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All-derivable points in continuous nest algebras ☆

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Abstract

Let $\mathcal{A}$ be an operator algebra on a Hilbert space. We say that an element $G \in \mathcal{A}$ is an all-derivable point of $\mathcal{A}$ for the strong operator topology if every strong operator topology continuous derivable linear mapping φ at G (i.e. $\varphi(ST) = \varphi(S)T + S\varphi(T)$ for any $S, T \in alg\mathcal{N}$ with $ST = G$) is a derivation. Let $\mathcal{N}$ be a continuous nest on a complex and separable Hilbert space H. We show in this paper that every orthogonal projection operator $P(M)$ $(0 \neq M \in \mathcal{N})$ is an all-derivable point of $alg\mathcal{N}$ for the strong operator topology.

Keywords: All-derivable point; Nest algebra; Derivable linear mapping at G

1. Introduction and preliminaries

Let $\mathcal{A}$ be an operator subalgebra in $B(H)$, where H is a complex and separable Hilbert space. We say that a linear mapping φ from $\mathcal{A}$ into itself is a derivable mapping at G if $\varphi(ST) = \varphi(S)T + S\varphi(T)$ for any $S, T \in \mathcal{A}$ with $ST = G$. We say that an element $G \in \mathcal{A}$ is an all-derivable point of $\mathcal{A}$ for strong operator topology if every strong operator topology continuous derivable linear mapping φ at G is a derivation.

We describe some of the results related to ours. Jin and Lu [7] showed that every derivable mapping φ at 0 with $\varphi(I) = 0$ on nest algebras is a derivation. Li, Pan and Xu [11] prove that every derivable mapping φ at 0 with $\varphi(I) = 0$ on CSL algebras is a derivation. Zhu and Xiong in [13–15] showed that

(1) every norm-continuous generalized derivable mapping at 0 on finite CSL algebras is a generalized derivation;
(2) every invertible operator of nest algebras is an all-derivable point for the strong operator topology; and
(3) $G \in \mathcal{TM}_2$ is an all-derivable point of $\mathcal{TM}_2$ if and only if $G \neq 0$, where $\mathcal{TM}_2$ is the algebra of all 2×2 upper triangular matrices.

For other results, see [1–3,8,9,12,16,17].

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It is the aim of this paper to prove the following statement. Let $\mathcal{N}$ be a continuous nest on a complex and separable Hilbert space H. If $\{0\} \neq M \in \mathcal{N}$, we show in this paper that the orthogonal projection operator $P(M)$ is an all-derivable point of the nest algebra $alg\mathcal{N}$ for the strong operator topology.

This paper is organized as follows: Section 2 concerns some results of all-derivable points in the algebra of 2×2 upper triangular operator matrices, and we obtain the major new result Theorem 2.2 in this paper. Using the results in Section 2, we give the proof of Theorem 3.2 in Section 3.

The symbols $B(H)$ and $F(H)$ stand for the algebra of all bounded linear operators on H and the algebra of all finite rank operators on H, respectively. We use the symbols I_H or I to denote the unit operator on H. If $\mathcal{N}$ is a complete nest on H, then the nest algebra $alg\mathcal{N}$ is the set of all operators which leave every member of $\mathcal{N}$ invariant. The algebra $alg\mathcal{N}$ is a Banach algebra. If $N \in \mathcal{N}$, we write N_- for $\vee\{M \in \mathcal{N}\colon\ M \subset N\}$. We say that $\mathcal{N}$ is a continuous nest if $N_- = N$ for any $N \in \mathcal{N}$. If N is a closed subspace in H, we write $P(N)$ for the orthogonal projection operator from H onto N. If $A \in B(H)$, then the kernel space and range space of A will be denoted by $N(A)$ and $R(A)$, respectively. We denote $\mathcal{C}$ for the complex number field.

2. All-derivable points in 2 × 2 upper triangular operator matrices

In this section, every 2×2 operator matrix is always represented as relative to the orthogonal decomposition $H \oplus H$. We use the symbols I to denote the unit operator on H.

Lemma 2.1. *Let $\mathcal{N}$ be a complete nest on a complex and separable Hilbert space H. If $\varphi : alg\mathcal{N} \to alg\mathcal{N}$ be a derivable mapping at 0. Then there exist two operators $C, D \in B(H)$ such that*

$$\varphi(X) = XC + DX$$

for any $X \in alg\mathcal{N}$.

