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Strichartz estimates for dispersive equations and solvability of the Kawahara equation

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Abstract

We first establish a series of Strichartz estimates for a general class of linear dispersive equations by applying the theory of oscillatory integrals established by Kenig, Ponce and Vega. Next we use such estimates to study solvability of the Cauchy problem of the Kawahara equation $\partial_t u + au\partial_x u + \beta\partial_x^3 u + \gamma\partial_x^5 u = 0$ in the class $C(R, H^s(R))$. Local existence is proved for $s > 1/4$ and global existence is proved for $s \geqslant 2$.

Keywords: Strichartz estimate; Dispersive equation; Kawahara equation; Global existence

1. Introduction

In this paper we shall first establish a series of Strichartz estimates for the dispersive equation

$$\partial_t u - iP(-i\partial_x)u = f(x,t), \quad x, t \in R, \tag{1.1}$$

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where $P(-i\partial_x)$ is a linear differential operator whose symbol $P(\xi)$ is a real polynomial of degree $\geqslant 3$. We shall next use such estimates to establish local and global existence of solutions for the initial value problem of the Kawahara equation:

$$\partial_t u + au\partial_x u + \beta\partial_x^3 u + \gamma\partial_x^5 u = 0 \quad \text{in } R^2, \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in R, \tag{1.3}$$

where a, β, γ are real constants ($a, \gamma \neq 0$), and u_0 is a given function.

Strichartz estimate is a very interesting topic in the field of dispersive-type (including hyperbolic-type) partial differential equations. It has wide applications in many other topics, such as well-posedness of initial value problems, regularity of solutions, large-time behavior of solutions, and so on. Due to this reason, this topic has drawn much attention since its first occurrence in 1970's. In this paper, we shall use the theory of oscillatory integrals developed by Kenig et al. [16,17] to establish a series of Strichartz estimates for the general linear dispersive equation (1.1) in one space variable. These estimates are generalizations of similar estimates established by Kenig et al. in [16–19] for the Airy equation

$$\partial_t u + \partial_x^3 u = f(x,t)$$

to the general dispersive equation (1.1). Note that if the operator $P(-i\partial_x)$ is homogeneous then such generalization is immediate. However, for inhomogeneous $P(-i\partial_x)$, which is the main objective of this work, there are many new difficulties for us to overcome. It is because of this fact that, unlike [16–19] where most estimates are global in the time variable, the Strichartz estimates established in this paper are local in the time variable.

Equation (1.2) was first proposed by Kawahara in 1972 [15], as a model equation describing solitary-wave propagation in media in which the first-order dispersion coefficient β is anomalously small (see also [1,9,14]). More specific physical background of this equation was introduced by Hunter and Scheurle in [12], where they used this equation to describe the evolution of solitary waves in fluids in which the Bond number is less than but close to $1/3$ and the Froude number is close to 1. In the literature this equation is also referred as fifth-order KdV equation or singularly perturbed KdV equation [8,10]. There has been much work on the solitary-wave solutions of this equation [1,2,8–10,12–15,22,23, 27]. An interesting property of its solitary-wave solutions is that their trails are oscillatory. However, well-posedness of its Cauchy problem (1.2)–(1.3) has not been well treated. As an application of the Strichartz estimates established in this paper, we shall prove that the problem (1.2)–(1.3) is locally well-posed in $H^s(R)$ for $s > 1/4$. This local result combined with the second and the third conservation laws of the Kawahara equation immediately implies that the problem (1.2)–(1.3) is globally well-posed in $H^s(R)$ for $s \geqslant 2$. The condition $s > 1/4$ for local well-posedness is perhaps not the weakest, and it might be weakened by following Bourgain's approach (see [5–7,20,21]). However, the method used here (which comes from Kenig et al. [16–19]) enables us to get more delicate messages on regularity of the solution, so that it cannot be covered by Bourgain's approach.

We emphasize that in addition to the application shown here, Strichartz estimates established in this paper have also many other applications. For instance, they can be used to study well-posedness of Cauchy problems of other equations with more general nonlinear terms than "bilinear" forms like $u\partial_x u$ treated here, such as those studied in [22,24], for which Bourgain's approach usually does not apply (see [21]). Such applications will be shown in our future work.

2. Strichartz estimates

Let $P(\xi)$ be a real polynomial in one variable ξ, of degree $m \geqslant 3$. Consider the linear dispersive equation

$$\partial_t u - iP(-i\partial_x)u = f(x,t) \quad \text{for } (x,t) \in R^2, \tag{2.1}$$

imposed with initial condition

$$u(x,0) = u_0(x) \quad \text{for } x \in R, \tag{2.2}$$

where f and u_0 are given functions. The fundamental solution of the problem (2.1)–(2.2) is given by the oscillatory integral (for $t \neq 0$)

$$G(x,t) = c\int_{-\infty}^{\infty} e^{i(tP(\xi)+x\xi)}\,d\xi,$$

where $c = (2\pi)^{-1}$. The integral on the right-hand side is understood as the limit $\lim_{M,N\to\infty}\int_{-N}^{M}$, which is convergent by the stationary phase argument (cf. [11, pp. 215–240] or [25, pp. 331–334]). This argument also shows that G is continuous in $R \times (R\setminus\{0\})$ and $\lim_{t\to 0} G(x,t) = \delta(x)$. In fact, for any $\alpha \in C$ satisfying $-1 < \operatorname{Re}\alpha < m-1$, $D_x^\alpha G(x,t)$ is continuous in $R \times (R\setminus\{0\})$, and

$$D_x^\alpha G(x,t) = c\int_{-\infty}^{\infty} |\xi|^\alpha e^{i(tP(\xi)+x\xi)}\,d\xi,$$

where D_x^α represents the αth order absolute derivative in the space variable, i.e., $D_x^\alpha\varphi(x) = F^{-1}(|\xi|^\alpha\tilde{\varphi}(\xi))$. Similarly as before, the integral on the right-hand side of the above equation (regarded as a similar limit) is meaningful by a similar argument.

For every $t \in R$, $t \neq 0$, let $W(t)$ be the operator defined by

$$W(t)\varphi = G(\cdot,t) * \varphi,$$

whenever the right-hand side makes sense, and let $W(0)\varphi = \varphi$. Then the unique solution of the problem (2.1)–(2.2) (for suitable f and u_0) is given by

$$u(\cdot,t) = W(t)u_0 + \int_0^t W(t-\tau)f(\cdot,\tau)\,d\tau \quad (t \in R).$$

Since $P(\xi)$ is real, it is obvious that for any $s \in R$, $\{W(t)\}_{t\in R}$ forms an one-parameter unitary group on the Sobolev space $H^s(R)$. Thus for any $\varphi \in H^s(R)$ and $t \in R$ we have

$$\|W(t)\varphi\|_{s,2} = \|\varphi\|_{s,2}, \tag{2.3}$$

where $\|\cdot\|_{s,2}$ represents the norm on $H^s(R)$. The aim of this section is to establish other estimates for $W(t)$ than this trivial one.

