J. Math. Anal. Appl. 358 (2009) 81–97

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the uniqueness of meromorphic functions that share four values in one angular domain ☆

Ting-Bin Cao [a,*], Hong-Xun Yi [b]

[a] Department of Mathematics, Nanchang University, Nanchang, Jiangxi 330031, China
[b] Department of Mathematics, Shandong University, Jinan, Shandong 250100, China

ARTICLE INFO

Article history:
Received 13 January 2009
Available online 3 May 2009
Submitted by A.V. Isaev

Keywords:
Meromorphic function
Uniqueness theorem
Shared values
Angular domain

ABSTRACT

The main purpose of this paper is to investigate the uniqueness of transcendental meromorphic functions that share four values in one angular domain which is an unbounded subset of the whole complex plane. From one of our main results, a question of J.H. Zheng [J.H. Zheng, On uniqueness of meromorphic functions with shared values in one angular domain, Complex Var. Elliptic Equ. 48 (9) (2003) 777–785] is completely answered. Furthermore, we give an example to explain the necessity of the condition

$$\lim_{r\to\infty}\frac{S_{\alpha,\beta}(r,f)}{\log(rT(r,f))}=\infty$$

in our results.

1. Introduction and main results

In this paper, a transcendental meromorphic function is meromorphic in the whole complex plane $\mathbb{C}$ and not rational. We assume familiarity with the Nevanlinna's theory of meromorphic functions and the standard notations such as $m(r,f)$, $T(r,f)$. For references, please see [6]. We say that two meromorphic functions f and g share the value a ($a\in\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$) in $X\subseteq\mathbb{C}$ provided that in X, we have $f(z)=a$ if and only if $g(z)=a$. We will state whether a shared value is by *CM* (counting multiplicities) or by *IM* (ignoring multiplicities). If a is shared *IM* by f and g and the multiplicities of zeros of $f-a$ and $g-a$ are different, then we say that the value a is shared *DM* by f and g.

R. Nevanlinna (see [8]) proved the following well-known theorems.

Theorem 1.1. *(See [8].) If f and g are two non-constant meromorphic functions that share five distinct values a_1, a_2, a_3, a_4, a_5 IM in $X=\mathbb{C}$, then $f(z)\equiv g(z)$.*

Theorem 1.2. *(See [8].) If f and g are two distinct non-constant meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 CM in $X=\mathbb{C}$, then f is a Möbius transformation of g, two of the shared values, say a_1 and a_2, are Picard values, and the cross ratio $(a_1,a_2,a_3,a_4)=-1$.*

After his very work, the uniqueness of meromorphic functions with shared values in the whole complex plane attracted many investigations (for references, see [13]). In [14], Zheng took into account of the uniqueness dealing with five shared

☆ This work was supported by the NNSF of China (No. 10771121) and the NSF of Jiangxi of China (No. 2008GQS0075).
* Corresponding author.
E-mail addresses: tbcao@ncu.edu.cn, ctb97@163.com (T.-B. Cao), hxyi@sdu.edu.cn (H.-X. Yi).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.04.043

values in some angular domains of $\mathbb{C}$. It is an interesting topic to investigate the uniqueness with shared values in the remaining part of the complex plane removing an unbounded closed set, see [14,15,1,9,7,11]. In [15], Zheng continued to investigate this subject. From the proof of Theorem 3 in [15], we deduce easily that the following result is true.

Theorem 1.3. *Let f and g be two transcendental meromorphic functions. Given one angular domain $X = \{z\colon \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leqslant 2\pi$, we assume that f and g share five distinct values a_1, a_2, a_3, a_4, a_5 IM in X. Then $f(z) \equiv g(z)$, provided that*

$$\lim_{r\to\infty} \frac{S_{\alpha,\beta}(r, f)}{\log(rT(r, f))} = \infty \quad (r \notin E).$$

Throughout, we denote by E a set of finite linear measure, not necessarily the same in each time. $S_{\alpha,\beta}(r, f)$ is Nevanlinna's angular characteristic and its definition can be found below. We may denote Theorems 1.1 and 1.3 by 5*IM* theorem. In [15], Zheng mentioned another result by a simple notation $3CM + 1IM = 4CM$ as follows.

Theorem 1.4. *(See [15].) Let f and g be two distinct transcendental meromorphic functions. Given one angular domain $X = \{z\colon \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leqslant 2\pi$, we assume that f and g share three distinct CM shared values a_j $(j = 1, 2, 3)$ and one IM shared value a_4 in X. Then a_4 is also one CM shared value in X of f and g, provided that*

$$\lim_{r\to\infty} \frac{S_{\alpha,\beta}(r, f)}{\log(rT(r, f))} = \infty \quad (r \notin E).$$

Zheng [15, p. 778] raised a question as follows.

Question 1.1. Whether does $2CM + 2IM = 4CM$ hold?

Also, we may raise a natural question

Question 1.2. What can be said to an analogous result as Theorem 1.2 in one angular domain?

In this paper, we shall answer these questions. Nevanlinna's theory on angular domain (see [3]) will play a key role in this paper. Let f be a meromorphic function on the angular domain $\overline{\Omega} = \{z\colon \alpha \leqslant \arg z \leqslant \beta\}$, where $0 < \beta - \alpha \leqslant 2\pi$. Following Nevanlinna define

$$A_{\alpha,\beta}(r, f) = \frac{\omega}{\pi} \int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right)\left\{\log^+\left|f\left(te^{i\alpha}\right)\right| + \log^+\left|f\left(te^{i\beta}\right)\right|\right\}\frac{dt}{t}, \tag{1}$$

$$B_{\alpha,\beta}(r, f) = \frac{2\omega}{\pi r^\omega} \int_\alpha^\beta \log^+\left|f\left(re^{i\theta}\right)\right| \sin\omega(\theta - \alpha)\, d\theta, \tag{2}$$

$$C_{\alpha,\beta}(r, f) = 2 \sum_{1<|b_n|<r} \left(\frac{1}{|b_n|^\omega} - \frac{|b_n|^\omega}{r^{2\omega}}\right) \sin\omega(\theta_n - \alpha), \tag{3}$$

$$D_{\alpha,\beta}(r, f) = A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f), \tag{4}$$

where $\omega = \frac{\pi}{\beta-\alpha}$, $1 \leqslant r < \infty$, and $b_n = |b_n|e^{i\theta_n}$ are the poles of f on $\overline{\Omega}$, appearing according their multiplicities. If we only consider the distinct poles of f, we denote the corresponding angular counting function by $\overline{C}_{\alpha,\beta}(r, f)$. Nevanlinna's angular characteristic is defined as follows

$$S_{\alpha,\beta}(r, f) = A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f) + C_{\alpha,\beta}(r, f). \tag{5}$$

Now we show one of our main results by a simple notation 4*CM* theorem similarly as Theorem 1.2, from which we can answer Question 1.2.

Theorem 1.5. *Let f and g be two distinct transcendental meromorphic functions. Given one angular domain $X = \{z\colon \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leqslant 2\pi$, we assume that f and g share four distinct values a_1, a_2, a_3, a_4 CM in X, and that*

$$\lim_{r\to\infty} \frac{S_{\alpha,\beta}(r, f)}{\log(rT(r, f))} = \infty \quad (r \notin E).$$

Then f is a Möbius transformation of g, two of the shared values, say a_1 and a_2, are Picard values in X, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.

Let f and g be two distinct transcendental meromorphic functions and let $X = \{z\colon \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leqslant 2\pi$. We denote by $\overline{C}^E_{\alpha,\beta}(r, f(z) = a = g(z))$ the counting function of those a-points in X where f and g have same multiplicities, each point in the counting function being counted only once. Throughout, we denote by $R(r,*)$ quantities satisfying

$$R(r,*) = O\big(\log\big(rT(r,*)\big)\big), \quad r \notin E.$$

We say that f and g share the value a "*CM*" in X if f and g share a *IM* in X, furthermore,

$$\overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) - \overline{C}^E_{\alpha,\beta}\big(r, f(z) = a = g(z)\big) = R(r, f)$$

and

$$\overline{C}_{\alpha,\beta}\left(r, \frac{1}{g-a}\right) - \overline{C}^E_{\alpha,\beta}\big(r, f(z) = a = g(z)\big) = R(r, g).$$

Remark 1.1. Obviously, if a is shared *CM* by f and g in X, then it must be shared "*CM*" by f and g in X.