Proof. Since φ is a derivable mapping at 0 on $alg\mathcal{N}$, we know from Theorem 5 in [7] that $\varphi(ST) = \varphi(S)T + S\varphi(T) - S\varphi(I)T$ for any $S, T \in alg\mathcal{N}$. We define a linear mapping $\psi : alg\mathcal{N} \to alg\mathcal{N}$ as

$$\psi(T) = \varphi(T) - T\varphi(I), \quad \forall T \in alg\mathcal{N}.$$

It is easy to verify that ψ is a derivation on $alg\mathcal{N}$. By Theorem 19.7 in [4], ψ is an inner derivation, i.e. there exists an operator $D \in B(H)$ such that $\psi(T) = TD - DT$ for any $T \in alg\mathcal{N}$. Furthermore $\varphi(T) = \psi(T) + T\varphi(I) = TD - DT + T\varphi(I)$ for any $T \in alg\mathcal{N}$. It is obvious that $C = \varphi(I) + D$ and D are desired in the lemma. □

Theorem 2.2. *Let $\mathcal{N}$ be a complete nest on a complex and separable Hilbert space H. If we write*

$$\mathcal{A} = \left\{ \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} :\ X, Y, Z \in alg\mathcal{N} \right\},$$

then $E_{11} = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is an all-derivable point of $\mathcal{A}$ for the strong operator topology.

Proof. Let φ be a strong operator topology continuous derivable linear mapping at E_{11} from $\mathcal{A}$ into itself. We only need to prove that φ is a derivation. For arbitrary $X, Y, Z \in alg\mathcal{N}$, we write

$$\begin{cases} \varphi\left(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} A_{11}(X) & A_{12}(X) \\ 0 & A_{22}(X) \end{bmatrix}, \\ \varphi\left(\begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} B_{11}(Y) & B_{12}(Y) \\ 0 & B_{22}(Y) \end{bmatrix}, \\ \varphi\left(\begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} C_{11}(Z) & C_{12}(Z) \\ 0 & C_{22}(Z) \end{bmatrix}. \end{cases}$$

Obviously, A_{ij}, B_{ij} and C_{ij} $(i, j = 1, 2, i \leqslant j)$ are strong operator topology continuous linear mappings on $alg\mathcal{N}$. Since φ is a derivable mapping at E_{11} on $\mathcal{A}$, we have

$$\begin{bmatrix} A_{11}(I) & A_{12}(I) \\ 0 & A_{22}(I) \end{bmatrix} = \varphi(E_{11}) = \varphi\left(E_{11}^2\right) = \varphi(E_{11})E_{11} + E_{11}\varphi(E_{11}) = \begin{bmatrix} 2A_{11}(I) & A_{12}(I) \\ 0 & 0 \end{bmatrix}.$$

Thus we have $A_{11}(I)=A_{22}(I)=0$.

For arbitrary $X\in alg\mathcal{N}$ and $S=\left[\begin{smallmatrix} I & 0\\ 0 & X\end{smallmatrix}\right]$, $T=\left[\begin{smallmatrix} I & 0\\ 0 & 0\end{smallmatrix}\right]$ in $\mathcal{A}$, then $ST=\left[\begin{smallmatrix} I & 0\\ 0 & 0\end{smallmatrix}\right]=E_{11}$. So $\varphi(E_{11})=\varphi(S)T+S\varphi(T)$. Thus we have

$$\begin{aligned}\begin{bmatrix} 0 & A_{12}(I)\\ 0 & 0\end{bmatrix}&=\varphi\left(\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}\right)=\varphi(E_{11})=\varphi(S)T+S\varphi(T)\\ &=\begin{bmatrix} C_{11}(X) & A_{12}(I)+C_{12}(X)\\ 0 & C_{22}(X)\end{bmatrix}\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix} I & 0\\ 0 & X\end{bmatrix}\begin{bmatrix} 0 & A_{12}(I)\\ 0 & 0\end{bmatrix}\\ &=\begin{bmatrix} C_{11}(X) & A_{12}(I)\\ 0 & 0\end{bmatrix}.\end{aligned}$$

It follows that $C_{11}(X)=0$ for any $X\in alg\mathcal{N}$.

For arbitrary $x,y,z,u,v,w\in\mathcal{C}$, we take $S=\left[\begin{smallmatrix} xX_1 & yI\\ 0 & zI\end{smallmatrix}\right]$ and $T=\left[\begin{smallmatrix} uX_2 & vI\\ 0 & wI\end{smallmatrix}\right]$ in $\mathcal{A}$ with $ST=\left[\begin{smallmatrix} I & 0\\ 0 & 0\end{smallmatrix}\right]=E_{11}$, then

$$ST=\begin{bmatrix} xX_1 & yI\\ 0 & zI\end{bmatrix}\begin{bmatrix} uX_2 & vI\\ 0 & wI\end{bmatrix}=\begin{bmatrix} xuX_1X_2 & xvX_1+ywI\\ 0 & zwI\end{bmatrix}=\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}=E_{11},$$

