

## NOTE

# A Note on the Propagation Speed of Travelling Waves for a Lotka–Volterra Competition Model with Diffusion

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This paper is concerned with the propagation speed of positive travelling waves for a Lotka–Volterra competition model with diffusion. We show that under a certain boundary condition, the propagation speed of the travelling wave is equal to 0. To do this, we employ the method of moving planes proposed by Gidas *et al.*

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## 1. INTRODUCTION

There have been many studies of reaction–diffusion equations

$$\mathbf{w}_t = D\mathbf{w}_{xx} + \mathbf{f}(\mathbf{w}), \quad x \in \mathbf{R}, \quad t > 0 \quad (1.1)$$

to explain phenomena which appear in various fields, where  $\mathbf{w}$  and  $\mathbf{f}$  are  $n$ -dimensional vectors, and  $D$  is a diagonal matrix whose elements are positive. One interesting phenomenon is the appearance of travelling waves which are represented as  $\mathbf{w}(t, x) = \mathbf{u}(\xi)$ ,  $\xi = x - st$ , where  $\mathbf{u}(\xi)$  is a  $C^2$ -class function, and  $s$  is a constant to be determined. Such waves necessarily satisfy the system of ODEs

$$0 = D\mathbf{u}_{\xi\xi} + s\mathbf{u}_\xi + \mathbf{f}(\mathbf{u}), \quad \xi \in \mathbf{R}. \quad (1.2a)$$

In order to determine the propagation speed  $s$ , we lay the boundary condition

$$\mathbf{u}(-\infty) = \mathbf{u}(+\infty) = \mathbf{u}_0 (\in \mathbf{R}^n) \quad (1.2b)$$

on the wave. Since

$$\mathbf{u}_\xi(\xi)/\xi = -D^{-1}\mathbf{f}(\mathbf{u}_0) + o(1) \quad \text{as } |\xi| \rightarrow +\infty$$

is satisfied, we find that  $\mathbf{u}_0$  must be a solution of  $\mathbf{f}(\mathbf{u}) = 0$ . Here we assume that

(H.1) the real part of every eigenvalue for the matrix  $\mathbf{f}_\mathbf{u}(\mathbf{u}_0)$  is negative,

which means that  $\mathbf{u}_0$  is an exponentially stable equilibrium point of  $\mathbf{u}_t = \mathbf{f}(\mathbf{u})$ .

Let us consider the case  $n = 1$ . Since  $f_u(u_0) < 0$  holds because of (H.1), we can easily check that every solution of (1.2) exponentially decays to  $u_0$  as  $|\xi| \rightarrow +\infty$ . Hence we have  $u_\xi(\pm\infty) = 0$ , and then obtain

$$s \int_{\mathbf{R}} u_\xi(\xi)^2 d\xi = - \int_{\mathbf{R}} (Du_{\xi\xi}(\xi) + f(u(\xi))) u_\xi(\xi) d\xi = 0,$$

which implies that the following property holds for (1.2): (P) If there exists a nonconstant solution for  $s = s_0 \in \mathbf{R}$ , then  $s_0 = 0$  must be satisfied. As the property (P) does not always hold for  $n \geq 2$ , one important problem for the travelling wave is the qualitative study on the propagation speed  $s$ .

In this paper, to approach the problem, we consider the propagation speed of travelling waves for the Lotka–Volterra competition model with diffusion

$$u_t = u_{xx} + uf_0(\mathbf{u}), \quad v_t = dv_{xx} + vg_0(\mathbf{u}), \quad x \in \mathbf{R}, \quad t > 0,$$

which describes the dynamics of the population  $\mathbf{u} = (u, v) \in \mathbf{R}^2$  of two competing species, where  $d$  is a positive constant, and  $\mathbf{f}_0(\mathbf{u}) = (f_0, g_0)(\mathbf{u})$  is a  $C^1$ -class function in  $\mathbf{u}$ . Since  $u$  and  $v$  are variables which indicate the population density, we restrict our discussion to nonconstant solutions which satisfy

$$0 = D\mathbf{u}_{\xi\xi} + s\mathbf{u}_\xi + \mathbf{f}(\mathbf{u}), \quad \xi \in \mathbf{R}, \quad (1.3a)$$

