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Existence of solutions for a class of third-order nonlinear boundary value problems ☆

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Abstract

In this paper, we are concerned with the following third-order ordinary differential equation:

$$x'''(t) + f\bigl(t, x(t), x'(t), x''(t)\bigr) = 0, \quad 0 < t < 1,$$

with the nonlinear boundary conditions

$$x(0) = 0, \quad g\bigl(x'(0), x''(0)\bigr) = A, \quad h\bigl(x'(1), x''(1)\bigr) = B,$$

where $A, B \in R$, $f : [0,1] \times R^3 \to R$ is continuous, $g, h : R^2 \to R$ are continuous. The existence result is given by using a priori estimate, Nagumo condition, upper and lower solutions and Leray–Schauder degree, and we give an example to demonstrate our result.

Keywords: Nonlinear boundary value problem; Nagumo condition; Upper and lower solutions; Leray–Schauder degree

1. Introduction

This paper deals with the existence of solutions for the nonlinear boundary value problem

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$$x'''(t) + f\big(t, x(t), x'(t), x''(t)\big) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$x(0) = 0, \quad g\big(x'(0), x''(0)\big) = A, \quad h\big(x'(1), x''(1)\big) = B, \tag{1.2}$$

where $A, B \in R$, $f : [0,1] \times R^3 \to R$ is continuous, $g, h : R^2 \to R$ are continuous.

Third-order boundary value problems (BVPs) were discussed in many papers in recent years, for instance, see [2–7,9] and references therein. However, the boundary conditions in the above-mentioned references are all linear and race works are done for nonlinear boundary conditions. Recently, Rovderová [8] established existence results for the boundary value problem

$$y''' = f(t, y, y', y''), \quad 0 < t < 1, \tag{1.3}$$

$$y(0) = A, \quad y''(0) = \sigma\big(y'(0)\big), \quad y'(T) = \tau\big(y(\tau)\big), \tag{1.4}$$

where f, $\partial f/\partial y$, $\partial f/\partial y'$, and $\partial f/\partial y''$ are continuous functions on $[0,1] \times R^3$, $\sigma(v) \in C^1(R, R)$, $\tau(v) \in C(R, R)$.

Motivated by the work of the above papers, the purpose of this article is to study the existence of solutions for boundary value problem (1.1)–(1.2) under the condition that $f(t, x, y, z)$ is continuous on $[0,1] \times R^3$ and increasing in x, which is weaker than the restriction imposed on f by Rovderová [8]. We also extend the result of Grossinho and Minhós [4], who studied third-order boundary value problems with linear separated boundary conditions. The tools we mainly used are the method of upper and lower solutions and Leray–Schauder degree theory [1].

2. Preliminary

In this section, we present some definitions and a lemma that are important to our main result.

Definition 1. Functions $\alpha(t), \beta(t) \in C^3[0,1]$ are called lower and upper solutions of BVP (1.1)–(1.2), respectively, if

$$\alpha'''(t) + f\big(t, \alpha(t), \alpha'(t), \alpha''(t)\big) \geqslant 0, \tag{2.1}$$

$$\beta'''(t) + f\big(t, \beta(t), \beta'(t), \beta''(t)\big) \leqslant 0, \tag{2.2}$$

and

$$\alpha(0) = \beta(0) = 0, \tag{2.3}$$

$$g\big(\alpha'(0), \alpha''(0)\big) \leqslant A \leqslant g\big(\beta'(0), \beta''(0)\big), \tag{2.4}$$

$$h\big(\alpha'(1), \alpha''(1)\big) \leqslant B \leqslant h\big(\beta'(1), \beta''(1)\big). \tag{2.5}$$

