}}. \end{aligned}$$

Hence, we obtain

$$\left(\int_M \frac{|u_q|^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} \geq \frac{1}{(\max_M K)^{\frac{p}{q}}} \left(\int_M \frac{1}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)} - \frac{p}{q}},$$

and

$$\frac{1}{(\max_M K)^{\frac{p}{q}}} \left(\int_M \frac{1}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)} - \frac{p}{q}} \leq (S(n, p, s)^p + \epsilon_0) \lambda_q + \tilde{B}_{\epsilon_0} \int_M |u_q|^p dv_g$$

from (4.10) with $\tilde{B}_{\epsilon_0} = B_{\epsilon_0} + (S(n, p, s)^p + \epsilon_0) \max_M |a|$. Since $\lambda_q \rightarrow \lambda$ and

$$\left(\int_M \frac{1}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)} - \frac{p}{q}} \rightarrow 0$$

as $q \rightarrow p^*(s)$, we get by passing to the limit as $q \rightarrow p^*(s)$ that

$$\frac{1}{(\max_M K)^{\frac{p}{p^*(s)}}} \leq (S(n, p, s)^p + \epsilon_0) \lambda + \tilde{B}_{\epsilon_0} \int_M |u_q|^p dv_g. \quad (4.11)$$

If $u \equiv 0$, we then get from (4.11) that

$$\frac{1}{(\max_M K)^{\frac{p}{p^*(s)}}} \leq (S(n, p, s)^p + \epsilon_0)\lambda.$$

This contradicts to (4.9), which implies $u \not\equiv 0$.

Step 5: We will prove that $u \in L^\infty(M) \cap C_{loc}^{1,\alpha}(M \setminus \{0\})$ for some $\alpha > 0$. Indeed, by Proposition 2.2 we know that $u \in L^\infty(M)$. Moreover, from standard elliptic regularity theory, the solutions to (1.1) are $C_{loc}^{1,\alpha}(M \setminus \{0\})$ for some $\alpha \in (0, 1)$, see, e.g., Druet [15] and Tolksdorf [34] for instance. \square

Remark 4.2. The complete regularity for problem (1.1) is a classical and of independent interest. It will be treated in a separated work.

Remark 4.3. By Step 2 we prove the almost everywhere convergence of the gradient for a sequence $\{u_q\}$. By applying the Brezis-Lieb lemma, we get that

$$\lim_{q \rightarrow p^*(s)} \left(\int_M |\nabla_g u_q|^p dv_g - \int_M |\nabla_g(u_q - u)|^p dv_g \right) = \int_M |\nabla_g u|^p dv_g.$$

Moreover, $u_q \rightarrow u$ in $H_1^r(M)$ for any $1 \leq r < p$. These results generalize Theorem 1.1 in De Valeriola and Willem [11]. It is worth noting that we don't use the concentration-compactness principle. The reader may refer to [11] and [27] for more related studies.

5. The role of the geometry on Hardy-Sobolev equation

We will prove Theorem 1.3 in this section. It relies on the estimates of a suitable test-function which enable us to establish the key inequalities (H2). Let

$$U(X) = \left(\frac{1}{1 + |X|^{\frac{p-s}{p-1}}} \right)^{\frac{n-p}{p-s}}$$

be a minimizer for the Hardy-Sobolev inequality in the Euclidean space \mathbb{R}^n such that

$$\|U\|_{L^{p^*(s)}(\mathbb{R}^n, |X|^{-s} dX)}^p = S^p(n, p, s) \|\nabla U\|_{L^p(\mathbb{R}^n)}^p.$$

For convenience, we denote $\Theta := S^{-p}(n, p, s)$ and

$$\begin{aligned} I_0 &= \int_{\mathbb{R}^n} |U|^p dX, \quad I_1 = \int_{\mathbb{R}^n} |\nabla U|^p dX, \quad I_2 = \int_{\mathbb{R}^n} |\nabla U|^p |X|^2 dX, \\ I_3 &= \int_{\mathbb{R}^n} \frac{|U|^{p^*(s)}}{|X|^s} dX, \quad I_4 = \int_{\mathbb{R}^n} \frac{|U|^{p^*(s)}}{|X|^s} |X|^2 dX, \end{aligned}$$

whenever the right-hand side makes sense, and clearly $I_1 = \Theta I_3^{\frac{p}{p^*(s)}}$.