Lemma 2.1. *Let $\alpha \in C$ and $-1 < \operatorname{Re}\alpha \leqslant (m-2)/2$. Then for any $\delta > 0$ there exists a constant $C > 0$ depending only on δ, $\operatorname{Re}\alpha$ and P such that for any $0 < |t| \leqslant \delta$,*

$$\sup_{x\in R}\left|D_x^\alpha G(x,t)\right| \leqslant C\big(1+|\operatorname{Im}\alpha|\big)^2|t|^{-\frac{1}{m}(\operatorname{Re}\alpha+1)}. \tag{2.4}$$

Proof. We denote $\mu = \operatorname{Re}\alpha$, $\nu = \operatorname{Im}\alpha$ and $\nu' = \nu/(m-2)$. For a sufficiently large positive number M_0 to be specified later, we write

$$\begin{aligned} D_x^\alpha G(x,t) &= c\Bigg(\int\limits_{|\xi|\leqslant M_0(\delta|t|^{-1})^{\frac{1}{m}}} + \int\limits_{|\xi|\geqslant M_0(\delta|t|^{-1})^{\frac{1}{m}}}\Bigg)|\xi|^\alpha e^{i(tP(\xi)+x\xi)}\,d\xi \\ &\equiv I_1(x,t)+I_2(x,t). \end{aligned}$$

Clearly,

$$\left|I_1(x,t)\right| \leqslant c\int\limits_{-M_0(\delta|t|^{-1})^{\frac{1}{m}}}^{M_0(\delta|t|^{-1})^{\frac{1}{m}}}|\xi|^\mu\,d\xi \leqslant C|t|^{-\frac{\mu+1}{m}}, \quad \forall x\in R,\ \forall t\in R\ (t\neq 0).$$

To estimate $I_2(x,t)$ we first take M_0 sufficiently large such that

$$\left|P''(\xi)\right| \geqslant C|\xi|^{m-2} \quad \text{for } |\xi|\geqslant M_0,$$

and for every $M > M_0$ we define

$$I_{2,M}^+(x,t) = c\int\limits_{M_0(\delta|t|^{-1})^{\frac{1}{m}}}^{M(\delta|t|^{-1})^{\frac{1}{m}}} e^{i(tP(\xi)+x\xi)}\left|P''(\xi)\right|^{\frac{1}{2}+i\nu'}\psi(\xi)\,d\xi,$$

$$I_{2,M}^-(x,t) = c\int\limits_{-M(\delta|t|^{-1})^{\frac{1}{m}}}^{-M_0(\delta|t|^{-1})^{\frac{1}{m}}} e^{i(tP(\xi)+x\xi)}\left|P''(\xi)\right|^{\frac{1}{2}+i\nu'}\psi(\xi)\,d\xi,$$

where $\psi(\xi) = |\xi|^{\mu+i\nu}|P''(\xi)|^{-\frac{1}{2}-i\nu'}$. By [17, Corollary 2.9] we have

$$\left|I_{2,M}^+(x,t)\right| \leqslant C\big(1+|\nu|\big)|t|^{-\frac{1}{2}}\Bigg\{\left|\psi\big(M\big(\delta|t|^{-1}\big)^{\frac{1}{m}}\big)\right| + \int\limits_{M_0(\delta|t|^{-1})^{\frac{1}{m}}}^{M(\delta|t|^{-1})^{\frac{1}{m}}}\left|\psi'(\xi)\right|d\xi\Bigg\}.$$

It is evident that the first term in the braces is dominated by

$$C|t|^{-\frac{1}{m}(\mu-\frac{m-2}{2})}, \tag{2.5}$$

where C is independent of M and ν. To show that the second term is also dominated by (2.5) we write $\psi(\xi) = \psi_1(\xi)\psi_2^{i\nu}(\xi)$, where

$$\psi_1(\xi) = |\xi|^\mu\left|P''(\xi)\right|^{-\frac{1}{2}}, \qquad \psi_2(\xi) = |\xi|\left|P''(\xi)\right|^{-\frac{1}{m-2}}.$$

Clearly,

$$\left|\psi'(\xi)\right| \leqslant \left|\psi_1'(\xi)\right| + |\nu|\left|\psi_1(\xi)\right|\left|\frac{\psi_2'(\xi)}{\psi_2(\xi)}\right|.$$

We now take M_0 further large such that both $\psi_1'(\xi)$ and $\psi_2'(\xi)/\psi_2(\xi)$ do not change sign on the intervals (M_0, ∞), $(-\infty, M_0)$. Then for any $0 < |t| \leqslant \delta$ we have

$$\begin{aligned}\int_{M_0(\delta|t|^{-1})^{\frac{1}{m}}}^{M(\delta|t|^{-1})^{\frac{1}{m}}} \left|\psi'(\xi)\right| d\xi &\leqslant \int_{M_0(\delta|t|^{-1})^{\frac{1}{m}}}^{M(\delta|t|^{-1})^{\frac{1}{m}}} \left|\psi_1'(\xi)\right| d\xi + |\nu| \int_{M_0(\delta|t|^{-1})^{\frac{1}{m}}}^{M(\delta|t|^{-1})^{\frac{1}{m}}} \left|\psi_1(\xi)\right|\left|\frac{\psi_2'(\xi)}{\psi_2(\xi)}\right| d\xi \\ &\leqslant C\big(1+|\nu|\big)|t|^{-\frac{1}{m}(\mu-\frac{m-2}{2})},\end{aligned}$$

where C is independent of M and ν. Hence

$$\left|I_{2,M}^{+}(x,t)\right| \leqslant C\big(1+|\nu|\big)^2|t|^{-\frac{\mu+1}{m}} \quad \text{for } 0 < |t| \leqslant \delta. \tag{2.6}$$

Similarly, we have also

$$\left|I_{2,M}^{-}(x,t)\right| \leqslant C\big(1+|\nu|\big)^2|t|^{-\frac{\mu+1}{m}} \quad \text{for } 0 < |t| \leqslant \delta. \tag{2.7}$$

Summing up (2.6), (2.7) and letting $M \to \infty$, we find that

$$\left|I_2(x,t)\right| \leqslant C\big(1+|\nu|\big)^2|t|^{-\frac{\mu+1}{m}} \quad \text{for } 0 < |t| \leqslant \delta.$$

Hence, the estimate (2.4) holds. □

Remark. In the case $0 \leqslant \alpha \leqslant (m-2)/2$ the inequality (2.4) is obtained by Ben-Artzi and Saut [3] with a different method.

Using Lemma 2.1 and Young's inequality we immediately get the following result.

Lemma 2.2. *Let $\alpha \in C$ and $-1 < \operatorname{Re}\alpha \leqslant (m-2)/2$. Then for any $\delta > 0$ there exists a constant $C > 0$ depending only on δ, $\operatorname{Re}\alpha$ and P such that for any $\varphi \in L^1(R)$ and $0 < |t| \leqslant \delta$,*

$$\left\|D^\alpha W(t)\varphi\right\|_\infty \leqslant C\big(1+|\operatorname{Im}\alpha|\big)^2|t|^{-\frac{\operatorname{Re}\alpha+1}{m}}\|\varphi\|_1, \tag{2.8}$$

where $\|\cdot\|_p$ $(1 \leqslant p \leqslant \infty)$ represents the norm on the space $L^p(R)$.

The next result follows easily from (2.3) (taking $s = 0$) and (2.8) by applying the Stein interpolation theorem [26, Theorem 4.1, pp. 205–209] to the analytic family of operators $T_z = D^{\alpha z}W(t)$ $(z \in C,\ 0 \leqslant \operatorname{Re} z \leqslant 1)$, where $0 < |t| \leqslant \delta$, $0 \leqslant \alpha \leqslant (m-2)/2$, and they are regarded as fixed.

Lemma 2.3. *Assume that $-1 < \alpha \leqslant (m-2)/2$ and $0 \leqslant \theta \leqslant 1$. Then for any $\delta > 0$ there exists a constant $C > 0$ depending only on δ, θ and P such that for any $\varphi \in L^{2/(1+\theta)}(R)$ and $0 < |t| \leqslant \delta$,*

$$\left\|D^{\theta\alpha}W(t)\varphi\right\|_{2/(1-\theta)} \leqslant C|t|^{-\frac{\theta(\alpha+1)}{m}}\|\varphi\|_{2/(1+\theta)}. \tag{2.9}$$

As in [16–21] we introduce notation $\|\cdot\|_{L_T^q L_x^p}$, $\|\cdot\|_{L_x^p L_T^q}$ $(1 \leqslant p, q \leqslant \infty)$ as follows: For finite p and q,

$$\|f\|_{L_T^q L_x^p} = \left(\int_{-T}^{T} \left(\int_{-\infty}^{\infty} \left|f(x,t)\right|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}},$$

$$\|f\|_{L_x^p L_T^q} = \left(\int_{-\infty}^{\infty} \left(\int_{-T}^{T} \left|f(x,t)\right|^q dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}};$$

for infinite p or q or both the left-hand sides are respectively defined as limits of the right-hand sides. The space $L_T^q L_x^p$ consists of measurable functions on $R \times [-T, T]$ such that $\|f\|_{L_T^q L_x^p} < \infty$; the space $L_x^p L_T^q$ is defined similarly. The notation p' (respectively q') denotes the dual number of p (respectively q).