Theorem 1.6. *Let f and g be two distinct transcendental meromorphic functions. Given one angular domain $X = \{z\colon \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leqslant 2\pi$, we assume that f and g share two distinct values a_1, a_2 "CM" and other two distinct values a_3, a_4 IM in X. Then a_1, a_2, a_3, a_4 are shared CM by f and g in X, provided that*

$$\lim_{r\to\infty} \frac{S_{\alpha,\beta}(r, f)}{\log(rT(r, f))} = \infty \quad (r \notin E).$$

We may denote the above result by a simple notation 2"*CM*" + 2*IM* = 4*CM*. Thus we can answer Question 1.1 above from the following corollary which is immediately deduced by Theorem 1.6.

Corollary 1.1. *Let f and g be two distinct transcendental meromorphic functions. Given one angular domain $X = \{z\colon \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leqslant 2\pi$, we assume that f and g share two distinct values a_1, a_2 CM and other two distinct values a_3, a_4 IM in X. Then a_1, a_2, a_3, a_4 are shared CM by f and g in X, provided that*

$$\lim_{r\to\infty} \frac{S_{\alpha,\beta}(r, f)}{\log(rT(r, f))} = \infty \quad (r \notin E).$$

By the following example, we explain the necessity of the condition

$$\lim_{r\to\infty} \frac{S_{\alpha,\beta}(r, f)}{\log(rT(r, f))} = \infty$$

in Theorems 1.3–1.6, Corollary 1.1 and Question 5.1 in the final section.

Example 1.1. Consider two entire functions $f(z) = e^{z+1} - 1$ and $g(z) = f^2(z) = (e^{z+1} - 1)^2$. Set $X = \{z \in \mathbb{C}\colon \frac{\pi}{2} = \alpha < \arg z < \beta = \frac{3\pi}{2}\}$. So $\omega = 1$. By the equality $|e^{z+1} - 1| = |e^{z+1}| + O(1)$, we have $\log^+ |f(re^{i\alpha})| = O(1)$, $\log^+ |f(re^{i\beta})| = O(1)$, $\log^+ |f(re^{i\theta})| = \max\{r\cos\theta, 0\} + O(1)$. Hence we have

$$A_{\alpha,\beta}(r, f) = O\left(1 + \frac{1}{r} + \frac{1}{r^2}\right), \qquad B_{\alpha,\beta}(r, f) = O\left(\frac{1}{r}\right), \qquad C_{\alpha,\beta}(r, f) \equiv 0,$$

and thus

$$S_{\alpha,\beta}(r, f) = A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f) + C_{\alpha,\beta}(r, f) = O\left(1 + \frac{1}{r} + \frac{1}{r^2}\right).$$

Noting that $T(r, f) = \frac{r}{\pi} + O(1)$, we have

$$\lim_{r\to\infty} \frac{S_{\alpha,\beta}(r, f)}{\log(rT(r, f))} < \infty. \tag{6}$$

On the other hand, if a is a real number with $a \geqslant 16$, then neither of the functions f and g attains this value a in the angular domain X because

$$f(z) = a \iff z \in -1 + \log(a+1) + i2\pi\mathbb{Z} \subset \mathbb{C} \setminus X,$$
$$g(z) = a \iff z \in -1 + \log(\sqrt{a} \pm 1) + i2\pi\mathbb{Z} \subset \mathbb{C} \setminus X.$$

However, the value 0 is shared *DM* by f and g in the angular domain X because

$$f(z) = g(z) = 0 \quad \text{in } X \qquad \Longleftrightarrow \qquad z \in -1 + 2\pi\mathbb{Z} \subset X.$$

(i) If take $a_1 = 16$, $a_2 = 17$, $a_3 = 18$, $a_4 = 19$, $a_5 = 20$, then a_1, a_2, a_3, a_4, a_5 are shared IM by f and g in X. However, $f(z) \not\equiv g(z)$.

(ii) If take $a_1 = 16$, $a_2 = 17$, $a_3 = 18$, $a_4 = 19$, then a_1, a_2, a_3, a_4 are shared CM by f and g in X. However, f is not a Möbius transformation of g.

(iii) If take $a_1 = 0$, $a_2 = 16$, $a_3 = 17$, $a_4 = 18$, then a_1 is shared IM, and a_2, a_3, a_4 are shared CM by f and g in X. However, a_1 is not shared CM by f and g in X.

(iv) If take $a_1 = 0$, $a_2 = 16$, $a_3 = 17$, $a_4 = 18$, then a_1 and a_2 are shared IM, and a_3, a_4 are shared CM by f and g in X. However, a_1 is not shared CM by f and g in X.

(v) If take $a_1 = 0$, $a_2 = 16$, $a_3 = 17$, $a_4 = 18$, then a_1, a_2, a_3 are shared IM, and a_4 is shared CM by f and g in X. However, a_1 is not shared CM by f and g in X.

2. Lemmas

Lemma 2.1. *(See [10,12,16].) Suppose that g is a non-constant meromorphic function in one angular domain $\overline{\Omega} = \{z\colon \alpha \leqslant \arg z \leqslant \beta\}$ with $0 < \beta - \alpha \leqslant 2\pi$. Then*

(i) *(see [3, Chapter 1]) for any complex number $a \neq \infty$,*

$$S_{\alpha,\beta}\left(r, \frac{1}{g-a}\right) = S_{\alpha,\beta}(r, g) + \varepsilon(r, a),$$

where $\varepsilon(r, a) = O(1)$ $(r \to \infty)$;

(ii) *(see [3, p. 138]) for any $1 \leqslant r < R$,*

$$A_{\alpha,\beta}\left(r, \frac{g'}{g}\right) \leqslant K\left\{\left(\frac{R}{r}\right)^{\omega} \int_1^R \frac{\log^+ T(t, g)}{t^{1+\omega}}\,dt + \log^+ \frac{r}{R-r} + \log\frac{R}{r} + 1\right\},$$

and

$$B_{\alpha,\beta}\left(r, \frac{g'}{g}\right) \leqslant \frac{4\omega}{r^{\omega}} m\left(r, \frac{g'}{g}\right),$$

where $\omega = \frac{\pi}{\beta-\alpha}$ and K is a positive constant not depending on r and R.

Remark 2.1. Nevanlinna conjectured that

$$D_{\alpha,\beta}\left(r, \frac{g'}{g}\right) = A_{\alpha,\beta}\left(r, \frac{g'}{g}\right) + B_{\alpha,\beta}\left(r, \frac{g'}{g}\right) = o\left(S_{\alpha,\beta}\left(r, \frac{1}{g-a}\right)\right) \tag{7}$$

when r tends to $+\infty$ outside an exceptional set of finite linear measure, and he proved that $D_{\alpha,\beta}(r, \frac{g'}{g}) = O(1)$ when the function g is meromorphic in $\mathbb{C}$ and has finite order. In 1974, Gol'dberg constructed a counter-example to show that (7) is not valid (see [2]). However, it follows from Lemma 2.1(ii) that

$$D_{\alpha,\beta}\left(r, \frac{g'}{g}\right) = A_{\alpha,\beta}\left(r, \frac{g'}{g}\right) + B_{\alpha,\beta}\left(r, \frac{g'}{g}\right) = R(r, g).$$

Lemma 2.2. *(See [15].) Suppose that f is a non-constant meromorphic function in one angular domain $\overline{\Omega} = \{z\colon \alpha \leqslant \arg z \leqslant \beta\}$ with $0 < \beta - \alpha \leqslant 2\pi$, then for arbitrary q distinct $a_j \in \overline{\mathbb{C}}$ $(1 \leqslant j \leqslant q)$, we have*

$$(q-2)S_{\alpha,\beta}(r, f) \leqslant \sum_{j=1}^{q} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + R(r, f),$$

where the term $\overline{C}_{\alpha,\beta}(r, \frac{1}{f-a_j})$ will be replaced by $\overline{C}_{\alpha,\beta}(r, f)$ when some $a_j = \infty$.