From now on we work in geodesic normal coordinates. Let (X^1, \dots, X^n) be a normal coordinate system centered at $x_0 \in M$. Given $\delta \in (0, i_g(M))$, where $i_g(M)$ is the injective radius of (M, g) , we consider a smooth radial cut-off function $\eta(x) \in C_0^\infty(B_g(x_0, \delta))$ satisfying

$$0 \leq \eta(x) \leq 1, \quad \eta(x) = \begin{cases} 1 & \text{if } x \in B_g(x_0, \delta/2), \\ 0 & \text{if } x \in M \setminus B_g(x_0, \delta). \end{cases}$$

Then, for $0 < \varepsilon \ll \delta$, we define the following radial smooth function

$$u_\epsilon(x) = \frac{\eta(r)}{(\epsilon + r^{\frac{p-s}{p-1}})^{\frac{n-p}{p-s}}} := v_\epsilon(r)\eta(r), \quad (5.1)$$

where $r = d_g(x, x_0)$ denotes the geodesic distance to the point x_0 . In order to prove Theorem 1.3, it suffices to show that, for ϵ small enough,

$$\frac{\int_M (|\nabla_g u_\epsilon|_g^p + a(x)u_\epsilon^p) dv_g}{\left(\int_M \frac{K(x)u_\epsilon^{p^*(s)}}{d_g(x, x_0)^s} dv_g\right)^{\frac{p}{p^*(s)}}} < \Theta K(x_0)^{-\frac{p}{p^*(s)}}$$

according to Theorem 1.1. With the help of the computation in Druet [15], we have the following estimates.

Lemma 5.1. *For ϵ small enough, the following estimates hold:*

$$\int_M a(x)u_\epsilon^p dv_g = a(x_0)I_0\epsilon^{\frac{p^2-n}{p-s}} + o(\epsilon^{\frac{p^2-n}{p-s}}), \text{ if } 1 < p^2 < n; \quad (5.2)$$

$$\int_M |\nabla_g u_\epsilon|_g^p dv_g = I_1\epsilon^{\frac{p-n}{p-s}} - \frac{Scal_g(x_0)}{6n}I_2\epsilon^{\frac{3p-n-2}{p-s}} + o(\epsilon^{\frac{3p-n-2}{p-s}}), \text{ if } 1 < p < \frac{n+2}{3}; \quad (5.3)$$

$$\int_M |\nabla_g u_\epsilon|_g^p dv_g = I_1\epsilon^{\frac{p-n}{p-s}} - w_{n-1}\left(\frac{n-p}{p-1}\right)^p \frac{Scal_g(x_0)}{6n}|\ln \epsilon| + o(|\ln \epsilon|), \text{ if } p = \frac{n+2}{3}; \quad (5.4)$$

$$\int_M \frac{K(x)u_\epsilon^{p^*(s)}}{d_g(x, x_0)^s} dv_g = K(x_0)I_3\epsilon^{-\frac{n-s}{p-s}} \quad (5.5)$$

$$\times \left[1 - \left(\frac{\Delta K(x_0)}{2nK(x_0)} + \frac{Scal_g(x_0)}{6n}\right)\frac{I_4}{I_3}\epsilon^{\frac{2(p-1)}{p-s}} + o(\epsilon^{\frac{2(p-1)}{p-s}})\right], \text{ if } 1 < p < \frac{n-s}{2} + 1.$$

Proof. Verification of (5.2). First, we write $\eta(x)^p = 1 + O(r^2)$, $dv_g = [1 + O(r^2)]dX$ and

$$a(x) = a(x_0) + \partial_i a(x_0)X^i + O(r^2).$$

Notice that $\int_{B(0,\delta)} (\epsilon + r^{\frac{p-s}{p-1}})^{-\frac{p(n-p)}{p-s}} X^i dX = 0$ for any $i \in \{1, \dots, n\}$, and then we get

$$\begin{aligned} \int_M a(x)u_\epsilon^p dv_g &= \int_{B(0,\delta)} [a(x_0) + \partial_i a(x_0)X^i + O(r^2)](\epsilon + r^{\frac{p-s}{p-1}})^{-\frac{p(n-p)}{p-s}} [1 + O(r^2)]dX \\ &= a(x_0) \int_{B(0,\delta)} (\epsilon + r^{\frac{p-s}{p-1}})^{-\frac{p(n-p)}{p-s}} dX + \int_{B(0,\delta)} (\epsilon + r^{\frac{p-s}{p-1}})^{-\frac{p(n-p)}{p-s}} O(r^2) dX. \end{aligned}$$

