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Periodic solutions for some nonautonomous second order Hamiltonian systems ☆

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Abstract

The existence and multiplicity of periodic solutions are obtained for nonautonomous second order Hamiltonian systems by the minimax methods in critical point theory.

Keywords: Periodic solution; Second order Hamiltonian system; Saddle Point Theorem; Generalized Mountain Pass Theorem; (PS) condition; Sobolev's inequality; Wirtinger's inequality

1. Introduction and main results

Consider the second order Hamiltonian systems

$$\begin{cases} \ddot{u}(t) = \nabla F\big(t, u(t)\big), & \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1}$$

where $T > 0$ and $F : [0, T] \times R^N \to R$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in R^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(R^+, R^+)$, $b \in L^1(0, T; R^+)$ such that

$$\big|F(t, x)\big| \leqslant a\big(|x|\big)b(t), \qquad \big|\nabla F(t, x)\big| \leqslant a\big(|x|\big)b(t)$$

for all $x \in R^N$ and a.e. $t \in [0, T]$.

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Many existence results are obtained for problem (1) by the least action principle, such as [2,10,11,15,16,18,23] and their references. Meanwhile, using the minimax methods, [4,7,9,14,21,22] consider the superquadratic second order Hamiltonian systems. The periodic potential (see [3,5,8,17,24]) and the subquadratic potential (see [5,6,16,19,20]) are also considered.

Especially, under the condition that $F(t,x) \to +\infty$ as $|x| \to +\infty$ uniformly for a.e. $t \in [0,T]$, Berger and Schechter [2] proved the existence of solutions for problem (1) (see Theorem 4.9 in [2]). Being based on [2], Tang and Wu [18] generalized the existence results to the locally coercive case. A natural question is whether problem (1) is also solvable under the opposite condition, that is, $F(t,x) \to -\infty$ as $|x| \to +\infty$ uniformly for a.e. $t \in [0,T]$. In general, we do not know whether the question is positive answer, but when $F(t,x) = G(x) + H(t,x)$, ∇H is bounded, that is, there exists $g \in L^1(0,T;R^+)$ such that

$$\left|\nabla H(t,x)\right| \leqslant g(t) \tag{2}$$

for all $x \in R^n$ and a.e. $t \in [0,T]$, and there exists $r < 4\pi^2/T^2$ such that

$$\left(\nabla G(x) - \nabla G(y), x - y\right) \geqslant -r|x-y|^2 \tag{3}$$

for all $x, y \in R^N$, Ahmad and Lazer [1] obtained the same results. In this paper, we suppose that ∇H is sublinear, that is, there exist $f, g \in L^1(0,T;R^+)$ and $\alpha \in [0,1)$ such that

$$\left|\nabla H(t,x)\right| \leqslant f(t)|x|^\alpha + g(t) \tag{4}$$

for all $x \in R^N$ and a.e. $t \in [0,T]$. Then the existence of periodic solutions, which generalizes Ahmad–Lazer's results mentioned above, are obtained by the minimax methods in critical point theory. Moreover, the multiplicity of periodic solutions is also obtained. Our main results are the following theorems.

Theorem 1. *Suppose that $F(t,x)$ satisfies assumption* (A)*,* (2) *and* (3)*. Assume that there exists $\gamma \in L^1(0,T)$ such that*

$$F(t,x) \leqslant \gamma(t) \tag{5}$$

for all $x \in R^N$ and a.e. $t \in [0,T]$, and that there exists a subset E of $[0,T]$ with $\operatorname{meas}(E) > 0$ *such that*

$$F(t,x) \to -\infty \quad \text{as } |x| \to \infty, \tag{6}$$

for a.e. $t \in E$. Then problem (1) *has at least one solution in H_T^1, where*

$$H_T^1 = \left\{ u : [0,T] \to R^N \;\middle|\; \begin{array}{l} u \text{ is absolutely continuous,} \\ u(0) = u(T) \text{ and } \dot{u} \in L^2(0,T;R^N) \end{array} \right\}$$

is a Hilbert space with the norm defined by

$$\|u\| = \left(\int_0^T \left|u(t)\right|^2 dt + \int_0^T \left|\dot{u}(t)\right|^2 dt \right)^{1/2}$$

for $u \in H_T^1$.

