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Locally convex quasi *-algebras with sufficiently many *-representations

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ABSTRACT

The main aim of this paper is the investigation of conditions under which a locally convex quasi *-algebra $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ attains sufficiently many (τ, t_w)-continuous *-representations in $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, to separate its points. Having achieved this, a usual notion of bounded elements on $\mathfrak{A}[\tau]$ rises. On the other hand, a natural order exists on $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ related to the topology τ, that also leads to a kind of bounded elements, which we call "order bounded". What is important is that under certain conditions the latter notion of boundedness coincides with the usual one. Several nice properties of order bounded elements are extracted that enrich the structure of locally convex quasi *-algebras.

1. Introduction

Quasi *-algebras form an important class of the so-called partial *-algebras introduced by J.-P. Antoine and W. Karwowski, in 1983 (for details, see [2]). Topological quasi *-algebras were initiated by G. Lassner, in 1981 (cf. [10,11]), aiming to the solution of questions that appear in quantum statistics and quantum dynamics, that the algebraic formulation of quantum theories presented in [9] by Haag and Kastler (1964), could not face. Nevertheless, the given definition of this kind of algebras, was lacking the "bimodule axiom", essential for several studies, like e.g., representation theory. The filling of this vacuous came only after almost 20 years by K. Schmüdgen in [12]. Partial and quasi *-algebras play an important role in the theory of unbounded operators and both have a special bearing in mathematical physics (see, for instance, [2,14]). A serious investigation of the theory and applications of quasi *-algebras has been done by the second named author, himself, and jointly with some of his collaborators (see, e.g., the literature of this article, as well as that of [18]). An easy example of a quasi *-algebra can be given by taking the completion of a locally convex *-algebra $\mathfrak{A}[\tau]$ with separately (but not jointly) continuous multiplication.

A particular class of locally convex quasi *-algebras is that of locally convex quasi C^*-algebras and/or locally convex quasi C^*-normed algebras, whose structure has been investigated in [4–6]. These quasi *-algebras are realized as locally convex quasi C^*-, respectively quasi C^*-normed algebras of operators and have several other nice properties similar to those of C^*-algebras.

The motivation for studying quasi *-algebras is quite clear from the above. In the present paper, the possibility of a locally convex quasi *-algebra $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ is examined, to admit sufficiently many (τ, t_w)-continuous *-representations, each of them taking values in some space $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ (where $\mathcal{H}$ is a Hilbert space and $\mathcal{D}$ is a dense subspace of $\mathcal{H}$), in order to separate its points. In the case of bounded *-representations, C^*-algebras have, of course, the preceding property. The achievement of the preceding goal leads to consideration of bounded elements on $(\mathfrak{A}[\tau], \mathfrak{A}_0)$, whose study gives interesting information about the structure of locally convex quasi *-algebras.

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More precisely, in Section 2, the basic definitions and notation are given, needed throughout the paper. In Section 3, the concept of positive elements of $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ is recalled, through which the notion of a positive linear functional on $\mathfrak{A}$ is given. If $\mathfrak{A}^+$ is the set of all positive elements of $\mathfrak{A}$, the property $\mathfrak{A}^+ \cap (-\mathfrak{A}^+) = \{0\}$ is characterized (Proposition 3.8) and plays an important role in the whole study. It is this condition together with a second one, related to a property of some elements of $\mathfrak{A}$ to become positive, that provides a locally convex quasi *-algebra $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ with sufficiently many unbounded *-representations, in order to separate its points (Corollary 3.11). The results of Section 3, lead to consideration of "fully representable quasi *-algebras" in Section 4. Examples of this sort of algebras are given and their structure is studied in Section 5. More precisely, Section 5 deals with bounded elements and their effects on fully representable locally convex quasi *-algebras. First, a usual notion of bounded elements is given due to the fact that Section 3 assures the existence of *-representations on a certain locally convex quasi *-algebra $(\mathfrak{A}[\tau], \mathfrak{A}_0)$. Secondly, the positive elements of $\mathfrak{A}$ lead, as we know, to the definition of an order on the real vector space of its self-adjoint elements. This in turn, leads to the notion of the "order boundedness", which on certain fully representable quasi *-algebras coincides with the "usual" boundedness (Theorem 5.5, Corollary 5.8). Furthermore, a "weak" partial multiplication $\Box$ is considered on a fully representable quasi *-algebra $(\mathfrak{A}[\tau], \mathfrak{A}_0)$, and an "unbounded" C^*-seminorm $\|\cdot\|_b$ is defined on $\mathfrak{A}$ (by means of the order boundedness) with domain the partial *-subalgebra $\mathfrak{A}_b$ of $\mathfrak{A}$ consisting of all order bounded elements. In this way, under certain conditions, $\mathfrak{A}_b$ becomes a C^*-algebra under the weak multiplication $\Box$ and the C^*-norm $\|\cdot\|_b$ (Theorem 5.16).

2. Preliminary definitions and notation

All algebras and vector spaces considered in this paper are over the field $\mathbb{C}$ of complexes. Moreover, by "locally convex space" we always mean a "Hausdorff locally convex space". Our basic definitions and notation mainly come from [2].

