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Support-type properties of generalized convex functions

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A B S T R A C T

Chebyshev systems induce in a natural way a concept of convexity. The functions convex in this sense behave in many aspects similarly to ordinary convex functions. In this paper support-type properties are investigated. Using osculatory interpolation, the existence of support-like functions is established for functions convex with respect to Chebyshev systems. Unique supports are determined. A characterization of the generalized convexity via support properties is presented.

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1. Introduction

During the 20th century mathematicians introduced and investigated many generalizations of convexity. As it is well known, the notion of the classical convexity can be expressed in terms of affine functions. Thus, an important direction for the generalization of convexity was to replace affine functions by another family of functions with, colloquially speaking, nice and reasonable properties. For instance, Beckenbach [1] introduced in 1937 the convexity in a new sense, based on the family of functions with the unique interpolation property with respect to two nodes: for any two points of the plane with distinct abscissas there exists exactly one function belonging to such a family with a graph joining these points. Tornheim [22] extended this idea by consideration of interpolation involving more than two nodes. Eleven years before Beckenbach, in 1926, Hopf in his PhD dissertation [16] treated the functions with nonnegative divided differences. This study was continued by Popoviciu [21], who introduced the name *higher-order convexity* (or *n-convexity*) and established many basic properties. In this case affine functions were replaced with polynomials of order n (i.e. of degree not greater than n). Then Hopf–Popoviciu's convexity concept is the special case of the Tornheim's one (this last author did not assume the linear structure of the considered family). A direct generalization of higher-order convexity seems to be convexity with respect to Chebyshev systems exposed in detail in the book [18]. Instead of polynomials the linear span of a Chebyshev system is considered. Then this is also the special case of convexity in Tornheim's sense.

In this paper we deal just with convexity with respect to Chebyshev systems.

It is well known that any convex function defined on a real interval admits an affine support at every interior point of a domain (the converse is also true). We extend this idea to convexity in the above mentioned sense.

The paper consists of six sections. Section 2 contains the preliminary information and some results useful in the proofs of our main theorems. Among others, we present Chebyshev systems and the convexity notion induced by them. Section 3 contains the first main result of the paper, which is Theorem 3.1 of support-type. Its special case concerning higher-order

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convexity was recently proved in [24, Theorem 2]. In Section 4 we present some applications of Theorem 3.1. Supports in the classical sense as well as so-called principal supports are considered. Section 5 is devoted to determining when the supports are unique (for the ordinary convex function affine support at an interior point of a domain is unique if and only if this is a point of differentiability; in the case of convexity induced by Chebyshev systems the situation is more complicated). The last Section 6 concerns the characterization of convexity in question in terms of support-type properties. The main results of this section are contained in Theorem 6.1 and Proposition 6.4. The classes of supports characterizing and not characterizing such a convexity are precisely determined.

The results are commented and compared with the other research in the field. In particular, our results are closely related to the works of Hungarian mathematicians Bessenyei and Páles.

2. Preliminaries

2.1. Difference operator

Let $f:\mathbb{R}\to\mathbb{R}$ be an arbitrary function, $x,h\in\mathbb{R}$, $n\in\mathbb{N}$. Then

$$\Delta_h f(x)=\Delta_h^1 f(x):=f(x+h)-f(x),\qquad \Delta_h^{n+1}f(x):=\Delta_h\Delta_h^n f(x).$$

The properties of this operator are well known. Below we recall two of them:

$$\Delta_h^n f(x)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(x+kh), \tag{2.1}$$

$$f(x)=x^n \quad\Longrightarrow\quad \Delta_h^n f(x)=n!h. \tag{2.2}$$

The proofs run by elementary induction. If $x=0$, $h=1$, then (2.1) and (2.2) yield

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^n=n!. \tag{2.3}$$

If $i\in\{0,\dots,n-1\}$ and $f(x)=x^i$, then (2.2) implies that $\Delta_h^i f(x)=i!h$. Therefore $\Delta_h^n f(x)=0$ and using (2.1) for $x=0$, $h=1$, we obtain

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^i=0,\quad i=0,\dots,n-1, \tag{2.4}$$

with the convention $0^0=1$, if $i=0$.

From now on I denotes the real interval.

Lemma 2.1. *Let I be an open interval, $f\in\mathscr{C}^{n-1}(I)$ and $x\in I$. If the finite right n-th derivative*

$$f_+^{(n)}(x):=\lim_{h\to 0^+}\frac{f^{(n-1)}(x+h)-f^{(n-1)}(x)}{h}$$

exists, then

$$\lim_{h\to 0^+}\frac{\Delta_h^n f(x)}{h^n}=f_+^{(n)}(x).$$

The analogous result is well known if the function f is n-times differentiable (see e.g. [17]; it follows also by the properties of divided differences—see e.g. [11] or [19, p. 375] together with [16, pp. 5, 10] or [21, p. 34]). However, in this paper we consider the convexity with respect to Chebyshev systems. This kind of convexity implies only the existence of the finite right n-th derivative on the interior of the domain (cf. Theorem 2.2 below). For that reason we give a proof, which requires a bit more attention than in the classical case of n-times differentiability.

Proof of Lemma 2.1. Fix $x\in I$. Consider $\Delta_h^n f(x)$ as a function of h, i.e. by (2.1)

$$\varphi(h):=\Delta_h^n f(x)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(x+kh).$$

Let $i \in \{0, \dots, n-1\}$. By (2.4)

$$\varphi^{(i)}(h) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^i f^{(i)}(x+kh) \xrightarrow[h\to 0^+]{} \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^i f^{(i)}(x) = 0.$$

Also $(h^n)^{(i)} \xrightarrow[h\to 0^+]{} 0$. Applying L'Hospital's rule $n-1$ times we obtain

$$\lim_{h\to 0^+}\frac{\Delta_h^n f(x)}{h^n} = \lim_{h\to 0^+}\frac{\varphi^{(n-1)}(h)}{(h^n)^{(n-1)}} = \lim_{h\to 0^+}\frac{1}{n!h}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^{n-1} f^{(n-1)}(x+kh),$$

if the last limit exists. But using (2.4) (for $i = n-1$) we may write

$$\begin{aligned}\frac{1}{n!h}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^{n-1} f^{(n-1)}(x+kh) &= \frac{1}{n!h}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^{n-1}\bigl(f^{(n-1)}(x+kh) - f^{(n-1)}(x)\bigr)\\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{k^n}{n!}\cdot\frac{f^{(n-1)}(x+kh) - f^{(n-1)}(x)}{kh}.\end{aligned}$$

If $h \to 0^+$, then this expression tends to

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{k^n}{n!}f_+^{(n)}(x) = f_+^{(n)}(x),$$

which equality follows by (2.3). This finishes the proof. □

2.2. Chebyshev systems

The extensive study of this topic is given in a book [18]. We recall some concepts needed in this paper.

Let $n \in \mathbb{N}$ and let the functions $u_0, u_1, \dots, u_n : I \to \mathbb{R}$ be continuous. An $(n+1)$-tuple $\mathscr{U} = (u_0, u_1, \dots, u_n)$ is called a *Chebyshev system* on I (*T-system* for brevity, this notation is adopted from [18]) provided

$$\begin{vmatrix} u_0(x_1) & u_0(x_2) & \dots & u_0(x_{n+1}) \\ u_1(x_1) & u_1(x_2) & \dots & u_1(x_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ u_n(x_1) & u_n(x_2) & \dots & u_n(x_{n+1}) \end{vmatrix} > 0$$

for any $x_1, x_2, \dots, x_{n+1} \in I$ such that $x_1 < x_2 < \dots < x_{n+1}$. A letter "T" is connected with formerly used spelling "Tchebycheff", as in the title of the book [18]. Nowadays the form "Chebyshev" is the most frequently used. If additionally $u_i \in \mathscr{C}^n(I)$, $i = 0, \dots, n$, then a T-system $\mathscr{U}$ is called an *extended Chebyshev system* (*ET-system*) on I provided

$$\begin{vmatrix} u_0(x_1) & \dots & u_0^{(\ell_1-1)}(x_1) & \dots & u_0(x_k) & \dots & u_0^{(\ell_k-1)}(x_k) \\ \vdots & & \vdots & & \vdots & & \vdots \\ u_n(x_1) & \dots & u_n^{(\ell_1-1)}(x_1) & \dots & u_n(x_k) & \dots & u_n^{(\ell_k-1)}(x_k) \end{vmatrix} > 0 \tag{2.5}$$

for any $k \leqslant n+1$, $x_1, \dots, x_k \in I$ such that $x_1 < \dots < x_k$ and any multiplicities $\ell_1, \dots, \ell_k \in \mathbb{N}$ summing up to $n+1$, i.e.