Remark 1. Theorem 1 extends the result in [1]. Ma and Tang [12] proved the same result replacing (6) by a weaker condition that

$$\int_0^T F(t,x)\,dt \to -\infty \quad \text{as } |x| \to \infty,$$

while adding another condition on G, that is, there exists $A \in C(R^N, R)$ such that

$$\left|\nabla G(x) - \nabla G(y)\right| \leqslant A(x-y) \tag{7}$$

for all $x, y \in R^N$ (see Theorem 3 in [12]). A natural question is whether Theorem 3 in [12] holds yet without (7). There are functions F satisfying the assumptions of our Theorem 1 and not satisfying those of the results mentioned above. For example, let $F(t,x) = G(x) + H(t,x)$ with $G(x) = -r\cos x_1$, which is bounded, and

$$H(t,x) = -|\sin \omega t| \ln\bigl(1 + |x|^2\bigr)$$

for all $x \in R^N$ and $t \in [0,T]$. Then F satisfies the assumptions of our Theorem 1. But F does not satisfy those of the results mentioned above, because that $F(t,x)$ is neither superquadratic in x, nor subquadratic in x, nor periodic in x.

Theorem 2. *Suppose that $F(t,x)$ satisfies assumption* (A), (3) *and* (4). *Assume that*

$$|x|^{-2\alpha} F(t,x) \to -\infty \tag{8}$$

as $|x| \to +\infty$ uniformly for a.e. $t \in [0,T]$, where α is the same as in (4). *Then problem* (1) *has at least one solution in H_T^1.*

Remark 2. Theorem 2 also generalizes the result in [1] which is the special case of our Theorem 2 corresponding to $\alpha = 0$. There are functions F satisfying the assumptions of our Theorem 2 and not satisfying those of the results mentioned above. For example, let $F(t,x) = G(x) + H(t,x)$ with $G(x) = -(r/2)|x_1|^2$, which is bounded from above, and

$$H(t,x) = -|x|^{1+\alpha}$$

where $0 < \alpha < 1$. Then F satisfies the assumptions of our Theorem 2. But F does not satisfy those of the results mentioned above, because that $F(t,x)$ is neither superquadratic in x, nor subquadratic in x, nor periodic in x.

We shall prove a more general result than Theorems 1 and 2.

Theorem 3. *Suppose that $F(t,x)$ satisfies assumption* (A), (3)–(5). *Assume that there exists a subset E of* $[0,T]$ *with* $\mathrm{meas}(E) > 0$ *such that*

$$|x|^{-2\alpha} F(t,x) \to -\infty \quad \text{as } |x| \to \infty, \tag{9}$$

for a.e. $t \in E$. Then problem (1) *has at lease one solution in H_T^1.*

Remark 3. Replacing condition (9) by a weaker condition that

$$|x|^{-2\alpha} \int_0^T F(t,x)\,dt \to -\infty \quad \text{as } |x| \to \infty,$$

Ma and Tang [12] proved the same result as Theorem 3 in addition to that there exist $M \geqslant 0$, $N \geqslant 0$ such that

$$\bigl|\nabla G(x) - \nabla G(y)\bigr| \leqslant M|x-y| + N \tag{10}$$

for all $x, y \in R^N$ (see Theorem 2 in [12]). A natural question is whether Theorem 2 in [12] holds yet without condition (10).

At last we give a corresponding multiplicity result.