Remark 2.2. If $R(r, f) = o(S_{\alpha,\beta}(r, f))$, then we can deduce from Lemma 2.2 that a meromorphic function f has at most two Picard values in X. Here, we explain the necessity of the condition $R(r, f) = o(S_{\alpha,\beta}(r, f))$. By Example 1.1, any value $d \in \{a \in \mathbb{R}\colon 16 \leqslant a\} \cup \{\infty\}$ is a Picard value of $f(z) = e^{z+1} - 1$ in $X = \{z \in \mathbb{C}\colon \frac{\pi}{2} < \arg z < \frac{3\pi}{2}\}$. However, there holds (6).

Lemma 2.3. *(See [1].) Suppose that f is a non-constant meromorphic function in the plane and that $X = \{z\colon \alpha < \arg z < \beta\}$ is an angular domain, where $0 < \beta - \alpha \leqslant 2\pi$. Let $P(f) = a_0 f^p + a_1 f^{p-1} + \cdots + a_p$ ($a_0 \neq 0$) be a polynomial of f with degree p, where the coefficients a_j ($j = 0, 1, \ldots, p$) are constants, and let b_j ($j = 1, 2, \ldots, q$) be q ($q \geqslant p + 1$) distinct finite complex numbers. Then*

$$D_{\alpha,\beta}\left(r, \frac{P(f)\cdot f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right) = R(r, f).$$

Lemma 2.4. *(See [1].) Let f and g be two distinct transcendental meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 IM in one angular domain $X = \{z\colon \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leqslant 2\pi$. Then*

(i) $S_{\alpha,\beta}(r, f) = S_{\alpha,\beta}(r, g) + R(r, f)$, $S_{\alpha,\beta}(r, g) = S_{\alpha,\beta}(r, f) + R(r, g)$;
(ii) $\sum_{j=1}^{4} \overline{C}_{\alpha,\beta}(r, \frac{1}{f-a_j}) = 2S_{\alpha,\beta}(r, f) + R(r, f)$;
(iii) $\overline{C}_{\alpha,\beta}(r, \frac{1}{f-b}) = S_{\alpha,\beta}(r, f) + R(r, f)$, $\overline{C}_{\alpha,\beta}(r, \frac{1}{g-b}) = S_{\alpha,\beta}(r, g) + R(r, g)$, *where* $b \neq a_j$ ($j = 1, 2, 3, 4$);
(iv) $C^{*}_{\alpha,\beta}(r, \frac{1}{f'}) = R(r, f)$, $C^{*}_{\alpha,\beta}(r, \frac{1}{g'}) = R(r, g)$, *where* $C^{*}_{\alpha,\beta}(r, \frac{1}{f'})$ *and* $C^{*}_{\alpha,\beta}(r, \frac{1}{g'})$ *are respectively the counting functions of the zeros of* f' *that are not zeros of* $f - a_j$ ($j = 1, 2, 3, 4$), *and the zeros of* g' *that are not zeros of* $g - a_j$ ($j = 1, 2, 3, 4$);
(v) $\sum_{j=1}^{4} C^{**}_{\alpha,\beta}(r, f(z) = a_j = g(z)) = R(r, f)$, *where* $C^{**}_{\alpha,\beta}(r, f(z) = a_j = g(z))$ *is the counting function for common multiple zeros of* $f - a_j$ *and* $g - a_j$ ($j = 1, 2, 3, 4$), *counting the smaller one of the two multiplicities at each of the points.*

Lemma 2.5. *Let f and g be two distinct transcendental meromorphic functions that share four distinct values $0, 1, \infty, c$ IM in one angular domain $X = \{z\colon \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leqslant 2\pi$. Let*

$$F = \left\{\frac{f''}{f'} - \left(\frac{2f'}{f} + \frac{f'}{f-1} + \frac{f'}{f-c}\right) - \frac{g''}{g'} - \left(\frac{2g'}{g} + \frac{g'}{g-1} + \frac{g'}{g-c}\right)\right\},$$
$$G = \left\{\frac{f''}{f'} - \left(\frac{f'}{f-1} + \frac{f'}{f-c} - \frac{2f'}{f}\right) - \frac{g''}{g'} - \left(\frac{g'}{g-1} + \frac{g'}{g-c} - \frac{2g'}{g}\right)\right\}.$$

If $F \not\equiv 0$, $G \not\equiv 0$, then

$$S_{\alpha,\beta}(r, F) \leqslant \overline{C}_{\alpha,\beta}(r, f) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = \infty = g(z)\big) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = 0 = g(z)\big) + R(r, f),$$

$$S_{\alpha,\beta}(r, G) \leqslant \overline{C}_{\alpha,\beta}(r, f) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = \infty = g(z)\big) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = 0 = g(z)\big) + R(r, f).$$

Proof. From Lemma 2.3 we have

$$D_{\alpha,\beta}(r, F) = R(r, f).$$

If $z_1 \in X$ is a zero of $f(z) - 1$ and $g(z) - 1$, with multiplicities q and p, respectively

$$f(z) = 1 + b_q(z - z_1)^q + b_{q+1}(z - z_1)^{q+1} + \cdots \quad (b_q \neq 0),$$
$$g(z) = 1 + c_p(z - z_1)^p + c_{p+1}(z - z_1)^{p+1} + \cdots \quad (c_p \neq 0),$$

then by computation,

$$F(z) = \left\{\frac{-1}{z - z_1} + O(1)\right\} - \left\{\frac{-1}{z - z_1} + O(1)\right\} = O(1).$$

Hence each zero of both $f(z) - 1$ and $g(z) - 1$ in X is not a pole of $F(z)$. Similarly, we get that each zero of both $f(z) - c$ and $g(z) - c$ in X is not a pole of $F(z)$. Obviously, any zero of both $f(z)$ and $g(z)$ with the same multiplicities in X is not a pole of $F(z)$. From the above discussion and Lemma 2.4(iv) we deduce that

$$\begin{aligned} C_{\alpha,\beta}(r, F) &\leqslant \overline{C}_{\alpha,\beta}(r, f) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = \infty = g(z)\big) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = 0 = g(z)\big) \\ &\quad + C^{*}_{\alpha,\beta}\left(r, \frac{1}{f'}\right) + C^{*}_{\alpha,\beta}\left(r, \frac{1}{g'}\right) + R(r, f) \\ &= \overline{C}_{\alpha,\beta}(r, f) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = \infty = g(z)\big) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = 0 = g(z)\big) + R(r, f). \end{aligned}$$

Using the same argument for $G(z)$ instead of $F(z)$, we can deduce the other inequality. Therefore the lemma follows. □

Lemma 2.6. *Let f and g be two distinct transcendental meromorphic functions that share four distinct values* 0, 1, ∞, c *IM in one angular domain* $X = \{z\colon \alpha < \arg z < \beta\}$ *with* $0 < \beta - \alpha \leqslant 2\pi$. *Let*

$$F_1 = \frac{g'(f-g)}{g(f-1)(g-c)}, \qquad G_1 = \frac{f'(f-g)}{f(g-1)(f-c)},$$
$$F_c = \frac{g'(f-g)}{g(g-1)(f-c)}, \qquad G_c = \frac{f'(f-g)}{f(f-1)(g-c)}.$$

Then

$$S_{\alpha,\beta}(r, F_1) \leqslant S_{\alpha,\beta}(r, f) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + R(r, f),$$
$$S_{\alpha,\beta}(r, G_1) \leqslant S_{\alpha,\beta}(r, g) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{g-1}\right) + R(r, g),$$
$$S_{\alpha,\beta}(r, F_c) \leqslant S_{\alpha,\beta}(r, f) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-c}\right) + R(r, f),$$
$$S_{\alpha,\beta}(r, G_c) \leqslant S_{\alpha,\beta}(r, g) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{g-c}\right) + R(r, g).$$