i.e. $xuX_1X_2=I$, $xvX_1+ywI=0$ and $zw=0$. Thus we have

$$\begin{aligned}\begin{bmatrix} 0 & A_{12}(I)\\ 0 & 0\end{bmatrix}&=\varphi(E_{11})=\varphi(S)T+S\varphi(T)\\ &=\varphi\left(\begin{bmatrix} xX_1 & yI\\ 0 & zI\end{bmatrix}\right)\begin{bmatrix} uX_2 & vI\\ 0 & wI\end{bmatrix}+\begin{bmatrix} xX_1 & yI\\ 0 & zI\end{bmatrix}\varphi\left(\begin{bmatrix} uX_2 & vI\\ 0 & wI\end{bmatrix}\right)\\ &=\begin{bmatrix} \begin{array}{l} xuA_{11}(X_1)X_2+yuB_{11}(I)X_2\\ \quad+zuC_{11}(I)X_2+xuX_1A_{11}(X_2)\\ \quad+xvX_1B_{11}(I)+xwX_1C_{11}(I)\end{array} & \begin{array}{l} xvA_{11}(X_1)+yvB_{11}(I)+zvC_{11}(I)\\ \quad+xwA_{12}(X_1)+ywB_{12}(I)+zwC_{12}(I)\\ \quad+xuX_1A_{12}(X_2)+xvX_1B_{12}(I)+xwX_1C_{12}(I)\\ \quad+yuA_{22}(X_2)+yvB_{22}(I)+ywC_{22}(I)\end{array}\\ 0 & \begin{array}{l} xwA_{22}(X_1)I+ywB_{22}(I)I+zwC_{22}(I)I\\ \quad+zuA_{22}(X_2)+zvB_{22}(I)+zwC_{22}(I)\end{array}\end{bmatrix}.\end{aligned}\tag{1}$$

Using Eq. (1) and $C_{11}(X)=0$, the following equations hold

$$0=xuA_{11}(X_1)X_2+yuB_{11}(I)X_2+xuX_1A_{11}(X_2)+xvX_1B_{11}(I),\tag{2}$$

$$\begin{aligned}A_{12}(I)&=xvA_{11}(X_1)+yvB_{11}(I)+xwA_{12}(X_1)+ywB_{12}(I)+xuX_1A_{12}(X_2)+xvX_1B_{12}(I)\\ &\quad+xwX_1C_{12}(I)+yuA_{22}(X_2)+yvB_{22}(I)+ywC_{22}(I),\end{aligned}\tag{3}$$

$$0=xwA_{22}(X_1)+ywB_{22}(I)+zuA_{22}(X_2)+zvB_{22}(I)\tag{4}$$

for any $xuX_1X_2=I$, $xvX_1+ywI=0$, $zw=0$.

If we take $x=u=1$ and $y=z=v=w=0$ in Eq. (2), then $A_{11}(X_1)X_2+X_1A_{11}(X_2)=0=A_{11}(I)$ for any $X_1,X_2\in alg\mathcal{N}$ with $X_1X_2=I$. So A_{11} is a derivable mapping at I on $alg\mathcal{N}$ for the strong operator topology. It follows from the main theorem in [14] that A_{11} is an inner derivation on $alg\mathcal{N}$. Thus there exists an operator $A\in B(H)$ such that

$$A_{11}(X)=XA-AX\tag{5}$$

for any $X\in alg\mathcal{N}$. It follows from Eq. (2) and $A_{11}(X_1)X_2+X_1A_{11}(X_2)=A_{11}(I)=0$ that

$$0=yuB_{11}(I)X_2+xvX_1B_{11}(I).$$

If we take $X_1=X_2=I$, $x=u=y=1$ and $z=v=w=0$ in the above equation, then $B_{11}(I)=0$. If we take $X_1=X_2=I$, $x=u=w=1$ and $y=z=v=0$ in Eq. (4), then $A_{22}(I)=0$.

For arbitrary $F\in alg\mathcal{N}$ with $F^2=F$, we write $F_\lambda=F+\lambda I$. If we take $\alpha,\beta\in\mathcal{C}$ with $\alpha\beta=1$ and $\alpha+\beta=-1$, then $F_\alpha F_\beta=F_\beta F_\alpha=I$ and $F_\alpha+F_\beta=2F-I$. Let $X_1=F_\alpha$, $X_2=F_\beta$, $x=u=w=1$ and $y=z=v=0$ in Eq. (4), then $A_{22}(F_\alpha)=0$. Furthermore $A_{22}(F)=A_{22}(F+\alpha I)=A_{22}(F_\alpha)=0$. Notice that every rank one operator in $alg\mathcal{N}$ may be written as a linear combination of at most four idempotents in $alg\mathcal{N}$ (see [6]), and every finite rank operator in $alg\mathcal{N}$ may be represented as a sum of rank one operators in $alg\mathcal{N}$ (see [5]). So $A_{22}(X)=0$ for any $X\in alg\mathcal{N}\cap F(H)$. It follows from Erdos Density Theorem (see Theorem 3.11 in [4]) that $A_{22}(X)=0$ for any $X\in alg\mathcal{N}$. It follows from Eq. (4) that

$$0=ywB_{22}(I)+zvB_{22}(I).$$

Taking $X_1=X_2=I$, $x=u=-1$, $y=v=w=1$ and $z=0$ in the above equation, then $B_{22}(I)=0$.