First we recall briefly what a partial *-algebra is. A complex vector space $\mathfrak{A}$ endowed with a conjugate linear involution * and a distributive partial multiplication $\cdot$ on a subset Γ of $\mathfrak{A} \times \mathfrak{A}$, such that

$$(x, y) \in \Gamma \quad \Leftrightarrow \quad \bigl(y^*, x^*\bigr) \in \Gamma \quad \text{and} \quad (x \cdot y)^* = y^* \cdot x^*,$$

is called a *partial *-algebra*.

The set of all right multipliers of an element $y \in \mathfrak{A}$ is denoted by $R(y)$; i.e., $R(y) := \{x \in \mathfrak{A}\colon\ (y, x) \in \Gamma\}$. Put now $R\mathfrak{A} := \{x \in \mathfrak{A}\colon\ (y, x) \in \Gamma,\ \forall y \in \mathfrak{A}\} = \bigcap_{y \in \mathfrak{A}} R_y$. The elements of $R\mathfrak{A}$ are called *universal right multipliers*. In the same way, L_y and $L\mathfrak{A}$ are defined.

Definition 2.1. Let $\mathfrak{A}$ be a complex vector space and $\mathfrak{A}_0$ a *-algebra contained in $\mathfrak{A}$. We say that $\mathfrak{A}$ is a *quasi *-algebra* with distinguished *-algebra $\mathfrak{A}_0$ (or, simply, over $\mathfrak{A}_0$) if

(i) the left multiplication ax and the right multiplication xa of an element a of $\mathfrak{A}$ and an element x of $\mathfrak{A}_0$, which extend the multiplication of $\mathfrak{A}_0$, are always defined and bilinear;
(ii) $x_1(x_2 a) = (x_1 x_2)a$ and $x_1(a x_2) = (x_1 a)x_2$, for each $x_1, x_2 \in \mathfrak{A}_0$ and $a \in \mathfrak{A}$;
(iii) an involution *, which extends the involution of $\mathfrak{A}_0$, is defined in $\mathfrak{A}$ with the property $(ax)^* = x^* a^*$ and $(xa)^* = a^* x^*$, for all $x \in \mathfrak{A}_0$ and $a \in \mathfrak{A}$.

We say that a quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is *unital*, if there is an element $e \in \mathfrak{A}_0$, such that $ae = a = ea$, for all $a \in \mathfrak{A}$; e is unique and called the *unit* of $(\mathfrak{A}, \mathfrak{A}_0)$.

Definition 2.2. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra and τ a locally convex topology on $\mathfrak{A}$. We say that $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ is *a locally convex quasi *-algebra* if

(i) the map $a \in \mathfrak{A} \to a^* \in \mathfrak{A}$ is continuous;
(ii) for every $x \in \mathfrak{A}_0$, the maps $a \mapsto ax$, $a \mapsto xa$ are continuous in $\mathfrak{A}[\tau]$;
(iii) $\mathfrak{A}_0$ is τ-dense in $\mathfrak{A}$.

Let now $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all (closable) linear operators X such that $D(X) = \mathcal{D}$, $D(X^*) \supseteq \mathcal{D}$, where $D(X)$ denotes the domain of X.

The set $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a partial *-algebra with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger = X^* \restriction \mathcal{D}$ and the (*weak*) partial multiplication $X_1 \Box X_2 = {X_1}^{\dagger *} X_2$. The latter is defined whenever X_2 is a weak right multiplier of X_1 (we shall write $X_2 \in R^{\mathrm{w}}(X_1)$ or $X_1 \in L^{\mathrm{w}}(X_2)$), that is, if and only if $X_2 \mathcal{D} \subset \mathcal{D}({X_1}^{\dagger *})$ and $X_1^\dagger \mathcal{D} \subset \mathcal{D}(X_2^*)$.

Let $\mathcal{L}^\dagger(\mathcal{D})$ be the subspace of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ consisting of all its elements which leave, together with their adjoints, the domain $\mathcal{D}$ invariant. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a *-algebra with respect to the usual algebraic operations.

Example 2.3. Let $\mathcal{L}^\dagger(\mathcal{D})_b$ denote the bounded part of $\mathcal{L}^\dagger(\mathcal{D})$; i.e.,

$$\mathcal{L}^\dagger(\mathcal{D})_b = \{X \in \mathcal{L}^\dagger(\mathcal{D}): \overline{X} \in \mathcal{B}(\mathcal{H})\}.$$

If $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is endowed with the strong *-topology t_{s^*}, defined by the set of seminorms

$$p_\xi(X) = \|X\xi\| + \|X^\dagger\xi\|, \quad \xi \in \mathcal{D},\ X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}),$$

then $(\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})[\mathrm{t}_{s^*}], \mathcal{L}^\dagger(\mathcal{D})_b)$ is a locally convex quasi *-algebra or, more precisely, a *locally convex quasi C^*-normed algebra*.

If $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is endowed with the weak topology t_w defined by the set of seminorms

$$p_{\xi,\eta}(X) = |\langle X\xi|\eta\rangle|, \quad \xi,\eta \in \mathcal{D},\ X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}),$$

then, again, $(\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})[\mathrm{t}_w], \mathcal{L}^\dagger(\mathcal{D})_b)$ is *a locally convex quasi *-algebra*.

