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Multiple Solutions and Eigenvalues for Third-Order Right Focal Boundary Value Problems

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We are concerned with the right focal boundary value problem

$$x'''(t) = f(t, x(t)), \qquad t_1 \le t \le t_3,$$

$$x(t_1) = x'(t_2) = x''(t_3) = 0,$$

and the associated eigenvalue problem

$$x'''(t) = \lambda a(t) f(x(t))$$

with the same boundary conditions. Under various assumptions on f, a, and λ we establish intervals of the parameter λ which yield the existence of a positive solution of the eigenvalue problem. By placing certain restrictions on the nonlinearity, we prove the existence of at least one, at least two, at least three, and infinitely many positive solutions of the boundary value problem by applying some known fixed-point theorems as well as some recent generalizations of these fixed-point theorems.
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1. INTRODUCTION

We are concerned with the existence of positive solutions to the third-order, three-point eigenvalue problem

$$x'''(t) = \lambda a(t) f(x(t)), \qquad t_1 \le t \le t_3, \tag{1.1}$$

$$x(t_1) = x'(t_2) = x''(t_3) = 0, \tag{1.2}$$

and the associated boundary value problem consisting of

$$x'''(t) = f(t, x(t)), \qquad t_1 \le t \le t_3, \tag{1.3}$$

and (1.2). Here, $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and nonnegative for $x \ge 0$. A solution of (1.3), (1.2) is nonnegative on $[t_1, t_3]$, nondecreasing on $[t_1, t_2]$, and nonincreasing on $[t_2, t_3]$.

The goal of this paper is to bring together a variety of recent results in fixed-point theory. Some of the fixed-point results we use have become standard in nonlinear analysis (Krasnoselskii's Theorem), some are becoming increasingly more common (Leggett–Williams Theorem), while others are newcomers (Five Functionals Fixed-Point Theorem). The unifying theme in the utilization of these theorems is the search for multiple solutions of various types of boundary value problems for ordinary differential equations. We want to use some of these techniques to establish the existence of at least one, at least two, at least three, and infinitely many positive solutions of (1.3), (1.2) which lie in a cone. Prior to that, we obtain eigenvalue intervals for which the problem (1.1), (1.2) has at least one positive solution in a cone.

A large part of the literature on multiple solutions to boundary value problems seems to be traced back to Krasnoselskii's work on nonlinear operator equations [26], especially the part dealing with the theory of cones in Banach spaces. In 1994, Erbe and Wang [16] applied Krasnoselskii's work to eigenvalue problems such as the one above to establish intervals of the parameter λ for which there is at least one positive solution. The upshot of their technique was assuming that the nonlinearity grew either superlinearly or sublinearly and $a(t)$ did not vanish identically on any subinterval of the domain. The growth assumptions and calculations involving the Green's function followed by an application of Krasnoselskii's Theorem yield the result. Many authors have used this approach or a variation thereof to obtain eigenvalue intervals; see [1, 3, 19]. Of course, taking $\lambda = 1$ turns this into a boundary value problem (with nonhomogeneity separated in space and time), and hence one gets a multiplicity result simultaneously. However, this is but one of the techniques for proving the existence of at least one solution; see [3, 30].

Recently, Davis et al. [13, 14] were able to provide significantly better eigenvalue intervals for classes of Sturm–Liouville-type problems using a new approach which did not require f to grow superlinearly or sublinearly and did not require $a(t)$ to be nonvanishing on every subinterval of the domain. We use this new approach on (1.1), (1.2), and that is the contribution of Section 2.

Next, various authors investigated the existence of at least two solutions [2, 3, 15, 30]. The main idea here is a variation on the Krasnoselskii theme: assume pairs of growth conditions on the nonlinearity and use this to fulfill the cone expansion/compression hypotheses of the fixed-point theorem. This results in distinct annular regions in the cone containing a fixed point, and these fixed points are the solutions of the boundary value problem. Some authors [2, 16] achieve the growth assumptions by assuming sublinear or superlinear growth in the nonlinearity or modify that slightly (as we do) to obtain a more general result assuming certain inequalities on f. By assuming more pairs of growth conditions on f, we are able to obtain the existence of at least n solutions for any $n \in \mathbb{N}$. Section 4 contains these results.

However, even more strides have been made in this direction. In 1998, Anderson [4], Avery [6], and Davis and Henderson [12] each applied seldom used fixed-point theorems due to Leggett and Williams [27] or Sun and Sun [32] to obtain the existence of at least three solutions for certain types of boundary value problems. Unlike earlier fixed-point applications, the theorems used here were not based on the Krasnoselskii theorem; rather, they were based on topological degree theory. A year later, Avery [7] generalized the Leggett–Williams Fixed-Point Theorem—again using degree theory—by replacing various inequalities in the hypotheses of their theorem with more general nonnegative continuous concave and convex functionals. This turned out to be a fruitful abstraction, and this is the strategy we employ in Section 6 for our problem (1.3), (1.2) after we provide sufficient background on Avery's theorem (called the Five Functionals Fixed-Point Theorem) in Section 5.

Finally, in Section 7 we outline some future paths that this area might take and discuss some other problems which should be accessible by some or all of the methods in this paper.

2. EIGENVALUE INTERVALS

Consider the nonlinear right focal eigenvalue problem (1.1), (1.2)

$$x'''(t) = \lambda a(t) f(x), \qquad t_1 \le t \le t_3, \tag{2.1}$$

$$x(t_1) = x'(t_2) = x''(t_3) = 0. \tag{2.2}$$

The corresponding Green's function for the homogeneous problem $x'''(t) = 0$ satisfying the boundary conditions (1.2) is given by [5]

$$G(t,s)=\begin{cases} s\in[t_1,t_2]: \begin{cases} \frac{1}{2}(t-t_1)(2s-t-t_1) & : \quad t\le s \\ \frac{1}{2}(s-t_1)^2 & : \quad t\ge s \end{cases} \\ s\in[t_2,t_3]: \begin{cases} \frac{1}{2}(t-t_1)(2t_2-t-t_1) & : \quad t\le s \\ \frac{1}{2}(t-t_1)(2t_2-t-t_1)+\frac{1}{2}(t-s)^2 & : \quad t\ge s. \end{cases} \end{cases} \tag{2.3}$$

Remark 2.1. By [5], if $t_2-t_1>t_3-t_2$,

$$G(t_2,s)\ge G(t,s)>0$$

for $t\in(t_1,t_3]$, $s\in[t_1,t_3]$. Thus throughout this paper we assume that

$$t_2-t_1>t_3-t_2. \tag{2.4}$$

Furthermore we have the assumptions

(A1) $a(t)$ is a nonnegative, measurable function defined on $[t_1,t_3]$ satisfying

$$0<\int_{t_2-h}^{t_2+h} G(t_2,s)a(s)ds<\infty, \tag{2.5}$$

for some $h\in(0,t_3-t_2)$ where, using (2.3),

$$G(t_2,s)=\begin{cases} \frac{1}{2}(s-t_1)^2: & s\in[t_1,t_2] \\ \frac{1}{2}(t_2-t_1)^2: & s\in[t_2,t_3]. \end{cases}$$

(A2) $f\colon[0,\infty)\to[0,\infty)$ is continuous such that both

$$f_0:=\lim_{x\to 0^+}\frac{f(x)}{x} \quad \text{and} \quad f_\infty:=\lim_{x\to\infty}\frac{f(x)}{x}$$

exist.