If we take $X_1=F_\alpha$, $X_2=F_\beta$, $x=u=1$ and $y=z=v=w=0$ in Eq. (3), then

$$A_{12}(I)=F_\alpha A_{12}(F_\beta).$$

Multiplying the above equation from left by F_β, we have $A_{12}(F_\beta)=F_\beta A_{12}(I)$. Similarly, $A_{12}(F_\alpha)=F_\alpha A_{12}(I)$. Adding two equations, we have $A_{12}(F)=FA_{12}(I)$. By imitating the proof of the above paragraph, we obtain

$$A_{12}(X)=XA_{12}(I)$$

for any $X\in alg\mathcal{N}$. If we write $B=A_{12}(I)$, then $A_{12}(X)=XB$. Taking $X_1=X_2=I$, $x=u=w=1$ and $y=z=v=0$ in Eq. (3), we have $A_{12}(I)+C_{12}(I)=0$. If we take $X_1=X_2=I$, $y=x=u=w=1$, $v=-1$ and $z=0$ in Eq. (3), then

$$-A_{11}(I)-B_{11}(I)+A_{12}(I)+B_{12}(I)+A_{12}(I)-B_{12}(I)+C_{12}(I)+C_{22}(I)=A_{12}(I).$$

So $C_{22}(I)=0$.

For arbitrary $X\in alg\mathcal{N}$, taking $S=\left[\begin{smallmatrix} xI & yX\\ 0 & zI\end{smallmatrix}\right]$ and $T=\left[\begin{smallmatrix} uI & vX\\ 0 & wI\end{smallmatrix}\right]$ in $\mathcal{A}$ with $ST=\left[\begin{smallmatrix} I & 0\\ 0 & 0\end{smallmatrix}\right]=E_{11}$, we have

$$ST=\begin{bmatrix} xI & yX\\ 0 & zI\end{bmatrix}\begin{bmatrix} uI & vX\\ 0 & wI\end{bmatrix}=\begin{bmatrix} xuI & (xv+yw)X\\ 0 & zwI\end{bmatrix}=\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix},$$

i.e. $xu=1$, $xv+yw=0$ and $zw=0$. It follows that

$$\begin{aligned}\begin{bmatrix} 0 & A_{12}(I)\\ 0 & 0\end{bmatrix}&=\varphi(E_{11})=\varphi(S)T+S\varphi(T)\\ &=\varphi\left(\begin{bmatrix} xI & yX\\ 0 & zI\end{bmatrix}\right)\begin{bmatrix} uI & vX\\ 0 & wI\end{bmatrix}+\begin{bmatrix} xI & yX\\ 0 & zI\end{bmatrix}\varphi\left(\begin{bmatrix} uI & vX\\ 0 & wI\end{bmatrix}\right)\\ &=\begin{bmatrix} yuB_{11}(X)+xvB_{11}(X) & *\\ 0 & ywB_{22}(X)+zvB_{22}(X)\end{bmatrix}.\end{aligned}\tag{6}$$

Using Eq. (6), we have

$$0=yuB_{11}(X)+xvB_{11}(X),\tag{7}$$

$$0=ywB_{22}(X)+zvB_{22}(X),\tag{8}$$

for any $X\in alg\mathcal{N}$, $xu=1$, $xv+yw=0$ and $zw=0$. If we take $x=y=u=1$ and $z=w=v=0$ in Eq. (7), then $B_{11}(X)=0$. If we take $x=y=u=v=1$, $w=-1$ and $z=0$ in Eq. (8), then $B_{22}(X)=0$. Hence $B_{11}(X)=B_{22}(X)=0$ for any $X\in alg\mathcal{N}$.

For arbitrary $X\in alg\mathcal{N}$, if we take $S=\left[\begin{smallmatrix} xI & yI\\ 0 & 0\end{smallmatrix}\right]$ and $T=\left[\begin{smallmatrix} uI & vX\\ 0 & wX\end{smallmatrix}\right]$ in $\mathcal{A}$ with $ST=\left[\begin{smallmatrix} I & 0\\ 0 & 0\end{smallmatrix}\right]=E_{11}$, then

$$ST=\begin{bmatrix} xI & yI\\ 0 & 0\end{bmatrix}\begin{bmatrix} uI & vX\\ 0 & wX\end{bmatrix}=\begin{bmatrix} xuI & (xv+yw)X\\ 0 & 0\end{bmatrix}=\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix},$$