Theorem 4. *Suppose that F satisfies* (A), (3)–(5) *and* (9). *Assume that there exist $\delta > 0$, $\varepsilon > 0$ and an integer $k > 0$ such that*

$$-\frac{1}{2}(k+1)^2\omega^2|x|^2 \leqslant F(t,x) - F(t,0) \tag{11}$$

for all $x \in R^N$ and a.e. $t \in [0,T]$, and

$$F(t,x) - F(t,0) \leqslant -\frac{1}{2}k^2\omega^2(1+\varepsilon)|x|^2 \tag{12}$$

for all $|x| \leqslant \delta$ and a.e. $t \in [0,T]$, where $\omega = \frac{2\pi}{T}$. Then problem (1) *has at least one nontrivial solution in H_T^1.*

2. Proof of theorems

For $u \in H_T^1$, let

$$\bar{u} = \frac{1}{T}\int_0^T u(t)\,dx \quad \text{and} \quad \tilde{u}(t) = u(t) - \bar{u}.$$

Then one has

$$\|\tilde{u}\|_\infty^2 \leqslant \frac{T}{12}\int_0^T \left|\dot{u}(t)\right|^2 dt \quad \text{(Sobolev's inequality)}$$

and

$$\int_0^T \left|\tilde{u}(t)\right|^2 dt \leqslant \frac{T^2}{4\pi^2}\int_0^T \left|\dot{u}(t)\right|^2 dt \quad \text{(Wirtinger's inequality).}$$

It follows from assumption (A) that the functional φ on H_T^1 given by

$$\varphi(u) = \frac{1}{2}\int_0^T \left|\dot{u}(t)\right|^2 dt + \int_0^T F\big(t, u(t)\big)\,dt$$

is continuously differentiable and weakly lower semi-continuous on H_T^1 (see [11]). Moreover, one has

$$\langle \varphi'(u), v\rangle = \int_0^T \big[\big(\dot{u}(t), \dot{v}(t)\big) + \big(\nabla F\big(t, u(t)\big), v(t)\big)\big]\,dt$$

for all $u, v \in H_T^1$. It is well known that the solutions of problem (1) correspond to the critical points of φ.

For convenience to quote we state an analog of Egorov's theorem (see Lemma 2 in [18]), in which we replace F by $-F$.

Lemma 1. *(See [18].) Suppose that F satisfies the assumption* (A) *and E is a measurable subset of* $[0, T]$. *Assume that*

$$F(t, x) \to -\infty \quad \textit{as } |x| \to \infty$$

for a.e. $t \in E$. Then for every $\delta > 0$ there exists a subset E_δ of E with $\operatorname{meas}(E \setminus E_\delta) < \delta$ *such that*

$$F(t, x) \to -\infty \quad \textit{as } |x| \to \infty$$

uniformly for all $t \in E_\delta$.

Lemma 2. *Assume that F satisfies assumption* (A), (3)–(5) *and* (9). *Then φ satisfies the* (PS) *condition, that is, u_n has a convergent subsequence whenever it satisfies $\varphi'(u_n) \to 0$ as $n \to \infty$ and $\varphi(u_n)$ is bounded.*

Proof. By Wirtinger's inequality, we have

$$\left(\int_0^T \left|\dot{u}_n(t)\right|^2 dt\right)^{1/2} \leqslant \|\tilde{u}_n\| \leqslant \left(\frac{T^2}{4\pi^2} + 1\right)^{1/2}\left(\int_0^T \left|\dot{u}_n(t)\right|^2 dt\right)^{1/2} \tag{13}$$

for all n.