LEMMA 2.1. *For all* $t,s\in[t_1,t_3]$,

$$g(t)G(t_2,s)\le G(t,s)\le G(t_2,s), \tag{2.6}$$

where

$$g(t):=\min\left\{\frac{t-t_1}{t_2-t_1},\frac{t_3-t}{t_3-t_2}\right\}. \tag{2.7}$$

Proof. As noted in the preceding remark, we have from [5] that $G(t,s) \le G(t_2,s)$ for all $t,s \in [t_1,t_3]$. For the lower bound, we proceed by cases on the branches of the Green's function (2.3).

(i) $t_1 \le t \le s \le t_2$: Here $G(t,s) = \frac{1}{2}(t-t_1)(2s-t-t_1)$, $G(t_2,s) = \frac{1}{2} \times (s-t_1)^2$. For these t, s we have

$$\frac{(s-t_1)^2}{t_2-t_1} \le s-t_1 \le 2s-t-t_1,$$

which implies

$$\left(\frac{t-t_1}{t_2-t_1}\right) G(t_2,s) \le G(t,s).$$

(ii) $t_1 \le s \le t \le t_2$: Since $G(t,s) = \frac{1}{2}(s-t_1)^2 = G(t_2,s)$, it follows that

$$\left(\frac{t-t_1}{t_2-t_1}\right) G(t_2,s) \le G(t,s).$$

(iii) $t_1 \le s \le t_2 \le t \le t_3$: As in case (ii), $G(t,s) = \frac{1}{2}(s-t_1)^2 = G(t_2,s)$; thus

$$\left(\frac{t_3-t}{t_3-t_2}\right) G(t_2,s) \le G(t,s).$$

(iv) $t_1 \le t \le t_2 \le s \le t_3$: In this case $G(t,s) = \frac{1}{2}(t-t_1)(2t_2-t-t_1)$ and $G(t_2,s) = \frac{1}{2}(t_2-t_1)^2$. Simple algebra yields

$$\left(\frac{t-t_1}{t_2-t_1}\right) G(t_2,s) \le G(t,s).$$

(v) $t_2 \le t \le s \le t_3$: As in (iv), $G(t,s) = \frac{1}{2}(t-t_1)(2t_2-t-t_1)$ and $G(t_2,s) = \frac{1}{2}(t_2-t_1)^2$. Define

$$\begin{aligned} w(t) &:= \frac{1}{2}(t-t_1)(2t_2-t-t_1) - \frac{1}{2}\left(\frac{t_3-t}{t_3-t_2}\right)(t_2-t_1)^2 \\ &= G(t,s) - \left(\frac{t_3-t}{t_3-t_2}\right) G(t_2,s). \end{aligned} \tag{2.8}$$

Now $w(t_2) = 0$, $w'(t_2) > 0$, and $w(t_3) = G(t_3,s) > 0$ by (2.4). Since w is concave down, $w(t) \ge 0$ on $[t_2,t_3]$; hence

$$\left(\frac{t_3-t}{t_3-t_2}\right) G(t_2,s) \le G(t,s).$$

(vi) $t_2 \le s \le t \le t_3$: Note that $G(t_2,s)=\frac{1}{2}(t_2-t_1)^2$, while $G(t,s)=\frac{1}{2}(t-t_1)(2t_2-t-t_1)+\frac{1}{2}(t-s)^2 \ge \frac{1}{2}(t-t_1)(2t_2-t-t_1)$; consequently, the employment of w as in (2.8) yields

$$\left(\frac{t_3-t}{t_3-t_2}\right)G(t_2,s) \le G(t,s).$$

∎

DEFINITION 2.1. Let E be a real Banach space. A nonempty closed convex set $\mathscr{P} \subset E$ is called a *cone* if it satisfies the following two conditions:

(i) $x \in \mathscr{P}$, $\lambda \ge 0$ implies $\lambda x \in \mathscr{P}$;

(ii) $x \in \mathscr{P}$, $-x \in \mathscr{P}$ implies $x = 0$.

Every cone $\mathscr{P} \subset E$ induces an ordering in E given by

$$x \preccurlyeq y \qquad \text{if and only if } y - x \in \mathscr{P}.$$

DEFINITION 2.2. An operator is *completely continuous* if it is continuous and maps bounded sets into precompact sets.

To establish eigenvalue intervals we will employ the following fixed-point theorem due to Krasnoselskii [26].

THEOREM 2.1. *Let E be a Banach space, let $K \subseteq E$ be a cone, and suppose that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $\mathscr{A}$: $K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is a completely continuous operator such that either*

(i) $\|\mathscr{A}u\| \le \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|\mathscr{A}u\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$, *or*

(ii) $\|\mathscr{A}u\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|\mathscr{A}u\| \le \|u\|$, $u \in K \cap \partial\Omega_2$ *holds. Then $\mathscr{A}$ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

Let $\mathscr{B}$ denote the Banach space $C[t_1,t_3]$ with the norm

$$\|x\| = \sup_{t \in [t_1,t_3]} |x(t)|.$$

Define the cone $\mathscr{P} \subset \mathscr{B}$ by

$$\mathscr{P} = \{x \in \mathscr{B}\colon\ x(t) \ge g(t)\|x\|, t \in [t_1,t_3]\},$$

where g is given in (2.7). Let $h \in (0, t_3 - t_2)$ be chosen such that

$$\int_{t_2-h}^{t_2+h} G(t_2,s)a(s)\,ds > 0. \tag{2.9}$$

For ease of notation, set

$$u(h) := 1 - \frac{h}{t_3 - t_2}. \tag{2.10}$$

THEOREM 2.2. *Suppose* (A1) *and* (A2) *hold. Then for each* λ *satisfying*

$$\frac{1}{u(h)f_\infty \int_{t_2-h}^{t_2+h} G(t_2,s)a(s)ds} < \lambda < \frac{1}{f_0 \int_{t_1}^{t_3} G(t_2,s)a(s)ds} \tag{2.11}$$

there exists at least one solution of (1.1), (1.2) *in* $\mathcal{P}$, *for* $u(h)$ *as in* (2.10).