Definition 2.4. Let $(\mathfrak{A},\mathfrak{A}_0)$ be a quasi *-algebra and $\mathcal{D}_\pi$ a dense domain in a certain Hilbert space $\mathcal{H}_\pi$. A linear map π from $\mathfrak{A}$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi,\mathcal{H}_\pi)$ is called a **-representation of* $\mathfrak{A}$, if the following properties are fulfilled:

(i) $\pi(a^*) = \pi(a)^\dagger$, $\forall a \in \mathfrak{A}$;
(ii) for $a \in \mathfrak{A}$ and $x \in \mathfrak{A}_0$, $\pi(a) \Box \pi(x)$ is well defined and $\pi(a) \Box \pi(x) = \pi(ax)$.

Moreover, if

(iii) $\pi(\mathfrak{A}_0) \subset \mathcal{L}^\dagger(\mathcal{D}_\pi)$,

then π is said to be a **-representation of the quasi *-algebra* $(\mathfrak{A},\mathfrak{A}_0)$. If $(\mathfrak{A},\mathfrak{A}_0)$ has a unit $e \in \mathfrak{A}_0$, we assume $\pi(e) = I$, where I is the identity operator on $\mathcal{D}_\pi$.

If π is a *-representation of $(\mathfrak{A},\mathfrak{A}_0)$, then the *closure* $\widetilde{\pi}$ of π is defined, for each $x \in \mathfrak{A}$, as the restriction of $\overline{\pi(x)}$ to the domain $\widetilde{\mathcal{D}_\pi}$, which is the completion of $\mathcal{D}_\pi$ under the *graph topology t_π* [12] defined by the seminorms $\xi \in \mathcal{D}_\pi \to \|\pi(a)\xi\|$, $a \in \mathfrak{A}$. If $\pi = \widetilde{\pi}$ the *-representation is said to be *closed*.

The *adjoint* of a *-representation π of a quasi *-algebra $(\mathfrak{A},\mathfrak{A}_0)$, denoted by π^*, is defined as follows; see [2,12]:

$$\mathcal{D}_{\pi^*} \equiv \bigcap_{x \in \mathfrak{A}} \mathcal{D}(\pi(x)^*) \quad \text{and} \quad \pi^*(x) = \pi(x^*)^* \upharpoonright \mathcal{D}_{\pi^*}, \quad x \in \mathfrak{A}.$$

The *-representation π is said to be *self-adjoint* if $\pi = \pi^*$.

The *-representation π is said to be *ultra-cyclic* if there exists $\xi_0 \in \mathcal{D}_\pi$ such that $\mathcal{D}_\pi = \pi(\mathfrak{A}_0)\xi_0$, while is said to be *cyclic* if there exists $\xi_0 \in \mathcal{D}_\pi$ such that $\pi(\mathfrak{A}_0)\xi_0$ is dense in $\mathcal{D}_\pi$ with respect to the graph topology t_π.

The following proposition, proved in [18, p. 53], extends the GNS construction to quasi *-algebras.

Proposition 2.5. *Let ω be a linear functional on $\mathfrak{A}$ satisfying the following requirements*:

(L1) $\omega(a^*a) \geqslant 0$, *for all $a \in \mathfrak{A}_0$*;
(L2) $\omega(b^*x^*a) = \overline{\omega(a^*xb)}$, *for all $a,b \in \mathfrak{A}_0$, $x \in \mathfrak{A}$*;
(L3) $\forall x \in \mathfrak{A}$ *there exists $\gamma_x > 0$ such that $|\omega(x^*a)| \leqslant \gamma_x \omega(a^*a)^{1/2}$, for all $a \in \mathfrak{A}_0$.*

Then there exists a triple $(\pi_\omega, \lambda_\omega, \mathcal{H}_\omega)$ such that:

- *π_ω is an ultra-cyclic *-representation of $\mathfrak{A}$ with ultra-cyclic vector ξ_ω*;
- *λ_ω is a linear map of $\mathfrak{A}$ into $\mathcal{H}_\omega$ with $\lambda_\omega(\mathfrak{A}_0) = \mathcal{D}_{\pi_\omega}$, $\xi_\omega = \lambda_\omega(e)$ and $\pi_\omega(x)\lambda_\omega(a) = \lambda_\omega(xa)$, for every $x \in \mathfrak{A}$, $a \in \mathfrak{A}_0$*;
- *$\omega(x) = \langle \pi_\omega(x)\xi_\omega|\xi_\omega\rangle$, for every $x \in \mathfrak{A}$.*

The *-representation π_ω satisfies the following properties:

$$\begin{aligned} &\pi_\omega(x)\lambda_\omega(a) = \lambda_\omega(xa), \quad x \in \mathfrak{A},\ a \in \mathfrak{A}_0, \\ &\pi_\omega^*(a)\lambda_\omega(x) = \lambda_\omega(ax), \quad x \in \mathfrak{A},\ a \in \mathfrak{A}_0, \end{aligned} \tag{2.1}$$

where π_ω^* is the adjoint of π_ω.

Definition 2.6. A linear functional ω on $\mathfrak{A}$ satisfying (L1)–(L3) will be called *representable.*

If ω is representable, $(\pi_\omega, \lambda_\omega, \mathcal{H}_\omega)$ will be called, as usual, the *GNS construction for* ω. The set of all representable functionals on $(\mathfrak{A}, \mathfrak{A}_0)$ will be denoted by $\mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$. Note that $\mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$ is a wedge. Moreover, given $\omega \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$ and $b \in \mathfrak{A}_0$, we define the functional ω_b by $\omega_b(x) = \omega(b^*xb)$, $x \in \mathfrak{A}$. It follows easily that ω_b is a representable functional on $\mathfrak{A}$. Thus,

$$\omega \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0) \quad \text{and} \quad b \in \mathfrak{A}_0, \quad \text{imply} \quad \omega_b \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0). \tag{2.2}$$

When $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ is a locally convex quasi *-algebra, let

$$\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0) = \bigl\{\omega \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0) \text{ such that } \omega \text{ is continuous}\bigr\}.$$

We say that a functional $\omega \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$ is *continuous* if there exists a continuous seminorm p on $\mathfrak{A}$, such that $|\omega(a)| \leqslant p(a)$, for every $a \in \mathfrak{A}$.

