Convergence of the binomial tree method for Asian options in jump-diffusion models ☆

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Abstract

The binomial tree methods (BTM), first proposed by Cox, Ross and Rubinstein [J. Cox, S. Ross, M. Rubinstein, Option pricing: A simplified approach, J. Finan. Econ. 7 (1979) 229–264] in diffusion models and extended by Amin [K.I. Amin, Jump diffusion option valuation in discrete time, J. Finance 48 (1993) 1833–1863] to jump-diffusion models, is one of the most popular approaches to pricing options. In this paper, we present a binomial tree method for Asian options in jump-diffusion models and show its equivalence to certain explicit difference scheme. Employing numerical analysis and the notion of viscosity solution, we prove the uniform convergence of the binomial tree method for European-style and American-style Asian options.
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1. Introduction

An Asian option gives the holder a payoff that depends on the average price of the underlying asset over a specified period of time (see [16]). The binomial tree method (BTM) is one of the most popular approaches to pricing vanilla options in diffusion model (see [6]). By introducing

an additional path-dependent variable at each node, BTM can be extended to the valuation of Asian option (see [9,11]).

In the paper, we study the BTM for Asian options in jump-diffusion models. It is well known that jump-diffusion models give a better explanation of sudden changes of asset prices in the market. Amin [2] first generalized Cox, Ross and Rubinstein's BTM to jump-diffusion models for vanilla options. Xu, Qian and Jiang [17] gave an optimal error estimation of European options in Amin's model. In essence, BTM belongs to the probabilistic one; however, it can be proved that the binomial tree method is consistent with certain explicit difference scheme (see [10,11, 14,17]). By virtue of the notion of viscosity solution (see [7]), Barles and Souganidis [4], Barles, Daher and Romano [3], and Jiang and Dai [11] presented a framework to prove the convergence of difference schemes for parabolic equations and convergence of BTM for vanilla and path-dependent options in diffusion models. Qian, Xu, Jiang and Bian [14] proved the convergence of BTM for American options in jump-diffusion models. In this paper, following the ideas of Amin [2] and Jiang and Dai [11], we develop a binomial tree method for Asian options in jump-diffusion models and use numerical analysis and the theory of viscosity solution to prove the convergence of this algorithm.

The rest of this paper is organized as follows. In the next section, we give the continuous model for Asian options in jump-diffusion models. Section 3 is devoted to the construction of BTM for Asian options in jump-diffusion models. In Section 4, we discuss the equivalence of the BTM and an explicit difference scheme and therefore give the convergence of the BTM for European-style Asian options. Finally, we prove the convergence of the BTM for some American-style Asian options.

2. Continuous model

Suppose there is a financial market with two assets (B_t, S_t). The first one is a risk-free asset whose price B_t is governed by the equation $dB_t = rB_t\,dt$ where r is the constant positive interest rate, and the other is a risky asset. In a given probability space $(\Omega, \mathcal{F}, P)$, the underlying asset price evolves according to the stochastic differential equation

$$\frac{dS_t}{S_t} = (\mu - q)\,dt + \sigma\,dW_t + U\,dN_t, \tag{2.1}$$

where the coefficients μ, q, σ are positive constants, q is the dividend yield, $(W_t)_{t \geqslant 0}$ is a standard Brownian motion, $(N_t)_{t \geqslant 0}$ is a Poisson process with constant intensity λ, and U is a square integrable random variable taking values in $(-1, +\infty)$ (since the price of a financial asset should be positive).

Consider an Asian option with the life time $[0, T]$ and the payoff

$$g(S, A) = \begin{cases} (S - A)^+ & \text{for floating strike call,} \\ (A - S)^+ & \text{for floating strike put,} \\ (A - K)^+ & \text{for fixed strike call,} \\ (K - A)^+ & \text{for fixed strike put,} \end{cases}$$

where constant K is the strike price and

$$A = \begin{cases} \frac{1}{t} \int_0^t S(\tau)\,d\tau & \text{for arithmetic average,} \\ \exp\left(\frac{1}{t} \int_0^t \ln S(\tau)\,d\tau\right) & \text{for geometric average.} \end{cases}$$

Let $V(S, A, t)$ be the Asian option price at time t with stock price S and path-dependent variable A. Because the market here is not complete, we may assume risk-neutrality for brevity (see [8,14]). Then we must have $\mu = r - \lambda k$ where $k = E[U]$ and $E[\cdot]$ is the expectation operator over the random variable U (see [13]). Using an argument similar to Pham [13], it can be shown that the European-style Asian option's price solves the following partial integro-differential equation:

$$\begin{cases} \mathcal{L}V(S, A, t) = 0, & t \in (0, T), \ (S, A) \in \mathcal{D} = (0, \infty) \times (0, \infty), \\ V(S, A, T) = g(S, A), & (S, A) \in \mathcal{D}, \end{cases} \tag{2.2}$$

where $\mathcal{L}$ is the parabolic integro-differential operator defined as

$$\mathcal{L}V = \begin{cases} \frac{\partial V}{\partial t} + \frac{S-A}{t}\frac{\partial V}{\partial A} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + (r - q - \lambda k)S\frac{\partial V}{\partial S} - (r + \lambda)V \\ \quad + \lambda \int_{-1}^{\infty} V(S(1 + y), A, t)\, d\mathcal{N}(y), \quad \text{for arithmetic average,} \\ \frac{\partial V}{\partial t} + \frac{A}{t}(\ln S - \ln A)\frac{\partial V}{\partial A} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + (r - q - \lambda k)S\frac{\partial V}{\partial S} - (r + \lambda)V \\ \quad + \lambda \int_{-1}^{\infty} V(S(1 + y), A, t)\, d\mathcal{N}(y), \quad \text{for geometric average} \end{cases}$$

and $\mathcal{N}(x)$ is the distribution function of random variable U.