$$\underbrace{x_1 = \dots = x_1}_{\ell_1 \text{ times}} < \dots < \underbrace{x_k = \dots = x_k}_{\ell_k \text{ times}}$$

and the above sequence contains $n+1$ terms. Let us give a simple example. If $u_0, u_1, u_2 \in \mathscr{C}^2(I)$, then (u_0, u_1, u_2) is an ET-system on I if it is a T-system on I and

$$\begin{vmatrix} u_0(x_1) & u_0'(x_1) & u_0(x_2) \\ u_1(x_1) & u_1'(x_1) & u_1(x_2) \\ u_2(x_1) & u_2'(x_1) & u_2(x_2) \end{vmatrix} > 0, \qquad \begin{vmatrix} u_0(x_1) & u_0(x_2) & u_0'(x_2) \\ u_1(x_1) & u_1(x_2) & u_1'(x_2) \\ u_2(x_1) & u_2(x_2) & u_2'(x_2) \end{vmatrix} > 0$$

for any $x_1, x_2 \in I$ such that $x_1 < x_2$ and

$$\begin{vmatrix} u_0(x_1) & u_0'(x_1) & u_0''(x_1) \\ u_1(x_1) & u_1'(x_1) & u_1''(x_1) \\ u_2(x_1) & u_2'(x_1) & u_2''(x_1) \end{vmatrix} > 0$$

for any $x_1 \in I$. Finally, $\mathscr{U}$ is called an *extended complete Chebyshev system* (*ECT-system*) on I provided for any $k \in \{0, 1, \dots, n\}$ a system $(u_0, \dots, u_k)$ is an ET-system on I. It is worth mentioning that $\mathscr{U}$ is an ECT-system if and only if all Wronskians $W(u_0)$, $W(u_0, u_1)$, $\dots$, $W(u_0, u_1, \dots, u_n)$ are positive (for details see [18]). However, this condition will not be used in this paper.

There are many examples of T-systems, ET-systems and ECT-systems. Let us notice that a polynomial system $(1, x, \dots, x^n)$ as well as an exponential system $(\mathrm{e}^{\alpha_0 x}, \mathrm{e}^{\alpha_1 x}, \dots, \mathrm{e}^{\alpha_n x})$ (where $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{R}$ are distinct), are the fundamental examples of ECT-systems.

2.3. Generalized convexity

A function $f : I \to \mathbb{R}$ is called *convex with respect to a T-system* $\mathscr{U} = (u_0, u_1, \dots, u_n)$ (briefly $\mathscr{U}$*-convex*), if

$$D(x_1, x_2, \dots, x_{n+2}; f) := \begin{vmatrix} u_0(x_1) & u_0(x_2) & \dots & u_0(x_{n+2}) \\ u_1(x_1) & u_1(x_2) & \dots & u_1(x_{n+2}) \\ \vdots & \vdots & & \vdots \\ u_n(x_1) & u_n(x_2) & \dots & u_n(x_{n+2}) \\ f(x_1) & f(x_2) & \dots & f(x_{n+2}) \end{vmatrix} \geqslant 0 \tag{2.6}$$

for all $x_1, x_2, \dots, x_{n+2} \in I$ such that $x_1 < x_2 < \dots < x_{n+2}$.

The convexity with respect to the polynomial ECT-system $(1, x, \dots, x^n)$ (called *n-convexity* or *convexity of order n*) was considered by Hopf [16] and Popoviciu [21], who established many basic properties. It was also extensively studied by many authors in the past and in present. Notice also that if $n = 1$ and $\mathscr{U} = (1, x)$, then $\mathscr{U}$-convexity, i.e. 1-convexity, reduces to the ordinary convexity.

Convex functions with respect to ECT-systems have nice regularity properties.

Theorem 2.2. *Let $\mathscr{U} = (u_0, u_1, \dots, u_n)$ be an ECT-system on I. If the function $f : I \to \mathbb{R}$ is $\mathscr{U}$-convex, then $f \in \mathscr{C}^{n-1}(\operatorname{Int} I)$ and for any $x \in \operatorname{Int} I$ there exist the finite one-sided n-th derivatives $f_+^{(n)}(x)$, $f_-^{(n)}(x)$. Moreover, the function $f_+^{(n)}$ is right-continuous on $\operatorname{Int} I$, while the function $f_-^{(n)}$ is left-continuous on $\operatorname{Int} I$.*

This result is not trivial. It can be found in [18, Chapter XI] (for the statement see §2, for the proof see §11). The above properties are well known, at least in the case of ordinary convex functions (classical) and convex functions of higher order (see e.g. [16,19–21]). In this special setting the proofs are not so difficult.

Let us recall some further property of higher-order convexity (see [19, Theorem 4, p. 392]). There is also the analogy for $\mathscr{U}$-convexity with respect to the arbitrary ECT-system (see [18, p. 381]).

Theorem 2.3. *Let $n \in \mathbb{N}$, $n > 1$, I be an open interval and let $f : I \to \mathbb{R}$ be a continuous function. Then f is n-convex if and only if $f \in \mathscr{C}^{n-1}(I)$ and the derivative $f^{(n-1)}$ is convex.*

As the immediate consequence we obtain (see [19, Theorem 6, p. 392]):

Theorem 2.4. *Let I be an open interval and let $f : I \to \mathbb{R}$ be $(n+1)$-times differentiable on I. Then f is n-convex if and only if $f^{(n+1)}(x) \geqslant 0$, $x \in I$.*

2.4. Osculatory interpolation

The above term is connected with interpolation with multiple nodes. The Latin word *osculare* means “to kiss”. Let $f \in \mathscr{C}^{n-1}(\operatorname{Int} I)$ and the finite $f_+^{(n)}$ exists on $\operatorname{Int} I$. To simplify the notation, whenever we write $f^{(n)}$, we will have in mind the right derivative $f_+^{(n)}$. This convention concerns only the derivative of n-th order, all derivatives of lower orders are ordinary two-sided ones.

Now let $\mathscr{U} = (u_0, u_1, \dots, u_n)$ be a T-system on I. Any linear combination $u = \sum_{i=0}^n \alpha_i u_i$ will be called the $\mathscr{U}$*-polynomial*.

The osculatory interpolation with respect to ET-systems is known and considered in many places. That is why the following result seems to be classical. We give a proof to make the paper self-contained.

Theorem 2.5. *Let $\mathscr{U} = (u_0, u_1, \dots, u_n)$ be an ET-system on I. Let $k \leqslant n+1$, $x_1, \dots, x_k \in I$, $x_1 < \dots < x_k$, and $\ell_1, \dots, \ell_k \in \mathbb{N}$ with $\ell_1 + \dots + \ell_k = n+1$. Additionally assume that if $x_1 = \inf I$, then $\ell_1 = 1$, if $x_k = \sup I$, then $\ell_k = 1$. If $f \in \mathscr{C}^{n-1}(\operatorname{Int} I)$ and the finite right n-th derivative $f_+^{(n)}$ exists on $\operatorname{Int} I$, then there exists exactly one $\mathscr{U}$-polynomial $u = \sum_{i=0}^n \alpha_i u_i$ such that*

$$f^{(i)}(x_j) = u^{(i)}(x_j), \quad j = 1, \dots, k, \ i = 0, \dots, \ell_j - 1. \tag{2.7}$$

Moreover, for any $x \in I$,

$$\begin{vmatrix} u_0(x_1) & \dots & u_0^{(\ell_1-1)}(x_1) & \dots & u_0(x_k) & \dots & u_0^{(\ell_k-1)}(x_k) & u_0(x) \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ u_n(x_1) & \dots & u_n^{(\ell_1-1)}(x_1) & \dots & u_n(x_k) & \dots & u_n^{(\ell_k-1)}(x_k) & u_n(x) \\ f(x_1) & \dots & f^{(\ell_1-1)}(x_1) & \dots & f(x_k) & \dots & f^{(\ell_k-1)}(x_k) & u(x) \end{vmatrix} = 0. \tag{2.8}$$

Proof. Using the interpolatory conditions (2.7) and the form $u = \sum_{i=0}^{n} \alpha_i u_i$, we arrive at the system of $n+1$ linear equations with unknowns $\alpha_0, \dots, \alpha_n$. Its determinant does not vanish due to an ET-property (2.5). Then the existence and uniqueness are proved. Eq. (2.8) holds because the last row of the involved determinant is a linear combination of the remaining rows (by (2.7), f can be replaced by $u = \sum_{i=0}^{n} \alpha_i u_i$). □

To clarify the determinantal form of an interpolating $\mathscr{U}$-polynomial we give two examples.