Definition 2. Let D be a subset of $[0,1] \times R^3$, we say that $f(t, x, y, z)$ satisfies Nagumo condition in D, if f is continuous and given any $a > 0$, there exists a positive function $\Phi : [0, \infty) \to [a, +\infty)$ such that

$$\big|f(t, x, y, z)\big| \leqslant \Phi\big(|z|\big) \tag{2.6}$$

for arbitrary $(t, x, y, z) \in D$ and such that

$$\int_0^{+\infty} \frac{s}{\Phi(s)}\, ds = +\infty. \tag{2.7}$$

Lemma 1 (see [4]). *Let $f(t, x, y, z) : [0, 1] \times R^3 \to R$ be a continuous function that satisfies Nagumo condition in*

$$D = \{(t, x, y, z) \in [0, 1] \times R^3\colon\ \Gamma_1(t) \leqslant x \leqslant \Gamma_2(t),\ \gamma_1(t) \leqslant y \leqslant \gamma_2(t)\},$$

where $\Gamma_1, \Gamma_2, \gamma_1, \gamma_2 : [0, 1] \to R$ are continuous functions such that $\Gamma_1(t) \leqslant \Gamma_2(t)$ and $\gamma_1(t) \leqslant \gamma_2(t)$ for every $t \in [0, 1]$. Then there exists a constant $r > 0$ (depending only on γ_1, γ_2 and Φ) such that every solution $x(t)$ of Eq. (1.1), verifying

$$\Gamma_1(t) \leqslant x(t) \leqslant \Gamma_2(t), \quad \gamma_1(t) \leqslant x'(t) \leqslant \gamma_2(t), \quad t \in [0, 1],$$

then satisfies $\|x''\|_\infty \leqslant r$.

3. Existence result

Theorem 1. *Assume that*

(i) *There exist lower and upper solutions of BVP (1.1)–(1.2), $\alpha(t), \beta(t)$, respectively, such that*

$$\alpha'(t) \leqslant \beta'(t), \quad t \in [0, 1]; \tag{3.1}$$

(ii) *$f(t, x, y, z)$ is continuous on $[0, 1] \times R^3$ and increasing in x;*

(iii) *$f(t, x, y, z)$ satisfies Nagumo condition in*

$$D_* = \{(t, x, y, z) \in [0, 1] \times R^3\colon\ \alpha(t) \leqslant x \leqslant \beta(t),\ \alpha'(t) \leqslant y \leqslant \beta'(t)\};$$

(iv) *$g(y, z)$, $h(y, z)$ are continuous on R^2, $g(y, z)$ is decreasing in z and $h(y, z)$ is increasing in z.*

Then BVP (1.1)–(1.2) has at least one solution $x(t) \in C^3[0, 1]$ such that

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \quad \alpha'(t) \leqslant x'(t) \leqslant \beta'(t), \quad t \in [0, 1].$$

Proof. Let $v_1, v_2, v_3 \in R$, such that $v_1 \leqslant v_3$, define

$$\omega(v_1, v_2, v_3) = \begin{cases} v_3 & \text{if } v_2 > v_3, \\ v_2 & \text{if } v_1 \leqslant v_2 \leqslant v_3, \\ v_1 & \text{if } v_2 < v_1. \end{cases}$$

For $\lambda \in [0, 1]$, we consider the auxiliary equation

$$\begin{aligned} &x'''(t) + \lambda f\bigl(t, \omega(\alpha(t), x(t), \beta(t)), \omega(\alpha'(t), x'(t), \beta'(t)), x''(t)\bigr) \\ &\quad = (1 - \lambda)x'(t) + \lambda\bigl[x'(t) - \omega(\alpha'(t), x'(t), \beta'(t))\bigr]\Phi\bigl(|x''(t)|\bigr), \end{aligned} \tag{3.2}$$

where Φ is decided by Nagumo condition, with the boundary conditions

$$x(0) = 0, \tag{3.3}$$

$$x'(0) = \lambda\big[A - g\big(\omega\big(\alpha'(0), x'(0), \beta'(0)\big), x''(0)\big) + \omega\big(\alpha'(0), x'(0), \beta'(0)\big)\big], \tag{3.4}$$

$$x'(1) = \lambda\big[B - h\big(\omega\big(\alpha'(1), x'(1), \beta'(1)\big), x''(1)\big) + \omega\big(\alpha'(1), x'(1), \beta'(1)\big)\big]. \tag{3.5}$$