For the American-style Asian options, (2.2) is replaced by a parabolic variational inequality given by

$$\begin{cases} \min\{-\mathcal{L}V(S, A, t), V(S, A, t) - g(S, A)\} = 0, & t \in (0, T), \ (S, A) \in \mathcal{D}, \\ V(S, A, T) = g(S, A), & (S, A) \in \mathcal{D}. \end{cases} \tag{2.3}$$

Remark 2.1. Note that $\mathcal{L}V$ is not well defined at $t = 0$. For European-style Asian options, it is easy to check that we can take the following transformation

$$I = \begin{cases} t A & \text{for arithmetic average,} \\ t \ln A & \text{for geometric average,} \end{cases}$$

to remove the singularity. But for American-style Asian options, one cannot remove the singularity in (2.3) at $t = 0$ because it is a nonlinear problem. In this paper, we always confine ourselves to the interval $(0, T]$ instead of $[0, T]$ for American-style Asian options.

3. Binomial tree method

In this section, we develop the BTM for Asian options in a jump-diffusion model. The idea stems from Amin [2]. In our discrete time market, trade occurs only on discrete dates in the interval $[0, T]$. Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, N be the number of discrete time points, $\Delta t = \frac{T}{N}$ and $t_i = i\, \Delta t$ for $i = 0, 1, 2, \ldots, N$. We assume that only two assets are traded in the market. The first is a bond B which has a riskless rate of return of ρ in every period. The second is a risky asset, for example, a stock. We assume that the underlying stock price S can take on values in a discrete set $\{u^l : l \in \mathbb{Z}\}$ with $u = e^{\sigma\sqrt{\Delta t}}$. We also assume that this stock pay a dividend $\eta = e^{q\Delta t}$ for positive constant dividend yield q in each period.

We now describe the stock price dynamics. In each period, the stock price undergoes either of two different types of price changes. In most periods, the stock price undergoes only "local" changes. Analogous to the Cox, Ross and Rubinstein's BTM [6], the stock price S moves up to Su or down to Su^{-1}. This price change is the discrete counterpart of the stock price changes due to the diffusion component in the continuous time case.

The stock price can also be changed due to the occurrence of a "rare event," which has a low probability of occurring in any given period. It corresponds to the arrival of important information which causes a large change in the stock price. When the "rare event" occurs, the stock price "jumps" to potentially any state Su^l ($l \in \mathbb{Z}$) at the next date. But we may assume that the two different kinds of changes in stock price are mutually exclusive, i.e., the stock price can not "jump" to the adjacent states $Su^{\pm 1}$. However, in the limit as $N \to \infty$, it does not matter whether we define the "jump" event in an adjacent state or over every point in the state space.

Let $V^n(S, A)$ be the option price at time t_n with stock price S and path-dependent variable A. Here we have

$$A = \begin{cases} \frac{1}{n} \sum_{i=1}^n S_{t_i} & \text{for arithmetic average,} \\ (\prod_{i=1}^n S_{t_i})^{\frac{1}{n}} & \text{for geometric average,} \end{cases}$$

where S_{t_i} stands for the stock price of such path at the time t_i, $i = 0, 1, \ldots, n$ (note $S_{t_n} = S$). If at time t_{n+1}, stock price S changes to Su^l ($l \in \mathbb{Z}$), A will consequently become A_l, where

$$A_l = \begin{cases} \frac{nA + Su^l}{n+1} & \text{for arithmetic average,} \\ (A^n Su^l)^{\frac{1}{n+1}} & \text{for geometric average.} \end{cases}$$

At time t_n, we consider a portfolio with one option, Δ share of stock and B dollars in the riskless bond. If we assume that the initial investment in this portfolio at time t_n is zero, then the portfolio value Π^n is given by

$$\Pi^n = \Delta S + B + V^n(S, A) = 0. \tag{3.1}$$

Suppose now the investor wishes to eliminate the risk of this portfolio due to the local changes of stock price in the interval $[t_n, t_{n+1}]$. Then the portfolio value must be equal (not necessarily zero) in the both adjoint states $Su^{\pm 1}$ at time t_{n+1}. This implies that

$$\begin{aligned} \Pi_{\pm 1}^{n+1} &= \Delta Su\eta + \rho B + V^{n+1}(Su, A_{+1}) \\ &= \Delta Su^{-1}\eta + \rho B + V^{n+1}(Su^{-1}, A_{-1}). \end{aligned} \tag{3.2}$$

Solving the above equation for Δ yields

$$\Delta = -\frac{V^{n+1}(Su, A_{+1}) - V^{n+1}(Su^{-1}, A_{-1})}{S\eta(u - u^{-1})}. \tag{3.3}$$

Now, eliminating Δ and B from (3.2) by using (3.1) and (3.3), we obtain

$$\Pi_{\pm 1}^{n+1} = \tilde{p}V^{n+1}(Su, A_{+1}) + (1 - \tilde{p})V^{n+1}(Su^{-1}, A_{-1}) - \rho V^n(S, A), \tag{3.4}$$

where

$$\tilde{p} = \frac{\rho/\eta - u^{-1}}{u - u^{-1}}. \tag{3.5}$$

Therefore, if there is no jump (rare event), the portfolio is riskless and the expression (3.4) must be equal to zero, which is the BTM for Asian options in diffusion models.