It follows from (4) and Sobolev's inequality that

$$\begin{aligned}
\left|\int_0^T \big(\nabla H\big(t,u(t)\big),\tilde{u}(t)\big)\,dt\right| &\leqslant \int_0^T f(t)\big|\bar{u}+\tilde{u}(t)\big|^{\alpha}\big|\tilde{u}(t)\big|\,dt+\int_0^T g(t)\big|\tilde{u}(t)\big|\,dt\\
&\leqslant \int_0^T 2f(t)\big(|\bar{u}|^{\alpha}+\big|\tilde{u}(t)\big|^{\alpha}\big)\big|\tilde{u}(t)\big|\,dt+\int_0^T g(t)\big|\tilde{u}(t)\big|\,dt\\
&\leqslant 2\big(|\bar{u}|^{\alpha}+\|\tilde{u}\|_\infty^{\alpha}\big)\|\tilde{u}\|_\infty\int_0^T f(t)\,dt+\|\tilde{u}\|_\infty\int_0^T g(t)\,dt\\
&\leqslant \frac{3(4\pi^2-rT^2)}{2\pi^2T}\|\tilde{u}\|_\infty^2+\frac{2\pi^2T}{3(4\pi^2-rT^2)}|\bar{u}|^{2\alpha}\left(\int_0^T f(t)\,dt\right)^2\\
&\quad+2\|\tilde{u}\|_\infty^{\alpha+1}\int_0^T f(t)\,dt+\|\tilde{u}\|_\infty\int_0^T g(t)\,dt\\
&\leqslant \frac{4\pi^2-rT^2}{8\pi^2}\int_0^T\big|\dot{u}(t)\big|^2\,dt+C_1|\bar{u}|^{2\alpha}\\
&\quad+C_2\left(\int_0^T\big|\dot{u}(t)\big|^2\,dt\right)^{(\alpha+1)/2}+C_3\left(\int_0^T\big|\dot{u}(t)\big|^2\,dt\right)^{1/2}
\end{aligned}$$

for all $u\in H_T^1$ and some positive constants C_1, C_2 and C_3. From (3) and Wirtinger's inequality we obtain

$$\int_0^T\big(\nabla G\big(u(t)\big),\tilde{u}(t)\big)\,dt=\int_0^T\big(\nabla G\big(u(t)\big)-\nabla G(\bar{u}),\tilde{u}(t)\big)\,dt\geqslant -r\int_0^T\big|\tilde{u}(t)\big|^2\,dt\geqslant -\frac{rT^2}{4\pi^2}\int_0^T\big|\dot{u}(t)\big|^2\,dt$$

for all $u\in H_T^1$. Hence one has

$$\begin{aligned}
\|\tilde{u}_n\| &\geqslant \big|\langle\varphi'(u_n),\tilde{u}_n\rangle\big|\\
&=\left|\int_0^T\big|\dot{u}_n(t)\big|^2\,dt+\int_0^T\big(\nabla F\big(t,u_n(t)\big),\tilde{u}_n(t)\big)\,dt\right|\\
&=\left|\int_0^T\big|\dot{u}_n(t)\big|^2\,dt+\int_0^T\big(\nabla G\big(u_n(t)\big),\tilde{u}_n(t)\big)\,dt+\int_0^T\big(\nabla H\big(t,u_n(t)\big),\tilde{u}_n(t)\big)\,dt\right|\\
&\geqslant \frac{4\pi^2-rT^2}{8\pi^2}\int_0^T\big|\dot{u}_n(t)\big|^2\,dt-C_1|\bar{u}|^{2\alpha}-C_2\left(\int_0^T\big|\dot{u}_n(t)\big|^2\,dt\right)^{(\alpha+1)/2}-C_3\left(\int_0^T\big|\dot{u}_n(t)\big|^2\right)^{1/2}
\end{aligned}$$

for large n. By (13) and the above inequality we have

$$C|\bar{u}_n|^{\alpha}\geqslant\left(\int_0^T\big|\dot{u}_n(t)\big|^2\,dt\right)^{1/2}-C_4 \tag{14}$$

for some constants $C > 0$, $C_4 > 0$ and all large n, which implies that

$$\|\tilde{u}_n\|_\infty \leqslant C_5\big(|\bar{u}_n|^\alpha + 1\big)$$

for all large n and some positive constant C_5 by Sobolev's inequality. Then one has

$$\big|u_n(t)\big| \geqslant |\bar{u}_n| - \big|\tilde{u}_n(t)\big| \geqslant |\bar{u}_n| - \|\tilde{u}_n\|_\infty \geqslant |\bar{u}_n| - C_5\big(|\bar{u}_n|^\alpha + 1\big)$$

for all large n and every $t \in [0, T]$, which implies that

$$\big|u_n(t)\big| \geqslant \frac{1}{2}|\bar{u}_n| \tag{15}$$

for all large n and every $t \in [0, T]$.