1. For $\mathscr{U} = (u_0, u_1, u_2, u_3)$ the $\mathscr{U}$-polynomial u such that $u(x_1) = f(x_1)$, $u(x_2) = f(x_2)$, $u'(x_2) = f'(x_2)$, $u''(x_2) = f''(x_2)$ (where $x_2 \in \operatorname{Int} I$) has the form

$$\begin{vmatrix} u_0(x_1) & u_0(x_2) & u_0'(x_2) & u_0''(x_2) & u_0(x) \\ u_1(x_1) & u_1(x_2) & u_1'(x_2) & u_1''(x_2) & u_1(x) \\ u_2(x_1) & u_2(x_2) & u_2'(x_2) & u_2''(x_2) & u_2(x) \\ u_3(x_1) & u_3(x_2) & u_3'(x_2) & u_3''(x_2) & u_3(x) \\ f(x_1) & f(x_2) & f'(x_2) & f''(x_2) & u(x) \end{vmatrix} = 0, \quad x \in I.$$

2. For $\mathscr{U} = (u_0, u_1, u_2)$ the $\mathscr{U}$-polynomial u such that $u(x_1) = f(x_1)$, $u'(x_1) = f'(x_1)$, $u''(x_1) = f''_+(x_1)$ (where $x_1 \in \operatorname{Int} I$) has the form

$$\begin{vmatrix} u_0(x_1) & u_0'(x_1) & u_0''(x_1) & u_0(x) \\ u_1(x_1) & u_1'(x_1) & u_1''(x_1) & u_1(x) \\ u_2(x_1) & u_2'(x_1) & u_2''(x_1) & u_2(x) \\ f(x_1) & f'(x_1) & f''_+(x_1) & u(x) \end{vmatrix} = 0, \quad x \in I.$$

3. Support-type theorem

In this section, because of regularity properties of $\mathscr{U}$-convex functions needed in the proof of Theorem 3.1 (cf. Theorem 2.2), we assume that $\mathscr{U}$ is an ECT-system on I.

Theorem 3.1. *Let $\mathscr{U} = (u_0, u_1, \dots, u_n)$ be an ECT-system on I. Let $k \leqslant n+1$, $x_1, \dots, x_k \in I$, $x_1 < \dots < x_k$, and $\ell_1, \dots, \ell_k \in \mathbb{N}$ with $\ell_1 + \dots + \ell_k = n+1$. Additionally assume that if $x_1 = \inf I$, then $\ell_1 = 1$, if $x_k = \sup I$, then $\ell_k = 1$. If the function $f : I \to \mathbb{R}$ is $\mathscr{U}$-convex, then there exists a $\mathscr{U}$-polynomial $u = \sum_{i=0}^{n} \alpha_i u_i$ such that $u(x_i) = f(x_i)$, $i = 1, \dots, k$, and*

$$(-1)^{n+1}\big(f(x) - u(x)\big) \geqslant 0 \quad \text{for } x < x_1,\ x \in I, \tag{3.1}$$

$$(-1)^{n+1-(\ell_1+\dots+\ell_j)}\big(f(x) - u(x)\big) \geqslant 0 \quad \text{for } x_j < x < x_{j+1},\ 1 \leqslant j \leqslant k-1, \tag{3.2}$$

$$f(x) - u(x) \geqslant 0 \quad \text{for } x > x_k,\ x \in I. \tag{3.3}$$

The numbers $\ell_1, \dots, \ell_k$ can be interpreted as the multiplicities of the points $x_1, \dots, x_k$, respectively. The $\mathscr{U}$-polynomial u from the assertion will be called the *support of* $(\ell_1, \dots, \ell_k)$*-type*. This abbreviation will be useful later in the paper.

Before we start the proof let us notice that the above inequalities have a simple geometrical interpretation. We omit a trivial verification.

Observation 3.2. *The $\mathscr{U}$-polynomial u described in Theorem* 3.1 *has the following properties*:

(i) $u(x) \leqslant f(x)$, $x > x_k$, $x \in I$,
(ii) *if ℓ_j, i.e. the multiplicity of x_j, is even, then the graph of u passing through x_j remains on the same side of the graph of f, while it changes the side, if ℓ_j is odd.*

Proof of Theorem 3.1. As a $\mathscr{U}$-convex function, f has the regularity properties needed in Theorem 2.5. Then there exists a $\mathscr{U}$-polynomial $u = \sum_{i=0}^{n} \alpha_i u_i$ fulfilling (2.7), in particular, interpolating f at the nodes $x_1, \dots, x_k$.

We shall prove the inequality (3.2). The remaining inequalities (3.1) and (3.3) are even easier to handle by a similar way, so we omit the details.

Fix $j \in \{1, \dots, k-1\}$ and $x \in I$ such that $x_j < x < x_{j+1}$. For sufficiently small $h > 0$ the sequence

$$\big(x_1, x_1 + h \dots, x_1 + (\ell_1 - 1)h, \dots, x_j, x_j + h, \dots, x_j + (\ell_j - 1)h, x,\\ x_{j+1}, x_{j+1} + h, \dots, x_{j+1} + (\ell_{j+1} - 1)h, \dots, x_k, x_k + h, \dots, x_k + (\ell_k - 1)h\big)$$

is increasingly ordered and it contains $n+2$ points. By $\mathscr{U}$-convexity and (2.6)

$$D\big(x_1, \dots, x_j + (\ell_j - 1)h, x, x_{j+1}, \dots, x_k + (\ell_k - 1)h; f\big) \geqslant 0.$$

In the above determinant we shift the column containing x to the last column. It is easy to see that $n + 1 - (\ell_1 + \dots + \ell_j)$ inversions are needed. Hence

$$(-1)^{n+1-(\ell_1+\dots+\ell_j)} D\big(x_1, \dots, x_j + (\ell_j - 1)h, x_{j+1}, \dots, x_k + (\ell_k - 1)h, x; f\big) \geqslant 0.$$

In the explicit form it reads as

$$(-1)^{n+1-(\ell_1+\dots+\ell_j)} \begin{vmatrix} u_0(x_1) & u_0(x_1+h) & \cdots & u_0(x_1+(\ell_1-1)h) & \cdots & u_0(x_k) & u_0(x_k+h) & \cdots & u_0(x_k+(\ell_k-1)h) & u_0(x) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ u_n(x_1) & u_n(x_1+h) & \cdots & u_n(x_1+(\ell_1-1)h) & \cdots & u_n(x_k) & u_n(x_k+h) & \cdots & u_n(x_k+(\ell_k-1)h) & u_n(x) \\ f(x_1) & f(x_1+h) & \cdots & f(x_1+(\ell_1-1)h) & \cdots & f(x_k) & f(x_k+h) & \cdots & f(x_k+(\ell_k-1)h) & f(x) \end{vmatrix} \geqslant 0.$$