Now, consider the portfolio value when a rare event occurs. Let U be a relative amplitude of jump on the stock when the rare event occurs and y be the index of the state induced by the jump at the next date. In other words, if the stock price at time t_n is S and a rare event occurs, the stock

price at time t_{n+1} will be equal to $S(1 + U) = Su^y$. Then the portfolio value Π_y^{n+1} can be written as

$$\Pi_y^{n+1} = \Delta S(1 + U)\eta + \rho B + V^{n+1}(Su^y, A_y). \tag{3.6}$$

Eliminating Δ and B from (3.6) by using (3.1) and (3.3), we get

$$\Pi_y^{n+1} = -\frac{V^{n+1}(Su, A_{+1}) - V^{n+1}(Su^{-1}, A_{-1})}{u - u^{-1}}\left(U + 1 - \frac{\rho}{\eta}\right)$$
$$+ V^{n+1}(Su^y, A_y) - \rho V^n(S, A). \tag{3.7}$$

Analogous to Merton [12] and Amin [2], we assume the jump risk is diversifiable, which implies that the expectation of the portfolio value in the next period with respect to the distribution of the rare event must be zero. Let the probability of a rare event in time interval Δt be equal to $\hat{\lambda}$ (corresponding to the Poisson jump component of the continuous time process in (2.1), we have $\hat{\lambda} = \lambda e^{-\lambda \Delta t} \Delta t = \lambda \Delta t + O(\Delta t^2)$). Let $E_U[\cdot]$ be the expectation operator with respect to the distribution of U. Taking the expectation of the portfolio value at time t_{n+1} with respect to the jump distribution and equating it to zero yields

$$0 = E_U[\Pi^{n+1}] = \hat{\lambda} E_U[\Pi_y^{n+1}] + (1 - \hat{\lambda})\Pi_{\pm 1}^{n+1}. \tag{3.8}$$

Substituting the portfolio values from (3.4) and (3.7) for those in (3.8) yields

$$\rho V^n(S, A) = \hat{\lambda}\left[E_U[V^{n+1}(Su^y, A_y)]\right.$$
$$- \frac{V^{n+1}(Su, A_{+1}) - V^{n+1}(Su^{-1}, A_{-1})}{u - u^{-1}}\left(E_U[U] + 1 - \frac{\rho}{\eta}\right)\right]$$
$$+ (1 - \hat{\lambda})[\tilde{p}V^{n+1}(Su, A_{+1}) + (1 - \tilde{p})V^{n+1}(Su^{-1}, A_{-1})]. \tag{3.9}$$

Further, replacing $\tilde{p}$ in (3.9) by (3.5), we have

$$\rho V^n(S, A) = (1 - \hat{\lambda})[pV^{n+1}(Su, A_{+1}) + (1 - p)V^{n+1}(Su^{-1}, A_{-1})]$$
$$+ \hat{\lambda} E_U[V^{n+1}(Su^y, A_y)], \tag{3.10}$$

where

$$p = \frac{\frac{\rho/\eta - \hat{\lambda}(E_U[U]+1)}{1 - \hat{\lambda}} - u^{-1}}{u - u^{-1}}. \tag{3.11}$$

Let the cumulative function of U be given by $\mathcal{N}(x)$ for $x > -1$ (noting that $k = E_U[U] = \int_{-1}^{+\infty} x\, d\mathcal{N}(x)$), and $\hat{p}_l$ ($l \in \mathbb{Z}$) represent the discrete probability distribution. Then, we have

$$\hat{p}_l = \text{Prob}\left\{\ln(1 + U) \in \left[\left(l - \frac{1}{2}\right)\sigma\sqrt{\Delta t}, \left(l + \frac{1}{2}\right)\sigma\sqrt{\Delta t}\right)\right\}$$
$$= \mathcal{N}(e^{(l+\frac{1}{2})\sigma\sqrt{\Delta t}}) - \mathcal{N}(e^{(l-\frac{1}{2})\sigma\sqrt{\Delta t}}) \tag{3.12}$$

and

$$E_U[V^{n+1}(Su^y, A_y)] = \sum_{l \in \mathbb{Z}} V^{n+1}(Su^l, A_l)\hat{p}_l. \tag{3.13}$$

Hence, from (3.10)–(3.13), we obtain the BTM for European-style Asian options as follows:

$$
\begin{cases}
V^n(S, A) = \dfrac{1}{\rho}\Big\{(1 - \hat{\lambda})\big[pV^{n+1}(Su, A_{+1}) + (1 - p)V^{n+1}\big(Su^{-1}, A_{-1}\big)\big] \\
\qquad\qquad + \hat{\lambda}\displaystyle\sum_{l\in\mathbb{Z}} V^{n+1}\big(Su^l, A_l\big)\hat{p}_l\Big\}, \\
V^N(S, A) = g(S, A).
\end{cases}
\tag{3.14}
$$

For American-style Asian options, the investor can choose to exercise the option if the current payoff of the option is worth more than its value being held till the next period. Thus, the BTM for American-style Asian options can be written instead of (3.14) as follows:

$$
\begin{cases}
V^n(S, A) = \max\Big\{\dfrac{1}{\rho}\Big\{(1 - \hat{\lambda})\big[pV^{n+1}(Su, A_{+1}) + (1 - p)V^{n+1}\big(Su^{-1}, A_{-1}\big)\big] \\
\qquad\qquad + \hat{\lambda}\displaystyle\sum_{l\in\mathbb{Z}} V^{n+1}\big(Su^l, A_l\big)\hat{p}_l\Big\}, g(S, A)\Big\}, \\
V^N(S, A) = g(S, A).
\end{cases}
\tag{3.15}
$$

Theorem 3.1. *The binomial tree method* (3.14) (*respectively* (3.15)) *is consistent with the corresponding partial integro-differential equation* (2.2) (*respectively* (2.3)).