Proof. Let λ be as in (2.11), let $u(h)$ be as in (2.10), and let $\epsilon > 0$ be such that

$$\frac{1}{(f_\infty - \epsilon)u(h) \int_{t_2-h}^{t_2+h} G(t_2,s)a(s)ds} \le \lambda \le \frac{1}{(f_0+\epsilon) \int_{t_1}^{t_3} G(t_2,s)a(s)ds.} \tag{2.12}$$

Since $x(t)$ is a solution of (1.1), (1.2) if and only if

$$x(t) = \lambda \int_{t_1}^{t_3} G(t,s)a(s)f(x(s))ds, \qquad t \in [t_1, t_3],$$

we define the operator $\mathcal{T}: \mathcal{P} \to \mathcal{B}$ by

$$(\mathcal{T}x)(t) := \lambda \int_{t_1}^{t_3} G(t,s)a(s)f(x(s))ds, \qquad x \in \mathcal{P}. \tag{2.13}$$

We seek a fixed point of $\mathcal{T}$ in $\mathcal{P}$ by establishing the hypotheses of Theorem 2.1. First, if $x \in \mathcal{P}$ then by (2.6) we have

$$\|\mathcal{T}x\| = \lambda \int_{t_1}^{t_3} G(t_2,s)a(s)f(x(s))ds$$

and

$$\begin{aligned}(\mathcal{T}x)(t) &= \lambda \int_{t_1}^{t_3} G(t,s)a(s)f(x(s))ds \\ &\ge \lambda g(t) \int_{t_1}^{t_3} G(t_2,s)a(s)f(x(s))ds \\ &\le g(t)\|\mathcal{T}x\|.\end{aligned}$$

Therefore $\mathcal{T}: \mathcal{P} \to \mathcal{P}$. Moreover, $\mathcal{T}$ is completely continuous by a typical application of the Arzela–Ascoli Theorem.

Now consider f_0. There exists an $H_1 > 0$ such that $f(x) \le (f_0+\epsilon)x$ for $0 < x \le H_1$ by the the definition of f_0. Pick $x \in \mathcal{P}$ with $\|x\| = H_1$. Using (2.6), we have

$$\begin{aligned}(\mathcal{T}x)(t) &= \lambda \int_{t_1}^{t_3} G(t,s)a(s)f(x(s))ds \\ &\le \lambda(f_0+\epsilon)\|x\| \int_{t_1}^{t_3} G(t_2,s)a(s)ds \\ &\le \|x\|\end{aligned}$$

from the right side of (2.12). As a result, $\|\mathcal{T}x\| \le \|x\|$. Thus, take

$$\Omega_1 := \{x \in \mathcal{B} : \|x\| < H_1\}$$

so that $\|\mathcal{T}x\| \le \|x\|$ for $x \in \mathcal{P} \cap \partial\Omega_1$.

Next consider f_∞. Again by definition there exists an $H_2 > H_1$ such that $f(x) \ge (f_\infty - \epsilon)x$ for $x \ge H_2$. If $x \in \mathcal{P}$ with $\|x\| = H_2$, then for $t \in [t_2 - h, t_2 + h]$, where h is as in (2.9) and $u(h)$ as in (2.10), we have

$$x(t) \ge g(t)\|x\| = H_2 g(t) \ge H_2 g(t_2 + h) = H_2 u(h). \tag{2.14}$$

Define $\Omega_2 := \{x \in \mathcal{B} : \|x\| < H_2\}$. Using (2.14), we get

$$\begin{aligned}
\|\mathcal{T}x\| &= \lambda \int_{t_1}^{t_3} G(t_2, s)a(s)f(x(s))ds \\
&\ge \lambda \int_{t_2-h}^{t_2+h} G(t_2, s)a(s)f(x(s))ds \\
&\ge \lambda(f_\infty - \epsilon) \int_{t_2-h}^{t_2+h} G(t_2, s)a(s)x(s)ds \\
&\ge \lambda(f_\infty - \epsilon)H_2 u(h) \int_{t_2-h}^{t_2+h} G(t_2, s)a(s)ds \\
&\ge H_2 \\
&= \|x\|,
\end{aligned}$$

where the penultimate line follows from the left side of (2.12). Hence we have shown that

$$\|\mathcal{T}x\| = \|x\|, \qquad x \in \mathcal{P} \cap \partial\Omega_2.$$

An application of Theorem 2.1 validates the conclusion of the theorem. ∎

THEOREM 2.3. *Suppose* (A1) *and* (A2) *hold. Then for each* λ *satisfying*

$$\frac{1}{u(h)f_0 \int_{t_2-h}^{t_2+h} G(t_2, s)a(s)ds} < \lambda < \frac{1}{f_\infty \int_{t_1}^{t_3} G(t_2, s)a(s)ds} \tag{2.15}$$

there exists at least one solution of (1.1), (1.2) *in* $\mathcal{P}$.

Proof. Let λ be as in (2.15) and let $\eta > 0$ be such that

$$\frac{1}{(f_0 - \eta)u(h) \int_{t_2-h}^{t_2+h} G(t_2, s)a(s)ds} \le \lambda \le \frac{1}{(f_\infty + \eta) \int_{t_1}^{t_3} G(t_2, s)a(s)ds} \tag{2.16}$$

Let $\mathcal{T}$ be the completely continuous, cone-preserving operator defined in (2.13). We seek a fixed point of $\mathcal{T}$ in $\mathcal{P}$ by establishing the hypotheses of Theorem 2.1.

First, consider f_0. There exists an $H_1 > 0$ such that $f(x) \le (f_0 - \eta)x$ for $0 < x \le H_1$ by the definition of f_0. Pick $x \in \mathcal{P}$ with $\|x\| = H_1$. For $t \in [t_2 - h, t_2 + h]$, where h is as in (2.9) and $u(h)$ is as in (2.10), we have

$$x(t) \ge g(t)\|x\| = H_1 g(t) \ge H_1 g(t_2 + h) = H_1 u(h). \tag{2.17}$$

Using the left side of (2.16) and (2.17), we get

$$\begin{aligned}
\|\mathcal{T}x\| &= \lambda \int_{t_1}^{t_3} G(t_2, s)a(s)f(x(s))\,ds \\
&\ge \lambda \int_{t_2-h}^{t_2+h} G(t_2, s)a(s)f(x(s))\,ds \\
&\ge \lambda(f_0 - \eta)\int_{t_2-h}^{t_2+h} G(t_2, s)a(s)x(s)\,ds \\
&\ge \lambda(f_0 - \eta)H_1 u(h)\int_{t_2-h}^{t_2+h} G(t_2, s)a(s)\,ds \\
&\ge H_1 \\
&= \|x\|.
\end{aligned}$$