We use the properties of determinants and the form (2.1) of a difference operator. From the second column of the above determinant we subtract the first one and then we divide the result by $h > 0$. From the third column we subtract two times the second column, then we add the first column and we divide the result by h^2 and so on. Then in the second column we obtain the Δ_h operator divided by h, in the third column the Δ_h^2 operator divided by h^2 and so on. After this operation the determinant changes the value but the sign is preserved. Hence

$$(-1)^{n+1-(\ell_1+\dots+\ell_j)} \begin{vmatrix} u_0(x_1) & \frac{\Delta_h u_0(x_1)}{h} & \cdots & \frac{\Delta_h^{\ell_1-1} u_0(x_1)}{h^{\ell_1-1}} & \cdots & u_0(x_k) & \frac{\Delta_h u_0(x_k)}{h} & \cdots & \frac{\Delta_h^{\ell_k-1} u_0(x_k)}{h^{\ell_k-1}} & u_0(x) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ u_n(x_1) & \frac{\Delta_h u_n(x_1)}{h} & \cdots & \frac{\Delta_h^{\ell_1-1} u_n(x_1)}{h^{\ell_1-1}} & \cdots & u_n(x_k) & \frac{\Delta_h u_n(x_k)}{h} & \cdots & \frac{\Delta_h^{\ell_k-1} u_n(x_k)}{h^{\ell_k-1}} & u_n(x) \\ f(x_1) & \frac{\Delta_h f(x_1)}{h} & \cdots & \frac{\Delta_h^{\ell_1-1} f(x_1)}{h^{\ell_1-1}} & \cdots & f(x_k) & \frac{\Delta_h f(x_k)}{h} & \cdots & \frac{\Delta_h^{\ell_k-1} f(x_k)}{h^{\ell_k-1}} & f(x) \end{vmatrix} \geqslant 0.$$

Because f is $\mathscr{U}$-convex, by Theorem 2.2 we infer that $f \in \mathscr{C}^{n-1}(\operatorname{Int} I)$ and $f_+^{(n)}$ exists on $\operatorname{Int} I$. Let $h \to 0^+$. Using Lemma 2.1 we arrive at

$$(-1)^{n+1-(\ell_1+\dots+\ell_j)} \begin{vmatrix} u_0(x_1) & u_0'(x_1) & \cdots & u_0^{(\ell_1-1)}(x_1) & \cdots & u_0(x_k) & u_0'(x_k) & \dots & u_0^{(\ell_k-1)}(x_k) & u_0(x) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ u_n(x_1) & u_n'(x_1) & \cdots & u_n^{(\ell_1-1)}(x_1) & \cdots & u_n(x_k) & u_n'(x_k) & \dots & u_n^{(\ell_k-1)}(x_k) & u_n(x) \\ f(x_1) & f'(x_1) & \cdots & f^{(\ell_1-1)}(x_1) & \cdots & f(x_k) & f'(x_k) & \dots & f^{(\ell_k-1)}(x_k) & f(x) \end{vmatrix} \geqslant 0. \quad (3.4)$$

On the other hand, we apply (2.8) to the $\mathscr{U}$-polynomial u constructed at the beginning of the proof. The obtained determinant differs from that of (3.4) only in one entry in the lower right corner:

$$(-1)^{n+1-(\ell_1+\dots+\ell_j)} \begin{vmatrix} u_0(x_1) & u_0'(x_1) & \cdots & u_0^{(\ell_1-1)}(x_1) & \cdots & u_0(x_k) & u_0'(x_k) & \dots & u_0^{(\ell_k-1)}(x_k) & u_0(x) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ u_n(x_1) & u_n'(x_1) & \cdots & u_n^{(\ell_1-1)}(x_1) & \cdots & u_n(x_k) & u_n'(x_k) & \dots & u_n^{(\ell_k-1)}(x_k) & u_n(x) \\ f(x_1) & f'(x_1) & \cdots & f^{(\ell_1-1)}(x_1) & \cdots & f(x_k) & f'(x_k) & \dots & f^{(\ell_k-1)}(x_k) & u(x) \end{vmatrix} = 0. \quad (3.5)$$

Note that if $k = 1$, then $\ell_1 = n+1$ and $f^{(n)}$ appears in the above two determinants. Then we use the convention $f^{(n)} = f_+^{(n)}$. If x_1 is the left endpoint of I, then $\ell_1 = 1$ by the assumption and in this case no differentiation is needed at x_1. Similarly, if x_k is the right endpoint of I, then $\ell_k = 1$ and in this case no differentiation is needed at x_k. So the regularity properties forced by $\mathscr{U}$-convexity (cf. Theorem 2.2) suffice to write down both determinants.

From the determinant at the inequality (3.4) we subtract that of (3.5). Using linearity with respect to the last row we arrive at

$$(-1)^{n+1-(\ell_1+\cdots+\ell_j)}\begin{vmatrix} u_0(x_1) & u_0'(x_1) & \cdots & u_0^{(\ell_1-1)}(x_1) & \cdots & u_0(x_k) & u_0'(x_k) & \cdots & u_0^{(\ell_k-1)}(x_k) & u_0(x) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ u_n(x_1) & u_n'(x_1) & \cdots & u_n^{(\ell_1-1)}(x_1) & \cdots & u_n(x_k) & u_n'(x_k) & \cdots & u_n^{(\ell_k-1)}(x_k) & u_n(x) \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & f(x)-u(x) \end{vmatrix} \geqslant 0.$$

Expanding the above determinant along the last row we obtain immediately

$$(-1)^{n+1-(\ell_1+\cdots+\ell_j)}\big(f(x)-u(x)\big) \geqslant 0.$$

Indeed, because $\mathscr{U}$ in an ECT-system, it is in particular an ET-system, therefore by (2.5) the minor at $f(x)-u(x)$ is positive. Thus the inequality (3.2) is checked and the proof is finished. $\square$

Remark 3.3. From the proof of Theorem 3.1 we derive even more than it was asserted in the statement: one of possible support-type $\mathscr{U}$-polynomials is the osculatory interpolation $\mathscr{U}$-polynomial fulfilling (2.7).

Now we would like to give some comments. First observe that if $n = 1$, $\mathscr{U} = (1, x)$ and $k = 1$, Theorem 3.1 reduces to the classical support theorem for convex functions. Specifying an ECT-system to the polynomial one $(1, x, \dots, x^n)$, we get immediately the following result (see [24, Theorem 2]).

Corollary 3.4. *Let k, $x_1, \dots, x_k$ and $\ell_1, \dots, \ell_k$ fulfill the assumptions of Theorem* 3.1. *If $f : I \to \mathbb{R}$ is n-convex (i.e. convex with respect to the ECT-system $(1, x, \dots, x^n)$), then f admits at the points $x_1, \dots, x_k$ the n-th order polynomial support of $(\ell_1, \dots, \ell_k)$-type, i.e. there exists a polynomial $u(x) = \sum_{i=0}^{n} \alpha_i x^i$ satisfying the assertions of Theorem* 3.1.

The method used in the proof given in [24] is based on boundedness of divided differences and the Newton's form of the interpolating polynomial. The constructed polynomial is in fact the same as the polynomial obtained in the proof of Theorem 3.1 by osculatory interpolation (in the special case of the polynomial ECT-system).

Remark 3.5. Theorem 3.1 was formulated in the form of the necessary condition. This is also a sufficient one. However, there are considerably weaker sufficient conditions of $\mathscr{U}$-convexity: some particular cases of Theorem 3.1 characterize $\mathscr{U}$-convex functions. This will be studied in detail later in Section 6. Now let us turn attention to two cases.

The first one we have in mind is obtained for $k = n + 1$ and $\ell_1 = \cdots = \ell_{n+1} = 1$ (the support of $(1, 1, \dots, 1)$-type). It says that f is $\mathscr{U}$-convex if and only if for any $x_1, \dots, x_{n+1} \in I$ such that $x_1 < \cdots < x_{n+1}$, the graph of the $\mathscr{U}$-polynomial u interpolating f at $x_1, \dots, x_{n+1}$, passing through the points $(x_1, f(x_1)), \dots, (x_{n+1}, f(x_{n+1}))$, alters the side of the graph of f starting with $u(x) \leqslant f(x)$, $x > x_{n+1}$, $x \in I$, next $u(x) \geqslant f(x)$, $x_n < x < x_{n+1}$, and so on. In the literature concerning higher-order convexity this condition is often treated as a definition, equally with (2.6) and nonnegativeness of divided differences. For $n = 1$ it expresses the fundamental geometrical interpretation of convexity: for any $x_1, x_2 \in I$, $x_1 < x_2$, the graph of a convex function $f : I \to \mathbb{R}$ lies on (x_1, x_2) below the chord joining the point $(x_1, f(x_1))$ with $(x_2, f(x_2))$ (and above both on the left of x_1 and on the right of x_2).