We now use Tomas argument [28] to derive a $L_T^q L_x^p$ estimate from (2.9).

Theorem 2.4. *Assume that* $-1 < \alpha \leqslant (m-2)/2$ *and* $0 \leqslant \theta \leqslant 1$. *Then for any* $\delta > 0$ *there exists a constant* $C > 0$ *depending only on* δ, α, θ *and* P *such that for any* $0 < T \leqslant \delta$ *and* $\varphi \in L^2(R)$,

$$\left\| D^{\frac{\theta\alpha}{2}} W(t)\varphi \right\|_{L_T^q L_x^p} \leqslant C\|\varphi\|_2, \tag{2.10}$$

where $p = 2/(1-\theta)$ *and* $q = 2m/\theta(\alpha+1)$.

Proof. From (2.9) it follows that for $p = 2/(1-\theta)$ we have

$$\begin{aligned}
&\left\| \int_{-T}^{T} D^{\theta\alpha} W(t-\tau) f(\cdot,\tau)\,d\tau \right\|_{L_T^q L_x^p} \\
&\quad \leqslant \left(\int_{-T}^{T} \left(\int_{-T}^{T} \left\| D^{\theta\alpha} W(t-\tau) f(\cdot,\tau)\,d\tau \right\|_p d\tau \right)^q dt \right)^{\frac{1}{q}} \\
&\quad \leqslant C \left(\int_{-T}^{T} \left(\int_{-T}^{T} |t-\tau|^{-\frac{\theta(\alpha+1)}{m}} \left\| f(\cdot,\tau) \right\|_{p'} d\tau \right)^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Since $0 \leqslant \theta(\alpha+1)/m \leqslant 1/2$ and

$$\frac{1}{q'} + \frac{\theta(\alpha+1)}{m} = 1 + \frac{1}{q} \quad \text{for } q = \frac{2m}{\theta(\alpha+1)},$$

using the Hardy–Littlewood–Sobolev inequality for Riesz potentials [11, Theorem 4.5.3] we obtain

$$\left\| \int_{-T}^{T} D^{\theta\alpha} W(t-\tau) f(\cdot,\tau)\,d\tau \right\|_{L_T^q L_x^p} \leqslant C\|f\|_{L_T^{q'} L_x^{p'}}.$$

Hence

$$
\begin{aligned}
&\left\| \int_{-T}^{T} D^{\frac{\theta\alpha}{2}} W(t) f(\cdot,t)\,dt \right\|_2^2 \\
&= \int_{-\infty}^{\infty} \left(\int_{-T}^{T} D^{\frac{\theta\alpha}{2}} W(t) f(\cdot,t)\,dt \right) \left(\int_{-T}^{T} \overline{D^{\frac{\theta\alpha}{2}} W(\tau) f(\cdot,\tau)\,d\tau} \right) dx \\
&= \int_{-\infty}^{\infty} \int_{-T}^{T} f(x,t) \left(\int_{-T}^{T} D^{\theta\alpha} W(t-\tau) \overline{f(\cdot,\tau)}\,d\tau \right) dt\,dx \\
&\leqslant \|f\|_{L_T^{q'} L_x^{p'}} \cdot \left\| \int_{-T}^{T} D^{\theta\alpha} W(t-\tau) \overline{f(\cdot,\tau)}\,d\tau \right\|_{L_T^q L_x^p} \leqslant C\|f\|^2_{L_T^{q'} L_x^{p'}},
\end{aligned}
$$

so that

$$
\left\| \int_{-T}^{T} D^{\frac{\theta\alpha}{2}} W(t) f(\cdot,t)\,dt \right\|_2 \leqslant C\|f\|_{L_T^{q'} L_x^{p'}}.
$$

Since

$$
\int_{-T}^{T} \int_{-\infty}^{\infty} D^{\frac{\theta\alpha}{2}} W(t)\varphi \cdot \overline{f(x,t)}\,dx\,dt = \int_{-\infty}^{\infty} \varphi(x) \overline{\left(\int_{-T}^{T} D^{\frac{\theta\alpha}{2}} W(-t) f(\cdot,t)\,dt \right)} dx,
$$

by a dual argument we get the desired estimate (2.10). □

In particular, by taking $p = q$ we obtain

Corollary 2.5. *Assume that* $-1/2 < s \leqslant (m-2)/6$*. Then for any* $\delta > 0$ *there exists a constant* $C > 0$ *depending only on* δ*,* s *and* P *such that for any* $\varphi \in L^2(R)$ *and* $0 < T \leqslant \delta$*,*

$$
\left\| D^s W(t)\varphi \right\|_{L^p(R_T)} \leqslant C\|\varphi\|_2, \tag{2.11}
$$

where $p = 2(m+1)/(2s+1)$ *and* $R_T = R \times [-T, T]$*. In particular,*

$$
\left\| W(t)\varphi \right\|_{L^{2(m+1)}(R_T)} \leqslant C\|\varphi\|_2, \tag{2.12}
$$

$$
\left\| D^{\frac{m-2}{6}} W(t)\varphi \right\|_{L^6(R_T)} \leqslant C\|\varphi\|_2. \tag{2.13}
$$

Next we consider $L_x^p L_T^q$ estimates.

Theorem 2.6. *For any* $\delta > 0$ *there exists a constant* $C > 0$ *depending only on* δ *and* P *such that for any* $\varphi \in L^2(R)$ *and* $0 < T \leqslant \delta$*,*

$$
\left\| D^{\frac{m-1}{2}} W(t)\varphi \right\|_{L_x^\infty L_T^2} \leqslant C\|\varphi\|_2. \tag{2.14}
$$

Proof. First we assume that $\varphi \in C_0^\infty(R)$. Denoting $\psi(x) = D_x^{\frac{m-1}{2}} \varphi(x)$, we see that

$$D_x^{\frac{m-1}{2}} W(t)\varphi(x) = W(t)\psi(x).$$

For $M > 0$ to be specified later we write

$$W(t)\psi(x) = c\left(\int_{|\xi|\leqslant M} + \int_{|\xi|>M}\right) e^{i(tP(\xi)+x\xi)}\tilde{\psi}(\xi)\,d\xi \equiv I_1'(x,t) + I_2'(x,t).$$

Clearly, for any $0 < T \leqslant \delta$ we have

$$\sup_{x\in R}\left(\int_{-T}^{T} \left|I_1'(x,t)\right|^2 dt\right)^{\frac{1}{2}} \leqslant \sqrt{2\delta}\, c \int_{-M}^{M} \left|\tilde{\psi}(\xi)\right| d\xi \leqslant C\|\tilde{\varphi}\|_2 = C\|\varphi\|_2.$$

We now take $M > 0$ sufficiently large such that

$$\left|P'(\xi)\right| \geqslant C|\xi|^{m-1} \quad \text{for } |\xi| \geqslant M.$$