Therefore $\|\mathcal{T}x\| \ge \|x\|$. This prompts us to define

$$\Omega_1 := \{x \in \mathcal{B} : \|x\| < H_1\},$$

whereby our work above confirms

$$\|\mathcal{T}x\| \ge \|x\|, \qquad x \in \mathcal{P} \cap \partial\Omega_1.$$

Next consider f_∞. Again by definition there exists an $\overline{H}_2 > H_1$ such that $f(x) \le (f_\infty + \eta)x$ for $x \ge \overline{H}_2$. If f is bounded, there exists $M > 0$ with $f(x) \le M$ for all $x \in (0, \infty)$. Let

$$H_2 := \max\left\{2\overline{H}_2, \lambda M \int_{t_1}^{t_3} G(t_2, s)a(s)\,ds\right\}.$$

If $x \in \mathcal{P}$ with $\|x\| = H_2$, then we have

$$\begin{aligned}
\|\mathcal{T}x\| &= \lambda \int_{t_1}^{t_3} G(t_2, s)a(s)f(x(s))\,ds \\
&\le \lambda M \int_{t_1}^{t_3} G(t_2, s)a(s)\,ds \\
&\le H_2 \\
&= \|x\|.
\end{aligned}$$

As a result, $\|\mathcal{T}x\| \le \|x\|$. Thus, take

$$\Omega_2 := \{x \in \mathcal{B} : \|x\| < H_2\}$$

such that $\|\mathcal{T}x\| \le \|x\|$ for $x \in \mathcal{P} \cap \partial\Omega_2$. If f is unbounded, take $H_2 := \max\{2H_1, \overline{H}_2\}$ such that $f(x) \le f(H_2)$ for $0 < x \le H_2$. If $x \in \mathcal{P}$ with $\|x\| = H_2$, then we have

$$\begin{aligned}
\|\mathcal{T}x\| &= \lambda \int_{t_1}^{t_3} G(t_2, s)a(s)f(x(s))\,ds \\
&\le \lambda \int_{t_1}^{t_3} G(t_2, s)a(s)f(H_2)\,ds \\
&\le \lambda(f_\infty + \eta)H_2 \int_{t_1}^{t_3} G(t_2, s)a(s)\,ds \\
&\le H_2 \\
&= \|x\|,
\end{aligned}$$

where the penultimate line follows from the left side of (2.16). Hence we have shown that

$$\|\mathcal{T}x\| \le \|x\|, \qquad x \in \mathcal{P} \cap \partial\Omega_2,$$

if we take

$$\Omega_2 := \{x \in \mathcal{B} : \|x\| < H_2\}.$$

Once again an application of Theorem 2.1 validates the conclusion of the theorem. ∎

3. EXISTENCE OF AT LEAST ONE POSITIVE SOLUTION

We are concerned with proving the existence of positive solutions of the third-order nonlinear right focal boundary value problem

$$x'''(t) = f(t, x(t)) \qquad \text{for all } t \in [t_1, t_3]$$

with boundary conditions

$$x(t_1) = x'(t_2) = x''(t_3) = 0$$

as in (1.2), where $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, and f is nonnegative for $x \ge 0$. The solutions of (1.3), (1.2) are the fixed points of the operator $\mathcal{A}$ defined by

$$\mathcal{A}x(t) = \int_{t_1}^{t_3} G(t, s)f(s, x(s))\,ds,$$

where $G(t,s)$ is the Green's function (2.3) for the homogeneous problem $x'''(t)=0$ satisfying the same boundary conditions (1.2). Many of the proofs use techniques employed in [22].

In the following discussion we will need the constants

$$k^{-1}:=\int_{t_1}^{t_3} G(t_2,s)ds=\tfrac{1}{6}(t_2-t_1)^2(3t_3-2t_2-t_1) \tag{3.1}$$

and

$$\begin{aligned} m^{-1}&:=u(h)\int_{t_2-h}^{t_2+h} G(t_2,s)ds \\ &=\left(1-\frac{h}{t_3-t_2}\right)\left[\frac{1}{6}(t_2-t_1)^3-\frac{1}{6}(t_2-t_1-h)^3+\frac{h}{2}(t_2-t_1)^2\right]. \end{aligned} \tag{3.2}$$

Then the growth restrictions on f which will yield the existence of positive and multiple solutions are as follows:

(C1) There exists a $p>0$ such that $f(t,x)\le kp$ for $t\in[t_1,t_3]$ and $0\le x\le p$.

(C2) There exists a $q>0$ such that $f(t,x)\ge mx$ for $t\in[t_2-h,t_2+h]$ and $qu(h)\le x\le q$, for $f\in(0,t_3-t_2)$.

THEOREM 3.1. *Suppose there exist positive numbers $p\ne q$ such that condition* (C1) *is satisfied with respect to p and condition* (C2) *is satisfied with respect to q. Then* (1.3), (1.2) *have a positive solution x such that $\|x\|$ lies between p and q.*

Proof. Without loss of generality, we may assume $0<p<q$. Define open sets

$$\Omega_p=\{x\in\mathscr{B}:\|x\|<p\},$$

and

$$\Omega_q=\{x\in\mathscr{B}:\|x\|<q\}.$$

Then $0\in\Omega_p\subset\Omega_q$. For $x\in\mathscr{P}\cap\partial\Omega_p$ such that $\|x\|=p$, we have

$$\begin{aligned} \|\mathscr{A}x\|&=\int_{t_1}^{t_3} G(t_2,s)f(s,x(s))ds \\ &\le kp\int_{t_1}^{t_3} G(t_2,s)ds \\ &=p \\ &=\|x\|, \end{aligned}$$

using (C1) and (3.1). Thus, $\|\mathscr{A}x\|\le\|x\|$ for $x\in\mathscr{P}\cap\partial\Omega_p$.

Similarly, let $x \in \mathscr{P} \cap \partial\Omega_q$, so that $\|x\| = q$. Then

$$\min_{t \in [t_2-h, t_2+h]} x(t) \geq \|x\| u(h),$$

since $g(t_2+h) \leq g(t_2-h)$ for all $h \in (0, t_3-t_2)$, g is as in (2.7), and $u(h)$ is as in (2.10). As a result, $qu(h) \leq x(s) \leq q$ for $s \in [t_2-h, t_2+h]$, and we have for $t \in [t_2-h, t_2+h]$

$$\begin{aligned} \|\mathscr{A}x\| &= \int_{t_1}^{t_3} G(t_2, s) f(s, x(s)) ds \\ &\geq \int_{t_2-h}^{t_2+h} G(t_2, s) f(s, x(s)) ds \\ &\geq m \int_{t_2-h}^{t_2+h} G(t_2, s) x(s) ds \\ &= q \\ &= \|x\| \end{aligned}$$

by (C2) and (3.2). Consequently, $\|\mathscr{A}x\| \geq \|x\|$ for $x \in \mathscr{P} \partial\Omega_q$. By Theorem 2.1, $\mathscr{A}$ has a fixed point $x \in \mathscr{P} \cap (\overline{\Omega} \backslash \Omega_p)$, which is a positive solution of (1.3), (1.2) such that $p \leq \|x\| \leq q$. ∎

Define

$$f_0(t) := \lim_{x \to 0^+} \frac{f(t,x)}{x}, \qquad f_\infty(t) := \lim_{x \to \infty} \frac{f(t,x)}{x}, \tag{3.3}$$

COROLLARY 3.1. *The boundary value problem* (1.3), (1.2) *has a positive solution provided either*

(C3) $f_0(t) < k$ *for* $t \in [t_1, t_3]$ *and* $f_\infty(t) > \frac{m}{u(h)}$ *for* $t \in [t_2-h, t_2+h]$, *or*

(C4) $f_0(t) > \frac{m}{u(h)}$ *for* $t \in [t_2-h, t_2+h]$ *and* $f_\infty(t) < k$ *for* $t \in [t_1, t_3]$, *where* $u(h)$ *is as in* (2.10), k *is as in* (3.1), m *is as in* (3.2), *and* f_0 *and* f_∞ *are as in* (3.3).