The second necessary and sufficient condition of $\mathscr{U}$-convexity we would like to mention, was proved in [23, Theorem 3]. In terms of our Theorem 3.1 we obtain it for $k = n$, $x_1, \dots, x_n \in \operatorname{Int} I$, $x_1 < \cdots < x_n$, $\ell_1 = \cdots = \ell_{n-1} = 1$ and $\ell_n = 2$. Thus the supports of $(1, \dots, 1, 2)$-type characterize $\mathscr{U}$-convexity.

4. Some applications of Theorem 3.1

In this section we present some support-type results which are the immediate consequences of Theorem 3.1. For some reasons they seem to be important in the theory of generalized convex functions. For instance they can be used to prove Hermite–Hadamard-type inequalities between quadrature operators (the well known Gauss–Legendre, Lobatto and Radau quadratures in the case of higher-order convexity) and the integral of the $\mathscr{U}$-convex function. For details see [24] (higher-order convexity) and [4] (the general $\mathscr{U}$-convexity).

4.1. Support theorem in the classical sense

We present a support theorem for convex functions with respect to ECT-systems with an even number of elements.

Corollary 4.1. *Let $\mathscr{U} = (u_0, u_1, \dots, u_{2n-1})$ be an ECT-system on I. If $f : I \to \mathbb{R}$ is $\mathscr{U}$-convex, then for any $x_1 \in \operatorname{Int} I$ there exists a $\mathscr{U}$-polynomial $u = \sum_{i=0}^{2n-1} \alpha_i u_i$ such that $u(x_1) = f(x_1)$ and $u \leqslant f$ on I.*

Proof. Use Theorem 3.1 for $2n-1$ instead of n and for $k=1$, $\ell_1=2n$. Since ℓ_1 is even and $u(x)\leqslant f(x)$, $x>x_1$, $x\in I$, then the graph of the $\mathscr{U}$-polynomial obtained by Theorem 3.1 remains below the graph of f on the whole I (see Observation 3.2). $\square$

Let us turn attention to the case $n=1$. In this setting, i.e. for convex functions with respect to an ECT-system (u_0,u_1), the above result was proved by Bessenyei and Páles [6, Theorem 4]. Obviously, if $\mathscr{U}=(1,x)$, then we obtain the classical support theorem for convex functions. Thus we can see that in our results it is necessary to assume that the multiplicities of (possible) endpoints of I are 1. For instance, the function $f(x)=-\sqrt{1-x^2}$ is convex on $[-1,1]$ with no affine support at the endpoints.

Return to the general case of Corollary 4.1. For polynomial ECT-systems its assertion was proved in [24, Corollary 6]. Ger [13] established a support theorem for $(2n-1)$-convex functions defined on an open and convex subset of a normed space. However, it was proved under $\mathscr{C}^{2n}$-class regularity assumption, which was not used in our results (notice that not every convex function with respect to an ECT-system possesses such regularity properties, e.g. for $n=1$, $f(x)=|x|$ is convex and not differentiable). Only the regularity properties forced by $\mathscr{U}$-convexity (cf. Theorem 2.2) are engaged in Theorem 3.1.

The assertion of Corollary 4.1 is no longer valid for ECT-systems with odd number of elements, that is for $\mathscr{U}=(u_0,u_1,\dots,u_{2n})$. If we try to repeat the proof, taking $k=1$ in Theorem 3.1, we must assign to x_1 an odd multiplicity. That is why the $\mathscr{U}$-polynomial resulting from this construction does not have a support property. Its graph passing through x_1 changes the side of the graph of f (cf. Observation 3.2). It is easy to give a counterexample. The function $f(x)=x^3$ is 2-convex (cf. Theorem 2.4), i.e. convex with respect to an ECT-system $(1,x,x^2)$, and there is no 2nd order polynomial supporting f on $\mathbb{R}$ (see [13]).

4.2. Principal supports

Now we would like to derive from Theorem 3.1 support-type properties involving more than one point. Let $\mathscr{U}=(u_0,u_1,\dots,u_n)$ be en ECT-system on I. In some circumstances, if we take $k=\lfloor\frac{n}{2}\rfloor+1$ points $x_1,\dots,x_k\in I$, for a $\mathscr{U}$-convex function $f:I\to\mathbb{R}$ there exists the $\mathscr{U}$-polynomial u interpolating f at $x_1,\dots,x_k$ with the graph lying below or above the graph of f. We have in mind the supports of types $(2,2,\dots,2)$, $(1,2,\dots,2,1)$ for odd n and of types $(1,2,\dots,2)$, $(2,\dots,2,1)$ for even n (with the reservation that the points of multiplicity 1 are the endpoints of I). Such supports, following Bessenyei [4], will be called *principal supports*. If $u\leqslant f$ on I, then u is the *lower principal support* (see Corollaries 4.2 and 4.4 below). If $u\geqslant f$ on I, then u is the *upper principal support* (cf. Corollaries 4.3 and 4.5).

Corollary 4.2. *Let $\mathscr{U}=(u_0,u_1,\dots,u_{2n-1})$ be an ECT-system on I. If $f:I\to\mathbb{R}$ is $\mathscr{U}$-convex, then for any n distinct points $x_1,\dots,x_n\in\operatorname{Int}I$, $x_1<\dots<x_n$, there exists a $\mathscr{U}$-polynomial $u=\sum_{i=0}^{2n-1}\alpha_i u_i$ such that $u(x_i)=f(x_i)$, $i=1,\dots,n$, and $u\leqslant f$ on I.*

Proof. In Theorem 3.1 take $2n-1$ in the role of n, further $k=n$, $\ell_1=\dots=\ell_n=2$. Then $\ell_1+\dots+\ell_n=2n=(2n-1)+1$. By Observation 3.2 the graph of the $\mathscr{U}$-polynomial u obtained by Theorem 3.1 remains on the same side of the graph of f (below since $u(x)\leqslant f(x)$, $x>x_n$). $\square$

The assertion of Corollary 4.2, i.e. the existence of a lower principal support at any n interior points of a domain, is in fact the necessary and sufficient condition of $\mathscr{U}$-convexity for $\mathscr{U}=(u_0,u_1,\dots,u_{2n-1})$. However, to prove the sufficiency, it is not needed to assume that $\mathscr{U}$ is an ECT-system, T-system is enough. That is why we formulate Corollary 4.2 only in the form of the necessary condition. The sufficiency is the immediate consequence of Theorem 6.1. Nevertheless, to announce the idea, we sketch the proof in this very special situation. Assume that f admits the lower principal support at any n interior points of I. To prove $(u_0,u_1,\dots,u_{2n-1})$-convexity, we have to check that (2.6) holds for any $x_1,\dots,x_{2n+1}\in I$ such that $x_1<\dots<x_{2n+1}$. For, take the lower principal support u at n points $x_2,x_4,\dots,x_{2n}$. Then $D(x_1,\dots,x_{2n+1};f)\geqslant D(x_1,\dots,x_{2n+1};u)=0$.

Corollary 4.3. *For $n\geqslant 2$ let $\mathscr{U}=(u_0,u_1,\dots,u_{2n-1})$ be an ECT-system on $[a,b]$. If $f:[a,b]\to\mathbb{R}$ is $\mathscr{U}$-convex, then for any $n-1$ distinct points $x_2,\dots,x_n\in(a,b)$, $x_2<\dots<x_n$, there exists a $\mathscr{U}$-polynomial $u=\sum_{i=0}^{2n-1}\alpha_i u_i$ such that $u(a)=f(a)$, $u(b)=f(b)$, $u(x_i)=f(x_i)$, $i=2,\dots,n$, and $u\geqslant f$ on $[a,b]$.*

Proof. The proof is similar to that of Corollary 4.2: use Theorem 3.1 for $2n-1$ instead of n, take $k=n+1$, $x_1=a$, $x_{n+1}=b$, $\ell_1=\ell_{n+1}=1$, $\ell_2=\dots=\ell_n=2$. $\square$

The above result remains valid also for $n=1$ with no interior nodes. Then (for $\mathscr{U}=(1,x)$) it states that a convex function $f:[a,b]\to\mathbb{R}$ admits only one upper principal support, which is a chord joining the points $(a,f(a))$ and $(b,f(b))$. Of course, this is not a characterization of convexity. In general, the existence of upper principal supports does not imply $\mathscr{U}$-convexity. We will return to this problem in Section 6.