Then by [17, Theorem 4.1] we have

$$\sup_{x\in R}\left(\int_{-T}^{T} \left|I_2'(x,t)\right|^2 dt\right)^{\frac{1}{2}} \leqslant \sup_{x\in R}\left(\int_{-\infty}^{\infty} \left|I_2'(x,t)\right|^2 dt\right)^{\frac{1}{2}}$$

$$\leqslant C\left(\int_{|\xi|\geqslant M} \left|\tilde{\psi}(\xi)\right|^2 \left|P'(\xi)\right|^{-1} d\xi\right)^{\frac{1}{2}} \leqslant C\left(\int_{|\xi|\geqslant M} \left|\tilde{\varphi}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \leqslant C\|\varphi\|_2.$$

Thus for $\varphi \in C_0^\infty(R)$ the estimate (2.14) holds. For a generic $\varphi \in L^2(R)$ this estimate follows from the denseness of $C_0^\infty(R)$ in $L^2(R)$. $\square$

Theorem 2.7. *For any $\delta > 0$ there exists a constant $C > 0$ depending only on δ and P such that for any $\varphi \in H^{\frac{1}{4}}(R)$ and $0 < T \leqslant \delta$,*

$$\left\|W(t)\varphi\right\|_{L_x^4 L_T^\infty} \leqslant C\|\varphi\|_{\frac{1}{4},2}. \tag{2.15}$$

Proof. Take $M > 0$ sufficiently large such that

$$\left|P''(\xi)\right| \geqslant C|\xi|^{m-2} \quad \text{for } |\xi| \geqslant M.$$

Then we have

$$\left|\frac{P'(\xi)}{P''(\xi)}\right| \leqslant C\big(1+|\xi|\big) \quad \text{for } |\xi| \geqslant M. \tag{2.16}$$

We write

$$W(t)\varphi(x) = c\left(\int_{|\xi|\leqslant M} + \int_{|\xi|>M}\right) e^{i(tP(\xi)+x\xi)}\tilde{\varphi}(\xi)\,d\xi \equiv I_1''(x,t) + I_2''(x,t).$$

By (2.16) and [17, Theorem 2.5] we have

$$\left(\int_{-\infty}^{\infty}\sup_{t\in R}\left|I_2''(x,t)\right|^4 dx\right)^{\frac{1}{4}} \leqslant C\left(\int_{|\xi|>M}\left|\tilde{\varphi}(\xi)\right|^2\left|\frac{P'(\xi)}{P''(\xi)}\right|^{\frac{1}{2}} d\xi\right)^{\frac{1}{2}} \leqslant C\|\varphi\|_{\frac{1}{4},2}.$$

To estimate $I_1''(x,t)$ we denote

$$K(x,t) = c\int_{|\xi|\leqslant M} e^{i(tP(\xi)+x\xi)}\,d\xi,$$

and let $A = \max_{|\xi|\leqslant M}|P'(\xi)|$. Since for $|x| \geqslant 2\delta A$, $|t| \leqslant T \leqslant \delta$ and $|\xi| \leqslant M$ there holds

$$\left|tP'(\xi)+x\right| \geqslant |x| - \delta A \geqslant \frac{1}{2}|x|,$$

using the stationary phase argument we easily get

$$\sup_{|t|\leqslant T}\left|K(x,t)\right| \leqslant C|x|^{-1} \quad \text{for } |x| \geqslant 2\delta A.$$

Since clearly $K(x,t)$ is bounded for $(x,t) \in R^2$, we have

$$\sup_{|t|\leqslant T}\left|K(x,t)\right| \leqslant C\big(1+|x|\big)^{-1} \quad \text{for } x \in R.$$

Therefore, using the Minkowski inequality we get

$$\begin{aligned}\left(\int_{-\infty}^{\infty}\sup_{|t|\leqslant T}\left|I_1''(x,t)\right|^4 dx\right)^{\frac{1}{4}} &= \left(\int_{-\infty}^{\infty}\sup_{|t|\leqslant T}\left|\int_{-\infty}^{\infty}K(x-y,t)\varphi(y)\,dy\right|^4 dx\right)^{\frac{1}{4}}\\ &\leqslant c\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\big(1+|x-y|\big)^{-1}\left|\varphi(y)\right|dy\right)^4 dx\right)^{\frac{1}{4}}\\ &\leqslant C\left(\int_{-\infty}^{\infty}\big(1+|x|\big)^{-\frac{4}{3}}\,dx\right)^{\frac{3}{4}}\|\varphi\|_2 = C\|\varphi\|_2.\end{aligned}$$

Hence (2.15) holds. □

Theorem 2.8. *For any $\delta > 0$ and $s > m/4$ there exists a constant $C > 0$ depending only on δ, s and P such that for any $\varphi \in H^s(R)$ and $0 < T \leqslant \delta$,*

$$\left\|W(t)\varphi\right\|_{L_x^2 L_T^\infty} \leqslant C\|\varphi\|_{s,2}. \tag{2.17}$$

Proof. Let $M > 0$ be a sufficiently large constant such that for any $|\xi| \geqslant M$,

$$C_1|\xi|^{m-k} \leqslant \left|\partial_\xi^k P(\xi)\right| \leqslant C_2|\xi|^{m-k}, \quad k = 0,1,\ldots,m. \tag{2.18}$$

Let k_0 be an integer such that $2^{k_0-1} \geqslant M$. Let $\{\psi_0\} \cup \{\psi_k\}_{k=k_0}^{\infty}$ be an inhomogeneous Littlewood–Paley decomposition of the unit in the following sense:

(i) $\psi_0 \in C_0^\infty(R)$, ψ_0 even, $\operatorname{supp}\psi_0 \subset \{\xi \in R\colon\ |\xi| \leqslant 2^{k_0}\}$, and $0 \leqslant \psi_0 \leqslant 1$.
(ii) For $k \geqslant k_0$, $\psi_k \in C_0^\infty(R)$, ψ_k even, $\operatorname{supp}\psi_k \subset \{\xi \in R\colon\ 2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}$, $0 \leqslant \psi_k \leqslant 1$, and $|\partial^j \psi_k| \leqslant C_j 2^{-jk}$ for $j = 1, 2, \ldots$.
(iii) $\psi_0(\xi) + \sum_{k=k_0}^\infty \psi_k(\xi) = 1$ for $\xi \in R$.

For each k, we denote by $\psi_k(D)$ the pseudo-differential operator with symbol ψ_k, and by $J_k(x,t)$ the kernel of the smoothing operator $\psi_k(D)W(t)$, namely,

$$J_k(x,t) = c\int_{-\infty}^{\infty} e^{i(tP(\xi)+x\xi)}\psi_k(\xi)\,d\xi = c\left(\int_{2^{k-1}}^{2^{k+1}} + \int_{-2^{k+1}}^{-2^{k-1}}\right)e^{i(tP(\xi)+x\xi)}\psi_k(\xi)\,d\xi$$
$$\equiv J_k^+(x,t) + J_k^-(x,t).$$

We assert that for any $k \geqslant k_0$ and $x \in R$ there holds

$$\sup_{|t|\leqslant\delta}\left|J_k(x,t)\right| \leqslant C h_k(x), \tag{2.19}$$

where

$$h_k(x) = \begin{cases} 2^{\frac{k}{2}}|x|^{-1/2}, & \text{for } |x| \leqslant C_0 2^{(m-1)k}, \\ 2^{-2k}|x|^{-2}, & \text{for } |x| \geqslant C_0 2^{(m-1)k} \end{cases}$$

($C_0 = 2^m\delta C_2$, cf. (2.18)). Clearly, to prove this estimate we only need to prove that both $J_k^+(x,t)$ and $J_k^-(x,t)$ satisfy a similar estimate. In the sequel we only consider $J_k^+(x,t)$; the argument for $J_k^-(x,t)$ is similar.