Let φ be a positive sesquilinear form defined on $\mathfrak{A}_0 \times \mathfrak{A}_0$. We say that φ is *closable* if for a net $\{x_\delta\}_{\delta\in\Delta}$ in $\mathfrak{A}_0$, one has

$$x_\delta \xrightarrow{\tau} 0 \quad \text{and} \quad \varphi(x_\delta - x_\gamma, x_\delta - x_\gamma) \to 0 \quad \Rightarrow \quad \varphi(x_\delta, x_\delta) \to 0.$$

Then, $|\varphi(x_\delta - x_\delta)^{1/2} - \varphi(x_\gamma - x_\gamma)^{1/2}| \leqslant \varphi(x_\delta - x_\gamma, x_\delta - x_\gamma)^{1/2} \to 0$, therefore $\{\varphi(x_\delta, x_\delta)\}_{\delta\in\Delta}$ is a Cauchy net. Thus, if φ is closable, then it can be extended to a positive sesquilinear form $\overline{\varphi}$ defined on $D(\overline{\varphi}) \times D(\overline{\varphi})$ by

$$\overline{\varphi}(a, a) = \lim_\delta \varphi(x_\delta, x_\delta),$$

where

$$D(\overline{\varphi}) = \bigl\{a \in \mathfrak{A}\colon \exists\{x_\delta\} \subset \mathfrak{A}_0 \text{ with } x_\delta \xrightarrow{\tau} a, \text{ and } \varphi(x_\delta - x_\gamma, x_\delta - x_\gamma) \to 0\bigr\}.$$

This definition extends in obvious way to pairs (a, b) with $a, b \in D(\overline{\varphi})$. If ω is a positive linear functional on $\mathfrak{A}_0$ then we can define a positive sesquilinear form φ_ω on $\mathfrak{A}_0 \times \mathfrak{A}_0$ by

$$\varphi_\omega(x, y) := \omega\bigl(y^*x\bigr), \quad x, y \in \mathfrak{A}_0.$$

Now we prove the following

Proposition 2.7. *Let $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$. Then φ_ω is closable.*

Proof. Let $x_\delta \xrightarrow{\tau} 0$ with $\varphi_\omega(x_\delta - x_\gamma, x_\delta - x_\gamma) \to 0$. Then $y^*x_\delta \to 0$, for every $y \in \mathfrak{A}_0$, since the multiplication is continuous. The continuity of ω then implies that $\omega(y^*x_\delta) \to 0$, $y \in \mathfrak{A}_0$. Put

$$N_\omega = \bigl\{x \in \mathfrak{A}_0\colon \omega\bigl(x^*x\bigr) = 0\bigr\}.$$

Then $\mathfrak{A}_0/N_\omega$ is a pre-Hilbert space under the well defined inner product

$$\bigl\langle\lambda_\omega(x)|\lambda_\omega(y)\bigr\rangle = \omega\bigl(y^*x\bigr), \quad x, y \in \mathfrak{A}_0,$$

where $\lambda_\omega(z) := z + N_\omega$, $z \in \mathfrak{A}_0$. Let $\mathcal{H}_\omega$ denote the Hilbert space completion of $\mathfrak{A}_0/N_\omega$. The net $\{\lambda_\omega(x_\delta)\}$ is Cauchy, since

$$\bigl\|\lambda_\omega(x_\delta) - \lambda_\omega(x_\gamma)\bigr\|^2 = \varphi_\omega(x_\delta - x_\gamma, x_\delta - x_\gamma) \to 0.$$

Hence it converges to some $\xi \in \mathcal{H}_\omega$ and

$$\bigl\langle\lambda_\omega(x_\delta)|\lambda_\omega(y)\bigr\rangle \to \bigl\langle\xi|\lambda_\omega(y)\bigr\rangle, \quad \forall y \in \mathfrak{A}_0.$$

Moreover,

$$\bigl\langle\lambda_\omega(x_\delta)|\lambda_\omega(y)\bigr\rangle = \omega\bigl(y^*x_\delta\bigr) \to 0, \quad \forall y \in \mathfrak{A}_0.$$

Thus $\langle\xi|\lambda_\omega(y)\rangle = 0$, for every $y \in \mathfrak{A}_0$. This implies that $\xi = 0$. Therefore

$$\varphi_\omega(x_\delta, x_\delta) \to 0. \qquad \square$$

Note that representability of ω is not used in the proof of Proposition 2.7.

Consider now the set

$$\mathfrak{A}_{\mathcal{R}} := \bigcap_{\omega\in\mathcal{R}_c(\mathfrak{A},\mathfrak{A}_0)} D(\overline{\varphi_\omega}).$$

If $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0) = \{0\}$, we put $\mathfrak{A}_{\mathcal{R}} = \mathfrak{A}$. Note that, if for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, φ_ω is jointly continuous with respect to τ, we get $\mathfrak{A}_{\mathcal{R}} = \mathfrak{A}$.