Corollary 4.4. *Assume that $a = \inf I \in I$ and $\mathscr{U} = (u_0, u_1, \ldots, u_{2n})$ is an ECT-system on I. If $f : I \to \mathbb{R}$ is $\mathscr{U}$-convex, then for any n distinct points $x_2, \ldots, x_{n+1} \in \operatorname{Int} I$, $x_2 < \cdots < x_{n+1}$, there exists a $\mathscr{U}$-polynomial $u = \sum_{i=0}^{2n} \alpha_i u_i$ such that $u(a) = f(a)$, $u(x_i) = f(x_i)$, $i = 2, \ldots, n+1$, and $u \leqslant f$ on I.*

Proof. In Theorem 3.1 take $2n$ instead of n, further $k = n+1$ and $x_1 = a$, $\ell_1 = 1$, $\ell_2 = \cdots = \ell_{n+1} = 2$. Then $\ell_1 + \cdots + \ell_{n+1} = 2n+1$ and the desired conclusion follows immediately by Observation 3.2. □

In the similar way we obtain:

Corollary 4.5. *Assume that $b = \sup I \in I$ and $\mathscr{U} = (u_0, u_1, \ldots, u_{2n})$ is an ECT-system on I. If $f : I \to \mathbb{R}$ is $\mathscr{U}$-convex, then for any n distinct points $x_1, \ldots, x_n \in \operatorname{Int} I$, $x_1 < \cdots < x_n$, there exists a $\mathscr{U}$-polynomial $u = \sum_{i=0}^{2n} \alpha_i u_i$ such that $u(x_i) = f(x_i)$, $i = 1, \ldots, n$, $u(b) = f(b)$ and $u \geqslant f$ on I.*

Let us turn attention to support-type results of the paper [4] by Bessenyei. He obtained the existence of principal supports in the setting of convexity with respect to so-called Beckenbach families, which includes $\mathscr{U}$-convexity as a special case (this is a kind of convexity considered by Tornheim [22]—cf. the Introduction). The proof is based on another method, not on the general support theorem like our Theorem 3.1. Till now, for convexity in this sense, such a result is not known.

Principal supports seem to be important in the theory of generalized convexity. They can be used to prove the Hermite–Hadamard-type inequalities between the quadrature operators and the integral of the function, which is convex in a desired sense. Using the support approach, for higher-order convexity it was done by the present author in [24], for (u_0, u_1)-convexity by Bessenyei and Páles [6] and for convexity with respect to Beckenbach families by Bessenyei [4]. There is also a series of papers, where the results of this kind were proved using another approach: [2] by Bessenyei and [5,7,8] by Bessenyei and Páles. Many results quoted in this paragraph are also collected in Bessenyei's PhD dissertation [3].

5. Unique supports

The classical result in the theory of convex functions states that any convex function defined on a real interval admits an affine support at every interior point of a domain. The support at some point is unique if and only if this is a point of differentiability. Green [14] investigated in this direction the functions, which are convex in the sense of Beckenbach. A natural question arises whether the support-like functions obtained by Theorem 3.1 are unique. The first impression could suggest the positive answer—in the proof of Theorem 3.1 the desired $\mathscr{U}$-polynomial was precisely pointed as the osculatory interpolation $\mathscr{U}$-polynomial. But some reflection shows that in general the supports need not to be unique. It is the case with ordinary convex functions: the function $f(x) = |x|$ admits at the point 0 infinitely many affine supports. Here the reason is the lack of differentiability. For $\mathscr{U}$-convexity with respect to $\mathscr{U} = (u_0, u_1, \ldots, u_n)$, where $n > 1$, the situation is more complicated: the supports of some types are unique, while supports of the other types are not unique. Here the non-uniqueness is not caused by the lack of smoothness (in the example below we consider the function of $\mathscr{C}^\infty$ class). It will be explained after the proof of Theorem 5.2.

Example 5.1. Let $f(x) = x^6$. It is easy to see that f is 5-convex, that is $(1, x, x^2, x^3, x^4, x^5)$-convex (use either (2.6), or Theorem 2.4). Take $x_1 = -1$, $x_2 = 0$, $x_3 = 1$ with multiplicities $\ell_1 = \ell_3 = 1$, $\ell_2 = 4$. Then using Theorem 3.1 (precisely Corollary 3.4) for $n = 5$ and $k = 3$, we obtain that there exists a polynomial u of degree at most 5 such that

$$u(-1) = f(-1) = 1, \qquad u(0) = f(0) = 0, \qquad u(1) = f(1) = 1, \tag{5.1}$$

$$u(x) \leqslant f(x) \quad \text{for } |x| > 1, \tag{5.2}$$

$$u(x) \geqslant f(x) \quad \text{for } |x| \leqslant 1. \tag{5.3}$$

The difference $f - u$ is a polynomial of sixth degree with zeros -1, 0, 1 and 0 is at least a double zero. Hence

$$f(x) - u(x) = x^6 - u(x) = \left(x^2 - 1\right)x^2\left(x^2 + ax + b\right)$$

for some $a, b \in \mathbb{R}$ and the necessary and sufficient condition for (5.1)–(5.3) is $x^2 + ax + b \geqslant 0$ for any $x \in \mathbb{R}$, i.e. $a^2 \leqslant 4b$. Therefore any polynomial

$$u(x) = -ax^5 + (1-b)x^4 + ax^3 + bx^2,$$

where $a^2 \leqslant 4b$, has the support-like properties expressed by (5.1)–(5.3). The same argument shows that the support-like polynomial obtained for the same nodes -1, 0, 1, but for multiplicities 1, 2, 3 (or 3, 2, 1), respectively, is not unique as well.

Notice that this example corrects a wrong one given in the author's earlier paper (see [24, Remark 4]).

Below we determine the situations, when the support-like functions are unique. It is the case only in a very special situation, where all multiplicities of the involved nodes are either 1, or 2.

Theorem 5.2. *Let $n > 1$. Under assumptions and notations of Theorem* 3.1*, if $\ell_1, \dots, \ell_k \in \{1, 2\}$, then the $\mathscr{U}$-convex function f admits at the points $x_1, \dots, x_k$ the unique support $u = \sum_{i=0}^{n} \alpha_i u_i$ of $(\ell_1, \dots, \ell_k)$-type.*

Proof. By Theorem 3.1 the function f admits at the points $x_1, \dots, x_k$ the support $u = \sum_{i=0}^{n} \alpha_i u_i$ of $(\ell_1, \dots, \ell_k)$-type. We shall show that u is unique.

Let $j \in \{1, \dots, k\}$. We have at the node x_j the interpolatory condition $u(x_j) = f(x_j)$. If $\ell_j = 2$, then $x_j \in \operatorname{Int} I$ by the assumption. Furthermore, support properties of u imply that the sign of the difference $u - f$ remains unchanged on some neighbourhood of x_j (see Observation 3.2). Then $u - f$ has the local extremum at x_j. By regularity properties of $\mathscr{U}$-convexity, f is differentiable on $\operatorname{Int} I$ (cf. Theorem 2.2). Hence $(u - f)'(x_j) = 0$ and we have at x_j an additional interpolatory condition $u'(x_j) = f'(x_j)$. Thus u is the osculatory interpolation $\mathscr{U}$-polynomial fulfilling (2.7). By Theorem 2.5, u is uniquely determined. □

The above argument is impossible to repeat for $n = 1$, i.e. for (u_0, u_1)-convex functions, which may be non-differentiable. In this case at some points there are non-unique supports (e.g. for the usual convexity and $f(x) = |x|$).