Putting $r = \epsilon^{\frac{p-1}{p-s}}t$, by direct calculation, we obtain for $p^2 < n$ that

$$\begin{aligned}
\int_M a(x) u_\epsilon^p dv_g &= a(x_0) w_{n-1} \epsilon^{\frac{p^2-n}{p-s}} \int_0^{\delta/\epsilon^{\frac{p-1}{p-s}}} (1+t^{\frac{p-s}{p-1}})^{-\frac{p(n-p)}{p-s}} t^{n-1} dt \\
&\quad + w_{n-1} \epsilon^{\frac{p^2-n}{p-s} + \frac{2(p-1)}{p-s}} \int_0^{\delta/\epsilon^{\frac{p-1}{p-s}}} (1+t^{\frac{p-s}{p-1}})^{-\frac{p(n-p)}{p-s}} t^{n-1} O(t^2) dt \\
&= a(x_0) w_{n-1} \epsilon^{\frac{p^2-n}{p-s}} \int_0^\infty (1+t^{\frac{p-s}{p-1}})^{-\frac{p(n-p)}{p-s}} t^{n-1} dt \\
&\quad + w_{n-1} \epsilon^{\frac{p^2-n}{p-s} + \frac{2(p-1)}{p-s}} \int_0^1 (1+t^{\frac{p-s}{p-1}})^{-\frac{p(n-p)}{p-s}} t^{n-1} O(t^2) dt + O(1) \\
&= a(x_0) I_0 \epsilon^{\frac{p^2-n}{p-s}} + o(\epsilon^{\frac{p^2-n}{p-s}}).
\end{aligned}$$

Thus, we get (5.2).

Verification of (5.3) and (5.4). We can write

$$\begin{aligned}
\int_M |\nabla_g u_\epsilon|_g^p dv_g &= \int_{B_g(x_0, \frac{\delta}{2})} |\nabla_g v_\epsilon|_g^p dv_g + \int_{B_g(x_0, \delta) \setminus B_g(x_0, \frac{\delta}{2})} |\nabla_g(\eta v_\epsilon)|_g^p dv_g \\
&= \int_{B_g(x_0, \frac{\delta}{2})} |\nabla v_\epsilon|^p dv_g + \int_{B_g(x_0, \delta) \setminus B_g(x_0, \frac{\delta}{2})} |\eta \nabla v_\epsilon + v_\epsilon \nabla \eta|^p dv_g,
\end{aligned}$$

since v_ϵ and η are radial with respect to the normal coordinates. We estimate the first integral. By direct computation, we get

$$|\nabla v_\epsilon|^p = \left(\frac{n-p}{p-1}\right)^p \left(\frac{1}{\epsilon + r^{\frac{p-s}{p-1}}}\right)^{\frac{p(n-s)}{p-s}} r^{\frac{p(1-s)}{p-1}}. \quad (5.6)$$

Notice that

$$\int_{B(0, \frac{\delta}{2})} |\nabla v_\epsilon|^p X^i X^j dX = \frac{\delta^{ij}}{n} \int_{B(0, \frac{\delta}{2})} |\nabla v_\epsilon|^p |X|^2 dX.$$

By Lemma 2.1, we have

$$\begin{aligned}
\int_{B_g(x_0, \frac{\delta}{2})} |\nabla v_\epsilon|^p dv_g &= \int_{B(0, \frac{\delta}{2})} |\nabla v_\epsilon|^p dX - \frac{Scal_g(x_0)}{6n} \int_{B(0, \frac{\delta}{2})} |\nabla v_\epsilon|^p |X|^2 dX \\
&\quad + \int_{B(0, \frac{\delta}{2})} |\nabla v_\epsilon|^p O(|X|^3) dX.
\end{aligned} \quad (5.7)$$