If $(|\bar{u}_n|)$ is unbounded, we may assume that, going to a subsequence if necessary,

$$|\bar{u}_n| \to \infty \quad \text{as } n \to \infty. \tag{16}$$

Set $\delta = \operatorname{meas} E/2$. It follows from (11) and Lemma 1 that there exists a subset E_δ of E with $\operatorname{meas}(E \setminus E_\delta) < \delta$ such that

$$|x|^{-2\alpha} F(t, x) \to -\infty \quad \text{as } |x| \to \infty$$

uniformly for all $t \in E_\delta$, which implies that

$$\operatorname{meas} E_\delta = \operatorname{meas} E - \operatorname{meas}(E \setminus E_\delta) > \delta > 0 \tag{17}$$

and for every $\beta > 0$, there exists $M \geqslant 1$ such that

$$|x|^{-2\alpha} F(t, x) \leqslant -\beta \tag{18}$$

for all $|x| \geqslant M$ and all $t \in E_\delta$. By (15) and (16), one has

$$\big|u_n(t)\big| \geqslant M \tag{19}$$

for large n and every $t \in [0, T]$. It follows from (14), (5), (17)–(19) and (15) that

$$\begin{aligned}
\varphi(u_n) &\leqslant \big(C|\bar{u}_n|^\alpha + C_4\big)^2 + \int\limits_{[0,T]\setminus E_\delta} \gamma(t)\,dt - \int\limits_{E_\delta} \beta\big|u_n(t)\big|^{2\alpha}\,dt \\
&\leqslant \big(C|\bar{u}_n|^\alpha + C_4\big)^2 + \int\limits_{[0,T]\setminus E_\delta} \gamma(t)\,dt - 2^{-2\alpha}|\bar{u}_n|^{2\alpha}\delta\beta
\end{aligned}$$

for all large n. Hence, we have

$$\limsup_{n\to\infty} |\bar{u}_n|^{-2\alpha}\varphi(u_n) \leqslant C^2 - 2^{-2\alpha}\delta\beta.$$

By the arbitrariness of $\beta > 0$, one has

$$\limsup_{n\to\infty} |\bar{u}_n|^{-2\alpha}\varphi(u_n) = -\infty,$$

which contradicts the boundedness of $\varphi(u_n)$. Hence $(|\bar{u}_n|)$ is bounded. Furthermore, (u_n) is bounded by (14) and (13). Arguing then as in Proposition 4.1 in [11], we conclude that the (PS) condition is satisfied. □

Now we prove our Theorem 3 first.

Proof of Theorem 3. It follows from Lemma 2 the φ satisfies the (PS) condition. We now prove that φ satisfies the other conditions of the Saddle Point Theorem (see Theorem 4.6 in [11]). Let $\widetilde{H}_T^1$ be the subspace of H_T^1 given by

$$\widetilde{H}_T^1 = \big\{u \in H_T^1 \mid \bar{u} = 0\big\}.$$

Then one has

$$\varphi(u) \to +\infty \tag{20}$$

as $\|u\| \to \infty$ in $\widetilde{H}_T^1$. In fact it follows from Sobolev's inequality and Wirtinger's inequality that

$$\begin{aligned}
\left| \int_0^T \left[H\big(t, u(t)\big) - H(t, 0)\right] dt \right| &= \left| \int_0^T \int_0^1 \big(\nabla H\big(t, su(t)\big), u(t)\big)\, ds\, dt \right| \\
&\leqslant \int_0^T \int_0^1 f(t)\big|su(t)\big|^\alpha \big|u(t)\big|\, ds\, dt + \int_0^T \int_0^1 g(t)\big|u(t)\big|\, ds\, dt \\
&\leqslant \int_0^T f(t)\big|u(t)\big|^\alpha \big|u(t)\big|\, dt + \int_0^T g(t)\big|u(t)\big|\, dt \\
&\leqslant \|\tilde{u}\|_\infty^{\alpha+1} \int_0^T f(t)\, dt + \|u\|_\infty \int_0^T g(t)\, dt \\
&\leqslant C_6 \left(\int_0^T \big|\dot{u}(t)\big|^2 dt \right)^{(\alpha+1)/2} + C_7 \left(\int_0^T \big|\dot{u}(t)\big|^2 dt \right)^{1/2}
\end{aligned}$$