i.e. $xu=1$ and $xv+yw=0$. Then the following matrix equation holds

$$\begin{aligned}\begin{bmatrix} 0 & A_{12}(I)\\ 0 & 0\end{bmatrix}&=\varphi(E_{11})=\varphi(S)T+S\varphi(T)\\ &=\varphi\left(\begin{bmatrix} xI & yI\\ 0 & 0\end{bmatrix}\right)\begin{bmatrix} uI & vX\\ 0 & wX\end{bmatrix}+\begin{bmatrix} xI & yI\\ 0 & 0\end{bmatrix}\varphi\left(\begin{bmatrix} uI & vX\\ 0 & wX\end{bmatrix}\right)\\ &=\begin{bmatrix} 0 & \begin{array}{l} xwA_{12}(I)X+ywB_{12}(I)X+xuA_{12}(I)\\ \quad +xvB_{12}(X)+xwC_{12}(X)+ywC_{22}(X)\end{array}\\ 0 & 0\end{bmatrix}\end{aligned}\tag{9}$$

for any $xu=1$ and $xv+yw=0$. If we take $x=u=w=1$ and $y=v=0$ in Eq. (9), then we have $A_{12}(I)X+C_{12}(X)=0$. Thus $C_{12}(X)=-A_{12}(I)X=-BX$. If we take $y=x=u=w=1$ and $v=-1$ in Eq. (9), then we have

$$B_{12}(I)X-B_{12}(X)+C_{22}(X)=0. \tag{10}$$

For arbitrary $S=\left[\begin{smallmatrix} xF_\alpha & yF_\alpha \\ 0 & zF_\alpha \end{smallmatrix}\right]$ and $T=\left[\begin{smallmatrix} uF_\beta & vI \\ 0 & wI \end{smallmatrix}\right]$ in $\mathcal{A}$ with $ST=\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]=E_{11}$, then

$$ST=\begin{bmatrix} xF_\alpha & yF_\alpha \\ 0 & zF_\alpha \end{bmatrix}\begin{bmatrix} uF_\beta & vI \\ 0 & wI \end{bmatrix}=\begin{bmatrix} xuI & (xv+yw)F_\alpha \\ 0 & zwF_\alpha \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

i.e. $xu=1$, $xv+yw=0$ and $zw=0$. Then the following matrix equation holds

$$\begin{aligned}
\begin{bmatrix} 0 & A_{12}(I) \\ 0 & 0 \end{bmatrix} &= \varphi(E_{11})=\varphi(S)T+S\varphi(T) \\
&= \varphi\left(\begin{bmatrix} xF_\alpha & yF_\alpha \\ 0 & zF_\alpha \end{bmatrix}\right)\begin{bmatrix} uF_\beta & vI \\ 0 & wI \end{bmatrix}+\begin{bmatrix} xF_\alpha & yF_\alpha \\ 0 & zF_\alpha \end{bmatrix}\varphi\left(\begin{bmatrix} uF_\beta & vI \\ 0 & wI \end{bmatrix}\right) \\
&= \begin{bmatrix} 0 & \begin{matrix} xvA_{11}(F_\alpha)+xwA_{12}(F_\alpha)+ywB_{12}(F_\alpha) \\ +xuF_\alpha A_{12}(F_\beta)+xvF_\alpha B_{12}(I)+xwF_\alpha C_{12}(I) \end{matrix} \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{11}$$

for any $xu=1$, $xv+yw=0$ and $zw=0$. Since $A_{12}(X)=XA_{12}(I)$ and $A_{12}(I)+C_{12}(I)=0$, $xwA_{12}(F_\alpha)+xwF_\alpha C_{12}(I)=xwF_\alpha(A_{12}(I)+C_{12}(I))=0$. If we take $y=x=u=w=1$, $v=-1$ and $z=0$ in Eq. (11) and note that $A_{12}(I)=F_\alpha A_{12}(F_\beta)$, then

$$-A_{11}(F_\alpha)+B_{12}(F_\alpha)-F_\alpha B_{12}(I)=0.$$

Similarly, we have

$$-A_{11}(F_\beta)+B_{12}(F_\beta)-F_\beta B_{12}(I)=0.$$

Adding the above two equations, we have $-A_{11}(F)+B_{12}(F)-FB_{12}(I)=0$. It follows that

$$-A_{11}(X)-XB_{12}(I)+B_{12}(X)=0 \tag{12}$$

for any $X\in alg\mathcal{N}$.

For arbitrary $Z_1, Z_2\in alg\mathcal{N}$ with $Z_1Z_2=0$, we take $S=\left[\begin{smallmatrix} I & 0 \\ 0 & Z_1 \end{smallmatrix}\right]$ and $T=\left[\begin{smallmatrix} I & 0 \\ 0 & Z_2 \end{smallmatrix}\right]$. Then $ST=E_{11}$. Thus we have

$$\begin{bmatrix} 0 & A_{12}(I) \\ 0 & 0 \end{bmatrix}=\varphi(E_{11})=\varphi(S)T+S\varphi(T)=\begin{bmatrix} 0 & * \\ 0 & C_{22}(Z_1)Z_2+Z_1C_{22}(Z_2) \end{bmatrix}.$$