First we assume that $|x| \leqslant C_0 2^{(m-1)k}$. By making change of integral variables $\eta = P(\xi)$, or $\xi = P^{-1}(\eta) \equiv Q(\eta)$, we get

$$J_k^+(x,t) = c\int_{a_k}^{b_k} e^{i(t\eta + xQ(\eta))}\psi_k\big(Q(\eta)\big)Q'(\eta)\,d\eta,$$

where $a_k = P(2^{k-1})$, $b_k = P(2^{k+1})$. By (2.18) we have

$$Q''(\eta) = -\frac{P''(Q(\eta))}{(P'(Q(\eta)))^3} \sim 2^{-(2m-1)k}, \qquad Q'(\eta) = \frac{1}{P'(Q(\eta))} \sim 2^{-(m-1)k}$$

for $\eta \in (a_k, b_k)$. Thus by using the Van der Corput Lemma [25, 332–334] we obtain

$$\sup_{t\in R}\left|J_k^+(x,t)\right| \leqslant C\big(2^{-(2m-1)k}|x|\big)^{-\frac{1}{2}}\int_{a_k}^{b_k}\left|\frac{d}{d\eta}\big\{\psi_k\big(Q(\eta)\big)Q'(\eta)\big\}\right|d\eta$$
$$\leqslant C\big(2^{-(2m-1)k}|x|\big)^{-\frac{1}{2}}\int_{a_k}^{b_k}\Big(\big|\psi_k'\big(Q(\eta)\big)\big|\big|Q'(\eta)\big|^2 + \big|\psi_k\big(Q(\eta)\big)\big|\big|Q''(\eta)\big|\Big)\,d\eta$$
$$\leqslant C\big(2^{-(2m-1)k}|x|\big)^{-\frac{1}{2}}\cdot\big(2^{mk}\cdot 2^{-(2m-1)k}\big) = C2^{\frac{k}{2}}|x|^{-\frac{1}{2}}.$$

Next we assume that $|x| \geqslant C_0 2^{(m-1)k} = 2^m \delta C_2 2^{(m-1)k}$. Since $|t| \leqslant \delta$ and on the integral interval there holds

$$\left|P'(\xi)\right| \leqslant C_2|\xi|^{m-1} \leqslant 2^{m-1} C_2 2^{(m-1)k},$$

we see that

$$\left|tP'(\xi)+x\right| \geqslant |x| - |t|\left|P'(\xi)\right| \geqslant \frac{1}{2}|x|. \tag{2.20}$$

Therefore, the operator $L = (tP'(\xi)+x)^{-1}\partial_\xi$ is well defined. Since clearly $L^2 e^{i(tP(\xi)+x\xi)} = e^{i(tP(\xi)+x\xi)}$, by integration by parts we get

$$J_k^+(x,t) = \int_{2^{k-1}}^{2^{k+1}} e^{i(tP(\xi)+x\xi)} \cdot {}^tL^2\psi_k(\xi)\,d\xi,$$

for $|x| \geqslant C_0 2^{(m-1)k}$ and $|t| \leqslant \delta$, where ${}^tL$ represents the transpose of L. Using this relation together with the estimates (2.18), (2.20) and the fact that $|\partial_\xi^j \psi_k(\xi)| \leqslant C_j 2^{-jk}$, we get

$$\left|J_k^+(x,t)\right| \leqslant C 2^{-2k}|x|^{-2},$$

for $|x| \geqslant C_0 2^{(m-1)k}$ and $|t| \leqslant \delta$. Hence the estimate (2.19) holds.

Consequently, by the Minkowski inequality we get, for any $k \geqslant k_0$ and $0 < T \leqslant \delta$,

$$\begin{aligned}
&\left\| \int_{-T}^{T} \psi_k(D)W(t-\tau)f(\cdot,\tau)\,d\tau \right\|_{L_x^2 L_T^\infty} \\
&\quad = \left\| \int_{-T}^{T}\int_{-\infty}^{\infty} J_k(x-y,t-\tau)f(y,\tau)\,dy\,d\tau \right\|_{L_x^2 L_T^\infty} \\
&\quad \leqslant C\left\| h_k * \int_{-T}^{T} \left|f(\cdot,\tau)\right|d\tau \right\|_2 \leqslant C 2^{\frac{mk}{2}}\|f\|_{L_x^2 L_T^1}.
\end{aligned}$$

Using the Tomas argument we then deduce that, for any $k \geqslant k_0$,

$$\begin{aligned}
&\left\| \int_{-T}^{T} \psi_k(D)W(t)f(\cdot,t)\,dt \right\|_2^2 \\
&\quad \leqslant \left\|\psi_k(D)f\right\|_{L_x^2 L_T^1} \cdot \left\| \int_{-T}^{T} \psi_k(D)W(t-\tau)\overline{f(\cdot,\tau)}\,d\tau \right\|_{L_x^2 L_T^\infty} \\
&\quad \leqslant C\|f\|_{L_x^2 L_T^1} \cdot C 2^{\frac{mk}{2}}\|f\|_{L_x^2 L_T^1} = C 2^{\frac{mk}{2}}\|f\|_{L_x^2 L_T^1}^2.
\end{aligned}$$

The inequality $\|\psi_k(D)f\|_{L_x^2L_T^1} \leqslant C\|f\|_{L_x^2L_T^1}$ follows from the fact that $\|\check{\psi}_k\|_1 \leqslant C$ (for every $k \geqslant k_0$), where $\check{\psi}_k$ stands for the inverse Fourier transformation of ψ_k. Therefore,

$$\left\| \int_{-T}^{T} \psi_k(D)W(t)f(\cdot,t)\,dt \right\|_2 \leqslant C2^{\frac{mk}{4}}\|f\|_{L_x^2L_T^1} \quad (k = k_0, k_0+1, \ldots),$$

and, by dual,

$$\left\|\psi_k(D)W(t)\varphi\right\|_{L_x^2L_T^\infty} \leqslant C2^{\frac{mk}{4}}\|\varphi\|_2 \quad (k = k_0, k_0+1, \ldots). \tag{2.21}$$

Next, it is immediate to see that

$$\sup_{|t|\leqslant\delta}\left(1+|x|^2\right)\left|J_0(x,t)\right| \leqslant C,$$

so that

$$\left\|\psi_0(D)W(t)\varphi\right\|_{L_x^2L_T^\infty} \leqslant C\|\varphi\|_2. \tag{2.22}$$

Now, since

$$\psi_k(D)W(t)\varphi = \sum_{j=-2}^{2}\psi_k(D)W(t)\psi_{k+j}(D)\varphi, \quad k \geqslant k_0$$

(where we define $\psi_{k+j}(D) = 0$ for $k+j < 0$), we have

$$\begin{aligned} W(t)\varphi &= \psi_0(D)W(t)\varphi + \sum_{k=k_0}^{\infty}\psi_k(D)W(t)\varphi \\ &= \psi_0(D)W(t)\varphi + \sum_{j=-2}^{2}\sum_{k=k_0}^{\infty}\psi_k(D)W(t)\psi_{k+j}(D)\varphi. \end{aligned}$$

Hence, using (2.21) and (2.22) we deduce that, for any $s > m/4$,

$$\begin{aligned} \left\|W(t)\varphi\right\|_{L_x^2L_T^\infty} &\leqslant \left\|\psi_0(D)W(t)\varphi\right\|_{L_x^2L_T^\infty} + \sum_{j=-2}^{2}\sum_{k=k_0}^{\infty}\left\|\psi_k(D)W(t)\psi_{k+j}(D)\varphi\right\|_{L_x^2L_T^\infty} \\ &\leqslant C\|\varphi\|_2 + C\sum_{j=-2}^{2}\sum_{k=k_0}^{\infty}2^{\frac{mk}{4}}\left\|\psi_{k+j}(D)\varphi\right\|_2 \\ &\leqslant C\|\varphi\|_2 + C\sum_{k'=k_0}^{\infty}2^{\frac{mk'}{4}}\left\|\psi_{k'}(D)\varphi\right\|_2 \\ &\leqslant C\|\varphi\|_2 + C\left(\sum_{k'=k_0}^{\infty}2^{-2(s-\frac{m}{4})k'}\right)^{\frac{1}{2}} \cdot \left(\sum_{k'=k_0}^{\infty}2^{2sk'}\left\|\psi_{k'}(D)\varphi\right\|_2^2\right)^{\frac{1}{2}} \leqslant C\|\varphi\|_{s,2}. \end{aligned}$$