Proposition 2.8. *$\mathfrak{A}_{\mathcal{R}}$ is a vector subspace of $\mathfrak{A}$ and $\mathfrak{A}_0 \subset \mathfrak{A}_{\mathcal{R}}$. Moreover, if $a \in \mathfrak{A}_{\mathcal{R}}$ and $x \in \mathfrak{A}_0$, then $xa \in \mathfrak{A}_{\mathcal{R}}$. Hence, if $\mathfrak{A}_{\mathcal{R}}$ is *-invariant, then $(\mathfrak{A}_{\mathcal{R}}, \mathfrak{A}_0)$ is a quasi *-algebra.*

Proof. We show that $a \in \mathfrak{A}_{\mathcal{R}}$ and $x \in \mathfrak{A}_0$ imply $xa \in \mathfrak{A}_{\mathcal{R}}$. Indeed, if $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ then also $\omega_x \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, by (2.2). This implies that $a \in D(\overline{\varphi_{\omega_x}})$ or, equivalently, $xa \in D(\overline{\varphi_\omega})$. Since ω is arbitrary, the statement is proved. □

3. Order structure

Given a locally convex quasi *-algebra $(\mathfrak{A}[\tau], \mathfrak{A}_0)$, we recall the concept of a positive element of $\mathfrak{A}$ (see e.g., [12, pp. 21–22]). This notion defines an order on the set $\mathfrak{A}_h$ of all self-adjoint elements of $\mathfrak{A}$. The condition $\mathfrak{A}^+ \cap (-\mathfrak{A}^+) = \{0\}$, on the positive elements $\mathfrak{A}^+$ of $\mathfrak{A}$, implies that every non-zero element in $\mathfrak{A}^+$ gives rise to a non-trivial continuous positive linear functional on $\mathfrak{A}[\tau]$ (Theorem 3.2). The preceding condition is characterized in Proposition 3.8 and it itself, together with another condition that forces an element of $\mathfrak{A}$ to be positive, show that $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ attains enough (τ, t_w)-continuous *-representations to separate its points (Corollary 3.11).

Coming back to the given locally convex quasi *-algebra, set

$$\mathfrak{A}_0^+ := \left\{ \sum_{k=1}^n x_k^* x_k,\ x_k \in \mathfrak{A}_0,\ n \in \mathbb{N} \right\}.$$

Then $\mathfrak{A}_0^+$ is a wedge in $\mathfrak{A}_0$ and we call the elements of $\mathfrak{A}_0^+$ *positive elements of* $\mathfrak{A}_0$. We call *positive elements of* $\mathfrak{A}$ the members of $\overline{\mathfrak{A}_0^+}^{\tau}$ and we denote by $\mathfrak{A}^+$. That is, $\mathfrak{A}^+ := \overline{\mathfrak{A}_0^+}^{\tau}$.

The set $\mathfrak{A}^+$ is a *qm-admissible wedge* (generalization of *m-admissible wedge* given by Schmüdgen [12, p. 22]), in the following sense:

(1) $e \in \mathfrak{A}^+$, if $(\mathfrak{A}, \mathfrak{A}_0)$ has a unit e;
(2) $x + y \in \mathfrak{A}^+$, $\forall x, y \in \mathfrak{A}^+$;
(3) $\lambda x \in \mathfrak{A}^+$, $\forall x \in \mathfrak{A}^+$, $\lambda \geqslant 0$;
(4) $a^* x a \in \mathfrak{A}^+$, $\forall x \in \mathfrak{A}^+$, $a \in \mathfrak{A}_0$.

Clearly, $\mathfrak{A}^+$ defines an order on the real vector space $\mathfrak{A}_h = \{x \in \mathfrak{A}\colon\ x = x^*\}$ by $x \leqslant y \Leftrightarrow y - x \in \mathfrak{A}^+$. For $x \in \mathfrak{A}^+$, we shall often use the notation $x \geqslant 0$.

The following proposition is straightforward.

Proposition 3.1. *If $x \geqslant 0$, then $\pi(x) \geqslant 0$, for every (τ, t_w)-continuous *-representation of $(\mathfrak{A}[\tau], \mathfrak{A}_0)$.*

The theorem that follows, shows that if the set $\mathfrak{A}^+$ is "proper", then $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ attains non-trivial continuous positive linear functionals, in the sense of Definition 3.3, below.

Theorem 3.2. *Assume that $\mathfrak{A}^+ \cap (-\mathfrak{A}^+) = \{0\}$. Let $a \in \mathfrak{A}^+$, $a \neq 0$. Then there exists a continuous linear functional ω on $\mathfrak{A}$ with the properties*:

(i) $\omega(x) \geqslant 0$, $\forall x \in \mathfrak{A}^+$;
(ii) $\omega(a) > 0$.

Proof. Consider the real vector space $\mathfrak{A}_h$ and $a \in \mathfrak{A}^+ \setminus \{0\}$. The set $\{a\}$ is obviously convex and compact and does not intersect $(-\mathfrak{A}^+)$. Hence by [8, Ch. 2, §5, Proposition 4], there exists a closed hyperplane separating these two sets. Let $g(x) = 0$ be the equation of this hyperplane. Then, either $g(a) > 0$ and $g(-\mathfrak{A}^+) < 0$ (in which case we take $\omega = g$), or the contrary (in this case we take $\omega = -g$). □

Theorem 3.2 leads to the following

Definition 3.3. A linear functional ω on $\mathfrak{A}$ is called *positive* if $\omega(x) \geqslant 0$, $\forall x \in \mathfrak{A}^+$.