$$u(\xi) \geq 0, \quad v(\xi) \geq 0, \quad \xi \in \mathbf{R}, \quad (1.3b)$$

$$\mathbf{u}(-\infty) = \mathbf{u}(+\infty) = \mathbf{u}_0 \quad (1.3c)$$

for some  $s \in \mathbf{R}$ , where  $D = \text{diag}(1, d)$ ,  $f(\mathbf{u}) = uf_0(\mathbf{u})$ ,  $g(\mathbf{u}) = vg_0(\mathbf{u})$ ,  $\mathbf{f}(\mathbf{u}) = (f, g)(\mathbf{u})$ , and  $\mathbf{u}_0$  is a solution of  $\mathbf{f}(\mathbf{u}) = 0$  in the first quadrant. From the competitive interaction, we assume the following:

(H.2) There exist  $\alpha_u > 0$  and  $\alpha_v > 0$  such that

$$\begin{aligned} f_0(u, 0)(u - \alpha_u) &< 0 \text{ for any } u \neq \alpha_u, \\ g_0(0, v)(v - \alpha_v) &< 0 \text{ for any } v \neq \alpha_v. \end{aligned}$$

(H.3)  $f_{0v}(\mathbf{u}) < 0$  and  $g_{0u}(\mathbf{u}) < 0$  are satisfied for any  $\mathbf{u}$ .

We should note that  $(0, \alpha_v)$  and  $(\alpha_u, 0)$  are solutions of  $\mathbf{f}(\mathbf{u}) = 0$  in the first quadrant.

Many authors have studied the existence of nonconstant solutions of the problem (1.3a), (1.3b) with a variety of boundary conditions (for instance, see Gardner [1], Hosono and Mimura [3], and Kan-on [4]). However, so far, we do not yet have enough results on the propagation speed of the travelling wave. The following is the main result in this paper.

**THEOREM 1.1.** *Let  $\mathbf{u}_0$  be either  $\mathbf{u}_0 = (0, \alpha_v)$  or  $\mathbf{u}_0 = (\alpha_u, 0)$ . If the assumptions (H.1), (H.2), and (H.3) are satisfied, then the property (P) holds for (1.3).*

In Section 3, we shall prove the above theorem by using the method of moving planes proposed by Gidas *et al.* [2].

## 2. APPLICATION

In this section, we consider the case where  $\mathbf{f}_0(\mathbf{u})$  is linear, that is,

$$f_0(\mathbf{u}) = 1 - u - cv, \quad g_0(\mathbf{u}) = a - bu - v,$$

where  $a$ ,  $b$ , and  $c$  satisfy  $0 < 1/c < a < b$ . We can easily check that  $(0, a)$  and  $(1, 0)$  are exponentially stable equilibrium points of  $\mathbf{u}_t = \mathbf{f}(\mathbf{u})$ .

**LEMMA 2.1** (Theorem 2.2 and Corollary 2.3 in [4]). *There exist a constant  $a_0 \in (1/c, b)$  and a  $C^1$ -class function  $\hat{\mathbf{u}}(\cdot, a)$  defined on  $(1/c, a_0)$  (resp.  $(a_0, b)$ ) such that for each  $a \in (1/c, a_0)$  (resp.  $a \in (a_0, b)$ ), (1.3) with  $s = 0$  and  $\mathbf{u}_0 = (0, a)$  (resp.  $\mathbf{u}_0 = (1, 0)$ ) has a unique nonconstant solution  $\hat{\mathbf{u}}(\xi, a)$  up to the translation.*

We find from [5] that for each  $a$ ,  $\hat{\mathbf{u}}(\xi, a)$  is an unstable travelling wave of (1.1) relative to the space of uniformly continuous functions from  $\mathbf{R}$  to  $\mathbf{R}^2$  with the supremum norm. Combining the above lemma with Theorem

1.1, we obtain the following:

**THEOREM 2.2.** *Let  $a \in (1/c, a_0)$  (resp.  $a \in (a_0, b)$ ), and let  $\mathbf{u}(\xi)$  be an arbitrary nonconstant solution of (1.3) with  $\mathbf{u}_0 = (0, a)$  (resp.  $\mathbf{u}_0 = (1, 0)$ ) for  $s = s_0 \in \mathbf{R}$ . Then  $s_0 = 0$  holds and there exists  $\tau \in \mathbf{R}$  such that  $\mathbf{u}(\xi) = \hat{\mathbf{u}}(\xi + \tau, a)$  for any  $\xi \in \mathbf{R}$ .*

### 3. PROOF OF THEOREM 1.1

We only show the proof for the case  $\mathbf{u}_0 = (0, \alpha_v)$ , because the other case can be proved in a similar manner. Contrary to the conclusion, we assume that (1.3) has a nonconstant solution  $\mathbf{u}(\xi) = (u, v)(\xi)$  for  $s = s_0$  ( $\neq 0$ ). Since  $\mathbf{u}(2\lambda - \xi)$  is a nonconstant solution of (1.3) with  $s = -s_0$  for each  $\lambda \in \mathbf{R}$ , we may assume  $s_0 > 0$  without loss of generality. Furthermore by (H.2), (H.3), and the comparison principle, we have  $0 < u(\xi) < \alpha_u$  and  $0 < v(\xi) < \alpha_v$  for any  $\xi \in \mathbf{R}$ .

The linearized operator of (1.3a) around  $\mathbf{u} = \mathbf{u}_0$  can be represented as

$$\begin{pmatrix} p_u\left(\frac{d}{d\xi}\right) & 0 \\ g_u(\mathbf{u}_0) & p_v\left(\frac{d}{d\xi}\right) \end{pmatrix},$$

where  $p_u(\gamma) = \gamma^2 + s_0\gamma + f_u(\mathbf{u}_0)$  and  $p_v(\gamma) = d\gamma^2 + s_0\gamma + g_v(\mathbf{u}_0)$ . Since

$$f_u(\mathbf{u}_0) < 0, \quad g_u(\mathbf{u}_0) < 0, \quad g_v(\mathbf{u}_0) < 0$$

are satisfied because of (H.1) and (H.3), we can define  $\gamma_v^\pm$  ( $\gamma_v^- < 0 < \gamma_v^+$ ) by the solutions of  $p_v(\gamma) = 0$  ( $v = u, v$ ). We set

$$\Gamma_1^\pm = \min\{\gamma_u^\pm, \gamma_v^\pm\}, \quad \Gamma_2^\pm = \max\{\gamma_u^\pm, \gamma_v^\pm\}, \quad m^\pm = 2 - \#\{\gamma_u^\pm, \gamma_v^\pm\},$$

where  $\#A$  is the number of elements of the set  $A$ . By definition, we have

$$\gamma_u^- + \gamma_u^+ = -s_0 (< 0), \quad \gamma_v^- + \gamma_v^+ = -s_0/d (< 0), \quad (3.1a)$$

and then obtain

$$\Gamma_1^+ + \Gamma_2^- \leq -\frac{s_0}{d} \min(1, d) (< 0). \quad (3.1b)$$

Let us obtain the estimate for  $\mathbf{u}(\xi)$  in a neighborhood of  $\xi = \pm\infty$ . Since  $u(\xi)$  satisfies  $(p_u(d/d\xi) + o(1))u(\xi) = 0$  as  $\xi \rightarrow -\infty$ , we have

$$u(\xi) = C_u^- e^{\gamma_u^+ \xi} (1 + o(1)) \quad \text{as } \xi \rightarrow -\infty, \quad (3.2a)$$

where  $C_u^-$  is a positive constant. From the above estimate, we have

$$\begin{aligned} 0 &= dv_{\xi\xi}(\xi) + s_0 v_{\xi}(\xi) + g(\mathbf{u}(\xi)) \\ &= \left( p_v \left( \frac{d}{d\xi} \right) + o(1) \right) (v(\xi) - \alpha_v) - C_1 e^{\gamma_u^+ \xi} (1 + o(1)) \end{aligned}$$