We will compute each term of the above expansion. Together with (5.6), we obtain

$$\begin{aligned}
 & \int_{B(0, \frac{\delta}{2})} |\nabla v_\epsilon|^p dX \\
 &= w_{n-1} \left(\frac{n-p}{p-1} \right)^p \epsilon^{\frac{p-n}{p-s}} \left(\int_0^\infty - \int_{\delta/2 \epsilon^{\frac{p-1}{p-s}}}^\infty \right) (1+t^{\frac{p-s}{p-1}})^{-\frac{p(n-s)}{p-s}} t^{\frac{p(1-s)}{p-1}+n-1} dt \\
 &= I_1 \epsilon^{\frac{p-n}{p-s}} + O(1),
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_{B(0, \frac{\delta}{2})} |\nabla v_\epsilon|^p |X|^2 dX \\
 &= w_{n-1} \left(\frac{n-p}{p-1} \right)^p \epsilon^{\frac{3p-n-2}{p-s}} \left(\int_0^\infty - \int_{\delta/2 \epsilon^{\frac{p-1}{p-s}}}^\infty \right) (1+t^{\frac{p-s}{p-1}})^{-\frac{p(n-s)}{p-s}} t^{\frac{p(1-s)}{p-1}+n+1} dt \\
 &= I_2 \epsilon^{\frac{3p-n-2}{p-s}} + O(1),
 \end{aligned}$$

where I_2 converges if and only if $1 < p < \frac{n+2}{3}$. For the third term in the right-hand side of (5.7), we have

$$\begin{aligned}
 \int_{B(0, \frac{\delta}{2})} |\nabla v_\epsilon|^p O(|X|^3) dX &= w_{n-1} \left(\frac{n-p}{p-1} \right)^p \epsilon^{\frac{4p-n-3}{p-s}} \int_0^{\delta/2 \epsilon^{\frac{p-1}{p-s}}} (1+t^{\frac{p-s}{p-1}})^{-\frac{p(n-s)}{p-s}} t^{\frac{p(1-s)}{p-1}+n+2} dt \\
 &= \begin{cases} O(\epsilon^{\frac{4p-n-3}{p-s}}), & \text{if } 1 < p < \frac{n+3}{4}, \\ O(\ln \epsilon), & \text{if } p = \frac{n+3}{4}, \\ O(\epsilon^{\frac{p-1}{p-s}}), & \text{if } p > \frac{n+3}{4}. \end{cases} \quad (5.8)
 \end{aligned}$$

We now estimate the second integral as follows

$$\begin{aligned}
 \int_{B_g(x_0, \delta) \setminus B_g(x_0, \frac{\delta}{2})} |\eta \nabla v_\epsilon + v_\epsilon \nabla \eta|^p dv_g &= O(1) \left[\int_{B(0, \delta) \setminus B(0, \frac{\delta}{2})} |\nabla v_\epsilon|^p + |v_\epsilon|^p dX \right] \\
 &= O(1). \quad (5.9)
 \end{aligned}$$

Therefore, if $1 < p < \frac{n+2}{3}$, we obtain

$$\int_M |\nabla_g u_\epsilon|_g^p dv_g = I_1 \epsilon^{\frac{p-n}{p-s}} - \frac{Scal_g(x_0)}{6n} \epsilon^{\frac{3p-n-2}{p-s}} I_2 + O(\epsilon^{\frac{3p-n-2}{p-s}}).$$

Thus we get (5.3).

On the other hand, if $p = \frac{n+2}{3}$, then we have

$$\begin{aligned}
 & \int_{B(0, \frac{\delta}{2})} |\nabla v_\epsilon|^p |X|^2 dX \\
 &= w_{n-1} \left(\frac{n-p}{p-1} \right)^p \left(\int_0^1 + \int_1^{\delta/2\epsilon^{\frac{p-1}{p-s}}} \right) (1+t^{\frac{p-s}{p-1}})^{-\frac{p(n-s)}{p-s}} t^{\frac{p(1-s)}{p-1}+n+1} dt \\
 &= w_{n-1} \left(\frac{n-p}{p-1} \right)^p \int_1^{\delta/2\epsilon^{\frac{p-1}{p-s}}} \frac{1}{t} \left[1 - \frac{p(n-s)}{p-s} t^{-\frac{p-s}{p-1}} + O(t^{-\frac{2(p-s)}{p-1}}) \right] + O(1) \\
 &= w_{n-1} \left(\frac{n-p}{p-1} \right)^p |\ln \epsilon| + o(|\ln \epsilon|).
 \end{aligned}$$

Anyway, in this case, the above formula combining with (5.7)-(5.9) yields

$$\int_M |\nabla_g u_\epsilon|_g^p dv_g = I_1 \epsilon^{\frac{p-n}{p-s}} - w_{n-1} \left(\frac{n-p}{p-1} \right)^p \frac{Scal_g(x_0)}{6n} |\ln \epsilon| + o(|\ln \epsilon|).$$

This proves (5.4).