Proof. We only give the proof for the case of European-type arithmetic average options since other cases follow similarly. Noting $u = e^{\sigma\sqrt{\Delta t}}$, $\rho = e^{r\Delta t}$, $\eta = e^{q\Delta t}$, $\hat{\lambda} = \lambda\Delta t + O(\Delta t^2)$ and letting $E_U[U] = k$, we need to show that for a sufficiently smooth function $\phi(S, A, t)$ and $(S_0, A_0, t_0) \in \mathcal{D} \times (0, T)$,

$$
\lim_{\substack{\Delta t\to 0 \\ (S,A,t)\to(S_0,A_0,t_0)}} \frac{1}{\Delta t}\big[\phi(S, A, t - \Delta t) - F_{\Delta t}\phi(S, A, t)\big] = -\mathcal{L}V\big|_{(S_0,A_0,t_0)},
$$

where

$$
F_{\Delta t}\phi(S, A, t) = e^{-r\Delta t}\Big\{(1 - \hat{\lambda})\big[p\phi(Su, A_{+1}, t) + (1 - p)\phi\big(Su^{-1}, A_{-1}, t\big)\big] \\
\qquad\qquad + \hat{\lambda}\sum_{l\in\mathbb{Z}}\phi\big(Su^l, A_l, t\big)\hat{p}_l\Big\},
\tag{3.16}
$$

$$
p = \frac{\frac{e^{(r-q)\Delta t} - \hat{\lambda}(1+k)}{1 - \hat{\lambda}} - u^{-1}}{u - u^{-1}} = \frac{1}{2} + \left(r - q - \lambda k - \frac{\sigma^2}{2}\right)\Delta t + O\big(\Delta t^{\frac{3}{2}}\big),
\tag{3.17}
$$

$$
A_l = \frac{(t - \Delta t)A + Su^l\Delta t}{t}.
\tag{3.18}
$$

It is easy to show that

$$
1 - e^{-r\Delta t}(1 - \hat{\lambda}) = (r + \lambda)\Delta t + O\big(\Delta t^2\big),
$$

$$
e^{-r\Delta t}(1 - \hat{\lambda})\big[p(u - 1) + (1 - p)\big(u^{-1} - 1\big)\big] = (r - q - \lambda k)\Delta t + O\big(\Delta t^2\big),
$$

$$
e^{-r\Delta t}(1 - \hat{\lambda})\big[p(u - 1)^2 + (1 - p)\big(u^{-1} - 1\big)^2\big] = \sigma^2\Delta t + O\big(\Delta t^2\big),
$$

$$
e^{-r\Delta t}(1 - \hat{\lambda})\big[p(u - 1)^3 + (1 - p)\big(u^{-1} - 1\big)^3\big] = O\big(\Delta t^2\big)
$$

and noting that $A^l - A = \frac{Su^l - A}{t} \Delta t$, we have

$$
e^{-r\Delta t} \hat{\lambda} \sum_{l \in \mathbb{Z}} \phi(Su^l, A_l, t) \hat{p}_l = \lambda \Delta t \int_{-\infty}^{\infty} \phi(Se^z, A, t) \, d\mathcal{N}(e^z - 1) + O\left(\Delta t^{\frac{3}{2}}\right)
$$

$$
= \lambda \Delta t \int_{-1}^{\infty} \phi(S(1+y), A, t) \, d\mathcal{N}(y) + O\left(\Delta t^{\frac{3}{2}}\right),
$$

$$
e^{-r\Delta t}(1 - \hat{\lambda}) \big[p(A_{+1} - A) + (1 - p)(A_{-1} - A) \big] = \frac{S - A}{t} \Delta t + O\left(\Delta t^2\right),
$$

$$
e^{-r\Delta t}(1 - \hat{\lambda}) \big[p(u - 1)(A_{+1} - A) + (1 - p)\left(u^{-1} - 1\right)(A_{-1} - A) \big] = O\left(\Delta t^2\right).
$$

Then, by Taylor expansions, we get from (3.16) and all the above equalities that

$$
\frac{1}{\Delta t} \big[\phi(S, A, t - \Delta t) - F_{\Delta t} \phi(S, A, t) \big]
$$

$$
= \bigg[-\frac{\partial \phi}{\partial t} - \frac{1}{t}(S - A)\frac{\partial \phi}{\partial A} - \frac{1}{2}\sigma^2 \frac{\partial^2 \phi}{\partial S}
$$

$$
- (r - q - \lambda k)S \frac{\partial \phi}{\partial S} + (r + \lambda)\phi - \lambda \int_{-1}^{\infty} \phi(S(1+y), A, t) \, d\mathcal{N}(y) \bigg]_{(S,A,t)} + O\left(\Delta t^{\frac{1}{2}}\right).
$$

This completes the proof. ☐

4. Finite difference method

In this section, we establish the relationship between BTM and finite difference methods for Asian options in jump-diffusion models. To illustrate the basic idea, we only consider the case for the European arithmetic average options since it is similar for other cases. The governing equation is

$$
\frac{\partial V}{\partial t} + \frac{S - A}{t} \frac{\partial V}{\partial A} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - q - \lambda k)S \frac{\partial V}{\partial S} - (r + \lambda)V
$$

$$
+ \lambda \int_{-1}^{\infty} V(S(1+y), A, t) \, d\mathcal{N}(y) = 0. \tag{4.1}
$$

Considering the characteristic line of the first-order partial differential equation $\frac{\partial V}{\partial t} + \frac{S-A}{t} \cdot \frac{\partial V}{\partial A} = 0$ in $[t_n, t_{n+1}]$, we have

$$
\begin{cases}
\dfrac{dt}{t} = \dfrac{dA}{S - A}, & t_n \leqslant t \leqslant t_{n+1}, \\
A(t_n) = A_k,
\end{cases}
$$

whose solution is given by

$$
A(t) = S - \frac{t_n}{t}(S - A_k), \quad t_n \leqslant t \leqslant t_{n+1}. \tag{4.2}
$$

It is well known that along this characteristic line, we have

$$\frac{\partial V}{\partial t} + \frac{S-A}{t} \cdot \frac{\partial V}{\partial A} = \frac{d}{dt} V\left(S, S - \frac{t_n}{t}(S - A_k), t\right).$$