The next Proposition 3.4, provides conditions under which continuous linear functionals on $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ are positive and hermitian.

Proposition 3.4. *Assume that $\mathfrak{A}_{\mathcal{R}} = \mathfrak{A}$ and that $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ has a unit. Then, every continuous linear functional ω on $\mathfrak{A}$ such that $\omega(a^* a) \geqslant 0$, for every $a \in \mathfrak{A}_0$, is positive and hermitian on $\mathfrak{A}$.*

Proof. Since ω is positive on $\mathfrak{A}_0$, it is hermitian on $\mathfrak{A}_0$. Thus, by continuity of ω and continuity of the involution (see Definition 2.2) we are done. $\square$

Definition 3.5. (See [3, p. 5].) Let $\mathfrak{A}$ be a partial *-algebra. A positive sesquilinear form φ is called *invariant* and, for brevity, we shall say that φ is an *ips-form*, if there exists a subspace $B(\varphi)$ of $\mathfrak{A}$ (called a *core for* φ) with the following properties:

(i) $B(\varphi) \subset R\mathfrak{A}$;
(ii) $\lambda_\varphi(B(\varphi))$ is dense in $\mathcal{H}_\varphi$ (where $\mathcal{H}_\varphi$ is the completion of the pre-Hilbert space $\mathfrak{A}/N(\varphi)$, with $N(\varphi)$ and inner product defined as in the proof of Proposition 3.6, below);
(iii) $\varphi(ax, y) = \varphi(x, a^*y)$, $\forall a \in \mathfrak{A}$ and $x, y \in B(\varphi)$;
(iv) $\varphi(a^*x, by) = \varphi(x, (ab)y)$, $\forall a \in L(b)$ and $x, y \in B(\varphi)$.

In particular, an ips-form is an *everywhere defined biweight* in the sense of [2].
The following proposition provides an ips-form on $\mathfrak{A}$.

Proposition 3.6. *Assume that $\mathfrak{A}_{\mathcal{R}} = \mathfrak{A}$. Then, for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, $\overline{\varphi_\omega}$ is an ips-form on $\mathfrak{A}$ with core $\mathfrak{A}_0$.*

Proof. Let $N(\overline{\varphi_\omega}) = \{x \in \mathfrak{A}\colon \overline{\varphi_\omega}(x, x) = 0\}$ and $\mathcal{H}_{\overline{\varphi_\omega}}$ the Hilbert space obtained by completing $\mathfrak{A}/N(\overline{\varphi_\omega})$ with respect to the (well defined) inner product

$$\left\langle \lambda_{\overline{\varphi_\omega}}(x) | \lambda_{\overline{\varphi_\omega}}(y)\right\rangle = \overline{\varphi_\omega}(x, y), \quad x, y \in \mathfrak{A},$$

where $\lambda_{\overline{\varphi_\omega}}(x) := x + N(\overline{\varphi_\omega})$. According to Definition 3.5 and the definition of $\overline{\varphi_\omega}$'s, we need to show that $\lambda_{\overline{\varphi_\omega}}(\mathfrak{A}_0)$ is dense in $\mathcal{H}_{\overline{\varphi_\omega}}$. Assume, on the contrary, that there exists $x \in \mathfrak{A}$ such that $\langle \lambda_{\overline{\varphi_\omega}}(a) | \lambda_{\overline{\varphi_\omega}}(x)\rangle = 0$ for every $a \in \mathfrak{A}_0$. Since $x \in \bigcap_{\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)} D(\overline{\varphi_\omega}) = \mathfrak{A}$, there exists a net $\{x_\delta\} \subset \mathfrak{A}_0$ such that $x_\delta \xrightarrow{\tau} x$ and $\overline{\varphi_\omega}(x_\delta - x, x_\delta - x) \to 0$. Hence,

$$\overline{\varphi_\omega}(x, x) = \lim_\delta \overline{\varphi_\omega}(x_\delta, x_\delta) = 0.$$

Consequently $\lambda_{\overline{\varphi_\omega}}(x) = 0$. The proof in now complete according to [18, Proposition 2.2]. $\square$

Definition 3.7. A family of positive linear functionals $\mathfrak{F}$ on $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ is called *sufficient* if for every $x \in \mathfrak{A}^+$, $x \neq 0$, there exists $\omega \in \mathfrak{F}$ such that $\omega(x) > 0$.

Proposition 3.8. *Let $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ be a locally convex quasi *-algebra. The following statements are equivalent*:

(i) $\mathfrak{A}^+ \cap (-\mathfrak{A}^+) = \{0\}$.
(ii) $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ *is sufficient.*

Proof. (i) $\Rightarrow$ (ii) This is Theorem 3.2.
(ii) $\Rightarrow$ (i) Let $x \in \mathfrak{A}^+ \cap (-\mathfrak{A}^+)$ and $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$. Then ω is a continuous positive linear functional on $\mathfrak{A}$, therefore $\omega(x) \geqslant 0$ and $\omega(-x) = -\omega(x) \geqslant 0$. Thus $\omega(x) = 0$. Since ω is arbitrary, we finally get $x = 0$. $\square$

It is clear from Theorem 3.2 and Definition 3.7 that if $\mathfrak{A}^+ \cap (-\mathfrak{A}^+) = \{0\}$, then *the family of all continuous positive linear functionals on* $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ *is sufficient*.