Then we can select $M_1 > 0$, such that for every $t \in [0, 1]$,

$$-M_1 < \alpha'(t) \leqslant \beta'(t) < M_1, \tag{3.6}$$

$$-f\big(t, \alpha(t), \alpha'(t), 0\big) - \big[M_1 + \alpha'(t)\big]\Phi(0) < 0, \tag{3.7}$$

$$-f\big(t, \beta(t), \beta'(t), 0\big) + \big[M_1 - \beta'(t)\big]\Phi(0) > 0, \tag{3.8}$$

$$\big|A - g\big(\beta'(0), 0\big) + \beta'(0)\big| < M_1, \qquad \big|A - g\big(\alpha'(0), 0\big) + \alpha'(0)\big| < M_1, \tag{3.9}$$

$$\big|B - h\big(\beta'(1), 0\big) + \beta'(1)\big| < M_1, \qquad \big|B - h\big(\alpha'(1), 0\big) + \alpha'(1)\big| < M_1. \tag{3.10}$$

In the following, we shall complete the proof by four steps.

Step 1. Every solution $x(t)$ of (3.2)–(3.5) satisfies

$$\big|x(t)\big| < M_1, \quad \big|x'(t)\big| < M_1, \quad t \in [0, 1],$$

and independently of λ.

We suppose that the estimate $|x'(t)| < M_1$ is not true. Then there exists $t \in [0, 1]$, such that $x'(t) \geqslant M_1$, or $x'(t) \leqslant -M_1$. Suppose that the first case holds. Define

$$\max_{t\in[0,1]} x'(t) := x'(t_0) \ (\geqslant M_1 > 0).$$

(1) If $t_0 \in (0, 1)$, then $x''(t_0) = 0$ and $x'''(t_0) \leqslant 0$. For $\lambda \in (0, 1]$, by condition (ii) and (3.8), we get the following contradiction:

$$\begin{aligned}
0 \geqslant x'''(t_0) &= -\lambda f\big(t_0, \omega\big(\alpha(t_0), x(t_0), \beta(t_0)\big), \omega\big(\alpha'(t_0), x'(t_0), \beta'(t_0)\big), x''(t_0)\big) \\
&\quad + (1-\lambda)x'(t_0) + \lambda\big[x'(t_0) - \omega\big(\alpha'(t_0), x'(t_0), \beta'(t_0)\big)\big]\Phi\big(\big|x''(t_0)\big|\big) \\
&= -\lambda f\big(t_0, \omega\big(\alpha(t_0), x(t_0), \beta(t_0)\big), \beta'(t_0), 0\big) \\
&\quad + (1-\lambda)x'(t_0) + \lambda\big[x'(t_0) - \beta'(t_0)\big]\Phi(0) \\
&\geqslant -\lambda f\big(t_0, \beta(t_0), \beta'(t_0), 0\big) + (1-\lambda)x'(t_0) + \lambda\big[M_1 - \beta'(t_0)\big]\Phi(0) \\
&\geqslant \lambda\big[-f\big(t_0, \beta(t_0), \beta'(t_0), 0\big) + \big(M_1 - \beta'(t_0)\big)\big]\Phi(0) > 0,
\end{aligned}$$

and for $\lambda = 0$, we have $0 \geqslant x'''(t_0) = x'(t_0) \geqslant M_1 > 0$. Thus $t_0 \notin (0, 1)$.