This proves (2.17). □

Since $W(t)$ is unitary on $L^2(R)$, we have

$$\left\|W(t)\varphi\right\|_{L_x^2 L_T^2} \leqslant \delta^{\frac{1}{2}}\|\varphi\|_2, \tag{2.23}$$

for any $\varphi \in L^2(R)$ and $0 < T \leqslant \delta$. Besides, since $\|f\|_{L_x^p L_T^q} \leqslant \|f\|_{L_T^q L_x^p}$ for $q \leqslant p$, from (2.10) we easily deduce that for $-1 < \alpha \leqslant (m-2)/2$ and $m/(m+\alpha+1) \leqslant \theta \leqslant 1$ there holds

$$\left\|D^{\frac{\theta\alpha}{2}} W(t)\varphi\right\|_{L_x^p L_T^q} \leqslant C\|\varphi\|_2, \tag{2.24}$$

where $p = 2/(1-\theta)$ and $2 \leqslant q \leqslant 2m/\theta(\alpha+1)$. Now let Ω be the quadrilateral in $(1/p, 1/q)$-plane with apices $O(0,0)$, $A_1(1/2,0)$, $A_2(1/2,1/2)$ and $A_3(0,1/2)$, comprising the two edges A_1A_2, A_2A_3 and the right half of the edge OA_1. From the estimates (2.14), (2.15), (2.17), (2.23) and (2.24) we can use the interpolation theory [4] to get, for every $(1/p, 1/q) \in \Omega$, an estimate of the form

$$\left\|D^{\alpha} W(t)\varphi\right\|_{L_x^p L_T^q} \leqslant \|\varphi\|_{s,2},$$

for suitable α and s (depending on p and q). We omit this tedious work here.

3. Solvability of the problem (1.2)–(1.3)

In this section we use the results obtained in the previous section to study solvability of the problem (1.2)–(1.3).

Taking $P(\xi) = -\gamma\xi^5 + \beta\xi^3$, we let $W(t)$ be the operator defined in the previous section. Then the problem (1.1)–(1.2) is equivalent to the following integral equation:

$$u(\cdot,t) = W(t)u_0 + \int_0^t W(t-\tau)(u\partial_x u)(\cdot,\tau)\,d\tau. \tag{3.1}$$

Using the estimates (2.3), (2.10), (2.14), (2.15) and (2.17) to the present particular operator $W(t)$ we respectively get

$$\left\|W(t)\varphi\right\|_{s,2} = \|\varphi\|_{s,2} \quad (s \in R), \tag{3.2}$$

$$\left\|D^{\frac{3}{4}} W(t)\varphi\right\|_{L_T^4 L_x^\infty} \leqslant C\|\varphi\|_2, \tag{3.3}$$

$$\left\|\partial_x^2 W(t)\varphi\right\|_{L_x^\infty L_T^2} \leqslant C\|\varphi\|_2, \tag{3.4}$$

$$\left\|W(t)\varphi\right\|_{L_x^4 L_T^\infty} \leqslant C\left\|\varphi\right\|_{\frac{1}{4},2}, \tag{3.5}$$

and

$$\left\|W(t)\varphi\right\|_{L_x^2 L_T^\infty} \leqslant C\|\varphi\|_{s,2} \quad \left(s > \frac{5}{4}\right). \tag{3.6}$$

Indeed, (3.2) and (3.5) are respectively duplications of (2.3) and (2.15); (3.3) is obtained from (2.10) by applying it to the special case $m = 5$, $\alpha = 3/2$, $\theta = 1$, $p = \infty$ and $q = 4$; (3.4) and (3.6) follow respectively from (2.14) and (2.17), by taking $m = 5$. We now prove

Lemma 3.1. *Let $P(\xi)$ and $W(t)$ be as above. Then for any $\delta > 0$ there exists a constant $C > 0$ depending only on δ, β and γ, such that for any $\varphi \in L^2(R)$ and $0 < T \leqslant \delta$,*

$$\left\| \partial_x W(t)\varphi \right\|_{L_x^4 L_T^2} \leqslant C\|\varphi\|_2. \tag{3.7}$$

Proof. Consider the analytic family of operators T_z ($z \in C$, $0 \leqslant \operatorname{Re} z \leqslant 1$) defined by

$$(T_z\varphi)(x,t) = D^{2z} W(t)\varphi(x).$$

Since $D^{i2\nu}$ ($\nu \in R$) is unitary on $L^2(R)$, we get from (2.14) and (2.23) that

$$\|T_{1+i\nu}\varphi\|_{L_x^\infty L_T^2} \leqslant C\|D^{i2\nu}\varphi\|_2 = C\|\varphi\|_2,$$
$$\|T_{i\nu}\varphi\|_{L_x^2 L_T^2} \leqslant C\|D^{i2\nu}\varphi\|_2 = C\|\varphi\|_2$$

($\nu \in R$). It follows from the Stein interpolation theorem that

$$\left\| D^1 W(t)\varphi \right\|_{L_x^4 L_T^2} = \|T_{\frac{1}{2}}\varphi\|_{L_x^4 L_T^2} \leqslant C\|\varphi\|_2.$$

Let H be the Hilbert transformation. Then $\partial_x = -D_x^1 H$. Since H is unitary on $L^2(R)$, the desired estimate follows immediately. $\square$

To prove the existence and uniqueness of a local solution for Eq. (3.1) we introduce a function space X_T^s (for given $s > 1/4$ and $T > 0$) as follows:

$$X_T^s = \left\{ u \in C\big([-T,T], H^s(R)\big):\ |||u|||_{s,T} \equiv \sum_{j=1}^{5} [u]_{j,s,T} < \infty \right\},$$

where

$$[u]_{1,s,T} = \sup_{|t|\leqslant T} \left\| u(\cdot,t) \right\|_{s,2},$$

$$[u]_{2,s,T} = [u]_{2,T} = \|\partial_x u\|_{L_T^4 L_x^\infty} = \left(\int_{-T}^{T} \left\| \partial_x u(\cdot,t) \right\|_\infty^4 dt \right)^{\frac{1}{4}},$$

$$[u]_{3,s,T} = \|D^{s+2}u\|_{L_x^\infty L_T^2} = \sup_{x\in R} \left(\int_{-T}^{T} \left| D_x^{s+2} u(x,t) \right|^2 dt \right)^{\frac{1}{2}},$$

$$[u]_{4,s,T} = \|D^s \partial_x u\|_{L_x^4 L_T^2} = \left(\int_{-\infty}^{\infty} \left(\int_{-T}^{T} \left| D^s \partial_x u(x,t) \right|^2 dt \right)^2 dx \right)^{\frac{1}{4}},$$

$$[u]_{5,s,T} = [u]_{5,T} = \|u\|_{L_x^4 L_T^\infty} = \left(\int_{-\infty}^{\infty} \sup_{|t|\leqslant T} \left| u(x,t) \right|^4 dx \right)^{\frac{1}{4}}.$$

Clearly, $(X_T^s, \|\!|\cdot|\!\|_{s,T})$ is a Banach space. For a given $u_0 \in H^s(R)$, let S be the mapping $u \to Su$ defined by

$$(Su)(\cdot,t) = W(t)u_0 + \int_0^t W(t-\tau)(u\partial_x u)(\cdot,\tau)\,d\tau, \quad \forall u \in X_T^s.$$

In the sequel we shall prove that S is well defined and it maps X_T^s into itself.