for all $u \in \widetilde{H}_T^1$ and some positive constants C_6 and C_7. By (3) and Wirtinger's inequality we have

$$\begin{aligned}
\int_0^T \left[G\big(u(t)\big) - G(0)\right] dt &= \int_0^T \int_0^1 \big(\nabla G\big(su(t)\big) - \nabla G(0), u(t)\big)\, ds\, dt \\
&= \int_0^T \int_0^1 \frac{1}{s} \big(\nabla G\big(su(t)\big) - \nabla G(0), su(t)\big)\, ds\, dt \\
&\geqslant \int_0^T \int_0^1 \frac{1}{s} \big(-rs^2 \big|u(t)\big|^2\big)\, ds\, dt \\
&\geqslant -\frac{rT^2}{8\pi^2} \int_0^T \big|\dot{u}(t)\big|^2 dt.
\end{aligned}$$

Hence one has

$$\begin{aligned}
\varphi(u) - \int_0^T F(t, 0)\, dt &= \frac{1}{2} \int_0^T \big|\dot{u}(t)\big|^2 dt + \int_0^T \left[F\big(t, u(t)\big) - F(t, 0)\right] dt \\
&= \frac{1}{2} \int_0^T \big|\dot{u}(t)\big|^2 dt + \int_0^T \left[G\big(u(t)\big) - G(0)\right] dt + \int_0^T \left[H\big(t, u(t)\big) - H(t, 0)\right] dt \\
&\geqslant \frac{4\pi^2 - rT^2}{8\pi^2} \int_0^T \big|\dot{u}(t)\big|^2 dt - C_6 \left(\int_0^T \big|\dot{u}(t)\big|^2 dt \right)^{(\alpha+1)/2} - C_7 \left(\int_0^T \big|\dot{u}(t)\big|^2 dt \right)^{1/2}
\end{aligned}$$

for all $u \in \widetilde{H}_T^1$, which implies (20) by (13) and $r < 4\pi^2/T^2$. Moreover, by (18) we have

$$\varphi(x) = \int_0^T F(t,x)\,dt \leqslant \int_{[0,T]\setminus E_\delta} \gamma(t)\,dt - \int_{E_\delta} \beta|x|^{2\alpha}\,dt \leqslant \int_{[0,T]\setminus E_\delta} \gamma(t)\,dt - \beta M^{2\alpha} \operatorname{meas} E_\delta$$
$$\leqslant \int_{[0,T]\setminus E_\delta} \gamma(t)\,dt - \beta \operatorname{meas} E_\delta$$

for all $|x| \geqslant M$, which implies that

$$\varphi(x) \to -\infty \tag{21}$$

as $|x| \to \infty$ in R^N by the arbitrariness of β.

Now Theorem 3 is proved by (20), (21) and the Saddle Point Theorem (see Theorem 4.6 in [11]). □

Then we prove our Theorems 1 and 2.

Proof of Theorems 1 and 2. Theorem 1 follows from Theorem 3 by letting $\alpha = 0$. Theorem 3 implies Theorem 2 because (5) follows from (8) and assumption (A). In fact, by (8) there exists $M > 0$ such that

$$|x|^{-2\alpha} F(t,x) \leqslant 0$$

for all $|x| \geqslant M$ and a.e. $t \in [0,T]$, which implies that

$$F(t,x) \leqslant 0$$

for all $|x| \geqslant M$ and a.e. $t \in [0,T]$. It follows from assumption (A) that

$$F(t,x) \leqslant a_0 b(t)$$

for all $|x| \leqslant M$ and a.e. $t \in [0,T]$, where $a_0 = \max_{0 \leqslant s \leqslant M} a(s)$. Now (5) holds with $\gamma(t) = a_0 b(t)$. Hence Theorem 2 follows from Theorem 3. □

At last we prove our Theorem 4.