(2) If $t_0 = 0$, then

$$\max_{t\in[0,1]} x'(t) := x'(0) \ (\geqslant M_1 > 0) \quad \text{and} \quad x''(0^+) = x''(0) \leqslant 0.$$

Therefore, from condition (iv), (3.4) and (3.9), the following contradiction is obtained:

$$\begin{aligned}
M_1 \leqslant x'(0) &= \lambda\big[A - g\big(\omega\big(\alpha'(0), x'(0), \beta'(0)\big), x''(0)\big) + \omega\big(\alpha'(0), x'(0), \beta'(0)\big)\big] \\
&= \lambda\big[A - g\big(\beta'(0), x''(0)\big) + \beta'(0)\big] \\
&\leqslant \lambda\big[A - g\big(\beta'(0), 0\big) + \beta'(0)\big] < M_1,
\end{aligned}$$

thus $t_0 \neq 0$.

(3) If $t_0 = 1$, then we get $\max_{t\in[0,1]} x'(t) := x'(1)$ ($\geqslant M_1 > 0$), and $x''(1^-) = x''(1) \geqslant 0$, from condition (iv), (3.5) and (3.10), we obtain the following contradiction:

$$\begin{aligned} M_1 \leqslant x'(1) &= \lambda\big[B - h\big(\omega\big(\alpha'(1), x'(1), \beta'(1)\big), x''(1)\big) + \omega\big(\alpha'(1), x'(1), \beta'(1)\big)\big] \\ &= \lambda\big[B - h\big(\beta'(1), x''(1)\big) + \beta'(1)\big] \\ &\leqslant \lambda\big[B - h\big(\beta'(1), 0\big) + \beta'(1)\big] < M_1, \end{aligned}$$

thus $t_0 \neq 1$. So we show that $x'(t) < M_1$ for every $t \in [0,1]$. Similar to the above argument, we can prove that $x'(t) > -M_1$ for every $t \in [0,1]$. Therefore, $|x'(t)| < M_1$ for $t \in [0,1]$. Since $x(0) = 0$, the estimate $|x(t)| < M_1$ is easily shown by integration.

Step 2. There exists $M_2 > 0$, such that every solution $x(t)$ of (3.2)–(3.5) satisfies

$$\big|x''(t)\big| < M_2 \quad \text{for } t \in [0,1],$$

independently of $\lambda \in [0,1]$.

If $x(t)$ is a solution of BVP (3.2)–(3.5), then

$$\begin{aligned} &x'''(t) + \lambda f\big(t, \omega\big(\alpha(t), x(t), \beta(t)\big), \omega\big(\alpha'(t), x'(t), \beta'(t)\big), x''(t)\big) \\ &\quad - (1-\lambda)x'(t) - \lambda\big[x'(t) - \omega\big(\alpha'(t), x'(t), \beta'(t)\big)\big]\Phi\big(\big|x''(t)\big|\big) = 0. \end{aligned}$$

Let

$$D_{M_1} = \big\{(t,x,y,z) \in [0,1] \times R^3\colon -M_1 \leqslant x \leqslant M_1,\ -M_1 \leqslant y \leqslant M_1\big\}.$$

Define the function $F_\lambda : D_{M_1} \to R$ as follows:

$$\begin{aligned} F_\lambda(t,x,y,z) := {}&\lambda f\big(t, \omega\big(\alpha(t), x, \beta(t)\big), \omega\big(\alpha'(t), y, \beta'(t)\big), z\big) \\ &- (1-\lambda)y - \lambda\big[y - \omega\big(\alpha'(t), y, \beta'(t)\big)\big]\Phi\big(|z|\big). \end{aligned}$$