Then (4.1) can be rewritten as

$$\frac{dV}{dt}\left(S, S - \frac{t_n}{t}(S - A_k), t\right) + \left[\frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - q - \lambda k) S \frac{\partial V}{\partial S} - (r + \lambda) V\right.$$

$$\left. + \lambda \int_{-1}^{\infty} V\big(S(1 + y), A, t\big) d\mathcal{N}(y)\right]_{A = S - \frac{t_n}{t}(S - A_k)} = 0. \tag{4.3}$$

Since our purpose is to derive the discrete scheme of (4.1), we may assume $S \frac{\partial^2 V}{\partial A^2}$, $S^2 \frac{\partial^2}{\partial S \partial A}$ and $S \frac{\partial V}{\partial A}$ are all bounded. Then, adding the following equation given by

$$\frac{\sigma^2}{2}\left(\frac{t - t_n}{t} S\right)^2 \frac{\partial^2 V}{\partial A^2} + \sigma^2 \frac{t - t_n}{t} S^2 \frac{\partial^2 V}{\partial A \partial S} + \left(r - \frac{\sigma^2}{2}\right) \frac{t - t_n}{t} S \frac{\partial V}{\partial A} \quad (t_n \leqslant t \leqslant t_n),$$

to (4.3) at $(S, S - \frac{t_n}{t}(S - A_n), t)$, we have

$$\left[\frac{dV}{dt} + \frac{\sigma^2}{2} S \frac{d}{dS}\left(S \frac{dV}{dS}\right) + \left(r - q - \lambda k - \frac{\sigma^2}{2}\right) S \frac{dV}{dS} - (r + \lambda) V\right]_{(S, S - \frac{t_n}{t}(S - A_k), t)}$$

$$+ \lambda \int_{-1}^{\infty} V\left(S(1 + y), S(1 + y) - \frac{t_n}{t}\big(S(1 + y) - A_k\big), t\right) d\mathcal{N}(y) + O(\Delta t) = 0, \tag{4.4}$$

where $\frac{d}{ds}$ is the differential operator with respect to S along the characteristic line defined in (4.2).

We now present an explicit difference scheme for (4.4). Given $\Delta x > 0$, let $S_i = e^{i \Delta x}$ ($i \in \mathbb{Z}$). Along the characteristic line (4.1) from (t_n, S_i, A_k), let $V_{i,k}^n$ and V_{i+l,k_l}^{n+1} represent, respectively, the values of numerical approximation of $V(S_i, A_k, t_n)$ and $V(S_{i+l}, A_{k_l}, t_{n+1})$ where $A_{k_l} = \frac{n A_k + S_{i+l}}{n+1}$. Note that the integral term in (4.4) can be changed to the following form:

$$\lambda \int_{-1}^{\infty} V\left(S(1 + y), S(1 + y) - \frac{t_n}{t}\big(S(1 + y) - A_k\big), t\right) d\mathcal{N}(y),$$

$$= \lambda \int_{-\infty}^{\infty} V\left(S e^z, S e^z - \frac{t_n}{t}\big(S e^z - A_k\big), t\right) d\mathcal{N}(e^z - 1),$$

$$= \lambda \sum_{l \in \mathbb{Z}} \int_{(l - \frac{1}{2})\Delta x}^{(l + \frac{1}{2})\Delta x} V\left(S e^z, \frac{t_n A_k + (t - t_n) S e^z}{t}, t\right) d\mathcal{N}(e^z - 1).$$

Then, taking the explicit difference scheme from (4.4), we have

$$\frac{V_{i,k_0}^{n+1} - V_{i,k}^n}{\Delta t} + \frac{\sigma^2}{2\Delta x^2}\big[V_{i+1,k_1}^{n+1} - 2V_{i,k_0}^{n+1} + V_{i-1,k_{-1}}^{n+1}\big] - (r + \lambda) V_{i,k}^n$$

$$+ \frac{(r - q - \lambda k - \sigma^2/2)}{2\Delta x} \left[V_{i+1,k_1}^{n+1} - V_{i-1,k_{-1}}^{n+1} \right] + \lambda \sum_{l \in \mathbb{Z}} V_{i+l,k_l}^{n+1} p_l + O\left(\Delta t^{\frac{1}{2}}\right) = 0, \quad (4.5)$$

where

$$p_l = \mathcal{N}\left(e^{(l+\frac{1}{2})\Delta x} - 1\right) - \mathcal{N}\left(e^{(l-\frac{1}{2})\Delta x} - 1\right). \tag{4.6}$$

Neglecting higher-order terms than Δt, we get

$$
\begin{aligned}
V_{i,k}^n = {} & \frac{1}{1 + r\Delta t} \Bigg\{ \frac{1}{1 + \frac{\lambda\Delta t}{1+r\Delta t}} \Bigg[\left(1 - \frac{\sigma^2 \Delta t}{\Delta x^2}\right) V_{i,k_0}^{n+1} \\
& + \left(\frac{\sigma^2 \Delta t}{2\Delta x^2} + \left(r - q - \lambda k - \frac{\sigma^2}{2}\right)\frac{\Delta t}{2\Delta x}\right) V_{i+1,k_1}^{n+1} \\
& + \left(\frac{\sigma^2 \Delta t}{2\Delta x^2} - \left(r - q - \lambda k - \frac{\sigma^2}{2}\right)\frac{\Delta t}{2\Delta x}\right) V_{i-1,k_{-1}}^{n+1} \Bigg] + \frac{\lambda\Delta t}{1 + \frac{\lambda\Delta t}{1+r\Delta t}} \sum_{l \in \mathbb{Z}} V_{i+l,k_l}^{n+1} p_l \Bigg\},
\end{aligned}
\tag{4.7}
$$

which is the explicit difference scheme along the characteristic line for (4.1).