Verification of (5.5). Since $K(x)$ achieves its maximum at x_0 , we have the expansion

$$K(x) = K(x_0) + \frac{1}{2} \partial_{ij} K(x_0) X^i X^j + O(r^3),$$

and $\eta^{p^*(s)} = 1 + O(r^3)$. Consequently, by Lemma 2.1, we have

$$\begin{aligned}
 \int_M \frac{K(x) u_\epsilon^{p^*(s)}}{d_g(x, x_0)^s} dv_g &= \int_{B(0, \delta)} \left[K(x_0) + \frac{1}{2} \partial_{ij} K(x_0) X^i X^j + O(r^3) \right] \\
 &\quad \times \left[1 - \frac{1}{6} R_{ij}(x_0) X^i X^j + O(r^3) \right] v_\epsilon^{p^*(s)} r^{-s} dX \\
 &= K(x_0) \int_{B(0, \delta)} v_\epsilon^{p^*(s)} r^{-s} dX + \left[\frac{1}{2} \partial_{ij} K(x_0) - \frac{1}{6} R_{ij}(x_0) \right] \int_{B(0, \delta)} v_\epsilon^{p^*(s)} r^{-s} X^i X^j dX \\
 &\quad + \int_{B(0, \delta)} v_\epsilon^{p^*(s)} r^{-s} O(r^3) dX
 \end{aligned}$$

Notice that $\Delta_g K(x_0) = \sum_{i=1}^n \partial_{ii} K(x_0)$ and for any i, j ,

$$\int_{B(0, \delta)} v_\epsilon^{p^*(s)} r^{-s} X^i X^j dX = \frac{\delta_{ij}}{n} \int_{B(0, \delta)} v_\epsilon^{p^*(s)} |X|^{2-s} dX.$$

Hence we obtain, for $1 < p < \frac{n-s}{2} + 1$

$$\begin{aligned}
 & \int_M \frac{K(x)u_\epsilon^{p^*(s)}}{d_g(x, x_0)^s} dv_g \\
 &= K(x_0) \int_{B(0, \delta)} v_\epsilon^{p^*(s)} |X|^{-s} dX - \frac{1}{2n} \left(\Delta K(x_0) + \frac{Scal_g(x_0)K(x_0)}{3} \right) \int_{B(0, \delta)} v_\epsilon^{p^*(s)} |X|^{2-s} dX \\
 & \quad + \int_{B(0, \delta)} v_\epsilon^{p^*(s)} |X|^{-s} O(|X|^3) dX \\
 &= K(x_0) I_3 \epsilon^{-\frac{n-s}{p-s}} - \frac{1}{2n} \left(\Delta K(x_0) + \frac{Scal_g(x_0)K(x_0)}{3} \right) I_4 \epsilon^{\frac{-n+2(p-1)+s}{p-s}} + o(\epsilon^{\frac{-n+2(p-1)+s}{p-s}}).
 \end{aligned}$$

Thus (5.5) is proved. \square

Remark 5.2. We note that for $s \in (0, p)$, $1 < p < \frac{n+2}{3}$ implies $1 < p < \frac{n-s}{2} + 1$.

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. By Lemma 5.1, we have, for $1 < p^2 < n$ and $1 < p < \frac{n+2}{3}$

$$\begin{aligned}
 & \int_M [|\nabla_g u_\epsilon|_g^p + a(x)u_\epsilon^p] dv_g \\
 &= \Theta I_3^{\frac{p}{p^*(s)}} \epsilon^{\frac{p-n}{p-s}} \left[1 - \Theta^{-1} \frac{Scal_g(x_0)}{6n} I_2 I_3^{-\frac{p}{p^*(s)}} \epsilon^{\frac{2(p-1)}{p-s}} \right. \\
 & \quad \left. + \Theta^{-1} a(x_0) I_0 I_3^{-\frac{p}{p^*(s)}} \epsilon^{\frac{p^2-p}{p-s}} + o(\epsilon^{\frac{2(p-1)}{p-s}}) + o(\epsilon^{\frac{p^2-p}{p-s}}) \right],
 \end{aligned} \tag{5.10}$$

and for $1 < p < \frac{n-s}{2} + 1$

$$\begin{aligned}
 \left(\int_M \frac{K(x)u_\epsilon^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}} &= (K(x_0) I_3)^{\frac{p}{p^*(s)}} \epsilon^{-\frac{n-p}{p-s}} \\
 & \times \left[1 - \left(\frac{\Delta K(x_0)}{2nK(x_0)} + \frac{Scal_g(x_0)}{6n} \right) \frac{I_4}{I_3} \epsilon^{\frac{2(p-1)}{p-s}} + o(\epsilon^{\frac{2(p-1)}{p-s}}) \right]^{\frac{p}{p^*(s)}}.
 \end{aligned} \tag{5.11}$$