Proof. First assume (C3) holds. Then, there exist sufficiently small $p > 0$ and sufficiently large $q > 0$ such that

$$\frac{f(t,x)}{x} \leq k, \qquad t \in [t_1, t_3], \quad 0 < x \leq p,$$

and

$$\frac{f(t,x)}{x} \geq \frac{m}{u(h)}, \qquad t \in [t_2-h, t_2+h], \quad x \geq qu(h).$$

Then

$$f(t,x) \leq kx \leq kp, \qquad t \in [t_1, t_3], \quad 0 \leq x \leq p,$$

and

$$f(t,x) \geq \frac{m}{u(h)} x \geq mq, \qquad t \in [t_2 - h, t_2 + h], \quad qu(h) \leq x \leq q.$$

In particular, both (C1) and (C2) hold, so that by Theory 3.1, (1.3), (1.2) has a positive solution.

Next assume (C4) holds. Then there exist $0 < p < q$ so that

$$\frac{f(t,x)}{x} \geq \frac{m}{u(h)}, \qquad t \in [t_2 - h, t_2 + h], \ 0 < x \leq p, \tag{3.4}$$

$$\frac{f(t,x)}{x} \leq k, \qquad t \in [t_1, t_3], \ x \geq q. \tag{3.5}$$

From (3.4) we have

$$f(t,x) \geq \frac{m}{u(h)} x \geq mp, \qquad t \in [t_2 - h, t_2 + h], \ pu(h) \leq x \leq p,$$

satisfying (C2) with respect to p.

Now consider (3.5); we wish to show that (C1) is satisfied. To that end, consider the two cases: (1) $f(t,x)$ is bounded, or (2) $f(t,x)$ is unbounded.

Case 1. Suppose there exists $N > 0$ such that $f(t,x) \leq N$, for $t \in [t_1, t_3]$ and $0 \leq x < \infty$. By (3.5), there is an $r \geq \max\{q, \frac{N}{k}\}$ such that $f(t,x) \leq N \leq kr$ for $t \in [t_1, t_3]$ and $0 \leq x \leq r$. Thus (C1) is satisfied with respect to r.

Case 2. If f is unbounded, there exist $t_0 \in [t_1, t_3]$ and $\bar{r} \geq q$ such that $f(t,x) \leq f(t_0, \bar{r})$ for $t \in [t_1, t_3]$ and $0 \leq x \leq \bar{r}$. Then, $f(t,x) \leq f(t_0, \bar{r}) \leq k\bar{r}$ for $t \in [t_1, t_3]$ and $0 \leq x \leq \bar{r}$, and (C1) is satisfied with respect to $\bar{r}$.

Thus in both cases condition (C1) is satisfied, and Theorem 3.1 yields the conclusion. ∎

4. EXISTENCE OF AT LEAST TWO (AND ARBITRARILY MANY) POSITIVE SOLUTIONS

In this section, we show that any number of positive solutions of (1.3), (1.2) can be obtained when appropriate combinations of assumptions like (C1), (C2), (C3), and (C4) are imposed on f. We begin the pattern by establishing the existence of at least two positive solutions.

THEOREM 4.1. *The boundary value problem* (1.3), (1.2) *has at least two positive solutions*, x_1 *and* x_2, *if* (C1) *is satisfied for some* $p > 0$ *and, in addi-*

tion, both

$$f_0(t) > \frac{m}{u(h)}, \quad t \in [t_2 - h, t_2 + h], \quad \textit{and}$$

$$f_\infty(t) > \frac{m}{u(h)}, \quad t \in [t_2 - h, t_2 + h]. \tag{4.1}$$

Moreover, $0 < \|x_1\| < p < \|x_2\|$.

Proof. Somewhat along the lines of the proof of Corollary 3.1, there exist $0 < p_1 < p < p_2$ for which

$$f(t,x) \geq mp_1, \qquad t \in [t_2 - h, t_2 + h], \quad p_1 u(h) \leq x \leq p_1,$$

and

$$f(t,x) \geq mp_2, \qquad t \in [t_2 - h, t_2 + h], \quad p_2 u(h) \leq x \leq p_2.$$

By Theorem 3.1, there exist solutions x_1 and x_2 of (1.3), (1.2) satisfying $0 < p_1 < \|x_1\| < p < \|x_2\| < p_2$. ∎

In a completely analogous manner, the next result is also obtained.

THEOREM 4.2. *The boundary value problem* (1.3), (1.2) *has at least two positive solutions,* x_1 *and* x_2, *if* (C2) *is satisfied for some* $q > 0$ *and, in addition, both*

$$f_0(t) < k, \quad t \in [t_1, t_3], \quad \textit{and} \quad f_\infty(t) < k, \quad t \in [t_1, t_3]. \tag{4.2}$$

Moreover, $0 < \|x_1\| < q < \|x_2\|$.

To set the pattern for the manner in which an arbitrary number of positive solutions are obtained, we state an existence result for at least three positive solutions.

THEOREM 4.3. *Suppose condition* (C3) (*or respectively, condition* (C4) *is satisfied, and suppose there exist* $0 < p_1 < p_2$ *such that* (C1) *holds with respect to* $p = p_2$ (*respectively,* $p = p_1$), *and* (C2) *holds with respect to* $q = p_1$ (*respectively,* $q = p_2$). *Then, the boundary value problem* (1.3), (1.2) *has at least three positive solutions,* x_1, x_2, *and* x_3, *satisfying* $0 < \|x_1\| < p_1 < \|x_2\| < p_2 < \|x_3\|$.

We now state sufficient conditions under which there are n positive solutions of (1.3), (1.2) for any $n \in \mathbb{N}$. We state these results in terms of whether n is odd or even.