Now, if we let $\frac{\sigma^2 \Delta t}{\Delta x^2} = 1$ and compare (3.12) and (4.6), we see

$$p_l = \hat{p}_l \tag{4.8}$$

and then neglecting higher-order terms than Δt again, (4.7) is reduced to

$$
\begin{aligned}
V_{i,k}^n = {} & \frac{1}{1 + r\Delta t} \Bigg\{ (1 - \lambda\Delta t)\Bigg[\left(\frac{1}{2} + \left(r - q - \lambda k - \frac{\sigma^2}{2}\right)\frac{\sqrt{\Delta t}}{2}\right) V_{i+1,k_1}^{n+1} \\
& + \left(\frac{1}{2} - \left(r - q - \lambda k - \frac{\sigma^2}{2}\right)\frac{\sqrt{\Delta t}}{2}\right) V_{i-1,k_{-1}}^{n+1} \Bigg] + \lambda\Delta t \sum_{l \in \mathbb{Z}} V_{i+l,k_l}^{n+1} p_l \Bigg\}.
\end{aligned}
\tag{4.9}
$$

Comparing the BTM (3.14) and the finite difference scheme (4.9) and noticing $u = e^{\sigma\sqrt{\Delta t}} = e^{\Delta x}$, $\rho = e^{r\Delta t} = 1 + r\Delta t + O(\Delta t^2)$, $\hat{\lambda} = \lambda\Delta t + O(\Delta t^2)$, (3.17) and (4.8), we deduce the following theorem.

Theorem 4.1. *The binomial tree method* (3.14) *is equivalent to the finite difference scheme* (4.5) *with* $\frac{\sigma^2 \Delta t}{\Delta x^2} = 1$ *in the sense of neglecting higher-order terms than* Δt.

It is easy to check that the scheme (4.9) is stable if

$$\frac{1}{2} - \left| r - q - \lambda k - \frac{\sigma^2}{2} \right| \frac{\sqrt{\Delta t}}{2} > 0,$$

when Δt is small enough. From Theorems 3.1 and 4.1, we can also deduce that the scheme (4.9) is consistent with (2.2). The next theorem follows from Lax theorem (see [15]) and Theorem 4.1.

Theorem 4.2. *As* $\Delta t \to 0$, *the solution of binomial tree method* (3.14) *converges to the solution of* (2.2) *in* $\mathcal{D} \times [0, T]$.

Remark 4.3. Similar equivalence and convergence results can also be obtained for geometric average options.

Remark 4.4. For American-style Asian options, we can also obtain similar equivalence results. But we have to use the notion of viscosity solution to prove the corresponding convergence results because of the nonlinearity of (2.3).

5. Convergence

In this section, we describe the convergence of BTM for some American-style Asian options. Without loss of generality, we assume

$$0 < p < 1, \tag{5.1}$$

which always holds for Δt small enough by (3.17). Now we investigate the properties of the BTM (3.15).

Lemma 5.1. *Let $V^n(S, A)$ be the function defined by* (3.15) *in $\mathcal{D}$ for American fixed strike put options with payoff $(K - A)^+$. Then for all $0 \leqslant n \leqslant N$,*

$$V^n(S, A) \leqslant K. \tag{5.2}$$

Proof. Clearly, $V^N(S, A) = (K - A)^+ \leqslant K$. If (5.2) holds for $n + 1$, then noting (5.1), we have

$$
\begin{aligned}
V^n(S, A) &= \max\Big\{ e^{-r\Delta t}(1 - \hat{\lambda})\big[p V^{n+1}(Su, A_{+1}) + (1 - p) V^{n+1}\big(Su^{-1}, A_{-1}\big) \big] \\
&\quad + e^{-r\Delta t}\hat{\lambda} \sum_{l \in \mathbb{Z}} V^{n+1}\big(Su^l, A_l\big)\hat{p}_l, (K - A)^+ \Big\}, \\
&\leqslant \max\Big\{ e^{-r\Delta t}(1 - \hat{\lambda})\big[pK + (1 - p)K \big] + e^{-r\Delta t}\hat{\lambda} \cdot K \sum_{l \in \mathbb{Z}} \hat{p}_l, (K - A)^+ \Big\}, \\
&= \max\big\{ K e^{-r\Delta t}, (K - A)^+ \big\}, \\
&\leqslant K.
\end{aligned}
$$

This completes the proof. $\quad\square$

Lemma 5.2. *Let $V^n(S, A)$ be the function defined by the BTM* (3.15) *in $\mathcal{D}$ for American floating strike call options with payoff $(S - A)^+$. Then for all $0 \leqslant n \leqslant N$,*

$$V^n(S, A) \leqslant S. \tag{5.3}$$

Proof. Clearly, $V^N(S, A) = (S - A)^+ \leqslant S$. Suppose (5.3) holds for $n + 1$, then

$$
\begin{aligned}
V^n(S, A) &= \max\Big\{ e^{-r\Delta t}(1 - \hat{\lambda})\big[p V^{n+1}(Su, A_{+1}) + (1 - p) V^{n+1}\big(Su^{-1}, A_{-1}\big) \big] \\
&\quad + e^{-r\Delta t}\hat{\lambda} \sum_{l \in \mathbb{Z}} V^{n+1}\big(Su^l, A_l\big)\hat{p}_l, (S - A)^+ \Big\}, \\
&\leqslant \max\Big\{ e^{-r\Delta t}(1 - \hat{\lambda})S\big[pu + (1 - p)u^{-1} \big] + e^{-r\Delta t}\hat{\lambda} \cdot S \sum_{l \in \mathbb{Z}} u^l \hat{p}_l, (S - A)^+ \Big\}.
\end{aligned}
$$