Proposition 3.9. *Let $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ be a locally convex quasi *-algebra with $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ sufficient.*
Assume that the following condition (P) *holds*:

(P) *$y \in \mathfrak{A}$ and $\omega(a^*ya) \geqslant 0$, for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ and $a \in \mathfrak{A}_0$, imply $y \in \mathfrak{A}^+$.*

Then, for an element $x \in \mathfrak{A}$, the following statements are equivalent:

(i) $x \in \mathfrak{A}^+$;
(ii) $\omega(x) \geqslant 0$, *for every* $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$;
(iii) $\pi(x) \geqslant 0$, *for every (τ, t_w)-continuous *-representation π of $(\mathfrak{A}[\tau], \mathfrak{A}_0)$.*

Proof. (i) $\Rightarrow$ (ii) is an easy consequence of the definition of positive elements and the continuity of the elements of $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ with respect to τ.

(ii) $\Rightarrow$ (iii) Let π be a (τ, t_w)-continuous *-representation of $(\mathfrak{A}[\tau], \mathfrak{A}_0)$. Define $\omega_\xi(x) := \langle \pi(x)\xi|\xi\rangle$, $x \in \mathfrak{A}$, with $\xi \in \mathcal{D}$, $\|\xi\| = 1$. Then $\omega_\xi \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, since

$$\left|\omega_\xi(x)\right| = \left|\left\langle \pi(x)\xi|\xi\right\rangle\right| \leqslant p(x), \quad \forall x \in \mathfrak{A},$$

for some τ-continuous seminorm p on $\mathfrak{A}$. Thus, if x satisfies (ii), $\langle \pi(x)\xi|\xi\rangle \geqslant 0$, for every $\xi \in \mathcal{D}$, which proves (iii).

(iii) $\Rightarrow$ (i) Let $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ and let π_ω be the corresponding GNS representation. Then, π_ω is (τ, t_w)-continuous. Indeed, due to the continuity of ω and that of multiplication, we get (see also (2.1))

$$\left|\left\langle \pi_\omega(x)\lambda_\omega(a)|\lambda_\omega(b)\right\rangle\right| = \left|\omega\left(b^*xa\right)\right| \leqslant p(x), \quad \forall x \in \mathfrak{A},\ a, b \in \mathfrak{A}_0,$$

for some τ-continuous seminorm p on $\mathfrak{A}$. Applying (iii) we have $\pi_\omega(x) \geqslant 0$. This implies that $\omega(a^*xa) \geqslant 0$, for every $a \in \mathfrak{A}_0$. The statement now follows from the assumption (P). $\square$

Remark 3.10. (1) If $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ has a unit, then the equivalence of (ii) and (iii) does not depend on (P). In this case, (P) is equivalent to the following

(P′) If $y \in \mathfrak{A}$ and $\omega(y) \geqslant 0$, for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, then $y \in \mathfrak{A}^+$.

Indeed, since we have unit (P) implies (P′). On the other hand, by (2.2) for any $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ and $a \in \mathfrak{A}_0$ we have that $\omega_a \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, where $\omega_a(y) := \omega(a^*ya)$, $y \in \mathfrak{A}$, so that (P′) implies (P).

(2) The condition (P) together with $\mathfrak{A}^+ \cap (-\mathfrak{A}^+) = \{0\}$ implies that for every $0 \neq x \in \mathfrak{A}$ there exists $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ such that $\omega(x) \neq 0$. Indeed, if $\omega(x) = 0$, for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, then (Proposition 3.9) $x \in \mathfrak{A}^+$ and $-x \in \mathfrak{A}^+$; hence $x = 0$, a contradiction.

From Remark 3.10(2) and the proof of Proposition 3.9 we now have the following

Corollary 3.11. *Let $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ be a locally convex quasi *-algebra with a unit e. Suppose that $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ is sufficient and that condition* (P) *of Proposition* 3.9 *is fulfilled. Then for every $0 \neq x \in \mathfrak{A}$, there is a (τ, t_w)-continuous *-representation π of $(\mathfrak{A}[\tau], \mathfrak{A}_0)$, namely $\pi = \pi_\omega$, $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, such that $\pi(x) \neq 0$.*

4. Fully representable quasi *-algebras

In Section 3, given a locally convex quasi *-algebra $(\mathfrak{A}[\tau], \mathfrak{A}_0)$, we have seen that the sufficiency of the set $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ and the condition $\mathfrak{A}_\mathcal{R} = \mathfrak{A}$ equip the given algebra with important properties (cf., for instance, Theorem 3.2, Proposition 3.4 and Corollary 3.11) that are very close to the properties that C^*-algebras enjoy and offer to them their rich structure. All these lead us to the following

Definition 4.1. A locally convex quasi *-algebra $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ is called *fully representable* if $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ is sufficient and $\mathfrak{A}_\mathcal{R} = \mathfrak{A}$.