In the following, we show that F_λ satisfies Nagumo condition in D_{M_1}, independently of $\lambda \in [0,1]$. In fact, since f satisfies Nagumo condition in D_{M_1}, then

$$\begin{aligned} \big|F_\lambda(t,x,y,z)\big| &\leqslant \big|f\big(t, \omega\big(\alpha(t), x, \beta(t)\big), \omega\big(\alpha'(t), y, \beta'(t)\big), z\big)\big| \\ &\quad + |y| + \big|y - \omega\big(\alpha'(t), y, \beta'(t)\big)\big|\Phi\big(|z|\big) \\ &\leqslant \Phi\big(|z|\big) + M_1 + \big(|y| + \big|\omega\big(\alpha'(t), y, \beta'(t)\big)\big|\big)\Phi\big(|z|\big) \\ &\leqslant M_1 + (1 + 2M_1)\Phi\big(|z|\big) := \Phi^*(z). \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned} \int_0^{+\infty} \frac{s}{\Phi^*(s)}\,ds &= \int_0^{+\infty} \frac{s}{M_1 + (1+2M_1)\Phi(s)}\,ds \geqslant \int_0^{+\infty} \frac{s}{(1 + 2M_1 + M_1/a)\Phi(s)}\,ds \\ &= \frac{1}{(1 + 2M_1 + M_1/a)} \int_0^{+\infty} \frac{s}{\Phi(s)}\,ds = +\infty, \end{aligned}$$

thus, F_λ satisfies Nagumo condition in D_{M_1}, independently of $\lambda \in [0,1]$. Let

$$\Gamma_1(t) = \gamma_1(t) = -M_1, \quad \Gamma_2(t) = \gamma_2(t) = M_1, \quad t \in [0,1].$$

In view of Step 1 and applying Lemma 1, then there exists $M_2 > 0$, such that $|x''(t)| < M_2$ for $t \in [0,1]$. Since M_1 and Φ do not depend on λ, we get that the estimate $|x''(t)| < M_2$ is also independently of λ.

Step 3. For $\lambda = 1$, BVP (3.2)–(3.5) has at least one solution $x_1(t)$.

Define the operators

$$L : C^2[0,1] \cap \operatorname{dom} L \to C[0,1] \times R^3$$

by

$$Lx = \bigl(x''', x(0), x'(0), x'(1)\bigr),$$

and

$$N_\lambda : C^2[0,1] \to C[0,1] \times R^3$$

by

$$\begin{aligned} N_\lambda x = \Bigl(&-\lambda f\bigl(t, \omega\bigl(\alpha(t), x(t), \beta(t)\bigr), \omega\bigl(\alpha'(t), x'(t), \beta'(t)\bigr), x''(t)\bigr) \\ &+ (1-\lambda)x' + \lambda\bigl[x' - \omega\bigl(\alpha'(t), x'(t), \beta'(t)\bigr)\bigr]\Phi\bigl(|x''|\bigr), 0, A_\lambda, B_\lambda\Bigr) \end{aligned}$$

with

$$\begin{aligned} A_\lambda &:= \lambda\bigl[A - g\bigl(\omega\bigl(\alpha'(0), x'(0), \beta'(0)\bigr), x''(0)\bigr) + \omega\bigl(\alpha'(0), x'(0), \beta'(0)\bigr)\bigr], \\ B_\lambda &:= \lambda\bigl[B - h\bigl(\omega\bigl(\alpha'(1), x'(1), \beta'(1)\bigr), x''(1)\bigr) + \omega\bigl(\alpha'(1), x'(1), \beta'(1)\bigr)\bigr]. \end{aligned}$$

As L^{-1} is compact, we can define the completely continuous operator

$$T_\lambda : \bigl(C^2[0,1], R\bigr) \to \bigl(C^2[0,1], R\bigr)$$

by

$$T_\lambda(x) = L^{-1}N_\lambda(x).$$

Consider the set

$$\Omega = \bigl\{x \in C^2[0,1]:\ \|x\|_\infty < M_1,\ \|x'\|_\infty < M_1,\ \|x''\|_\infty < M_2\bigr\}.$$