Noting that

$$pu + (1 - p)u^{-1} = \frac{e^{(r-q)\Delta t} - \hat{\lambda}(1 + k)}{1 - \hat{\lambda}},$$

$$\sum_{l \in \mathbb{Z}} u^l \hat{p}_l = 1 + k + O\big(\Delta t^{\frac{1}{2}}\big),$$

we see for sufficiently small Δt,

$$V^n(S, A) = \max\big\{S\big[e^{-q\Delta t} + O\big(\Delta t^{\frac{3}{2}}\big)\big], (S - A)^+\big\} \leqslant S,$$

and thus the result of this theorem follows. $\quad\square$

Employing the notion of viscosity solution, we will show the convergence of the BTM for some American-style Asian options. Firstly, we recall the definition of viscosity solution and it is convenient to have the following notations:

$$USC\big(\mathbf{R} \times [0, T]\big) = \big\{\text{upper semicontinuous functions } u : \mathbf{R} \times [0, T] \to \mathbf{R}\big\},$$

$$LSC\big(\mathbf{R} \times [0, T]\big) = \big\{\text{lower semicontinuous functions } u : \mathbf{R} \times [0, T] \to \mathbf{R}\big\}.$$

Definition 5.3. A locally bounded function $u \in USC(\mathcal{D} \times (0, T])$ (respectively $u \in LSC(\mathcal{D} \times (0, T])$) is a viscosity subsolution (respectively supersolution) of (2.3) if, for all $(S, A) \in \mathcal{D}, u(S, A, T) \leqslant g(S, A)$ (respectively $u(S, A, T) \geqslant g(S, A)$) and, for all $(S, A, t) \in \mathcal{D} \times (0, T), \phi \in C^2(\mathcal{D} \times (0, T))$ such that $u(S, A, t) = \phi(S, A, t)$, and $u < \phi$ (respectively $u > \phi$) on $\mathcal{D} \times (0, T]/(S, A, t)$, we have

$$\min\big\{-\mathcal{L}\phi(S, A, t), \phi(S, A, t) - g(S, A)\big\} \leqslant 0 \quad (\text{respectively } \geqslant 0).$$

Further, we call $u \in C(\mathcal{D} \times (0, T])$ is a viscosity solution of (2.3) if it is simultaneously a subsolution and a supersolution.

The proof for convergence needs the strong comparison principle which holds for (2.3) (see Theorem 3.5 of Qian et al. [14], Alvarez et al. [1] and the references therein).

Lemma 5.4 (*Comparison principle*). *Suppose u and v are, respectively, viscosity subsolution and supersolution of problem* (2.3)*, then $u \leqslant v$.*

Remark 5.5. From Lemma 5.4, the uniqueness of the solution for (2.3) follows immediately.

Let $V^n(S, A)$ be the function defined by the BTM (3.15) in $\mathcal{D}$. We now define the extension function $V_{\Delta t}(S, A, t)$ as follows:

$$V_{\Delta t}(S, A, t) = \frac{(n + 1)\Delta t - t}{\Delta t} V^n(S, A) + \frac{t - n\Delta t}{\Delta t} V^{n+1}(S, A),$$

where $t \in [n\Delta t, (n + 1)\Delta t]$ for $n = 0, 1, \ldots, N - 1$.

Theorem 5.6. *Suppose that $V(S, A, t)$ is the viscosity solution of* (2.3) *for American-style Asian options with payoff $(S - A)^+$ or $(K - A)^+$. Then, as $\Delta t \to 0$, we have $V_{\Delta t}(S, A, t)$ converges uniformly to $V(S, A, t)$ in any bounded closed subdomain of $\mathcal{D} \times (0, T)$.*

Proof. Denote

$$V^*(S, A, t) = \limsup_{\Delta t \to 0, \, (x,y,z) \to (S,A,t)} V_{\Delta t}(S, A, t),$$

$$V_*(S, A, t) = \liminf_{\Delta t \to 0, \, (x,y,z) \to (S,A,t)} V_{\Delta t}(S, A, t).$$

Owing to Lemmas 5.1 and 5.2, V^* and V_* are well defined for American fixed strike call options and American floating strike call options. Obviously, $V^* \in USC(\mathcal{D} \times (0, T])$, $V_* \in LSC(\mathcal{D} \times (0, T])$ and $V_*(S, A, t) \leqslant V^*(S, A, t)$. If we can show V^* and V_* are subsolution and supersolution of (2.3), respectively, then in terms of comparison principle (Lemma 5.4), we deduce $V^* \leqslant V_*$ and thus $V^*(S, A, t) = V_*(S, A, t) = V(S, A, t)$, which guarantees that the whole sequence converges to the unique viscosity solution $V(S, A, t)$.

We only need to show that V^* is a viscosity subsolution of (2.3). It can be shown that $V^*(S, A, T) = g(S, A)$. Suppose that for $\phi \in C^2(\mathcal{D} \times (0, T))$, $V^* - \phi$ attains a strict global maximum at $(S_0, A_0, t_0) \in \mathcal{D} \times (0, T)$ and $(V^* - \phi)(S_0, A_0, t_0) = 0$. Set $\Phi = \phi - \varepsilon$, $\varepsilon > 0$, then $V^* - \Phi$ attains a strict global maximum at (S_0, A_0, t_0) and

$$(V^* - \Phi)(S_0, A_0, t_0) > 0. \tag{5.4}$$

By the definition of V^*, there exists a sequence $V_{\Delta t_m}(S_m, A_m, t_m)$ such that

$$\Delta t_m \to 0, \qquad (S_m, A_m, t_m) \to (S_0, A_0, t_0), \qquad V_{\Delta t_m}(S_m, A_m, t_m) \to V^*(S_0, A_0, t_0)$$