Proof of Theorem 4. Let $E = H_T^1$,

$$H_k = \left\{ \sum_{j=0}^{k} (a_j \cos j\omega t + b_j \sin j\omega t) \;\middle|\; a_j,\ b_j \in R^N,\ j = 0, \ldots, k \right\}$$

and

$$\psi(u) = -\varphi(u) + \int_0^T F(t,0)\,dt = -\frac{1}{2}\int_0^T \left|\dot u(t)\right|^2 dt - \int_0^T \left[F(t,x) - F(t,0)\right] dt.$$

Then $\psi \in C^1(E,R)$ satisfies the (PS) condition. By the Generalized Mountain Pass Theorem (see Theorem 5.29 and Example 5.26 in [13]), we only need to proof

(ψ_1) $\liminf \|u\|^{-2}\psi(u) > 0$ as $u \to 0$ in H_k.
(ψ_2) $\psi(u) \leqslant 0$ for all $u \in H_k^\perp$, and
(ψ_3) $\psi(u) \to -\infty$ as $\|u\| \to \infty$ in $H_{k-1}^\perp$.

Notice that

$$F(t,x) - F(t,0) = \int_0^1 \left(\nabla F(t,sx), x\right) ds$$

for all $x \in R^N$ and a.e. $t \in [0, T]$. By (4) we have

$$F(t,x) - F(t,0) \leqslant \frac{f(t)}{1+\alpha}|x|^{1+\alpha} + g(t)|x| \leqslant h(t)|x|^3$$

for all $|x| \geqslant \delta$, a.e. $t \in [0, T]$ and some $h \in L^1(0, T; R^+)$ given by

$$h(t) = \frac{\delta^{\alpha-2}}{1+\alpha} f(t) + \delta^{-2} g(t).$$

Now it follows from (12) that

$$F(t,x) - F(t,0) \leqslant -\frac{1}{2}k^2\omega^2(1+\varepsilon)|x|^2 + h(t)|x|^3$$

for all $x \in R^N$ and a.e. $t \in [0, T]$. Hence we obtain

$$\begin{aligned}
\psi(u) &\geqslant -\frac{1}{2}\int_0^T \left|\dot{u}(t)\right|^2 dt + \frac{1}{2}k^2\omega^2(1+\varepsilon)\int_0^T \left|u(t)\right|^2 dt - \int_0^T h(t)\left|u(t)\right|^3 dt \\
&\geqslant \frac{1}{2}\varepsilon \int_0^T \left|u(t)\right|^2 dt + \frac{1}{2}k^2\omega^2(1+\varepsilon)|\bar{u}|^2 T - \|u\|_\infty^3 \int_0^T h(t)\, dt \\
&\geqslant C_8\|u\|^2 - C_9\|u\|^3
\end{aligned}$$

for all $u \in H_k$, where $C_8 = \min\{\frac{\varepsilon}{2}, \frac{1}{2}k^2\omega^2(1+\varepsilon)T\}$, $C_9 = (C_{10})^3 \int_0^T h(t)\, dt$ and the constant C_{10} follows from the inequality

$$\|u\|_\infty \leqslant |\bar{u}| + \|\tilde{u}\|_\infty \leqslant T^{-1}\int_0^T \left|u(t)\right| dt + \|\tilde{u}\|_\infty \leqslant T^{-1/2}\|u\|_{L^2} + \left(\frac{T}{12}\right)^{1/2} \|\dot{u}\|_{L^2} \leqslant C_{10}\|u\|$$

for all $u \in H_T^1$, where we have used Hölder's inequality and Sobolev's inequality. Now (ψ_1) follows from the above inequality. For $u \in H_k^\perp$, by (11) one has

$$\psi(u) \leqslant -\frac{1}{2}\int_0^T \left|\dot{u}(t)\right|^2 dt + \frac{1}{2}(k+1)^2\omega^2 \int_0^T \left|u(t)\right|^2 dt \leqslant 0,$$

which is (ψ_2). At last (ψ_3) follows from (20). Hence the proof of Theorem 4 is completed. □

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