By Steps 1 and 2, the degree $\deg(I - T_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0,1]$ and by homotopy invariance, we get

$$\deg(I - T_0, \Omega, 0) = \deg(I - T_1, \Omega, 0).$$

As the equation $x = T_0(x)$ has only the trivial solution, by degree theory,

$$\deg(T_0, \Omega, 0) = 1.$$

Hence, the equation $x = T_1(x)$ has at least one solution. That is, the problem

$$x'''(t) + f\big(t, \omega\big(\alpha(t), x(t), \beta(t)\big), \omega\big(\alpha'(t), x'(t), \beta'(t)\big), x''(t)\big)$$
$$= \big[x'(t) - \omega\big(\alpha'(t), x'(t), \beta'(t)\big)\big]\Phi\big(\big|x''(t)\big|\big), \tag{3.11}$$

with the boundary conditions

$$x(0) = 0, \tag{3.12}$$
$$x'(0) = A - g\big(\omega\big(\alpha'(0), x'(0), \beta'(0)\big), x''(0)\big) + \omega\big(\alpha'(0), x'(0), \beta'(0)\big), \tag{3.13}$$
$$x'(1) = B - h\big(\omega\big(\alpha'(1), x'(1), \beta'(1)\big), x''(1)\big) + \omega\big(\alpha'(1), x'(1), \beta'(1)\big), \tag{3.14}$$

has at least one solution $x_1(t)$ in Ω.

Step 4. The $x_1(t)$ is a solution of BVP (1.1)–(1.2).

In fact, the above solution $x_1(t)$ of BVP (3.11)–(3.14) will be a solution of the BVP (1.1)–(1.2), too, since it satisfies in [0, 1],

$$\alpha(t) \leqslant x_1(t) \leqslant \beta(t), \qquad \alpha'(t) \leqslant x_1'(t) \leqslant \beta'(t).$$

If the assertion is not true, then there exists $t \in [0, 1]$, such that $x_1'(t) > \beta'(t)$, we define

$$\max_{t\in[0,1]}\big[x_1'(t) - \beta'(t)\big] := x_1'(t_1) - \beta'(t_1) > 0.$$

If $t_1 \in (0, 1)$, then

$$x_1''(t_1) = \beta''(t_1) \quad \text{and} \quad x_1'''(t_1) \leqslant \beta'''(t_1).$$

By condition (ii), we get the following contradiction:

$$\begin{aligned}
0 &\geqslant x_1'''(t_1) - \beta'''(t_1) \\
&\geqslant -f\big(t_1, \omega\big(\alpha(t_1), x_1(t_1), \beta(t_1)\big), \omega\big(\alpha'(t_1), x_1'(t_1), \beta'(t_1)\big), x_1''(t_1)\big) \\
&\quad + \big[x_1'(t_1) - \omega\big(\alpha'(t_1), x_1'(t_1), \beta'(t_1)\big)\big]\Phi\big(\big|x_1''(t_1)\big|\big) \\
&\quad + f\big(t, \beta(t_1), \beta'(t_1), \beta''(t_1)\big) \\
&= -f\big(t_1, \omega\big(\alpha(t_1), x_1(t_1), \beta(t_1)\big), \beta'(t_1), \beta''(t_1)\big) \\
&\quad + \big[x'(t_1) - \beta'(t_1)\big]\Phi\big(\big|x_1''(t_1)\big|\big) + f\big(t, \beta(t_1), \beta'(t_1), \beta''(t_1)\big) \\
&> -f\big(t_1, \beta(t_1), \beta'(t_1), \beta''(t_1)\big) + f\big(t_1, \beta(t_1), \beta'(t_1), \beta''(t_1)\big) = 0,
\end{aligned}$$

thus $t_1 \notin (0, 1)$.

If $t_1 = 0$, then

$$\max_{t\in[0,1]}\big[x_1'(t) - \beta'(t)\big] := x_1'(0) - \beta'(0) > 0$$

and

$$x_1''(0^+) - \beta''(0^+) = x_1''(0) - \beta''(0) \leqslant 0,$$

therefore, from condition (iv), (3.13) and (2.4), the following contradiction is obtained:

$$\begin{aligned}\beta'(0) \leqslant x_1'(0) &= A - g\big(\omega\big(\alpha'(0), x_1'(0), \beta'(0)\big), x_1''(0)\big) + \omega\big(\alpha'(0), x_1'(0), \beta'(0)\big)\\ &= A - g\big(\beta'(0), x_1''(0)\big) + \beta'(0)\\ &\leqslant A - g\big(\beta'(0), \beta''(0)\big) + \beta'(0) < \beta'(0),\end{aligned}$$

thus $t_1 \neq 0$.