Proof. We rewrite F_1 and get

$$F_1 = \frac{1}{f-1}\left\{\frac{g'}{g(g-c)} - \frac{g'}{g-c}\right\} + \frac{g'}{g(g-c)}.$$

Thus from Lemma 2.3 we get

$$D_{\alpha,\beta}(r, F_1) \leqslant D_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + R(r, f).$$

If $z_c \in X$ is a zero of $f(z) - c$ and $g(z) - c$, then it must be a simple pole of $\frac{g'}{g-c}$, and be a zero of $f - g$. Hence z_c is not a pole of F_1. Similarly, any zero of f and g in X is not a pole of F_1. Let $z^* \in X$ be a pole of $f(z)$ and $g(z)$ with multiplicities p and q, respectively, then z^* must be a pole of $f(z) - g(z)$ with multiplicity at most $\max\{p, q\}$. Hence we have

$$\begin{aligned} F_1(z) &= O\left((z - z^*)^{(2q+p)-(q+1+\max\{p,q\})}\right) \\ &= O\left((z - z^*)^{(q+p-1-\max\{p,q\})}\right). \end{aligned}$$

So z^* is not a pole of F_1. If $z_1 \in X$ is a zero of $f - 1$ with multiplicity p, and is a zero of $g - 1$, then z_1 is also a zero of $f - g$. Then z_1 is a pole of F_1 with multiplicities at most $p - 1$. From the above discussion we obtain

$$C_{\alpha,\beta}(r, F_1) \leqslant C_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right).$$

Hence we have

$$\begin{aligned} S_{\alpha,\beta}(r, F_1) &= D_{\alpha,\beta}(r, F_1) + C_{\alpha,\beta}(r, F_1) \\ &\leqslant D_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + C_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + R(r, f) \\ &= S_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + R(r, f). \end{aligned}$$

With a similar argument as above, we can get other three inequalities. Therefore the lemma follows. $\square$

Lemma 2.7. *Let f and g be two distinct transcendental meromorphic functions that share four distinct values* 0, 1, ∞, c *IM in one angular domain* $X = \{z\colon \alpha < \arg z < \beta\}$ *with* $0 < \beta - \alpha \leqslant 2\pi$. *Set*

$$\begin{aligned} \gamma &= F^2 - (1+c)^2\Psi, \\ \delta &= G^2 - (1+c)^2\Psi, \end{aligned}$$

where Ψ is defined by

$$\Psi = \frac{f'g'(f-g)^2}{(f-a_1)(f-a_2)(f-a_3)(g-a_1)(g-a_2)(g-a_3)}, \tag{8}$$

F and G are the functions defined in Lemma 2.5. If $z_0 \in X$ is a simple zero of both f and g, and if $z_\infty \in X$ is a simple pole of both f and g, then $\gamma(z_0)=0$, $\delta(z_\infty)=0$.

Proof. Set

$$f(z) = a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \quad (a_1 \neq 0),$$
$$g(z) = b_1(z-z_0) + b_2(z-z_0)^2 + \cdots \quad (b_1 \neq 0).$$

By computation we get

$$\Psi(z_0) = \frac{1}{c^2}(a_1-b_1)^2,$$
$$F(z_0) = \left(1+\frac{1}{c}\right)(a_1-b_1).$$

Hence we have

$$\gamma(z_0) = \big(F(z_0)\big)^2 - (1+c)^2\Psi(z_0) = 0.$$

Set

$$f(z) = \frac{c_1}{z-z_\infty} + c_2 + O(z-z_\infty) \quad (c_1 \neq 0),$$
$$g(z) = \frac{d_1}{z-z_\infty} + d_2 + O(z-z_\infty)^2 \quad (d_1 \neq 0).$$

By computation we get

$$\Psi(z_\infty) = \left(\frac{1}{c_1} - \frac{1}{d_1}\right)^2,$$
$$G(z_\infty) = (1+c)\left(\frac{1}{c_1} - \frac{1}{d_1}\right).$$

Hence we have

$$\delta(z_\infty) = \big(G(z_\infty)\big)^2 - (1+c)^2\Psi(z_\infty) = 0.$$

Therefore the lemma follows. □

Lemma 2.8. *Under the assumption of Lemma 2.7, we have*

$$F(z_0) = (1+c)F_1(z_0) = (1+c)G_1(z_0) = (1+c)F_c(z_0) = (1+c)G_c(z_0),$$
$$G(z_\infty) = (1+c)F_1(z_\infty) = (1+c)G_1(z_\infty) = (1+c)F_c(z_\infty) = (1+c)G_c(z_\infty),$$

where F_1, G_1, F_c, G_c are the functions defined in Lemma 2.7.

Proof. Using the same notations as in the proof of Lemma 2.7, we have

$$F_1(z_0) = G_1(z_0) = F_c(z_0) = G_c(z_0) = \frac{1}{c}(a_1-b_1),$$
$$F_1(z_\infty) = G_1(z_\infty) = F_c(z_\infty) = G_c(z_\infty) = \frac{1}{c_1} - \frac{1}{d_1}.$$

Hence we can obtain the conclusion of the lemma. □

We denote by $\overline{C}^{1)}_{\alpha,\beta}(r, f(z)=a=g(z))$ the counting function of simple zeros of both $f(z)-a$ and $g(z)-a$ in X, by $\overline{C}^{1)}_{\alpha,\beta}(r, \frac{1}{f-a})$ the counting function of simple zeros of $f(z)-a$ in X, by $\overline{C}^{(2}_{\alpha,\beta}(r, \frac{1}{f-a})$ the counting function of zeros of $f(z)-a$ in X with multiplicities at least two, and by $\overline{C}^{(2}_{\alpha,\beta}(r, f)$ the counting function of those poles of f in X with multiplicities at least two, each point is counted in the counting functions only once. One can obtain the following lemma by Lemma 2.4(v).

Lemma 2.9. *Let f and g be two distinct transcendental meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 IM in one angular domain $X = \{z\colon\ \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leqslant 2\pi$. Then for $j = 1, 2, 3, 4$, we have*

$$\overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = a_j = g(z)\big) = \overline{C}^{1)}_{\alpha,\beta}\big(r, f(z) = a_j = g(z)\big) + R(r, f).$$

Lemma 2.10. *Let f and g be two distinct transcendental meromorphic functions. Given one angular domain $X = \{z\colon\ \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leqslant 2\pi$, we assume that f and g share four distinct values a_1, a_2, a_3, a_4 "CM" in X, then a_1, a_2, a_3, a_4 are shared CM in X by f and g, provided that*

$$\lim_{r\to\infty} \frac{S_{\alpha,\beta}(r, f)}{\log(rT(r, f))} = \infty \quad (r \notin E).$$

Proof. Without loss of generality, we assume that $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$, $a_4 = c$. From Lemma 2.4(i) we see that $R(r, f) = R(r, g)$. We assume that there exist three of $\overline{C}_{\alpha,\beta}(r, \frac{1}{f-a_j})$ $(j = 1, 2, 3, 4)$, say $\overline{C}_{\alpha,\beta}(r, \frac{1}{f-a_j})$ $(j = 1, 2, 3)$, such that $\overline{C}_{\alpha,\beta}(r, \frac{1}{f-a_j}) = R(r, f)$, then we deduce by Lemma 2.2 that

$$S_{\alpha,\beta}(r, f) \leqslant \sum_{j=1}^{3} \overline{C}_{\alpha,\beta}\left(\frac{1}{f - a_j}\right) + R(r, f) = R(r, f),$$

a contradiction with the condition of the lemma. Hence there are at least two of $\overline{C}_{\alpha,\beta}(r, \frac{1}{f-a_j})$ $(j = 1, 2, 3, 4)$, say $\overline{C}_{\alpha,\beta}(r, \frac{1}{f-a_j})$ $(j = 1, 3)$, such that

$$\overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) \neq R(r, f), \qquad \overline{C}_{\alpha,\beta}(r, f) \neq R(r, f). \tag{9}$$