THEOREM 4.4. *Let* $n = 2j+1$, *where* $j \in \mathbb{N}$, *be given. Suppose condition* (C3) (*or, respectively, condition* (C4)) *is satisfied, and suppose there exist* $0 < p_1 < \cdots < p_{n-1}$ *such that* (C2) (*respectively,* (C1)) *holds with respect to* p_{2i-1}, $1 \leq i \leq j$, *and* (C1) (*respectively,* (C2)) *holds with respect to* p_{2i}, $1 \leq i \leq j$. *Then, the boundary value problem* (1.3), (1.2) *has at least n positive solutions* $x_1, x_2, \ldots, x_n$ *satisfying* $0 < \|x_1\| < p_1 < \|x_2\| < p_2 < \cdots < \|x_{n-1}\| < p_{n-1} < \|x_n\|$.

THEOREM 4.5. *Let $n=2j$, where $j\in\mathbb{N}$, be given. Suppose* (4.1) (*or, respectively*, (4.2)) *is satisfied, and suppose there exist* $0<p_1<\cdots<p_{n-1}$ *such that* (C1) (*respectively*, (C2)) *holds with respect to* p_{2i-1}, $1\le i\le j$, *and* (C2) *holds* (*respectively*, (C1) *holds*) *with respect to* p_{2i}, $1\le i\le j-1$. *Then, the boundary value problem* (1.3), (1.2) *has at least n positive solutions*, $x_1, x_2,\ldots,x_n$ *satisfying* $0<\|x_1\|<p_1<\|x_2\|<p_2<\cdots<\|x_{n-1}\|<p_{n-1}<\|x_n\|$.

5. THE FIVE FUNCTIONALS FIXED-POINT THEOREM

In this section we will set up and state the Five Functionals Fixed-Point Theorem [7], a generalization of the Leggett–Williams theorem, which will be used to prove the existence of at least three solutions.

DEFINITION 5.1. A map α is said to be a *nonnegative continuous concave functional* on a cone $\mathcal{P}$ of a real Banach space E if

$$\alpha\colon \mathcal{P}\to[0,\infty)$$

is continuous and

$$\alpha(tx+(1-t)y)\ge t\alpha(x)+(1-t)\alpha(y)$$

for all $x,y\in\mathcal{P}$ and $t\in[0,1]$. Similarly we say the map β is a *nonnegative continuous convex functional* on a cone $\mathcal{P}$ of a real Banach space E if

$$\beta\colon \mathcal{P}\to[0,\infty)$$

is continuous and

$$\beta(tx+(1-t)y)\le t\beta(x)+(1-t)\beta(y)$$

for all $x,y\in\mathcal{P}$ and $t\in[0,1]$.

Let γ,β,θ be nonnegative continuous convex functionals on $\mathcal{P}$ and let α,ψ be nonnegative continuous concave functionals on $\mathcal{P}$. For nonnegative real numbers a,b,c,d, and h we define the following convex sets:

$$P(\gamma,c)=\{x\in\mathcal{P}:\gamma(x)<c\},$$

$$P(\gamma,\alpha,a,c)=\{x\in\mathcal{P}:a\le\alpha(x),\gamma(x)\le c\},$$

$$Q(\gamma,\beta,d,c)=\{x\in\mathcal{P}:\beta(x)\le d,\gamma(x)\le c\},$$

$$P(\gamma,\theta,\alpha,a,b,c)=\{x\in\mathcal{P}:a\le\alpha(x),\theta(x)\le b,\gamma(x)\le c\},$$

$$Q(\gamma,\beta,\psi,h,d,c)=\{x\in\mathcal{P}:h\le\psi(x),\beta(x)\le d,\gamma(x)\le c\}.$$

The following is the so-called Five Functionals Fixed-Point Theorem [7], a generalization of the Leggett–Williams Fixed-Point Theorem [27].

THEOREM 5.1. *Let $\mathcal{P}$ be a cone in a real Banach space E; let c, q be positive numbers; let α, ψ be nonnegative continuous concave functionals on $\mathcal{P}$; and let γ, β, θ be nonnegative continuous convex functionals on $\mathcal{P}$ with*

$$\alpha(x) \le \beta(x) \quad \text{and} \quad \|x\| \le q\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose

$$\mathcal{A}\colon \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$$

is completely continuous and there exist nonnegative numbers h, d, a, b with $0 < d < a$ such that

(i) $\{x \in P(\gamma, \theta, \alpha, a, b, c) : \alpha(x) > a\} \neq \varnothing$ *and* $\alpha(\mathcal{A}x) > a$ *for* $x \in P(\gamma, \theta, \alpha, a, b, c)$;

(ii) $\{x \in Q(\gamma, \beta, \psi, h, d, c) : \beta(x) < d\} \neq \varnothing$ *and* $\beta(\mathcal{A}x) < d$ *for* $x \in Q(\gamma, \beta, \psi, h, d, c)$;

(iii) $\alpha(\mathcal{A}x) > a$ *for* $x \in P(\gamma, \alpha, a, c)$ *with* $\theta(\mathcal{A}x) > b$;

(iv) $\beta(\mathcal{A}x) < d$ *for* $x \in Q(\gamma, \beta, d, c)$ *with* $\psi(\mathcal{A}x) < h$.

Then $\mathcal{A}$ has at least three fixed points, x_1, x_2, $x_3 \in \overline{P(\gamma, c)}$, such that

$$\beta(x_1) < d, \quad a < \alpha(x_2), \quad \text{and} \quad d < \beta(x_3) \quad \text{with } \alpha(x_3) < a.$$

LEMMA 5.1. *Let $h \in [0, t_3 - t_2]$. Then $G(t_2 - h, s) \le G(t_2 + h, s)$ for all $s \in [t_1, t_3]$.*

Proof. We use the various branches of the Green's function given in (2.3). For $s \in [t_1, t_2 - h]$, we know that $t_2 + h \ge t_2 - h \ge s$. As a result,

$$G(t_2 + h, s) - G(t_2 - h, s) \equiv 0.$$

If $s \in [t_2 - h, t_2]$ it follows that

$$\begin{aligned} &G(t_2 + h, s) - G(t_2 - h, s) \\ &= \tfrac{1}{2}(s - t_1)^2 - \tfrac{1}{2}(t_2 - h - t_1)(2s - t_2 + h - t_1) \\ &= \tfrac{1}{2}(s - t_1)^2 - \tfrac{1}{2}(t_2 - h - t_1)[(s - t_1) + (s - t_2 + h)] \\ &= \tfrac{1}{2}(s - t_2 + h)^2 \\ &\ge 0 \end{aligned}$$