as  $\xi \rightarrow -\infty$  and then obtain

$$\begin{aligned} v(\xi) - \alpha_v &= \begin{cases} \frac{C_1 e^{\gamma_u^+ \xi}}{p_v(\gamma_u^+)} (1 + o(1)) + C_2 e^{\gamma_v^+ \xi} (1 - o(1)) & \text{if } \gamma_u^+ \neq \gamma_v^+, \\ -\frac{C_1 |\xi| e^{\gamma_u^+ \xi}}{\frac{d}{d\gamma} p_v(\gamma_u^+)} (1 + o(1)) & \text{if } \gamma_u^+ = \gamma_v^+ \end{cases} \end{aligned}$$

as  $\xi \rightarrow -\infty$ , where  $C_1 = -g_u(\mathbf{u}_0)C_u^- (> 0)$ , and  $C_2$  is a suitable constant. By the definition of  $\gamma_u^+$  and  $\gamma_v^+$ , we have

$$\frac{d}{d\gamma} p_v(\gamma_v^+) > 0, \quad p_v(\gamma_u^+) \begin{cases} < 0 & \text{if } \gamma_u^+ < \gamma_v^+, \\ > 0 & \text{if } \gamma_u^+ > \gamma_v^+. \end{cases}$$

Since  $0 < v(\xi) < \alpha_v$  holds for any  $\xi \in \mathbf{R}$ , we see that  $C_2 < 0$  must be satisfied when  $\gamma_u^+ > \gamma_v^+$  holds. Hence we have

$$v(\xi) = \alpha_v - C_v^- |\xi|^{m^+} e^{\Gamma_1^+ \xi} (1 + o(1)) \quad \text{as } \xi \rightarrow -\infty, \quad (3.2b)$$

where  $C_v^-$  is a positive constant. In a similar manner, we can obtain

$$\begin{aligned} u(\xi) &= C_u^+ e^{\gamma_u^- \xi} (1 + o(1)), \\ v(\xi) &= \alpha_v - C_v^+ |\xi|^{m^-} e^{\Gamma_2^- \xi} (1 + o(1)) \end{aligned} \quad (3.2c)$$

as  $\xi \rightarrow +\infty$ , where  $C_u^+$  and  $C_v^+$  are positive constants. By (3.1) and (3.2), we obtain

$$\frac{u(2\lambda - \xi)}{u(\xi)} = \frac{C_u^+ e^{2\gamma_u^+ \lambda}}{C_u^-} e^{s_0 \xi} (1 + o(1)) < 1,$$

$$\frac{v(2\lambda - \xi) - \alpha_v}{v(\xi) - \alpha_v} = \frac{C_v^+ |2\lambda - \xi|^{m^-} e^{2\Gamma_2^- \lambda}}{C_v^- |\xi|^{m^+}} e^{-(\Gamma_2^- + \Gamma_1^+) \xi} (1 + o(1)) < 1$$

as  $\xi \rightarrow -\infty$  for any fixed  $\lambda \in \mathbf{R}$ , which implies that

$$u(2\lambda - \xi) < u(\xi),$$

$$v(2\lambda - \xi) > v(\xi) \text{ in a neighborhood of } \xi = -\infty \quad (3.3)$$

hold for any fixed  $\lambda \in \mathbf{R}$ .

We set  $\mathbf{U}(\xi, \lambda) (= (U, V)(\xi, \lambda)) \equiv \mathbf{u}(\xi) - \mathbf{u}(2\lambda - \xi)$ . Let us show that there exists  $\lambda_0 \in \mathbf{R}$  such that  $V(\xi, \lambda) \leq 0 \leq U(\xi, \lambda)$  holds on  $(-\infty, \lambda]$  for any  $\lambda \geq \lambda_0$ . Since  $\mathbf{u}(\xi)$  is not oscillatory near  $\xi = \pm\infty$ , we obtain  $-\infty < \xi_- \leq \xi_+ < +\infty$ , where  $\xi_- = \min(\xi_-^u, \xi_-^v)$ ,  $\xi_+ = \max(\xi_+^u, \xi_+^v)$ ,