So $C_{22}(Z_1)Z_2+Z_1C_{22}(Z_2)=0=C_{22}(0)$, i.e. C_{22} is a derivable mapping at 0 from $alg\mathcal{N}$ into itself. By Lemma 2.1, there exist two operators $C, D\in B(H)$ such that

$$C_{22}(X)=XC+DX. \tag{13}$$

It follows from Eqs. (10) and (13) that

$$B_{12}(X)=B_{12}(I)X+C_{22}(X)=B_{12}(I)X+XC+DX.$$

If we take $X=I$ in the above equation, then we obtain $C+D=0$. So $C_{22}(X)=XC-CX$. Furthermore we have

$$B_{12}(X)=B_{12}(I)X+XC-CX=XC-\big(C-B_{12}(I)\big)X. \tag{14}$$

It follows from Eqs. (12) and (5) that

$$B_{12}(X)=XB_{12}(I)+A_{11}(X)=XB_{12}(I)+XA-AX=X\big(A+B_{12}(I)\big)-AX. \tag{15}$$

Subtracting Eqs. (14) and (15), we get

$$X\big(C-A-B_{12}(I)\big)-\big(C-A-B_{12}(I)\big)X=0.$$

So $C-A-B_{12}(I)\in(alg\mathcal{N})'=\mathcal{C}I$. Thus there exists $\lambda\in\mathcal{C}$ such that $C-A-B_{12}(I)=\lambda I$. Hence $B_{12}(I)=C-A-\lambda I$.

For arbitrary $X \in alg\mathcal{N}$, we have

$$A_{22}(X) = B_{11}(X) = B_{22}(X) = C_{11}(X) = 0,$$
$$A_{11}(X) = XA - AX,$$
$$A_{12}(X) = XB,$$
$$C_{12}(X) = -BX,$$
$$C_{22}(X) = XC - CX,$$
$$B_{12}(X) = XC - (A + \lambda I)X = XC - AX - \lambda X.$$

Thus we have

$$\varphi\left(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} A_{11}(X) & A_{12}(X) \\ 0 & A_{22}(X) \end{bmatrix} = \begin{bmatrix} XA - AX & XB \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} - \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix},$$

$$\varphi\left(\begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} B_{11}(Y) & B_{12}(Y) \\ 0 & B_{22}(Y) \end{bmatrix} = \begin{bmatrix} 0 & YC - AY - \lambda Y \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} - \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} - \lambda\begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix},$$

$$\varphi\left(\begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} C_{11}(Z) & C_{12}(Z) \\ 0 & C_{22}(Z) \end{bmatrix} = \begin{bmatrix} 0 & -BZ \\ 0 & ZC - CZ \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix}\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} - \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix}$$

for any $X, Y, Z \in alg\mathcal{N}$. Hence we get

$$\varphi\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} - \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} - \lambda\begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\begin{bmatrix} A + \frac{1}{2}\lambda I & B \\ 0 & C - \frac{1}{2}\lambda I \end{bmatrix} - \begin{bmatrix} A + \frac{1}{2}\lambda I & B \\ 0 & C - \frac{1}{2}\lambda I \end{bmatrix}\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}.$$

Hence φ is an inner derivation. This completes the proof. □

3. All-derivable points in continuous nest algebras

In this section, we always use $\mathcal{N}$ to denote a continuous nest on a complex and separable Hilbert space H. If $M \in \mathcal{N}$ with $\{0\} \neq M \neq H$, then all 2×2 operator matrices always are represented as relative to the orthogonal decomposition $H = M \oplus M^{\perp}$.

Lemma 3.1. *Let $\mathcal{N}$ be a complete nest on a complex and separable Hilbert space H. Let $\varphi : B(H) \to B(H)$ be a strong operator topology continuous linear mapping such that $\varphi(Y)W = 0$ for any $Y \in B(H)$ and $W \in alg\mathcal{N}$ with $YW = 0$. Then there exists an operator $D \in B(H)$ such that $\varphi(Y) = DY$ for any $Y \in B(H)$.*

Proof. *Case* 1. Suppose $H_- \neq H$. For arbitrary $Y \in B(H)$ and $x \in N(Y)$, we claim that $\varphi(Y)x \in R(Y)$. In fact, if we take $y \in (H_-)^{\perp}$ with $\| y \| = 1$, then $W = x \otimes y \in alg\mathcal{N}$ and $YW = 0$. So we have

$$\varphi(Y)W = \varphi(Y)x \otimes y = 0,$$

i.e. $\varphi(Y)N(Y) = 0 \in R(Y)$. Obviously $\varphi(Y)N(Y) \subseteq R(Y)$. By Theorem 3 in [18], there exist two operators $D', F' \in B(H)$ such that

$$\varphi(Y) = YF' + D'Y,$$

for any $Y \in B(H)$. For arbitrary closed subspace $L \subset H$, then $0 = \varphi(P(L^{\perp}))L = (P(L^{\perp})F' + D'P(L^{\perp}))L = P(L^{\perp})F'L$. Thus $L \in LatF'$. Hence there exists $\lambda \in \mathcal{C}$ such that $F' = \lambda I$. Thus we have

$$\varphi(Y) = (\lambda I + D')Y$$

for any $Y \in B(H)$. Hence $D = \lambda I + D'$ is desired as in the lemma.