for these s. Next let $s \in [t_2, t_2 + h]$. Then

$$G(t_2 + h, s) - G(t_2 - h, s) = \tfrac{1}{2}(t_2 + h - s)^2 \ge 0.$$

Finally, for $s \in [t_2 + h, t_3]$,

$$G(t_2 + h, s) - G(t_2 - h, s) \equiv 0.$$

∎

Define the Banach space E by

$$E=\{x|x\in C[t_1,t_3]\}$$

with the sup norm, and define the cone $\mathscr{P}\subseteq E$ by

$$\mathscr{P}=\left\{x\in E\;\middle|\begin{array}{l}x \text{ is nondecreasing on } [t_1,t_2]\\ x \text{ is nonincreasing on } [t_2,t_3];\\ x \text{ is nonnegative valued on } [t_1,t_3];\\ x(t_2+h)\geq x(t_2-h) \quad \text{and}\\ \quad x(t_2+h)\geq m_2\|x\|\\ \text{for all } h\in[0,t_3-t_2].\end{array}\right\}, \tag{5.1}$$

where m_2 is given in (5.3). For real numbers h,k_1,k_2, with

$$0\leq h\leq t_3-t_2$$

and

$$0<k_1\leq k_2<t_3-t_2,$$

define the concave functionals on the cone $\mathscr{P}$,

$$\alpha(x):=\min_{t\in[t_2-k_2,t_2-k_1]\cup[t_2+k_1,t_2+k_2]}x(t)=x(t_2-k_2)$$

$$\psi(x):=\min_{t\in[t_2-h,t_2+h]}x(t)=x(t_2-h),$$

and the convex functionals on the cone $\mathscr{P}$,

$$\beta(x):=\max_{t\in[t_2-h,t_2+h]}x(t)=x(t_2),$$

$$\gamma(x):=\max_{t\in[t_1,t_2-h]\cup[t_2+h,t_3]}x(t)=x(t_2+h),$$

$$\theta(x):=\max_{t\in[t_2-k_2,t_2-k_1]\cup[t_2+k_1,t_2+k_2]}x(t)=x(t_2+k_1).$$

We will make use of various properties and constants associated with the Green's function, which include the easily calculated values

$$C_1:=\int_{t_1}^{t_3}G(t_2+h,s)ds=\tfrac{1}{6}(t_2-t_1)^3+\tfrac{1}{6}h^3+\tfrac{1}{2}(t_2-t_1+h)(t_2-t_1-h)(t_3-t_2),$$

$$C_2:=\int_{t_1}^{t_2-h}G(t_2,s)ds=\tfrac{1}{6}(t_2-h-t_1)^3,$$

$$C_3:=\int_{t_2+h}^{t_3}G(t_2,s)ds=\tfrac{1}{2}(t_2-t_1)^2(t_3-t_2-h),$$

$$C_4:=\int_{t_2-h}^{t_2+h}G(t_2,s)ds=\tfrac{1}{2}h(t_2-t_1)^2+\tfrac{1}{6}(t_2-t_1)^3-\tfrac{1}{6}(t_2-h-t_1)^3,$$

$$C_5:=\left(\int_{t_2-k_2}^{t_2-k_1}+\int_{t_2+k_1}^{t_2+k_2}\right)G(t_2-k_2,s)ds$$

$$=\tfrac{1}{2}(t_2-k_2-t_1)[(t_2-k_1)^2-(t_2-k_2)^2+2(k_2-t_1)]$$

and the constants

$$M := \max_{t_1 \le s \le t_3} \frac{G(t_2, s)}{G(t_2 - h, s)} = \frac{(t_2 - t_1)^2}{(t_2 - h - t_1)(t_2 + h - t_1)}$$

$$m_1 := \min_{t_1 \le s \le t_3} \frac{G(t_2 - k_2, s)}{G(t_2 + k_1, s)} = \frac{(t_2 - k_2 - t_1)(t_2 + k_2 - t_1)}{(t_2 - t_1)^2} \tag{5.2}$$

$$m_2 := \min_{t_1 \le s \le t_3} \frac{G(t_2 + h, s)}{G(t_2, s)} = \frac{(t_2 + h - t_1)(t_2 - h - t_1)}{(t_2 - t_1)^2} \tag{5.3}$$

for all $h \in [0, t_3 - t_2]$ and $0 < k_1 \le k_2 < t_3 - t_2$.

6. EXISTENCE OF AT LEAST THREE POSITIVE SOLUTIONS

In this section we state and prove a theorem on the existence of three positive solutions to the boundary value problem (1.3), (1.2) using Theorem 5.1. By a positive solution of the boundary value problem (1.3), (1.2) we mean a solution which is in the cone defined in the proof of the following theorem.

THEOREM 6.1. *Suppose a, b, and c are real numbers with $0 < a < b < b/m_1 \le c$ such that nonnegative continuous f satisfies the following conditions*:

(i) $f(x) < (a - c(C_2 + C_3)/C_1)/C_4$ *for all* $x \in [\frac{a}{M}, a]$,

(ii) $f(x) > b/C_5$ *for* $x \in [b, b/m_1]$,

(iii) $f(x) \le c/C_1$ *for* $x \in [0, c/m_2]$.

Then, the third-order boundary value problem (1.3), (1.2) *has three positive solutions*, $x_1, x_2, x_3 \in P(\gamma, c)$.

Proof. Define the completely continuous operator $\mathcal{A}$ by

$$\mathcal{A}x(t) = \int_{t_1}^{t_3} G(t, s) f(x(s)) ds.$$

We seek fixed points of $\mathcal{A}$ which satisfy the conclusion of the theorem. We note first, if $x \in \mathcal{P}$ for $\mathcal{P}$ given by (5.1), then from properties of $G(t, s)$,

$$\mathcal{A}x(t) \ge 0,$$

$$\frac{d}{dt}\mathcal{A}x(t) \ge 0 \qquad \text{for } t \in [t_1, t_2],$$

$$\frac{d}{dt}\mathcal{A}x(t) \le 0 \qquad \text{for } t \in [t_2, t_3];$$

and, for $h \in [0, t_3 - t_2]$,

$$\mathscr{A}x(t_2 - h) \le \mathscr{A}x(t_2 + h),$$
$$\mathscr{A}x(t_2 + h) \ge m_2 \mathscr{A}x(t_2) = m_2 \|\mathscr{A}x\|.$$

Consequently, $\mathscr{A}x \in \mathscr{P}$, that is, $\mathscr{A}: \mathscr{P} \to \mathscr{P}$.