Remark 5.3. It follows immediately by Theorem 5.2 that for $\mathscr{U} = (u_0, u_1, \dots, u_n)$ with $n > 1$, the principal supports are unique.

We could see in Example 5.1 that if some ℓ_j, i.e. the multiplicity of x_j, is greater than 2, then the support-like function obtained by Theorem 3.1 need not to be unique. Indeed, only the multiplicities 1 or 2 carry the full information on interpolatory properties. If $\ell_j = 1$, then the difference $u - f$ has at x_j a zero, which forces one interpolatory condition. If $\ell_j = 2$, then $u - f$ has at x_j at least a double zero and this case produces two interpolatory conditions. For greater multiplicities some degrees of freedom are still left and the information is lost. In this case it is impossible to distinguish between simple or double zero and a multiple zero. Only the parity is preserved: odd multiplicities yield one interpolatory condition, while even multiplicities give two conditions.

6. Characterization of generalized convexity via support properties

It is well known that the existence of an affine support at every interior point of a domain characterizes the ordinary convexity. In this section we extend this idea to convexity with respect to Chebyshev systems. It was previously said (cf. the comment after the proof of Corollary 4.2) that $(u_0, u_1, \dots, u_{2n-1})$-convexity follows by the existence of lower principal supports. It will be shown at the end of this section that it is not the case for $(u_0, u_1, \dots, u_{2n})$-convexity. Also the existence of upper principal supports does not imply $\mathscr{U}$-convexity. It was proved in [23] that the supports of type $(1, \dots, 1, 2)$ characterize $(u_0, \dots, u_n)$-convexity (for all $n \in \mathbb{N}$). The class of supports with the characterization property is significantly richer. Theorem 3.1 states that the $\mathscr{U}$-convex function admits the $\mathscr{U}$-polynomial supports of any type $(\ell_1, \dots, \ell_k)$. Some types characterize $\mathscr{U}$-convexity, while the other do not characterize it. We will determine both these types of supports.

6.1. Supports characterizing $\mathscr{U}$-convexity

Below we shall prove that $\mathscr{U}$-convexity follows by the existence of supports of $(\ell_1, \dots, \ell_k)$-type with $\ell_1, \dots, \ell_k \in \{1, 2\}$.

Theorem 6.1. *Let $\mathscr{U} = (u_0, u_1, \dots, u_n)$ be a T-system on I. Let $k \leqslant n + 1$, $\ell_1, \dots, \ell_k \in \{1, 2\}$ with $\ell_1 + \dots + \ell_k = n + 1$. If the function $f : I \to \mathbb{R}$ admits the support of $(\ell_1, \dots, \ell_k)$-type at any points $x_1, \dots, x_k \in I$, then f is $\mathscr{U}$-convex.*

Proof. Take the arbitrary $x_1, \dots, x_{n+2} \in I$ such that $x_1 < \dots < x_{n+2}$. We shall prove that (2.6) holds. Define the set $S := \{i_1, \dots, i_k\} \subset \{1, \dots, n + 2\}$ choosing its elements in the following way:

$$i_1 := 2, \qquad i_{j+1} := i_j + \ell_j, \quad j = 1, \dots, k - 1.$$

Then $i_{j+1} \in \{i_j + 1, i_j + 2\}$, $j = 1, \dots, k - 1$. Moreover, if $i \in \{1, \dots, n + 1\}$, then $\{i, i + 1\} \cap S \neq \emptyset$.

Let u be a support of $(\ell_1, \dots, \ell_k)$-type at the points $x_{i_1}, \dots, x_{i_k}$. The numbers $\ell_1, \dots, \ell_k$ are called the multiplicities of the points $x_{i_1}, \dots, x_{i_k}$, respectively. These points induce a partition of I into $k + 1$ subintervals.

Claim 1. *If $i \in S$, then $(-1)^{n+i}(f(x) - u(x)) \leqslant 0$ for all x belonging to the subinterval of the above partition with the right endpoint x_i.*

Claim 1 holds for $i = 2$, which is the first element of S. Indeed, by the support property (3.1), $(-1)^{n+1}(f(x) - u(x)) \geqslant 0$, $x < x_2$, $x \in I$. For $i \in S$, $i > 2$, suppose that Claim 1 is true for all $i' \in S$ with $i' < i$. Let $j := \max\{i' \in S\colon\ i' < i\}$. By the previous assumption we get

$$(-1)^{n+j}\big(f(x)-u(x)\big)\leqslant 0 \tag{6.1}$$

on the left neighbourhood of x_j. If the multiplicity of x_j equals 1, then by Observation 3.2 the inequality in (6.1) reverts on (x_j, x_i). But the construction of the set S yields $i = j+1$, whence i fulfills the inequality of Claim 1. If the multiplicity of x_j equals 2, then we derive from Observation 3.2 that the inequality in (6.1) is preserved on (x_j, x_i). Since $i = j+2$ (we use once more the construction of S), then i fulfills Claim 1.

Claim 2. *If $i \notin S$, then $(-1)^{n+i}(f(x_i)-u(x_i)) \geqslant 0$.*

If $n+2 \notin S$, then $n+1 \in S$ and by virtue of (3.3), $f(x)-u(x) \geqslant 0$, $x > x_{n+1}$, $x \in I$. In particular, $f(x_{n+2}) - u(x_{n+2}) \geqslant 0$ and Claim 2 holds for $i = n+2$. If $i \notin S$, $i < n+2$, then $i+1 \in S$ and Claim 2 follows immediately by Claim 1.

Finally we check (2.6), i.e. the inequality $D(x_1, \dots, x_{n+2}; f) \geqslant 0$. Observe that $D(x_1, \dots, x_{n+2}; u) = 0$, because $u = \sum_{i=0}^{n} \alpha_i u_i$ and the last row of this determinant is a linear combination of the remaining rows. Notice also that $D(x_1, \dots, x_{n+2}; \cdot)$ is a linear functional. Using the Laplace expansion along the last row we arrive at the desired conclusion:

$$D(x_1,\dots,x_{n+2}; f) = D(x_1,\dots,x_{n+2}; f-u) = \sum_{i=1}^{n+2}(-1)^{n+i}v_i\big(f(x_i)-u(x_i)\big) \geqslant 0.$$

Indeed, the minors $v_1, \dots, v_{n+2}$ are positive, because $\mathscr{U}$ is a T-system. The components of the above sum with $i \in S$ vanish due to the interpolation property of u, while the components with $i \notin S$ are nonnegative by Claim 2. □

Remark 6.2. For $n = 1$, $k = 1$ and $\mathscr{U} = (1, x)$, Theorem 6.1 reduces to the classical result characterizing convexity mentioned at the beginning of this section.

Note that under the continuity assumption there are also another characterizations of convexity and generalized convexity, which involve Hermite–Hadamard-type inequalities. For ordinary convex functions they are well known (e.g. [9,12, 15], [19, Exercise 8, p. 205]). For (u_0, u_1)-convex functions see [3,9]. For higher-order convexity see [10]. Till now such a characterization of $\mathscr{U}$-convexity for non-polynomial T-systems consisting of more than two functions is not known.

6.2. Supports, which do not characterize $\mathscr{U}$-convexity

We will show that only the supports of $(\ell_1, \dots, \ell_k)$-type with $\ell_1, \dots, \ell_k \in \{1, 2\}$ characterize $\mathscr{U}$-convexity. The existence of supports of $(\ell_1, \dots, \ell_k)$-type with at least one multiplicity $\ell_j > 2$ does not imply $\mathscr{U}$-convexity. In the construction of a counterexample for simplicity we restrict ourselves to n-convexity, i.e. $(1, x, \dots, x^n)$-convexity. Let $x_+ := \max\{x, 0\}$.

Proposition 6.3. *Let $m \in \mathbb{N}$, $m > 1$. The function $f(x) = (x_+)^m$, $x \in \mathbb{R}$, is m-convex and it is not n-convex for any $n > m$.*

Proof. We have $f^{(m-1)}(x) = m!x_+$, whence $f^{(m-1)}$ is a convex function. Then the first assertion follows by Theorem 2.3. The second assertion follows by Theorem 2.2, because f is not m times differentiable at 0. □

Recall that the polynomial of order n means the polynomial of degree at most n.