Then, it follows from (5.10), (5.11) and $I_1 = \Theta I_3^{\frac{p}{p^*(s)}}$, that

$$\begin{aligned}
 & \frac{\int_M [|\nabla_g u_\epsilon|_g^p + a(x)u_\epsilon^p] dv_g}{\left(\int_M \frac{K(x)u_\epsilon^{p^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{p}{p^*(s)}}} = \Theta (K(x_0))^{-\frac{p}{p^*(s)}} \\
 & \times \left[1 - \Theta^{-1} \frac{Scal_g(x_0)}{6n} I_2 I_3^{-\frac{p}{p^*(s)}} \epsilon^{\frac{2(p-1)}{p-s}} + \Theta^{-1} a(x_0) I_0 I_3^{-\frac{p}{p^*(s)}} \epsilon^{\frac{p^2-p}{p-s}} + o(\epsilon^{\frac{2(p-1)}{p-s}}) + o(\epsilon^{\frac{p^2-p}{p-s}}) \right] \\
 & \times \left[1 - \left(\frac{\Delta K(x_0)}{2nK(x_0)} + \frac{Scal_g(x_0)}{6n} \right) \frac{I_4}{I_3} \epsilon^{\frac{2(p-1)}{p-s}} + o(\epsilon^{\frac{2(p-1)}{p-s}}) \right]^{-\frac{p}{p^*(s)}} \\
 &= \Theta (K(x_0))^{-\frac{p}{p^*(s)}} \left\{ 1 - \left[\left(\frac{I_2}{I_1} - \frac{p}{p^*(s)} \frac{I_4}{I_3} \right) \frac{Scal_g(x_0)}{6n} \epsilon^{\frac{2(p-1)}{p-s}} - \frac{p}{p^*(s)} \frac{\Delta K(x_0)}{2nK(x_0)} \frac{I_4}{I_3} \epsilon^{\frac{2(p-1)}{p-s}} \right. \right. \\
 & \quad \left. \left. - a(x_0) \frac{I_0}{I_1} \epsilon^{\frac{p^2-p}{p-s}} \right] + o(\epsilon^{\frac{2(p-1)}{p-s}}) + o(\epsilon^{\frac{p^2-p}{p-s}}) \right\}.
 \end{aligned} \tag{5.12}$$

If $1 < p < 2$ and $n + 2 > 3p$, then $p^2 - p < 2(p - 1)$ and $1 < p < \frac{n-s}{2} + 1$. Moreover, the brace in the equality (5.12) is equivalent to

$$1 + a(x_0) \frac{I_0}{I_1} \epsilon^{\frac{p^2-p}{p-s}}.$$

Thus, if $a(x_0) < 0$, then we get that

$$1 + a(x_0) \frac{I_0}{I_1} \epsilon^{\frac{p^2-p}{p-s}} < 1.$$

If $p = 2$, the brace in the equality (5.12) reads as follows:

$$\begin{aligned} & 1 - \left[\left(\frac{I_2}{I_1} - \frac{p}{p^*(s)} \frac{I_4}{I_3} \right) \frac{Scal_g(x_0)}{6n} - \frac{p}{p^*(s)} \frac{\Delta K(x_0)}{2nK(x_0)} \frac{I_4}{I_3} - a(x_0) \frac{I_0}{I_1} \right] \epsilon^{\frac{p^2-p}{p-s}} \\ &= 1 - \frac{1}{2n} \frac{I_4}{I_3} \frac{p}{p^*(s)} \left[\left(\frac{I_2 I_3}{I_1 I_4} \frac{p^*(s)}{p} - 1 \right) \frac{Scal_g(x_0)}{3} - \frac{\Delta K(x_0)}{K(x_0)} - 2na(x_0) \frac{I_0 I_3}{I_1 I_4} \frac{p^*(s)}{p} \right] \epsilon^{\frac{p^2-p}{p-s}}. \end{aligned} \quad (5.13)$$