$$\xi_-^u = \sup\{\tau | u_\xi(\xi) \geq 0 \text{ on } (-\infty, \tau]\},$$

$$\xi_+^u = \inf\{\tau | u_\xi(\xi) \leq 0 \text{ on } [\tau, +\infty)\},$$

$$\xi_-^v = \sup\{\tau | v_\xi(\xi) \leq 0 \text{ on } (-\infty, \tau]\},$$

$$\xi_+^v = \inf\{\tau | v_\xi(\xi) \geq 0 \text{ on } [\tau, +\infty)\}.$$

We assume that there exists  $\tau (> \xi_+) such that  $u_\xi(\tau)v_\xi(\tau) = 0$  holds. By (H.3), we have$

$$u_{\xi\xi}(\tau) = 0, \quad 0 \geq u_{\xi\xi\xi}(\tau) = -f_v(\mathbf{u}(\tau))v_\xi(\tau) \geq 0$$

when  $u_\xi(\tau) = 0$ ,

$$v_{\xi\xi}(\tau) = 0, \quad 0 \leq dv_{\xi\xi\xi}(\tau) = -g_u(\mathbf{u}(\tau))u_\xi(\tau) \leq 0$$

when  $v_\xi(\tau) = 0$ ,

which implies that both of  $\mathbf{u}_\xi(\tau) = 0$  and  $\mathbf{u}_{\xi\xi}(\tau) = 0$  hold. Since  $\mathbf{u}_\xi(\xi)$  is a solution of

$$0 = D\mathbf{w}_{\xi\xi} + s_0 \mathbf{w}_\xi + \mathbf{f}_u(\mathbf{u}(\xi))\mathbf{w}, \quad \xi \in \mathbf{R}, \quad (3.4)$$

we find that  $u(\xi)$  must be a constant function. This is a contradiction. Hence we obtain

$$u_{\xi}(\xi) < 0 < v_{\xi}(\xi) \quad \text{for any } \xi > \xi_+. \quad (3.5)$$

Since

$$U_{\xi}(\xi, \lambda) = u_{\xi}(\xi) + u_{\xi}(2\lambda - \xi) < 0,$$

$$V_{\xi}(\xi, \lambda) = v_{\xi}(\xi) + v_{\xi}(2\lambda - \xi) > 0$$

holds on  $[\xi_+, \lambda)$  for any  $\lambda \geq \xi_+$ , we have

$$V(\xi, \lambda) < 0 < U(\xi, \lambda) \quad \text{on } [\xi_+, \lambda) \quad (3.6a)$$

for any  $\lambda \geq \xi_+$  because of  $U(\lambda, \lambda) = 0$ . It follows from (3.3) that there exists  $\xi_1 (\leq \xi_-)$  such that  $V(\xi, \xi_+) < 0 < U(\xi, \xi_+)$  on  $(-\infty, \xi_1]$ . Since

$$U_{\lambda}(\xi, \lambda) = -2u_{\xi}(2\lambda - \xi) > 0, \quad V_{\lambda}(\xi, \lambda) = -2v_{\xi}(2\lambda - \xi) < 0$$

are satisfied on  $(-\infty, \lambda)$  for any  $\lambda \geq \xi_+$  because of (3.5), we have

$$V(\xi, \lambda) < 0 < U(\xi, \lambda) \quad \text{on } (-\infty, \xi_1] \quad (3.6b)$$

for any  $\lambda \geq \xi_+$ . By (3.5), we can take  $\xi_2 (\geq \xi_+)$  as satisfying

$$u(\xi_2) \leq \min_{\xi \in [\xi_1, \xi_+]} u(\xi), \quad v(\xi_2) \geq \max_{\xi \in [\xi_1, \xi_+]} v(\xi),$$

and then obtain

$$\begin{aligned} V(\xi, \lambda) &\leq v(\xi) - v(\xi_2) \leq 0 \leq u(\xi) - u(\xi_2) \\ &\leq U(\xi, \lambda) \quad \text{on } [\xi_1, \xi_+] \end{aligned} \quad (3.6c)$$

for any  $\lambda \geq \xi_2$ . Hence we arrive at  $V(\xi, \lambda) \leq 0 \leq U(\xi, \lambda)$  on  $(-\infty, \lambda]$  for any  $\lambda \geq \xi_2$ . Setting

$$\lambda_0 = \inf\{\lambda \mid V(\xi, \tau) \leq 0 \leq U(\xi, \tau) \text{ on } (-\infty, \tau] \text{ for any } \tau \geq \lambda\},$$