Since $0, 1, \infty, c$ are shared "CM" by f and g in X, we obtain from (iv) and (v) in Lemma 2.4 that

$$\overline{C}^{(2}_{\alpha,\beta}(r, f) + \overline{C}^{(2}_{\alpha,\beta}(r, g) = R(r, f) \tag{10}$$

and

$$\overline{C}_{\alpha,\beta}\left(r, \frac{1}{f'}\right) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{g'}\right) = R(r, f). \tag{11}$$

Set

$$H = \frac{f''}{f'} - \frac{g''}{g'}. \tag{12}$$

Then we have

$$D_{\alpha,\beta}(r, H) = R(r, f)$$

and

$$\begin{aligned} C_{\alpha,\beta}(r, H) &\leqslant \overline{C}_{\alpha,\beta}(r, f) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = \infty = g(z)\big) + \overline{C}_{\alpha,\beta}(r, g) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = \infty = g(z)\big) \\ &\quad + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f'}\right) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{g'}\right) \\ &= R(r, f). \end{aligned}$$

Hence we have

$$S_{\alpha,\beta}(r, H) = R(r, f).$$

If $z_0 \in X$ is a simple pole of f and g, then form (12) we see that z_0 must be a zero of H. Hence we can deduce by (10) that

$$\overline{C}_{\alpha,\beta}(r, f) - R(r, f) \leqslant C_{\alpha,\beta}\left(r, \frac{1}{H}\right) \leqslant S_{\alpha,\beta}(r, H) + O(1) = R(r, f).$$

Thus we have $\overline{C}_{\alpha,\beta}(r, f) = R(r, f)$, a contradiction with (9). So $H \equiv 0$. It follows from (12) that

$$f(z) \equiv Ag(z) + B, \tag{13}$$

where $A(\neq 0)$, B are constants. From (9) and (13) we get $B=0$. Hence we have

$$f(z) \equiv Ag(z).$$

Since $f(z) \not\equiv g(z)$, we get $A \neq 1$. This means that $1, c$ are Picard values of f and g in X. Again by (13), A and Ac also are Picard values of f and g in X. Therefore we have

$$A = c, \qquad Ac = 1.$$

From this we obtain $c=-1$ and $f(z) \equiv -g(z)$. We now get that 0, 1, ∞, c are shared *CM* by f and g in X. Therefore the lemma follows. □

Lemma 2.11. *Let f and g be two distinct transcendental meromorphic functions. Given one angular domain $X = \{z\colon \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leqslant 2\pi$, we assume that f and g share four distinct values a_1, a_2, a_3, a_4 IM in X, and that $\overline{C}_{\alpha,\beta}(r, \frac{1}{f-a_j}) = R(r, f)$ $(j = 1, 2)$. Then a_1, a_2, a_3, a_4 are shared CM in X by f and g, provided that*

$$\lim_{r\to\infty} \frac{S_{\alpha,\beta}(r, f)}{\log(rT(r, f))} = \infty \quad (r \notin E).$$

Proof. Without loss of generality, we assume that $a_1 = 0$, $a_2 = \infty$, $a_3 = 1$, $a_4 = c$. Then

$$\overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) = R(r, f), \qquad \overline{C}_{\alpha,\beta}(r, f) = R(r, f).$$

Hence 0, ∞ are shared "*CM*" by f and g in X. Hence from Lemma 2.2 we have

$$\begin{aligned} S_{\alpha,\beta}(r, f) &\leqslant \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) + \overline{C}_{\alpha,\beta}(r, f) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + R(r, f) \\ &= \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + R(r, f). \end{aligned}$$

From Lemma 2.1(i) we have

$$\begin{aligned} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) &= \overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + \overline{C}^{(2}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) \\ &\leqslant \overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + \frac{1}{2} C^{(2}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) \\ &\leqslant \frac{1}{2}\overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + \frac{1}{2} C_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) \\ &\leqslant \frac{1}{2}\overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + \frac{1}{2} S_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) \\ &\leqslant \frac{1}{2}\overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + \frac{1}{2} S_{\alpha,\beta}(r, f) + O(1). \end{aligned}$$

From the above inequalities and the condition of the lemma, we have

$$S_{\alpha,\beta}(r, f) \leqslant \overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + R(r, f) \leqslant S_{\alpha,\beta}(r, f) + R(r, f) \leqslant S_{\alpha,\beta}(r, f).$$

Hence we obtain

$$S_{\alpha,\beta}(r, f) = \overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + R(r, f),$$

$$\overline{C}^{(2}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) = R(r, f).$$

By a similar discussion, we have

$$\overline{C}^{(2}_{\alpha,\beta}\left(r, \frac{1}{g-1}\right) = R(r, g) = R(r, f).$$

Hence

$$\overline{C}^{(2}_{\alpha,\beta}\big(r, f(z)=1=g(z)\big)=R(r,f).$$

Therefore we have

$$\begin{aligned}\overline{C}^{1)}_{\alpha,\beta}\big(r, f(z)=1=g(z)\big)+R(r,f)&=\overline{C}^{1)}_{\alpha,\beta}\big(r, f(z)=1=g(z)\big)+\overline{C}^{(2}_{\alpha,\beta}\big(r, f(z)=1=g(z)\big)+R(r,f)\\&=\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)+R(r,f).\end{aligned}$$

From these equalities and Lemma 2.9, we have

$$\begin{aligned}\overline{C}^{E}_{\alpha,\beta}\big(r, f(z)=1=g(z)\big)&=\overline{C}^{1)}_{\alpha,\beta}\big(r, f(z)=1=g(z)\big)+R(r,f)\\&=\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)+R(r,f).\end{aligned}$$

This means that 1 is shared "*CM*" by f and g in X.

Using a similar discussion, we can deduce that c is also shared "*CM*" by f and g in X. Thus 0, ∞, 1, c are "*CM*" shared values of f and g in X. By Lemma 2.10, we get that 0, ∞, 1, c are *CM* shared values of f and g in X. Therefore the lemma follows. □

3. Proof of Theorem 1.5

Using the same argument as in the proof of Lemma 2.10, we get that $R(r,f)=R(r,g)$, and that there are at least two of $\overline{C}_{\alpha,\beta}(r,\frac{1}{f-a_j})$ $(j=1,2,3,4)$, say $\overline{C}_{\alpha,\beta}(r,\frac{1}{f-a_j})$ $(j=3,4)$, such that

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_3}\right)\neq R(r,f),\qquad \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_4}\right)\neq R(r,f). \tag{14}$$

Set

$$L(z)=\frac{z-a_3}{z-a_4}\cdot\frac{a_2-a_4}{a_2-a_3}.$$

Then $L(a_3)=0$, $L(a_4)=\infty$, $L(a_2)=1$, and

$$L(a_1)=\frac{a_1-a_3}{a_1-a_4}\cdot\frac{a_2-a_4}{a_2-a_3}=(a_1,a_2,a_3,a_4)$$

which is the cross ratio of a_1, a_2, a_3, a_4. Let

$$F(z)=L\big(f(z)\big),\qquad G(z)=L\big(g(z)\big).$$

We get from $f(z)\not\equiv g(z)$ that $F(z)\not\equiv G(z)$. Since a_j $(j=1,2,3,4)$ are shared CM by f and g in X, $L(a_j)$ $(j=1,2,3,4)$ are shared CM by F and G in X. Hence c, 1, 0, ∞ are CM shared values of F and G in X, where $c=L(a_1)$. Obviously, $R(r,F)=R(r,G)$. We obtain by (14) that

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{F}\right)\neq R(r,f),\qquad \overline{C}_{\alpha,\beta}(r,F)\neq R(r,f). \tag{15}$$

Set

$$H=\frac{F'}{F(F-1)(F-c)}-\frac{G'}{G(G-1)(G-c)}. \tag{16}$$

Assume that $H(z)\not\equiv 0$, we get from Lemma 2.3 that

$$D_{\alpha,\beta}(r,H)=R(r,F).$$

If $z_0\in X$ is a point such that $F(z_0)=G(z_0)=L(a_j)$ for some $j=1,2,3,4$, then from (16) we see that H has no pole in X. Hence we have

$$S_{\alpha,\beta}(r,H)=D_{\alpha,\beta}(r,H)+C_{\alpha,\beta}(r,H)=R(r,F).$$

If $z_1\in X$ is a pole of F with multiplicity p, then it must be a pole of G with multiplicity p. Thus from (16) we see that z_1 is a zero of H with multiplicities at least $3p-(p+1)=2p-1$. Therefore

$$\overline{C}_{\alpha,\beta}\left(r, \frac{1}{F}\right) \leqslant C_{\alpha,\beta}\left(r, \frac{1}{H}\right) \leqslant S_{\alpha,\beta}(r, H) + O(1) = R(r, f),$$

a contradiction with (15). So we have $H(z) \equiv 0$.