as $m \to +\infty$. Assuming that $(\hat{S}_m, \hat{A}_m, \hat{t}_m)$ is a global maximum point of $V_{\Delta t_m} - \Phi$ on $\mathcal{D} \times (0, T)$, we can easily deduce by reduction to absurdity that there is a subsequence $V_{\Delta t_{m_i}}(\hat{S}_{m_i}, \hat{A}_{m_i}, \hat{t}_{m_i})$ such that

$$\Delta t_{m_i} \to 0, \qquad (\hat{S}_{m_i}, \hat{A}_{m_i}, \hat{t}_{m_i}) \to (S_0, A_0, t_0),$$
$$(V_{\Delta t_{m_i}} - \Phi)(\hat{S}_{m_i}, \hat{A}_{m_i}, \hat{t}_{m_i}) \to (V^* - \Phi)(S_0, A_0, t_0) \tag{5.5}$$

as $m_i \to +\infty$. Therefore,

$$V_{\Delta t_{m_i}}(\cdot, \cdot, \hat{t}_{m_i} + \Delta t_{m_i}) \leqslant \Phi(\cdot, \cdot, \hat{t}_{m_i} + \Delta t_{m_i}) + (V_{\Delta t_{m_i}} - \Phi)(\hat{S}_{m_i}, \hat{A}_{m_i}, \hat{t}_{m_i}) \quad \text{in } \mathcal{D}. \tag{5.6}$$

Then, by (3.15), (5.1) and (5.4)–(5.6), we can obtain

$$V_{\Delta t_{m_i}}(\hat{S}_{m_i}, \hat{A}_{m_i}, \hat{t}_{m_i}) \leqslant \max\{F_{\Delta t_{m_i}} \Phi(\hat{S}_{m_i}, \hat{A}_{m_i}, \hat{t}_{m_i}), g(\hat{S}_{m_i}, \hat{A}_{m_i})\}$$
$$+ (V_{\Delta t_{m_i}} - \Phi)(\hat{S}_{m_i}, \hat{A}_{m_i}, \hat{t}_{m_i}),$$

where the operator $F_{\Delta t_{m_i}}$ is defined by (3.16). Thus we have

$$\min\{\Phi - F_{\Delta t_{m_i}} \Phi, \ \Phi - g\}_{(\hat{S}_{m_i}, \hat{A}_{m_i}, \hat{t}_{m_i})} \leqslant 0. \tag{5.7}$$

Dividing the first term of the minimum in (5.7) by Δt_{m_i} and letting $m_i \to \infty$, $\varepsilon \to 0$, it follows from Theorem 3.1 that

$$\min\{-\mathcal{L}\phi, \ \phi - g\}_{(S_0, A_0, t_0)} \leqslant 0.$$

Hence, it follows from Definition 5.3 that V^* is a subsolution of (2.3). Similarly, we can show that V_* is a supersolution of (2.3). Thus, we have proven $V_{\Delta t}(S, A, t)$ converges to $V(S, A, t)$ as $\Delta t \to 0$, and this convergence is locally uniform (see [10,14]). $\quad\square$

Remark 5.7. It is well known that the BTM is computationally infeasible for pricing arithmetic average options because the number of possible arithmetic average values increases exponentially with the number of timesteps. Barraquand and Pudet [5] proposed the forward shooting grid method, a modified BTM to restrict the possible average values to a set of predetermined values. We can also deduce the forward shooting grid method in jump-diffusion models and show the convergence of it. For Δt given, let

$$\Delta Y = \gamma \sigma \sqrt{\Delta t},$$

where γ is a quantization parameter depending on Δt. Let discrete values of the stock price S and the arithmetic average price A be given by

$$S_j^n = e^{j\sigma\sqrt{\Delta t}}, \qquad A_k^n = e^{k\Delta Y}$$

for $n = 0, \ldots, N$ and $j, k \in \mathbb{Z}$. If at next timestep, S_j^n changes to S_{j+l}^{n+1} ($l \in \mathbb{Z}$), A_k^n will consequently become $A_{k_l}^{n+1}$, where

$$A_{k_l}^{n+1} = \frac{nA_k^n + S_{j+l}^{n+1}}{n+1}.$$

Here $A_{k_l}^{n+1}$ may not coincide with $A_{k_l}^{n+1} = e^{k_l \Delta Y}$ for some integer k_l. Using Barraquand and Pudet's technique [5], we define a integer k_l by the nearest lattice point

$$k_l = \text{nearest}\left[\frac{\ln(A_{k_l}^{n+1})}{\Delta Y}\right].$$

Let $V^n(S_j^n, A_k^n)$ denote the option price at time $t = n\Delta t$, $S = S_j^n$ and $A = A_k^n$. We have

$$
\begin{cases}
V^n(S_j^n, A_k^n) = \max\Big\{ e^{-r\Delta t}\big\{(1 - \lambda\Delta t)\big[pV^{n+1}(S_{j+1}^{n+1}, A_{k_1}^{n+1}) \\
\qquad\qquad + (1-p)V^{n+1}(S_{j-1}^{n+1}, A_{k_{-1}}^{n+1})\big] \\
\qquad\qquad + \lambda\Delta t \sum_{l\in\mathbb{Z}} V^{n+1}(S_{j+l}^{n+1}, A_{k_l}^{n+1}) p_l \big\}, g(S_j^n, A_k^n) \Big\}, \\
V^N(S_j^N, A_k^N) = g(S_j^N, A_k^N)
\end{cases}
\tag{5.8}
$$

for $n = 0, 1, \ldots, N-1$ and $j, k \in \mathbb{Z}$. Under the condition of

$$\gamma = o\big(\Delta t^{\frac{1}{2}}\big),$$

it is not difficult to verify that the scheme (5.8) is consistent with the corresponding equation (2.3) and all the preceding proof of this paper can be applied to prove the convergence of the scheme (5.8).

Acknowledgment

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