Case 2. Suppose $H_- = H$. Then there exists a sequence of closed subspaces $\{N_n\} \subseteq \mathcal{N}$ in H such that $P(N_n) \to I$ for the strong operator topology. We define mappings $\varphi_n : B(N_n) \to B(N_n)$ as

$$\varphi_n(T) = P(N_n)\varphi(T)P(N_n)$$

for any $T \in B(N_n)$. For arbitrary $T \in B(N_n) \subseteq B(H)$, take $x \in N(T) \cap N_n$. If $y \in (N_n)^\perp$ with $\| y \| = 1$, then $W = x \otimes y \in alg\mathcal{N}$ and $TW = 0$. So we have

$$0 = \varphi(T)x \otimes y = \varphi_n(T)x \otimes y,$$

i.e. $\varphi_n(T)N(T) = 0 \in R(T)$. By imitating the proof in the Case 1, there exists $D_n \in B(N_n)$ such that

$$\varphi_n(T) = D_n T.$$

Note that $\lim_{n\to+\infty} D_n = \lim_{n\to+\infty} D_n P(N_n) = \lim_{n\to+\infty} \varphi(P(N_n)) = \varphi(I)$ for the strong operator topology. We claim that $D = \varphi(I)$ is desired as in the lemma. In fact, for arbitrary $Y \in B(H)$, taking $Y_n = P(N_n)YP(N_n) \in B(N_n)$, we have

$$P(N_n)\varphi(Y_n)P(N_n) = \varphi_n(Y_n) = D_n Y_n = D_n P(N_n)YP(N_n).$$

Let $n \to +\infty$ in the above equation. Then we have

$$\varphi(Y) = DY.$$

This completes the proof. $\square$

Theorem 3.2. *Let $\mathcal{N}$ be a continuous nest on a complex and separable Hilbert space H. If $\{0\} \neq M \in \mathcal{N}$, then $P(M)$ is an all-derivable point of alg$\mathcal{N}$ for the strong operator topology.*

Proof. Suppose that $0 \neq M \in \mathcal{N}$. We claim that $P(M)$ is an all-derivable point of $alg\mathcal{N}$ for the strong operator topology. By the main theorem in [14], we may assume that $M \neq H$.

Let φ be a strong operator topology continuous derivable mapping at $P(M)$ on $alg\mathcal{N}$, and write $\mathcal{N}_M = \{N \cap M\colon \forall N \in \mathcal{N}\}$. Note that $\mathcal{N}$ is a continuous nest. So both $\mathcal{N}_M$ and $\mathcal{N}_{M^\perp}$ are continuous nests on infinite dimension separable Hilbert M and $M^\perp$, respectively. By Theorem 2.10 in [10], given $\varepsilon > 0$, we know that there exist a position invertible operator on $M^\perp$ and a unitary operator U from $M^\perp$ into M with $T - I_M$ compact and $\| T - I \| < \varepsilon$ such that $UT\mathcal{M}_M = \mathcal{M}_{M^\perp}$. Then $alg\mathcal{N}_{M^\perp} = UT alg\mathcal{N}_M (UT)^{-1}$.

We write $P = UT$ and

$$\mathcal{A} = \left\{ \begin{bmatrix} X & YP^{-1} \\ 0 & PZP^{-1} \end{bmatrix} \colon X, Y, Z \in alg\mathcal{N}_M \right\}.$$

Obviously $\mathcal{A} \subseteq \left\{ \left[\begin{smallmatrix} X & YP^{-1} \\ 0 & PZP^{-1} \end{smallmatrix}\right] \colon X, Z \in alg\mathcal{N}_M, Y \in B(M) \right\} = alg\mathcal{N}$. We know from the main theorem in [14] that I_M is an all-derivable point of $alg\mathcal{N}_M$ for the strong operator topology. By imitating the proof of Theorem 2.2, we may get that $E_{11}(I_M) = \left[\begin{smallmatrix} I_M & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is an all-derivable point of $\mathcal{A}$ for strong operator topology and there exists $Q = \left[\begin{smallmatrix} A & BP^{-1} \\ 0 & PCP^{-1} \end{smallmatrix}\right] \in B(H)$ (where $A, B, C \in B(M)$) such that

$$\varphi(S) = SQ - QS$$

for any $S \in \mathcal{A}$. For arbitrary $Y \in B(M)$ and $W \in alg\mathcal{N}_M$, we write

$$\varphi\left(\begin{bmatrix} 0 & YP^{-1} \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} B_{11}(Y) & B_{12}(Y)P^{-1} \\ 0 & PB_{22}(Y)P^{-1} \end{bmatrix}.$$