Set

$$Q = \frac{FF'}{(F-1)(F-c)} - \frac{GG'}{(G-1)(G-c)}. \tag{17}$$

Assume that $Q(z) \not\equiv 0$, we get from Lemma 2.3 that

$$D_{\alpha,\beta}(r, Q) = R(r, F).$$

If $z_0 \in X$ is a point such that $F(z_0) = G(z_0) = L(a_j)$ for some $j = 1, 2, 3, 4$, then from (17) we see that Q has no pole in X. Hence we have

$$S_{\alpha,\beta}(r, Q) = D_{\alpha,\beta}(r, Q) + C_{\alpha,\beta}(r, Q) = R(r, F).$$

If $z_1 \in X$ is a zero of F with multiplicity p, then it must be a zero of G with multiplicities p. Thus from (17) we see that z_1 is a zero of H with multiplicity at least $3p + (p-1) = 2p - 1$. Therefore

$$\overline{C}_{\alpha,\beta}\left(r, \frac{1}{F}\right) \leqslant C_{\alpha,\beta}\left(r, \frac{1}{Q}\right) \leqslant S_{\alpha,\beta}(r, Q) + O(1) = R(r, f),$$

a contradiction with (15). So we have $Q(z) \equiv 0$.

From $F(z) \equiv G(z) \equiv 0$ we have

$$F^2(z) \equiv G^2(z).$$

Since $F(z) \not\equiv G(z)$, we have $F(z) \equiv -G(z)$. Thus both 1 and -1 are Picard values of F and G in X. It follows from Lemma 2.4(iii) that $c = -1$. Hence we have

$$L(a_1) = (a_1, a_2, a_3, a_4) = -1.$$

Therefore we obtain that both a_1 and a_2 are Picard values of f and g in X and that

$$L\big(f(z)\big) = -L\big(g(z)\big).$$

It means that f is a Möbius transformation of g.

Therefore Theorem 1.5 follows.

4. Proof of Theorem 1.6

Without loss of generality, we assume $a_1 = \infty$, $a_2 = 0$, $a_3 = 1$, $a_4 = c$. Using the notations of the lemmas in Section 2, we deal with four cases as follows.

Case 1. Assume that $\gamma \not\equiv 0$, $\delta \not\equiv 0$.

Since ∞, 0 are shared "*CM*" in X by f and g, we can get from Lemmas 2.5, 2.7 and 2.9 that

$$\begin{aligned}
\overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) &= \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = 0 = g(z)\big) \\
&= \overline{C}^{1)}_{\alpha,\beta}\big(r, f(z) = 0 = g(z)\big) + R(r, f) \\
&\leqslant C_{\alpha,\beta}\left(r, \frac{1}{\gamma}\right) \\
&\leqslant S_{\alpha,\beta}(r, \gamma) + O(1) \\
&\leqslant S_{\alpha,\beta}\big(r, \alpha^2 - (1+c)^2\Psi\big) + O(1) \\
&\leqslant 2S_{\alpha,\beta}(r, \alpha) + S_{\alpha,\beta}(r, \Psi) + O(1) \\
&= 2\left[\overline{C}_{\alpha,\beta}(r, f) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = \infty = g(z)\big) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = 0 = g(z)\big)\right] + R(r, f) \\
&= R(r, f).
\end{aligned}$$

Similarly, we can get from Lemmas 2.5, 2.7 and 2.9 that

$$\overline{C}_{\alpha,\beta}(r, f) = R(r, f).$$

Hence from Lemma 2.11 we get that 0, ∞, 1, c are shared *CM* by f and g in X.

Case 2. Assume that $\gamma \not\equiv 0$, $\delta \equiv 0$.

Since $\gamma \not\equiv 0$, we can also get similarly to Case 1 that

$$\overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) = R(r, f). \tag{18}$$

Subcase 2.1. $c \neq -1$.

If $F_1 \equiv G_1$, then

$$\frac{f'(f-1)}{f(f-c)} \equiv \frac{g'(g-1)}{g(g-c)}.$$

From the equality, we see that 0, 1, ∞, c are shared *CM* by f and g in X. Similarly, if $F_c \equiv G_c$, then we also see that 0, 1, ∞, c are shared *CM* by f and g in X. We now assume that $F_1 \not\equiv G_1$, and $F_c \not\equiv G_c$. From $F_1 \not\equiv G_1$, we get that at least one of the two functions

$$\beta - (1+c)F_1, \qquad \beta - (1+c)G_1$$

are not identically equal to 0. From Lemmas 2.1, 2.4–2.6, 2.8 and 2.9, we have

$$\begin{aligned}
\overline{C}_{\alpha,\beta}(r, f) &= \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = \infty = g(z)\big) \\
&= \overline{C}^{1)}_{\alpha,\beta}\big(r, f(z) = \infty = g(z)\big) + R(r, f) \\
&\leqslant C_{\alpha,\beta}\left(r, \frac{1}{\beta - (1+c)F_1}\right) \\
&\leqslant S_{\alpha,\beta}(r, \beta) + S_{\alpha,\beta}(r, F_1) + O(1) \\
&= \overline{C}_{\alpha,\beta}(r, f) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = \infty = g(z)\big) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = 0 = g(z)\big) \\
&\quad + S_{\alpha,\beta}(r, f) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + R(r, f) \\
&= S_{\alpha,\beta}(r, f) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + R(r, f),
\end{aligned}$$

or

$$\begin{aligned}
\overline{C}_{\alpha,\beta}(r, f) &= \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = \infty = g(z)\big) \\
&= \overline{C}^{1)}_{\alpha,\beta}\big(r, f(z) = \infty = g(z)\big) + R(r, f) \\
&\leqslant C_{\alpha,\beta}\left(r, \frac{1}{\beta - (1+c)G_1}\right) \\
&\leqslant \overline{C}_{\alpha,\beta}(r, f) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = \infty = g(z)\big) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) - \overline{C}^{E}_{\alpha,\beta}\big(r, f(z) = 0 = g(z)\big) \\
&\quad + S_{\alpha,\beta}(r, f) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + R(r, f) \\
&= S_{\alpha,\beta}(r, f) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + R(r, f).
\end{aligned}$$

Similarly, from functions

$$\beta - (1+c)F_c, \qquad \beta - (1+c)G_c,$$

we also have

$$\overline{C}_{\alpha,\beta}(r,f) \leqslant S_{\alpha,\beta}(r,f) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-c}\right) + R(r,f).$$

Hence we can deduce by Lemma 2.4(ii) that

$$\begin{aligned} 2\overline{C}_{\alpha,\beta}(r,f) &\leqslant 2S_{\alpha,\beta}(r,f) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-c}\right) + R(r,f) \\ &= \overline{C}_{\alpha,\beta}(r,f) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) + R(r,f), \end{aligned}$$

namely,

$$\overline{C}_{\alpha,\beta}(r,f) \leqslant \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) + R(r,f). \tag{19}$$

From (18) and (19) we get

$$\overline{C}_{\alpha,\beta}(r,f) = R(r,f). \tag{20}$$

Again making use of Lemma 2.11, we get from (18) and (20) that 0, ∞, 1, c are shared *CM* by f and g in X.

Subcase 2.2. $c = -1$.