we have  $\lambda_0 \leq \xi_2$ , and  $V(\xi, \lambda_0) \leq 0 \leq U(\xi, \lambda_0)$  on  $(-\infty, \lambda_0]$ . By the definition of  $\xi_+^u$ , we find that there exists  $\xi_3 (< \xi_+^u)$  such that  $u_{\xi}(\xi) < 0$  for any  $\xi \in (\xi_3, \xi_+^u)$ . Since  $U(\lambda, \lambda) = 0$  and  $U_{\xi}(\lambda, \lambda) = 2u_{\xi}(\lambda) < 0$  hold for any  $\lambda \in (\xi_3, \xi_+^u)$ , we see that for each  $\lambda \in (\xi_3, \xi_+^u)$ ,  $U(\xi, \lambda) < 0$  is satisfied for some  $\xi < \lambda$ . By the definition of  $\lambda_0$ , we obtain  $\xi_+^u \leq \lambda_0$ . Similarly we can prove  $\xi_+^v \leq \lambda_0$ . Hence we have  $\xi_+ \leq \lambda_0$ .

We consider the case  $\lambda_0 = \xi_+^u$ . By definition, we have

$$\begin{aligned} u_\xi(\lambda_0) &= 0, & u_{\xi\xi}(\lambda_0) &\leq 0, & v_\xi(\lambda_0) &\geq 0, \\ U(\lambda_0, \lambda_0) &= 0, & U_\xi(\lambda_0, \lambda_0) &= 0, & U_{\xi\xi}(\lambda_0, \lambda_0) &= 0. \end{aligned}$$

By  $s_0 > 0$  and (H.3), we obtain

$$0 \geq U_{\xi\xi\xi}(\lambda_0, \lambda_0)/2 = u_{\xi\xi\xi}(\lambda_0) = -s_0 u_{\xi\xi}(\lambda_0) - f_v(\mathbf{u}(\lambda_0))v_\xi(\lambda_0) \geq 0$$

and then find that  $u_{\xi\xi}(\lambda_0) = 0$  and  $v_\xi(\lambda_0) = 0$  are satisfied. Similarly we can prove  $v_{\xi\xi}(\lambda_0) = 0$  by the change of the role of  $u$  and  $v$  in the above argument. Hence we have  $\mathbf{u}_\xi(\lambda_0) = 0$  and  $\mathbf{u}_{\xi\xi}(\lambda_0) = 0$ . Since  $\mathbf{u}_\xi(\xi)$  is a solution of (3.4), we see from uniqueness that  $\mathbf{u}(\xi)$  must be a constant function. This contradiction implies that  $\lambda_0 > \xi_+^u$  holds. Analogously we can prove  $\lambda_0 > \xi_+^v$ . Therefore we obtain  $\lambda_0 > \xi_+^*$ .

We see from (3.6) that there exists  $\xi_4 \in [\xi_1, \xi_+]$  such that  $U(\xi_4, \lambda_0)V(\xi_4, \lambda_0) = 0$  holds. If  $U(\xi_4, \lambda_0) = 0$  is satisfied, then we have  $U_\xi(\xi_4, \lambda_0) = 0$  and

$$\begin{aligned} 0 &\leq U_{\xi\xi}(\xi_4, \lambda_0) \\ &= s_0 u_\xi(2\lambda_0 - \xi_4) - s_0 u_\xi(\xi_4) + f(\mathbf{u}(2\lambda_0 - \xi_4)) - f(\mathbf{u}(\xi_4)) \\ &= 2s_0 u_\xi(2\lambda_0 - \xi_4) - C_3 V(\xi_4, \lambda_0) < 0 \end{aligned}$$

because of  $s_0 > 0$ , (H.3), and (3.5), where

$$C_3 = \int_0^1 f_v(\theta \mathbf{u}(\xi_4) + (1 - \theta)\mathbf{u}(2\lambda_0 - \xi_4)) d\theta (< 0).$$

Similarly we can derive a contradiction when  $V(\xi_4, \lambda_0) = 0$  is satisfied. These contradictions imply that  $s_0 = 0$  holds. Thus the proof of Theorem 1.1 is completed. ■

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