Proposition 6.4. *Let $n > 1$, $k \leqslant n+1$, $\ell_1, \dots, \ell_k \in \mathbb{N}$ with $\ell_1 + \dots + \ell_k = n+1$. If at least one multiplicity $\ell_j > 2$, then there exists a non-n-convex function, which admits the n-th order polynomial support of $(\ell_1, \dots, \ell_k)$-type at any k points of a domain.*

Proof. Define the new multiplicities ℓ_i', $i = 1, \dots, k$, by

$$\ell_i' := \begin{cases} 1, & \text{if } \ell_i \geqslant 3 \text{ is odd},\\ 2, & \text{if } \ell_i \geqslant 4 \text{ is even},\\ \ell_i, & \text{if } \ell_i \in \{1,2\}.\end{cases}$$

If $\ell_j > 2$ for some $j \in \{1, \dots, k\}$, then $m := \ell_1' + \dots + \ell_k' - 1 \leqslant (n+1) - 1 - 2 < n-1$. The function $f(x) = (x_+)^m$, $x \in \mathbb{R}$, is m-convex and it is not n-convex (see Proposition 6.3 above). By m-convexity and Theorem 3.1 (precisely Corollary 3.4), f admits at any points $x_1, \dots, x_k \in \mathbb{R}$ with $x_1 < \dots < x_k$, the m-th order polynomial support of $(\ell_1', \dots, \ell_k')$-type. This is also the n-th order polynomial support of $(\ell_1, \dots, \ell_k)$-type. It follows by the character of inequalities (3.1)–(3.3). The new multiplicities $\ell_1', \dots, \ell_k'$ are obtained by subtracting the even numbers from the old ones $\ell_1, \dots, \ell_k$, so the signs of the expressions in these inequalities remain unchanged. □

Remark 6.5. The similar counterexample can be also given for any ECT-system $\mathscr{U} = (u_0, u_1, \dots, u_n)$. In this general case the construction requires the ECT-property (recall that the characterization of $\mathscr{U}$-convexity given Theorem 6.1 is given for T-systems). Let $m < n$. In [18, Lemma 2.1, p. 382] we can find the function, which is $(u_0, \dots, u_m)$-convex but it is not $(u_0, \dots, u_m, \dots, u_n)$-convex. This last assertion (concerning nonconvexity) is not explicitly written, but it can be read between the lines. Then we could perform exactly the same construction as in the proof of Proposition 6.4.

6.3. Upper principal supports do not characterize $\mathscr{U}$-convexity

By Theorem 6.1 we obtain immediately that the existence of lower principal supports characterizes $(u_0, u_1, \dots, u_{2n-1})$-convexity (in this case such supports are of $(2, 2, \dots, 2)$-type). As we announced at the beginning of this section, now we shall show that the existence of lower principal supports does not characterize $(u_0, u_1, \dots, u_{2n})$-convexity. In this case such supports are of $(1, 2, \dots, 2)$-type and the first support point is the left endpoint of I (compare with Corollary 4.4). Notice that by Theorem 6.1 the characterization holds, if the support exists at any k points. The placement of even one support point in a fixed position (as in a discussed case) causes the loss of the characterization property.

Example 6.6. Let $\mathscr{U} = (1, x, x^2)$, i.e. we deal with 2-convexity. Consider the function $f : [-1, \infty) \to \mathbb{R}$ given by

$$f(x) = (x_+)\mathrm{e}^x.$$

Because f is not differentiable at 0, it is not 2-convex (such functions are of $\mathscr{C}^1$ class on the interior of a domain, see Theorem 2.2).

We will show that f admits lower principal supports, that is for any $a > -1$ there exists a polynomial u of order 2, which is a support of $(1, 2)$-type at the points -1 and a. Precisely, u fulfills the conditions

$$u(-1) = f(-1) = 0, \qquad u(a) = f(a) \quad \text{and} \quad u \leqslant f \quad \text{on } [-1, \infty).$$

If $a \in (-1, 0]$, then $u(x) = 0$ has the desired properties. For $a > 0$ we will demonstrate that it is enough to take the polynomial u given by

$$u(x) = \frac{\mathrm{e}^a}{(a+1)^2}(x+1)\big[x\big(a^2 + a + 1\big) - a^3\big].$$

This is a polynomial interpolating f at a simple node -1 and at a double node a, so it can be determined either by Newton's interpolation formula, or by (2.8) (having in mind the method of the proof of Theorem 3.1, osculatory interpolation polynomial is the natural object of research).

Obviously $u(-1) = f(-1) = 0$ and $u(a) = a\mathrm{e}^a = f(a)$. Now the inequality $u \leqslant f$ on $[-1, \infty)$ is left to check.

The zeros of u are -1 and $\frac{a^3}{a^2+a+1} > 0$. Then, in particular, $u(x) \leqslant 0 = f(x)$ for $x \in [-1, 0)$.

Let $x \in [0, \infty)$. Consider the auxiliary function $\varphi(x) = f(x) - u(x)$. Since $\varphi'''(x) = (x+3)\mathrm{e}^x > 0$, then by Theorem 2.4, φ is 2-convex on $[0, \infty)$. Using Theorem 3.1 (in fact Corollary 3.4) together with Remark 3.3, we conclude φ has at the points 0 and a the quadratic (precisely at most quadratic) support ψ of $(1, 2)$-type, which is additionally the osculatory interpolation polynomial, i.e. together with $\psi \leqslant \varphi$ on $[0, \infty)$, ψ fulfills the conditions $\psi(0) = \varphi(0)$, $\psi(a) = \varphi(a)$, $\psi'(a) = \varphi'(a)$. Then ψ is uniquely determined by the interpolatory conditions. Using once more the Newton's interpolation formula or (2.8) we have

$$\psi(x) = \frac{a\mathrm{e}^a}{(a+1)^2}(x-a)^2.$$

Because $\psi \leqslant \varphi$ on $[0, \infty)$, using the above equation we arrive at

$$0 \leqslant \psi(x) \leqslant \varphi(x) = f(x) - u(x), \quad x \in [0, \infty).$$

This shows that $u \leqslant f$ on $[-1, \infty)$.

Remark 6.7. It turns out that for $\mathscr{U} = (u_0, u_1, \dots, u_{2n})$ upper principal supports do not characterize $\mathscr{U}$-convexity as well. To convince ourselves of it, consider a function $g : (-\infty, 1] \to \mathbb{R}$ given by $g(x) = -f(-x)$, where f is given in Example 6.6. Then g is a non-2-convex function admitting upper principal supports, i.e. for any $a < 1$ the function g admits the 2nd order polynomial support p of $(2, 1)$-type at the points a and 1 ($p(a) = g(a)$, $p(1) = g(1)$ and $p \geqslant g$ on $(-\infty, 1]$). Indeed, let u be a lower principal support for f at the points -1 and $-a$. Then it is enough to take $p(x) = -u(-x)$. □

We considered above the case of $(u_0, u_1, \dots, u_{2n})$-convexity. There is also no characterization of $(u_0, u_1, \dots, u_{2n-1})$-convexity via upper principal supports. It is easy to see for $n = 1$ and the usual convexity (cf. the comment after the proof of Corollary 4.3). Below we announce the nontrivial example.

Example 6.8. Let $\mathscr{U} = (1, x, x^2, x^3)$—now we deal with 3-convexity. The function $f : [-1, 1] \to \mathbb{R}$ given by

$$f(x) = (x_+)^2 e^x$$

is not 3-convex, while it admits upper principal supports, that is for any $a \in (-1, 1)$ there exists a polynomial u of 3rd order, which is a support of $(1, 2, 1)$-type for f at the points -1, a and 1 ($u(-1) = f(-1)$, $u(a) = f(a)$, $u(1) = f(1)$ and $u \geqslant f$ on $[-1, 1]$; compare with Corollary 4.3). This is the osculatory interpolation polynomial satisfying apart from the above interpolatory conditions the additional one $u'(a) = f'(a)$.

As we can see, the idea is the same as in Example 6.6. However, to check the above assertions we need considerably longer and more complicated computations, which we would like to omit.

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