Since $\delta \equiv 0$, then $\beta \equiv 0$. By integration, we have

$$\frac{f'f^2}{f^2-1} \equiv A \cdot \frac{g'g^2}{g^2-1}, \tag{21}$$

where $A(\neq 0)$ is an integral constant. If both -1 and 1 are Picard values of $f(z)$ in X, then from Lemma 2.11 we get that 1, -1, 0, ∞ are shared *CM* by f and g in X. Without loss of generality, we now assume that 1 is not a Picard value of f in X. Hence we can assume that $z_1 \in X$ such that $f(z_1) = 1 = g(z_1)$ and

$$\begin{aligned} f(z) &= 1 + b_p(z-z_1)^p + b_{p+1}(z-z_1)^{p+1} + \cdots \quad (b_p \neq 0), \\ g(z) &= 1 + c_q(z-z_1)^q + c_{q+1}(z-z_1)^{q+1} + \cdots \quad (c_q \neq 0). \end{aligned}$$

From (21), we deduce by computation that $A = \frac{p}{q}$. Hence

$$\frac{f'f^2}{f^2-1} \equiv \frac{p}{q} \cdot \frac{g'g^2}{g^2-1}. \tag{22}$$

Set

$$\lambda = \frac{f'}{f(f^2-1)} - \frac{p}{q} \cdot \frac{g'}{g(g^2-1)}. \tag{23}$$

If $\lambda \equiv 0$, then we have

$$\frac{f'}{f(f^2-1)} \equiv \frac{p}{q} \cdot \frac{g'}{g(g^2-1)}. \tag{24}$$

Combining (22) and (24), we get $f^3 \equiv g^3$. Hence we have

$$f(z) \equiv B \cdot g(z), \tag{25}$$

where B is a constant such that $B^3 = 1$. Since $f \not\equiv g$, then $B \neq 1$. Hence B is either $\exp\{\frac{2i\pi}{3}\}$ or $\exp\{\frac{4i\pi}{3}\}$. From (25) we obtain that 1, -1, B, $-B$ are Picard values of f in X. From Remark 2.2, we see that this is a contradiction. Therefore we have $\lambda \not\equiv 0$. By Lemma 2.3, we have

$$D_{\alpha,\beta}(r,\lambda) = R(r,f) + R(r,g) = R(r,f).$$

It is obvious that each pole of both f and g in X is not a pole of λ. If $z^* \in X$ is a zero of both $f(z)-1$ and $g(z)-1$ and

$$\begin{aligned} f(z) &= 1 + b_m(z-z^*)^m + b_{m+1}(z-z^*)^{m+1} + \cdots \quad (b_m \neq 0), \\ g(z) &= 1 + c_n(z-z^*)^n + c_{n+1}(z-z^*)^{n+1} + \cdots \quad (c_n \neq 0). \end{aligned}$$

From (22) we have $\frac{m}{n} = \frac{p}{q}$. Hence from (23) we have

$$\begin{aligned}\lambda(z) &= \left(\frac{2m}{z-z^*}+O(1)\right)-\frac{p}{q}\cdot\left(\frac{2n}{z-z^*}+O(1)\right)\\ &= O(1).\end{aligned}$$

So z^* is not a pole of λ. Similarly, each zero of both $f(z)+1$ and $g(z)+1$ in X is not pole of λ. Hence we get

$$C_{\alpha,\beta}(r,\lambda)\leqslant \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right). \tag{26}$$

Combining (18) and (26), we have

$$C_{\alpha,\beta}(r,\lambda)=R(r,f).$$

Hence

$$S_{\alpha,\beta}(r,\lambda)=R(r,f).$$

If $z^{**}\in X$ is a pole of f and g with same multiplicity t, then from (23) we see that z^{**} is a zero of λ with multiplicity at least $2t-1$. Hence we have

$$\begin{aligned}\overline{C}_{\alpha,\beta}(r,f) &= \overline{C}^E_{\alpha,\beta}\big(r,f\big(z=\infty=g(z)\big)\big)\\ &\leqslant C_{\alpha,\beta}\left(r,\frac{1}{\lambda}\right)+R(r,f)\\ &\leqslant S_{\alpha,\beta}(r,\lambda)+O(1)\\ &= R(r,f).\end{aligned}$$

Therefore from Lemma 2.11 we get that $0, \infty, 1, c$ are shared *CM* by f and g in X.

Case 3. Assume that $\gamma\equiv 0$, $\delta\not\equiv 0$.

Since $\delta\not\equiv 0$, we can also get similarly to Case 1 that

$$\overline{C}_{\alpha,\beta}(r,f)=R(r,f). \tag{27}$$

Subcase 3.1. $c\neq -1$.

If $F_1\equiv G_1$ or $F_c\equiv G_c$, then we can get similarly to Subcase 2.1 that $0, 1, \infty, c$ are shared *CM* by f and g in X. We now assume that $F_1\not\equiv G_1$, and $F_c\not\equiv G_c$. From $F_1\not\equiv G_1$, we get that at least one of the two functions

$$\alpha-(1+c)F_1,\qquad \alpha-(1+c)G_1$$

are not identically equal to 0. From Lemmas 2.1, 2.4–2.6, 2.8 and 2.9, we have

$$\begin{aligned}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) &= \overline{C}^E_{\alpha,\beta}\big(r,f(z)=0=g(z)\big)\\ &= \overline{C}^{1)}_{\alpha,\beta}\big(r,f(z)=0=g(z)\big)+R(r,f)\\ &\leqslant C_{\alpha,\beta}\left(r,\frac{1}{\alpha-(1+c)F_1}\right)\\ &\leqslant S_{\alpha,\beta}(r,\alpha)+S_{\alpha,\beta}(r,F_1)+O(1)\\ &= \overline{C}_{\alpha,\beta}(r,f)-\overline{C}^E_{\alpha,\beta}\big(r,f(z)=\infty=g(z)\big)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)-\overline{C}^E_{\alpha,\beta}\big(r,f(z)=0=g(z)\big)\\ &\quad +S_{\alpha,\beta}(r,f)-\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)+R(r,f)\\ &= S_{\alpha,\beta}(r,f)-\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)+R(r,f),\end{aligned}$$

or

$$\begin{aligned}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)&=\overline{C}^{E}_{\alpha,\beta}\big(r,f(z)=0=g(z)\big)\\&=\overline{C}^{1)}_{\alpha,\beta}\big(r,f(z)=0=g(z)\big)+R(r,f)\\&\leqslant C_{\alpha,\beta}\left(r,\frac{1}{\alpha-(1+c)G_1}\right)\\&\leqslant S_{\alpha,\beta}(r,\alpha)+S_{\alpha,\beta}(r,G_1)+R(r,f)\\&=\overline{C}_{\alpha,\beta}(r,f)-\overline{C}^{E}_{\alpha,\beta}\big(r,f(z)=\infty=g(z)\big)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)-\overline{C}^{E}_{\alpha,\beta}\big(r,f(z)=0=g(z)\big)\\&\quad+S_{\alpha,\beta}(r,f)-\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)+R(r,f)\\&=S_{\alpha,\beta}(r,f)-\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)+R(r,f).\end{aligned}$$

Similarly, from functions

$$\beta-(1+c)F_c,\qquad \beta-(1+c)G_c,$$

we also have

$$\overline{C}_{\alpha,\beta}(r,f)\leqslant S_{\alpha,\beta}(r,f)-\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-c}\right)+R(r,f).$$

Hence we can deduce by Lemma 2.4(ii) that

$$\begin{aligned}2\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)&\leqslant 2S_{\alpha,\beta}(r,f)-\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)-\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-c}\right)+R(r,f)\\&=\overline{C}_{\alpha,\beta}(r,f)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)+R(r,f),\end{aligned}$$

namely,

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)\leqslant\overline{C}_{\alpha,\beta}(r,f)+R(r,f).\tag{28}$$

From (27) and (28) we get

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)=R(r,f).\tag{29}$$

Again making use of Lemma 2.11, we get from (27) and (29) that 0, ∞, 1, c are shared *CM* by f and g in X.

Subcase 3.2. $c=-1$.