With the aid of the *Gamma* and *Beta* functions, we have that,

$$\int_0^\infty (1+x^t)^{-s} x^r dx = \frac{1}{t\Gamma(s)} \Gamma\left(s - \frac{r+1}{t}\right) \Gamma\left(\frac{r+1}{t}\right),$$

where t, s, r are real numbers such that the integral makes sense. By direct computation it yields

$$\begin{aligned} I_0 &= \omega_{n-1} \int_0^\infty \frac{t^{n-1}}{(1+t^{\frac{p-s}{p-1}})^{\frac{p(n-p)}{p-s}}} dt \\ &= \omega_{n-1} \frac{p-1}{p-s} \Gamma\left(\frac{n-p^2}{p-s}\right) \Gamma\left(\frac{n(p-1)}{p-s}\right) \left(\Gamma\left(\frac{p(n-p)}{p-s}\right)\right)^{-1}, \\ I_1 &= \omega_{n-1} \left(\frac{n-p}{p-1}\right)^p \int_0^\infty \frac{t^{\frac{p-ps}{p-1}+n-1}}{(1+t^{\frac{p-s}{p-1}})^{\frac{p(n-s)}{p-s}}} dt \\ &= \omega_{n-1} \left(\frac{n-p}{p-1}\right)^p \frac{p-1}{p-s} \Gamma\left(\frac{n-p}{p-s}\right) \Gamma\left(\frac{pn-sp-n+p}{p-s}\right) \left(\Gamma\left(\frac{p(n-s)}{p-s}\right)\right)^{-1}, \\ I_2 &= \omega_{n-1} \left(\frac{n-p}{p-1}\right)^p \int_0^\infty \frac{t^{\frac{p-ps}{p-1}+n+1}}{(1+t^{\frac{p-s}{p-1}})^{\frac{p(n-s)}{p-s}}} dt \\ &= \omega_{n-1} \left(\frac{n-p}{p-1}\right)^p \frac{p-1}{p-s} \Gamma\left(\frac{n-3p+2}{p-s}\right) \Gamma\left(\frac{p(n-s)-n+3p-2}{p-s}\right) \left(\Gamma\left(\frac{p(n-s)}{p-s}\right)\right)^{-1}, \\ I_3 &= \omega_{n-1} \int_0^\infty \frac{t^{n-1-s}}{(1+t^{\frac{p-s}{p-1}})^{\frac{p(n-s)}{p-s}}} dt \\ &= \omega_{n-1} \frac{p-1}{p-s} \Gamma\left(\frac{n-s}{p-s}\right) \Gamma\left(\frac{p(n-s)-n+s}{p-s}\right) \left(\Gamma\left(\frac{p(n-s)}{p-s}\right)\right)^{-1}, \\ I_4 &= \omega_{n-1} \int_0^\infty \frac{t^{n+1-s}}{(1+t^{\frac{p-s}{p-1}})^{\frac{p(n-s)}{p-s}}} dt \end{aligned}$$

$$= \omega_{n-1} \frac{p-1}{p-s} \Gamma\left(\frac{n-s-2p+2}{p-s}\right) \Gamma\left(\frac{p(n-s)-n+2p+s-2}{p-s}\right) \left(\Gamma\left(\frac{p(n-s)}{p-s}\right)\right)^{-1}.$$

Hence, we have (note that $p = 2$)

$$\frac{I_2 I_3}{I_1 I_4} = \frac{(n-2)(n-s+2)}{(n-4)(n-s)}, \quad (5.14)$$

and

$$\frac{I_0 I_3}{I_1 I_4} = \frac{2(2n-2-s)}{(n-4)(n-s)n}. \quad (5.15)$$

Taking (5.14) and (5.15) into (5.13), we get

$$\begin{aligned} & \left(\frac{I_2 I_3}{I_1 I_4} \frac{p^*(s)}{p} - 1\right) \frac{Scal_g(x_0)}{3} - \frac{\Delta K(x_0)}{K(x_0)} - 2na(x_0) \frac{I_0 I_3}{I_1 I_4} \frac{p^*(s)}{p} \\ &= \frac{6-s}{3(n-4)} Scal_g(x_0) - \frac{\Delta K(x_0)}{K(x_0)} - \frac{4(2n-2-s)}{(n-4)(n-2)} a(x_0). \end{aligned}$$