Note that for all $x \in \mathscr{P}$,

$$\alpha(x) = x(t_2 - k_2) \le x(t_2) = \beta(x)$$

and

$$\|x\| \le \frac{1}{m_2} x(t_2 + h) = \frac{1}{m_2} \gamma(x).$$

If $x \in \overline{P(\gamma, c)}$, then $\|x\| \le (1/m_2)\gamma(x) \le c/m_2$, and by assumption (iii) we have

$$\begin{aligned}
\gamma(\mathscr{A}x) &= \max_{t \in [t_1, t_2 - h] \cup [t_2 + h, t_3]} \int_{t_1}^{t_3} G(t, s) f(x(s)) ds \\
&= \int_{t_1}^{t_3} G(t_2 + h, s) f(x(s)) ds \\
&\le \left(\frac{c}{C_1} \right) \int_{t_1}^{t_3} G(t_2 + h, s) ds \\
&= c.
\end{aligned}$$

Therefore,

$$\mathscr{A}: \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}.$$

Let

$$x_P(t) \equiv \frac{b}{2} \left(1 + \frac{1}{m_1} \right) \quad \text{and} \quad x_Q(t) \equiv \frac{a}{2} \left(1 + \frac{1}{M} \right)$$

for all $t \in [t_1, t_3]$. It is immediate that

$$x_P \in \left\{ x \in P\left(\gamma, \theta, \alpha, b, \frac{b}{m_1}, c \right) : \alpha(x) > b \right\}$$

and

$$x_Q \in \left\{ x \in Q\left(\gamma, \beta, \psi, \frac{a}{M}, a, c \right) : \beta(x) < a \right\}.$$

Hence both sets are nonempty.

In the following claims, we verify the remaining conditions of the Five Functionals Fixed-Point Theorem.

Claim 1. If $x \in P(\gamma, \theta, \alpha, b, b/m_1, c)$ then $\alpha(\mathcal{A}x) > b$.

$$\begin{aligned}
\alpha(\mathcal{A}x) &= \min_{t\in[t_2-k_2,\,t_2-k_1]\cup[t_2+k_1,\,t_2+k_2]} \int_{t_1}^{t_3} G(t,s)f(x(s))ds \\
&\geq \left(\int_{t_2-k_2}^{t_2-k_1} + \int_{t_2+k_1}^{t_2+k_2}\right) G(t_2-k_2,s)f(x(s))ds \\
&> \left(\frac{b}{C_5}\right)\left(\int_{t_2-k_2}^{t_2-k_1} + \int_{t_2+k_1}^{t_2+k_2}\right) G(t_2-k_2,s)ds \\
&= \left(\frac{b}{C_5}\right) C_5 \\
&= b.
\end{aligned}$$

Claim 2. If $x \in Q(\gamma, \beta, \psi, \frac{a}{M}, a, c)$ then $\beta(\mathcal{A}x) < a$.

$$\begin{aligned}
\beta(\mathcal{A}x) &= \min_{t\in[t_2-h,\,t_2+h]} \int_{t_1}^{t_3} G(t,s)f(x(s))ds \\
&= \left(\int_{t_1}^{t_2-h} + \int_{t_2-h}^{t_2+h} + \int_{t_2+h}^{t_3}\right) G(t_2,s)f(x(s))ds \\
&< \left(\frac{c}{C_1}\right)\left(\int_{t_1}^{t_2-h} + \int_{t_2+h}^{t_3}\right) G(t_2,s)ds \\
&\quad + \left(\frac{a - c(C_2+C_3)/C_1}{C_4}\right) \int_{t_2-h}^{t_2+h} G(t_2,s)ds \\
&= \left(\frac{c}{C_1}\right)(C_2+C_3) + \left(\frac{a - c(C_2+C_3)/C_1}{C_4}\right) C_4 \\
&= a.
\end{aligned}$$

Claim 3. If $x \in P(\gamma, \alpha, b, c)$ with $\theta(\mathcal{A}x) > b/m_1$ then $\alpha(\mathcal{A}x) > b$.

$$\begin{aligned}
\alpha(\mathcal{A}x) &= \min_{t\in[t_2-k_2,\,t_2-k_1]\cup[t_2+k_1,\,t_2+k_2]} \int_{t_1}^{t_3} G(t,s)f(x(s))ds \\
&= \int_{t_1}^{t_3} G(t_2-k_2,s)f(x(s))ds \\
&= \int_{t_1}^{t_3} \left(\frac{G(t_2-k_2,s)}{G(t_2+k_1,s)}\right) G(t_2+k_1,s)f(x(s))ds \\
&\geq m_1 \int_{t_1}^{t_3} G(t_2+k_1,s)f(x(s))ds \\
&= m_1\theta(\mathcal{A}x) \\
&> b.
\end{aligned}$$

Claim 4. If $x \in Q(\gamma, \beta, a, c)$ with $\psi(\mathcal{A}x) < \frac{a}{M}$ then $\beta(\mathcal{A}x) < a$.

$$\begin{aligned}
\beta(\mathcal{A}x) &= \max_{t \in [t_2-h, t_2+h]} \int_{t_1}^{t_3} G(t,s) f(x(s)) ds \\
&= \int_{t_1}^{t_3} G(t_2, s) f(x(s)) ds \\
&= \int_{t_1}^{t_3} \frac{G(t_2, s)}{G(t_2-h, s)} G(t_2-h, s) f(x(s)) ds \\
&\leq M \int_{t_1}^{t_3} G(t_2-h, s) f(x(s)) ds \\
&= M\psi(\mathcal{A}x) \\
&< a.
\end{aligned}$$

Therefore the hypothesis of the Five Functionals Fixed-Point Theorem are satisfied, and there exist three positive solutions, $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$, for the third-order focal boundary value problem (1.3), (1.2) such that

$$\alpha(x_1) > b,$$
$$\beta(x_2) < a,$$

and

$$\alpha(x_3) < b \qquad \text{with} \qquad \beta(x_3) > a.$$

∎

7. FUTURE DIRECTIONS

We outlined in Section 1 that the study of multiple fixed-point theorems is proceeding very rapidly. No doubt other multiple fixed-point theorems will be proved which allow four or five or more distinct fixed points, possibly (as in those presented here) providing some insight into their location. Even now, there are some results in the literature hinting at this, although the techniques are very different from paper to paper. For example, Habets and Gaudenzi [21] as well as Korman and Ouyang [25] used bifurcation methods to prove that certain second-order problems have *exactly* four solutions. Stanek [31] uses a very general approach (making use of Bihari's Lemma, the topological homotopy method, degree theory, and Borsuk's Theorem) to provide sufficient conditions for the existence of at least four solutions to problems of the form $x''(t) = (Fx)(t)$. Lu [29] studied an equation modeling laminar flow in a uniformly porous pipe. For sufficiently large radii, he

obtains at least four solutions. Finally, Castro et al. [9] even show that a certain semilinear elliptic boundary value problem has *five* solutions when the range of the derivative of the nonlinearity includes at least the first two eigenvalues.

There are several interesting problems which could be attacked using the methods of Sections 2–6. Closest to this paper would be the nth-order differential equation

$$y^{(n)}(t) = f(y(t)), \qquad a \le t \le b,$$

satisfying either k-point (multipoint) right focal boundary conditions or k-point conjugate boundary conditions. Special cases of certain of these multiplicity results can already be found in the literature; the two-point problems are most common, probably because they are the most accessible. Discrete analogs could of course be pursued, as well as the so-called dynamic equations on time scales of this type, which we only allude to here.

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