Lemma 3.2. *Assume that $s > 1/4$ and $T > 0$. Given $u_0 \in H^s(R)$ and $v \in L^1([-T,T], H^s(R))$, let*

$$w(\cdot,t) = W(t)u_0 + \int_0^t W(t-\tau)v(\cdot,\tau)\,d\tau$$

$(|t| \leqslant T)$. Then $w \in X_T^s$. Moreover, there exists a constant $C_T > 0$ depending only on s, β, γ and T such that

$$\|\!|w|\!\|_{s,T} \leqslant C_T\|u_0\|_{s,2} + C_T\int_{-T}^{T} \big\|v(\cdot,t)\big\|_{s,2}\,dt. \tag{3.8}$$

If $0 < T \leqslant \delta$ for some $\delta > 0$ then $C_T \leqslant C_\delta$.

Proof. By (3.2) it is clear that $w \in C([-T,T], H^s(R))$, and

$$[w]_{1,s,T} = \sup_{|t|\leqslant T} \big\|w(\cdot,t)\big\|_{s,2} \leqslant \|u_0\|_{s,2} + \int_{-T}^{T} \big\|v(\cdot,t)\big\|_{s,2}\,dt. \tag{3.9}$$

To estimate $[w]_{2,T}$ we write

$$\partial_x w(\cdot,t) = -D^{\frac{3}{4}}W(t)HD^{\frac{1}{4}}u_0 - \int_0^t D^{\frac{3}{4}}W(t-\tau)HD^{\frac{1}{4}}v(\cdot,\tau)\,d\tau.$$

By (3.3) we have, for $0 < T \leqslant \delta$,

$$\begin{aligned}
\|\partial_x w\|_{L_T^4 L_x^\infty} \leqslant{}& C\|HD^{\frac{1}{4}}u_0\|_2 + \left(\int_0^T\left(\int_0^t \big\|D^{\frac{3}{4}}W(t-\tau)HD^{\frac{1}{4}}v(\cdot,\tau)\big\|_\infty\,d\tau\right)^4 dt\right)^{\frac{1}{4}}\\
&+ \left(\int_{-T}^0\left(\int_t^0 \big\|D^{\frac{3}{4}}W(t-\tau)HD^{\frac{1}{4}}v(\cdot,\tau)\big\|_\infty\,d\tau\right)^4 dt\right)^{\frac{1}{4}}\\
\leqslant{}& C\|u_0\|_{s,2} + \int_0^T\left(\int_\tau^T \big\|D^{\frac{3}{4}}W(t-\tau)HD^{\frac{1}{4}}v(\cdot,\tau)\big\|_\infty^4\,dt\right)^{\frac{1}{4}} d\tau
\end{aligned}$$

$$+\int_{-T}^{0}\left(\int_{-T}^{\tau}\left\|D^{\frac{3}{4}}W(t-\tau)HD^{\frac{1}{4}}v(\cdot,\tau)\right\|_{\infty}^{4}dt\right)^{\frac{1}{4}}d\tau.$$

To estimate the second term we make change of variables $t = t' + \tau$, and get

$$\begin{aligned}
&\int_{0}^{T}\left(\int_{\tau}^{T}\left\|D^{\frac{3}{4}}W(t-\tau)HD^{\frac{1}{4}}v(\cdot,\tau)\right\|_{\infty}^{4}dt\right)^{\frac{1}{4}}d\tau \\
&\quad\leqslant \int_{0}^{T}\left(\int_{0}^{T}\left\|D^{\frac{3}{4}}W(t')HD^{\frac{1}{4}}v(\cdot,\tau)\right\|_{\infty}^{4}dt'\right)^{\frac{1}{4}}d\tau \\
&\quad\leqslant C\int_{0}^{T}\left\|HD^{\frac{1}{4}}v(\cdot,\tau)\right\|_{2}d\tau \leqslant \int_{-T}^{T}\left\|v(\cdot,t)\right\|_{s,2}dt.
\end{aligned}$$

The third term can be estimated in the same way. It follows that

$$[w]_{2,T} = \|\partial_x w\|_{L_T^4 L_x^\infty} \leqslant C\|u_0\|_{s,2} + C\int_{-T}^{T}\left\|v(\cdot,t)\right\|_{s,2}dt. \tag{3.10}$$

To estimate $[w]_{3,s,T}$ we note that since $D_x^{s+2} = -\partial_x^2 D_x^s$, by using the estimate (3.4) and treating the integral term with a similar argument as above we get

$$\begin{aligned}
[w]_{3,s,T} &= \|D^{s+2}w\|_{L_x^\infty L_T^2} \\
&\leqslant \left\|\partial_x^2 W(t)D^s u_0\right\|_{L_x^\infty L_T^2} + \sup_{x\in R}\left(\int_{-T}^{T}\left|\int_{0}^{t}\partial_x^2 W(t-\tau)D^s v(\cdot,\tau)\,d\tau\right|^2 dt\right)^{\frac{1}{2}} \\
&\leqslant C\|D^s u_0\|_2 + C\int_{-T}^{T}\left\|D^s v(\cdot,\tau)\right\|_2 d\tau \\
&\leqslant C\|u_0\|_{s,2} + C\int_{-T}^{T}\left\|v(\cdot,t)\right\|_{s,2}dt.
\end{aligned} \tag{3.11}$$

To estimate $[w]_{4,s,T}$ we use the estimate (3.7) and treat the integral term similarly as above. Then we get

$$\begin{aligned}
[w]_{4,s,T} &= \|D^s\partial_x w\|_{L_x^4 L_T^2} \\
&\leqslant \left\|\partial_x W(t)D^s u_0\right\|_{L_x^4 L_T^2} + \left\|\left(\int_{-T}^{T}\left|\int_{0}^{t}\partial_x W(t-\tau)D^s v(\cdot,\tau)\,d\tau\right|^2 dt\right)^{\frac{1}{2}}\right\|_4
\end{aligned}$$

$$\leqslant C\|D^s u_0\|_2 + C\int_{-T}^{T} \left\|D^s v(\cdot,\tau)\right\|_2 d\tau$$

$$\leqslant C\|u_0\|_{s,2} + C\int_{-T}^{T} \left\|v(\cdot,t)\right\|_{s,2} dt. \tag{3.12}$$

Finally, using the estimate (3.5) we obtain

$$\begin{aligned}[w]_{5,T} &= \|w\|_{L_x^4 L_T^\infty} \\ &\leqslant \left\|W(t)u_0\right\|_{L_x^4 L_T^\infty} + \left\|\sup_{|t|\leqslant T}\left|\int_0^t W(t-\tau)v(\cdot,\tau)\,d\tau\right|\right\|_4 \\ &\leqslant C\|u_0\|_{s,2} + \int_{-T}^{T}\left\|\sup_{|t'|\leqslant T}\left|W(t')v(\cdot,\tau)\right|\right\|_4 d\tau \\ &\leqslant C\|u_0\|_{s,2} + C\int_{-T}^{T}\left\|v(\cdot,t)\right\|_{s,2} dt. \end{aligned} \tag{3.13}$$

Summing up (3.9)–(3.13), we get the desired result. □

Lemma 3.3. *Assume that $s > 1/4$ and $T > 0$. Then for any $u \in X_T^s$ we have $u\partial_x u \in L^2([-T,T], H^s(R))$, and*

$$\left(\int_{-T}^{T}\left\|(u\partial_x u)(\cdot,t)\right\|_{s,2}^2 dt\right)^{\frac{1}{2}} \leqslant C_T |||u|||_{s,T}^2. \tag{3.14}$$

If $0 < T \leqslant \delta$ for some $\delta > 0$ then $C_T \leqslant C_\delta$.