Now, if $p > 2$, then $p^2 - p > 2(p-1)$. It suffices to prove

$$\begin{aligned} & \left(\frac{I_2}{I_1} - \frac{p}{p^*(s)} \frac{I_4}{I_3}\right) \frac{Scal_g(x_0)}{6n} - \frac{p}{p^*(s)} \frac{\Delta K(x_0)}{2nK(x_0)} \frac{I_4}{I_3} \\ &= \frac{1}{2n} \frac{I_4}{I_3} \frac{p}{p^*(s)} \left[\left(\frac{I_2 I_3}{I_1 I_4} \frac{p^*(s)}{p} - 1\right) \frac{Scal_g(x_0)}{3} - \frac{\Delta K(x_0)}{K(x_0)}\right] > 0. \end{aligned}$$

By careful computation, we have

$$\frac{I_2 I_3}{I_1 I_4} = \frac{(n-p)(n-s+2)}{(n-3p+2)(n-s)},$$

and then

$$\frac{1}{3} \left(\frac{I_2 I_3}{I_1 I_4} \frac{p^*(s)}{p} - 1\right) = \frac{1}{3} \frac{3p-s}{n-3p+2}.$$

This completes the proof. \square

Proof of Theorem 1.6 and Theorem 1.7. In order to show that the Hardy-Sobolev inequality (1.9) is false for the given values of q stated in Theorem 1.6, it suffices to prove that there exists a sequence of functions $\{u_\varepsilon\}$ such that

$$\frac{\|u_\varepsilon\|_{p^*(s),s}^p - S(n,p,s)^p \|\nabla_g u_\varepsilon\|_p^p}{\|u_\varepsilon\|_q^p} \rightarrow \infty \quad (5.16)$$

as $\varepsilon \rightarrow 0$, according to (1.5). Let u_ε be as in (5.1). With the same computation as in Lemma 5.1, we have

$$\|u_\varepsilon\|_q^p = \begin{cases} O\left(\epsilon^{\frac{p[n(p-1)-q(n-p)]}{q(p-s)}}\right), & \text{if } q > \frac{n(p-1)}{n-p}, \\ O(|\ln \epsilon|^{\frac{p}{q}}), & \text{if } q = \frac{n(p-1)}{n-p}, \\ O(1), & \text{if } 1 < q < \frac{n(p-1)}{n-p}. \end{cases} \quad (5.17)$$

If $1 < p < \frac{n+2}{3}$ and $q < \frac{np}{n-p+2}$, then $\frac{np}{n-p+2} > \frac{n(p-1)}{n-p}$. Moreover, by Lemma 5.1 we have

$$\begin{aligned} & \|u_\varepsilon\|_{p^*(s),s}^p - S(n,p,s)^p \|\nabla_g u_\varepsilon\|_p^p \\ &= \frac{\text{Scal}_g(x_0)}{6n} \frac{p}{p^*(s)} \frac{I_4}{I_3} I_3^{\frac{p}{p^*(s)}} \left(\frac{I_2 I_3}{I_1 I_4} \frac{p^*(s)}{p} - 1 \right) \varepsilon^{\frac{3p-n-2}{p-s}} + o\left(\varepsilon^{\frac{3p-n-2}{p-s}}\right). \end{aligned} \quad (5.18)$$

Therefore, (5.16) follows from (5.17), (5.18) and the fact that

$$\frac{3p-n-2}{p-s} - \frac{p[n(p-1)-q(n-p)]}{q(p-s)} = (p-1)(n-p+2) \left(q - \frac{np}{n-p+2} \right) < 0.$$

If $p = \frac{n+2}{3}$, $n > 4$ and $q \leq \frac{np}{n-p+2}$, then $\frac{np}{n-p+2} = \frac{n(p-1)}{n-p}$. Again with Lemma 5.1, we get

$$\|u_\varepsilon\|_{p^*(s),s}^p - S(n,p,s)^p \|\nabla_g u_\varepsilon\|_p^p = O(|\ln \varepsilon|). \quad (5.19)$$

Together with (5.17), (5.19) and note here that $q = p + \frac{p}{2} - 1 > p$, we thus proved (5.16).

Using exactly the same arguments as in the previous proof, Theorem 1.7 follows a tedious but straightforward computation. In order to save space we omit the details. \square

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