Note that φ is a strong operator topology continuous derivable mapping at $E_{11}(I_M)$ on $alg\mathcal{N}$. We take $S = \left[\begin{smallmatrix} I & YP^{-1} \\ 0 & 0 \end{smallmatrix}\right]$ and $T = \left[\begin{smallmatrix} I & 0 \\ 0 & PWP^{-1} \end{smallmatrix}\right]$ in $alg\mathcal{N}$ with $ST = E_{11}(I_M)$, i.e. $YW = 0$. Note that $E_{11}(I_M) = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} I & 0 \\ 0 & PWP^{-1} \end{smallmatrix}\right]$ in $\mathcal{A}$, so we have

$$\begin{bmatrix}0 & BP^{-1}\\0 & 0\end{bmatrix} = \varphi\big(E_{11}(I_M)\big) = \varphi(S)T + S\varphi(T)$$
$$= \varphi\left(\begin{bmatrix}I & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & YP^{-1}\\0 & 0\end{bmatrix}\right)\begin{bmatrix}I & 0\\0 & PWP^{-1}\end{bmatrix} + \begin{bmatrix}I & YP^{-1}\\0 & 0\end{bmatrix}\varphi\left(\begin{bmatrix}I & 0\\0 & PWP^{-1}\end{bmatrix}\right)$$
$$= \left(\begin{bmatrix}0 & BP^{-1}\\0 & 0\end{bmatrix} + \begin{bmatrix}B_{11}(Y) & B_{12}(Y)P^{-1}\\0 & PB_{22}(Y)P^{-1}\end{bmatrix}\right)\begin{bmatrix}I & 0\\0 & PWP^{-1}\end{bmatrix}$$
$$+ \begin{bmatrix}I & YP^{-1}\\0 & 0\end{bmatrix}\begin{bmatrix}0 & BP^{-1} - BWP^{-1}\\0 & P(WC - CW)P^{-1}\end{bmatrix}$$
$$= \begin{bmatrix}B_{11}(Y) & (BW + B_{12}(Y)W + B - BW + Y(WC - CW))P^{-1}\\0 & PB_{22}(Y)WP^{-1}\end{bmatrix}.$$

It follows from the above matrix equation that $(B_{12}(Y) - YC)W = 0$ and $B_{22}(Y)W = 0$ for any $Y \in B(M)$ and $W \in alg\mathcal{N}$ with $YW = 0$. We also know from the above matrix equation that $B_{11}(Y) \equiv 0$ for any $Y \in B(M)$. By Lemma 3.1, there exist two operators $F_1, F_2 \in B(M)$ such that $B_{12}(Y) - YC = F_1Y$ and $B_{22}(Y) = F_2Y$. Hence $B_{12}(Y) = YC + F_1Y$ for any $Y \in B(M)$. Thus, for arbitrary $Y \in alg\mathcal{N}_M$, we have

$$\begin{bmatrix}0 & (YC + F_1Y)P^{-1}\\0 & PF_2YP^{-1}\end{bmatrix} = \begin{bmatrix}B_{11}(Y) & B_{12}(Y)P^{-1}\\0 & PB_{22}(Y)P^{-1}\end{bmatrix} = \varphi\left(\begin{bmatrix}0 & YP^{-1}\\0 & 0\end{bmatrix}\right)$$
$$= \begin{bmatrix}0 & YP^{-1}\\0 & 0\end{bmatrix}Q - Q\begin{bmatrix}0 & YP^{-1}\\0 & 0\end{bmatrix}$$
$$= \begin{bmatrix}0 & YP^{-1}\\0 & 0\end{bmatrix}\begin{bmatrix}A & BP^{-1}\\0 & PCP^{-1}\end{bmatrix} - \begin{bmatrix}A & BP^{-1}\\0 & PCP^{-1}\end{bmatrix}\begin{bmatrix}0 & YP^{-1}\\0 & 0\end{bmatrix}$$
$$= \begin{bmatrix}0 & (YC - AY)P^{-1}\\0 & 0\end{bmatrix}.$$

So $YC + FY = YC - AY$ and $F_2 = 0$. Hence $F_1 = -A$, i.e. $B_{12}(Y) = YC - AY$ and $B_{22}(Y) = 0$ for any $Y \in B(M)$. Thus we have

$$\varphi\left(\begin{bmatrix}0 & YP^{-1}\\0 & 0\end{bmatrix}\right) = \begin{bmatrix}0 & YP^{-1}\\0 & 0\end{bmatrix}\begin{bmatrix}A & BP^{-1}\\0 & PCP^{-1}\end{bmatrix} - \begin{bmatrix}A & BP^{-1}\\0 & PCP^{-1}\end{bmatrix}\begin{bmatrix}0 & YP^{-1}\\0 & 0\end{bmatrix}$$

for any $Y \in B(M)$. Furthermore we have $\varphi(T) = TQ - QT$ for any $T = \left[\begin{smallmatrix}X & YP^{-1}\\0 & PZP^{-1}\end{smallmatrix}\right] \in alg\mathcal{N}$. Hence φ is an inner derivation. This completes the proof. □

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