Proof. Since $\|\cdot\|_{s,2} \sim \|\cdot\|_2 + \|D^s(\cdot)\|_2$, we only need to prove that

$$\|u\partial_x u\|_{L_T^2 L_x^2} \leqslant C_T |||u|||_{s,T}^2, \qquad \left\|D^s(u\partial_x u)\right\|_{L_T^2 L_x^2} \leqslant C_T |||u|||_{s,T}^2.$$

In fact, it is clear that

$$\|u\partial_x u\|_{L_T^2 L_x^2} \leqslant \|u\|_{L_T^\infty L_x^2}\|\partial_x u\|_{L_T^2 L_x^\infty} \leqslant [u]_{1,s,T}\cdot T^{\frac{1}{2}}[u]_{2,T} \leqslant T^{\frac{1}{2}}|||u|||_{s,T}^2,$$

and, by [19, Theorem A.8],

$$\begin{aligned}\left\|D^s(u\partial_x u)\right\|_{L_T^2 L_x^2} &= \left\|D^s(u\partial_x u)\right\|_{L_x^2 L_T^2} \\ &\leqslant \|uD^s\partial_x u\|_{L_x^2 L_T^2} + \|D^s u\partial_x u\|_{L_x^2 L_T^2} + C\|u\|_{L_x^4 L_T^\infty}\|D^s\partial_x u\|_{L_x^4 L_T^2} \\ &\leqslant C\|u\|_{L_x^4 L_T^\infty}\|D^s\partial_x u\|_{L_x^4 L_T^2} + C\|D^s u\|_{L_T^\infty L_x^2}\|\partial_x u\|_{L_T^2 L_x^\infty} \\ &\leqslant C\|u\|_{L_x^4 L_T^\infty}\|D^s\partial_x u\|_{L_x^4 L_T^2} + C[u]_{1,s,T}\cdot T^{\frac{1}{2}}[u]_{2,T} \leqslant C(T)|||u|||_{s,T}^2.\end{aligned}$$

Hence the desired assertion follows. □

Since $L^2([-T,T],H^s(R))\subset L^1([-T,T],H^s(R))$ and

$$\int_{-T}^{T}\|v(\cdot,t)\|_{s,2}\,dt\leqslant T^{\frac{1}{2}}\left(\int_{-T}^{T}\|v(\cdot,t)\|_{s,2}^2\,dt\right)^{\frac{1}{2}},\tag{3.15}$$

by Lemmas 3.2 and 3.3 it follows that the mapping S is well defined and it maps X_T^s into itself. Moreover, these lemmas and (3.15) ensure that for any $\delta>0$ there exists a constant $C>0$ depending only on s, δ, β and γ, such that for any $0<T\leqslant\delta$,

$$|||Su|||_{s,T}\leqslant C\|u_0\|_{s,2}+CT^{\frac{1}{2}}|||u|||_{s,T}^2,\quad \forall u\in X_T^s.\tag{3.16}$$

We now take $\delta=1$, and let $M>0$ be a sufficiently large number such that

$$M\geqslant C\|u_0\|_{s,2}+1,$$

where C is the constant in (3.16). Then for $T>0$ sufficiently small such that $T\leqslant T_0$, where $T_0=(CM^2)^{-\frac{1}{2}}$, we infer from (3.16) that S maps the closed ball

$$X_{T,M}^s=\left\{u\in X_T^s\colon\ |||u|||_{s,T}\leqslant M\right\}$$

into itself.

Lemma 3.4. *Let $s>1/4$, and let M, T_0 be as above. Then there exists $0<T_0'\leqslant T_0$ such that for any $0<T\leqslant T_0'$, S is a contraction mapping on $X_{T,M}^s$.*

Proof. Given $u_1,u_2\in X_{T,M}^s$, we have

$$(Su_1)(\cdot,t)-(Su_2)(\cdot,t)=\int_0^t W(t-\tau)\left\{(u_1-u_2)\partial_x u_1+u_2\partial_x(u_1-u_2)\right\}(\cdot,\tau)\,d\tau.$$

A similar argument as in the proof of Lemma 3.2 shows that

$$\begin{aligned}&\left(\int_{-T}^{T}\left\|(u_1-u_2)\partial_x u_1+u_2\partial_x(u_1-u_2)\right\|_{s,2}^2\,dt\right)^{\frac{1}{2}}\\&\quad\leqslant C|||u_1-u_2|||_{s,T}\left(|||u_1|||_{s,T}+|||u_2|||_{s,T}\right),\end{aligned}$$

so that by (3.15) and Lemma 3.1 we have

$$\begin{aligned}|||Su_1-Su_2|||_{s,T}&\leqslant CT^{\frac{1}{2}}|||u_1-u_2|||_{s,T}\left(|||u_1|||_{s,T}+|||u_2|||_{s,T}\right)\\&\leqslant 2CMT^{\frac{1}{2}}|||u_1-u_2|||_{s,T}.\end{aligned}$$

Hence, if we take $0<T_0'\leqslant T_0$ sufficiently small such that $2CM(T_0')^{1/2}<1$, then S is a contraction mapping on $X_{T,M}^s$. $\square$

By Lemma 3.3 and the Banach fixed point theorem, we conclude that S has a unique fixed point in $X_{T,M}^s$ provided $0<T\leqslant T_0'$. Thus we have proved the following result.

Theorem 3.5. *Assume that $u_0 \in H^s(R)$ and $s > 1/4$. Then there exists $T > 0$ depending only on a, β, γ and the upper bound of $\|u_0\|_{s,2}$, such that the problem* (1.2)–(1.3) *has a solution u on $R \times [-T, T]$, satisfying*

$$u \in C\big([-T,T], H^s(R)\big) \cap L_x^4 L_T^\infty,$$
$$\partial_x u \in L_T^4 L_x^\infty, \quad D_x^s \partial_x u \in L_x^4 L_T^2, \quad D_x^{s+2} u \in L_x^\infty L_T^2.$$

Moreover, the solution is unique under these conditions.

In addition to the usual momentum and energy conservation laws, the Kawahara (1.1) equation has a third conservation law which reads as follows:

$$\frac{\partial}{\partial t}\left\{\frac{1}{3}au^3 - \beta(\partial_x u)^2 + \gamma\big(\partial_x^2 u\big)^2\right\} + \frac{\partial}{\partial x}F\big(u, \partial_t u, \partial_x u, \ldots, \partial_t \partial_x^3 u, \partial_x^4 u\big) = 0,$$

where F represents a polynomial in its arguments. Hence, using Theorem 3.5 and a standard argument, we obtain the following result.

Theorem 3.6. *Assume that $u_0 \in H^s(R)$ and $s \geqslant 2$. Then the problem* (1.2)–(1.3) *has a unique global solution u on R^2, satisfying the conditions* (i) *and* (ii) *below*:

(i) *$u \in C(R, H^s(R)) \cap L^\infty(R, H^2(R))$, and for $t \in R$,*

$$\big\|u(\cdot,t)\big\|_2 = \text{const},$$
$$\big\|\partial_x^2 u(\cdot,t)\big\|_2^2 - \frac{\beta}{\gamma}\big\|\partial_x u(\cdot,t)\big\|_2^2 + \frac{a}{3\gamma}\int_{-\infty}^{\infty} u^3(x,t)\,dx = \text{const}.$$

(ii) *There exists constant $C > 0$ depending only on a, β and γ such that for any finite $T > 0$,*

$$\sup_{|t|\leqslant T}\big\|u(\cdot,t)\big\|_{s,2} \leqslant \|u_0\|_{s,2}\exp\left(C\int_{-T}^{T}\big\|\partial_x u(\cdot,t)\big\|_\infty\,dt\right).$$

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