If $t_1 = 1$, similar to the above argument, we can deduce that $t_1 \neq 1$. So we show that $x_1'(t) \leqslant \beta'(t)$ for every $t \in [0,1]$. Similarly, we can prove that $\alpha'(t) \leqslant x_1'(t)$ for every $t \in [0,1]$. Therefore,

$$\alpha'(t) \leqslant x_1'(t) \leqslant \beta'(t), \quad t \in [0,1].$$

Since $\alpha(0) = \beta(0) = 0$, by integrating the above inequalities on $[0,t]$, we obtain

$$\alpha(t) \leqslant x_1(t) \leqslant \beta(t), \quad t \in [0,1].$$

Thus $x_1(t)$ is a solution of BVP (1.1)–(1.2). □

Remark 1. In Theorem 1, the condition that f is decreasing in x for $(t,x,y,z) \in [0,1] \times R^3$ can be relaxed as f is increasing in x for $(t,y,z) \in [0,1] \times R^2$ and $\alpha(t) \leqslant x(t) \leqslant \beta(t)$. Since the condition is only used in Step 1 to prove the inequality

$$-\lambda f\big(t_0, \omega\big(\alpha(t_0), x(t_0), \beta(t_0)\big), \beta'(t_0), 0\big) \geqslant -\lambda f\big(t_0, \beta(t_0), \beta'(t_0), 0\big),$$

and Step 4 to prove the inequality

$$-f\big(t_1, \omega\big(\alpha(t_1), x_1(t_1), \beta(t_1)\big), \beta'(t_1), \beta''(t_1)\big) \geqslant -f\big(t_1, \beta(t_1), \beta'(t_1), \beta''(t_1)\big).$$

4. Example

Example. We consider the following third-order boundary value problem:

$$x''' - (t-x)^2 - t(4+t^2)x' - (x')^2 \sin(x'') = 0, \quad 0 < t < 1, \tag{4.1}$$

$$x(0) = 0{,}5\big(x'(0)\big)^2 - \frac{1}{2}x''(0) = 5, \qquad \big(x'(1)\big)^2 + \big(x''(1)\big)^3 = 1, \tag{4.2}$$

let

$$\begin{aligned}&f(t,x,y,z) = -(t-x)^2 - t(4+t^2)y - y^2 \sin z,\\ &g(y,z) = 5y^2 - \frac{1}{2}z, \qquad h(y,z) = y^2 + z^3.\end{aligned}$$

It is easily to prove that $\alpha(t) = -t$, $\beta(t) = t$ are lower and upper solutions of BVP (4.1)–(4.2), respectively. f is continuous on $[0,1] \times R^3$ and increasing in x when $\alpha(t) \leqslant x_1(t) \leqslant \beta(t)$, $t \in [0,1]$. g, h are continuous on R^2, $g(y,z)$ is decreasing in z, $h(y,z)$ is increasing in z. Furthermore, we obtain f satisfies Nagumo condition in

$$D = \big\{(t,x,y,z) \in [0,1] \times R^3\colon\ -t \leqslant x \leqslant t,\ -1 \leqslant x' \leqslant 1\big\}.$$

Therefore, by Theorem 1, there exists at least one solution $x(t)$ for BVP (4.1)–(4.2) such that

$$-t \leqslant x(t) \leqslant t, \quad -1 \leqslant x'(t) \leqslant 1, \quad t \in [0,1].$$

Remark 2. We extend the result in Ref. [4] since our boundary conditions are more generalized. Obviously the result given in [4] is not available to our example.

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