Since $\gamma\equiv 0$, then $\alpha\equiv 0$. By integration, we have

$$\frac{f'}{f^2(f^2-1)}\equiv A\cdot\frac{g'}{g^2(g^2-1)},\tag{30}$$

where $A(\neq 0)$ is an integral constant. Set

$$\mu=\frac{f'f}{f^2-1}-A\cdot\frac{g'g}{g^2-1}.\tag{31}$$

If $\mu\equiv 0$, then we have

$$\frac{f'f}{f^2-1}\equiv A\cdot\frac{g'g}{g^2-1}.\tag{32}$$

Combining (30) and (32), we get $f^3\equiv g^3$. Using a similar discussion to Subcase 2.2, we can also have a contradiction. Therefore, we have $\lambda\not\equiv 0$.

By Lemma 2.3, we have

$$D_{\alpha,\beta}(r,\mu)=R(r,f)+R(r,g)=R(r,f).$$

Using similar argument to Subcase 2.2, we get from (30) and (31) that

$$C_{\alpha,\beta}(r,\mu) \leqslant \overline{C}_{\alpha,\beta}(r,f). \tag{33}$$

Combining (27) and (33), we have

$$C_{\alpha,\beta}(r,\mu) = R(r,f).$$

Hence

$$S_{\alpha,\beta}(r,\mu) = R(r,f).$$

Obviously, each zero of f and g in X is a zero of μ. Hence we have

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) \leqslant C_{\alpha,\beta}\left(r,\frac{1}{\mu}\right) \leqslant S_{\alpha,\beta}(r,\mu) + O(1) = R(r,f).$$

Therefore from Lemma 2.11 we get that 0, ∞, 1, c are shared *CM* by f and g in X.

Case 4. Assume that $\gamma \equiv 0$, $\delta \equiv 0$. Then $\gamma - \delta \equiv 0$.

Since

$$\gamma - \delta = \alpha^2 - \beta^2 = (\alpha+\beta)(\alpha-\beta).$$

Hence we have $\alpha + \beta \equiv 0$, or $\alpha - \beta \equiv 0$.

If $\alpha - \beta \equiv 0$, then from

$$\alpha - \beta = \frac{-4f'}{f} + \frac{4g'}{g},$$

we get

$$\frac{f'}{f} \equiv \frac{g'}{g}.$$

By integration, we have

$$f(z) \equiv A \cdot g(z),$$

where $A(\neq 0)$ is an integral constant. Since $f(z) \not\equiv g(z)$, we have $A \neq 1$. Using similar argument to the proof of Lemma 2.10, we get that $A = c = -1$, and 0, ∞, 1, c are shared *CM* by f and g in X.

If $\alpha - \beta \not\equiv 0$, then from

$$\alpha + \beta = \left\{\frac{2f''}{f'} - 2\left(\frac{f'}{f-1} + \frac{f'}{f-c}\right)\right\} - \left\{\frac{2g''}{g'} - 2\left(\frac{g'}{g-1} + \frac{g'}{g-c}\right)\right\},$$

we get

$$\frac{f''}{f'} - \frac{f'}{f-1}\frac{f'}{f-c} \equiv \frac{g''}{g'} - \frac{g'}{g-1} - \frac{g'}{g-c}.$$

By integration, we have

$$\frac{f'}{(f-1)(f-c)} \equiv A \cdot \frac{g'}{(g-1)(g-c)}, \tag{34}$$

where $A(\neq 0)$ is an integral constant. If 1, c are Picard values of f and g in X, then from Lemma 2.11 we get that 0, ∞, 1, c are shared *CM* by f and g in X. Without loss of generality, let $z_1 \in X$ such that $f(z_1) = 1 = g(z_1)$ and

$$f(z) = 1 + b_p(z-z_1)^p + b_{p+1}(z-z_1)^{p+1} + \cdots \quad (b_p \neq 0),$$
$$g(z) = 1 + c_q(z-z_1)^q + c_{q+1}(z-z_1)^{q+1} + \cdots \quad (c_q \neq 0).$$

From (34) we have $A = \frac{p}{q}$. Hence we have

$$\frac{q \cdot f'}{(f-1)(f-c)} \equiv \frac{p \cdot g'}{(g-1)(g-c)}.$$

From integration, it becomes

$$\left(\frac{f-1}{f-c}\right)^q \equiv B \cdot \left(\frac{g-1}{g-c}\right)^p, \tag{35}$$

where B ($\neq 0$) is an integral constant. From (35), we have

$$qS_{\alpha,\beta}(r, f) = pS_{\alpha,\beta}(r, g) + O(1).$$

From this and Lemma 2.4(i), we have $p = q$. Hence

$$\left(\frac{f-1}{f-c}\right) \equiv B \cdot \left(\frac{g-1}{g-c}\right).$$

Hence we can deduce that 1 and c are shared *CM* (of course "*CM*") by f and g in X. Therefore we get from Lemma 2.10 that 0, ∞, 1, c are shared *CM* by f and g in X. This completes the proof of Theorem 1.6.

5. Concluding remark

It is well known that there exists an example, which shows that the four values CM cannot be replaced by the four values *IM* in Theorem 1.1 if $X = \mathbb{C}$ (see [5]). So we may raise the following question by a simple notation $1CM + 3IM = 4CM$ similarly as the open question in the uniqueness theory of meromorphic functions that share four values in the plane [4].

Question 5.1. Let f and g be two distinct transcendental meromorphic functions that share three values *IM* and share a fourth value *CM* in one angular domain $X = \{z\colon\ \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leqslant 2\pi$. Suppose that

$$\lim_{r\to\infty} \frac{S_{\alpha,\beta}(r, f)}{\log(rT(r, f))} = \infty \quad (r \notin E).$$

Then do f and g necessarily share the four values *CM* in X?

Acknowledgments

The authors would like to thank the referee for making valuable suggestions and comments, and also thank Professor Wei-Chuan Lin for his useful help.

References

[1] T.-B. Cao, H.-X. Yi, Analytic functions sharing three values DM in one angular, J. Korean Math. Soc. 45 (6) (2008) 1523–1534.
[2] A. Eremenko, I.V. Ostrovskii, M. Sodin, Anatolii Asirovich Gol'dberg, Complex Var. Elliptic Equ. 37 (1–4) (1998) 1–51.
[3] A.A. Goldberg, I.V. Ostrovskii, The Distribution of Values of Meromorphic Functions, Nauka, Moscow, 1970 (in Russian).
[4] G.G. Gundersen, Meromorphic functions that share three values IM and a fourth value CM, Complex Var. Elliptic Equ. 20 (1992) 99–106.
[5] G.G. Gundersen, Meromorphic functions that share three or four values, J. London Math. Soc. 20 (2) (1979) 457–466.
[6] W.K. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.
[7] W.-C. Lin, S. Mori, K. Tohge, Uniqueness theorems in an angular domain, Tohoku Math. J. 58 (4) (2006) 509–527.
[8] R. Nevanlinna, Le théorème de Picard–Borel et la théorie des fonctions méromorphes, Reprinting of the 1929 original, Chelsea Publishing Co., New York, 1974 (in French).
[9] Z.-J. Wu, A remark on uniqueness theorems in an angular domain, Proc. Japan Acad. Ser. A 64 (6) (2008) 73–77.
[10] S.-J. Wu, On the location of zeros of solutions of $f'' + A(z)f = 0$ where $A(z)$ is entire, Math. Scand. 74 (1994) 293–312.
[11] J.-F. Xu, H.-X. Yi, On uniqueness of meromorphic functions with shared four values in some angular domains, Bull. Malays. Math. Sci. Soc. 31 (1) (2008) 1–9.
[12] L. Yang, C.-C. Yang, Angular distribution of values of ff', Sci. China Ser. A 37 (3) (1994) 284–294.
[13] H.-X. Yi, C.-C. Yang, Uniqueness Theory of Meromorphic Functions, Science Press, 1995; Kluwer, 2003.
[14] J.-H. Zheng, On uniqueness of meromorphic functions with shared values in some angular domains, Canad. Math. Bull. 47 (1) (2004) 152–160.
[15] J.-H. Zheng, On uniqueness of meromorphic functions with shared values in one angular domain, Complex Var. Elliptic Equ. 48 (9) (2003) 777–785.
[16] J.-H. Zheng, On transcendental meromorphic functions with radially distributed values, Sci. China Ser. A 47 (3) (2003) 401–416.