Example 4.2. Let $I = [0, 1]$. Consider the normed quasi *-algebra $(L^p(I), L^\infty(I))$. Then every $\omega \in \mathcal{R}_c(L^p(I), L^\infty(I))$ has the form

$$\omega(f) = \int_0^1 f(x)w(x)\,dx, \quad f \in L^p(I),$$

with $w \in L^{\bar{p}}(I)$, $w \geqslant 0$, $\frac{1}{p} + \frac{1}{\bar{p}} = 1$. One readily checks that ω satisfies (L1) and (L2). It is easily seen that, if $p \geqslant 2$ and $w \in L^{p/p-2}(I)$, then (L3) is fulfilled. Conversely, assume that (L3) is satisfied; i.e., for every $f \in L^p(I)$ there exists $\gamma > 0$ such that

$$\left|\omega\left(f^*\alpha\right)\right| = \left|\int_0^1 \overline{f(x)}\alpha(x)w(x)\,dx\right| \leqslant \gamma\left(\int_0^1 \left|\alpha(x)\right|^2 w(x)\,dx\right)^{1/2}, \quad \forall \alpha \in L^\infty(I).$$

This implies that $f \in L^2(I, w\,dx)$ and $\gamma = \|f\|_{2,w}$. Hence, in order that (L3) be satisfied for every $f \in L^p(I)$, we must have $w \in L^{p/p-2}(I)$. Hence, if $p \geqslant 1$, ω is representable if, and only if, $w \in L^{p/p-2}(I)$. If $w \in L^{p/p-1}(I) \setminus L^{p/p-2}(I)$, then ω is continuous but not representable.

If $1 \leqslant p < 2$, the condition $L^p(I) \subset L^2(I, w\,dx)$ is not satisfied for every non-zero w. In this case there are no continuous representable functionals.

Let us now come back to the case $p \geqslant 2$. Let $w \in L^{p/p-2}(I)$. We want to determine $\overline{\varphi_\omega}$, where

$$\varphi_\omega(\alpha, \beta) = \int_0^1 \alpha(x)\overline{\beta(x)}w(x)\,dx, \quad \alpha, \beta \in L^\infty(I).$$

Let $f \in D(\overline{\varphi_\omega})$. Then there exists $\{\alpha_n\} \subset L^\infty(I)$ such that

$$\alpha_n \xrightarrow{p} f \quad \text{and} \quad \int_0^1 \left|\alpha_n(x) - \alpha_m(x)\right|^2 w(x)\,dx \to 0.$$

Hence there exists $v \in L^2(I, w\,dx)$ such that $\alpha_n \to v$ in the $L^2(I, w\,dx)$-norm. This, in turn, implies that $f = v$ almost everywhere. Thus, $D(\overline{\varphi_\omega}) = L^p(I) \cap L^2(I, w\,dx)$. Therefore,

$$L^p(I)_{\mathcal{R}} = \bigcap_{\omega \in \mathcal{R}_c(L^p, L^\infty)} D(\overline{\varphi_\omega}) = L^p(I) \cap \left(\bigcap_{w \in L^{p/p-2}(I)} L^2(I, w\,dx) \right).$$

But $f \in L^2(I, w\,dx)$, for every $w \in L^{p/p-2}(I)$ if and only if $f \in L^p(I)$.

In conclusion, for $p \geqslant 2$, $L^p(I)_{\mathcal{R}} = L^p(I)$ and $(L^p(I), L^\infty(I))$ is fully representable.

Example 4.3. The space $\mathcal{S}'(\mathbb{R})$ of tempered distributions may be regarded as a locally convex quasi *-algebra over the *-algebra $\mathcal{S}(\mathbb{R})$. $\mathcal{S}'(\mathbb{R})$ is the dual of $\mathcal{S}(\mathbb{R})$ when the latter is endowed with the locally convex topology t defined by the family of seminorms

$$p_{k,r}(f) = \sup_{x \in \mathbb{R}} \left| x^k D^r f(x) \right|, \quad f \in \mathcal{S}(\mathbb{R});\ k, r \in \mathbb{N}.$$

The (partial) multiplication in $\mathcal{S}'(\mathbb{R})$ is defined by

$$(F \cdot f)(g) = (f \cdot F)(g) = F(fg), \quad F \in \mathcal{S}'(\mathbb{R}),\ f, g \in \mathcal{S}(\mathbb{R}).$$

The space $\mathcal{S}'(\mathbb{R})$ is endowed with the strong dual topology t$'$. Since $\mathcal{S}'(\mathbb{R})[\mathrm{t}']$ is reflexive, every continuous functional ω on $\mathcal{S}'(\mathbb{R})[\mathrm{t}']$ has the form $\omega(F) = \omega_f(F) := F(f)$, for some $f \in \mathcal{S}(\mathbb{R})$. Also in this case there are no non-trivial continuous representable functionals on $\mathcal{S}'(\mathbb{R})$. Indeed, (L3) is never satisfied by non-zero positive functionals ω_f, $f \geqslant 0$, since, if for every $F \in \mathcal{S}'(\mathbb{R})$ there exists $\gamma_F > 0$ such that

$$\omega_f\left(F^* \cdot g\right) \leqslant \gamma_F \omega_f\left(g^* g\right)^{1/2}, \quad \forall g \in \mathcal{S}(\mathbb{R}),$$

then

$$\left|F^*(gf)\right| \leqslant \gamma_F \left(\int_{\mathbb{R}} g^*(x) g(x) f(x)\,dx \right)^{1/2} = \gamma_F \|g\|_{2,f},$$

where $\|\cdot\|_{2,f}$ denotes the norm of $L^2(\mathbb{R}, f\,dx)$. This implies that there exists $h \in L^2(\mathbb{R}, f\,dx)$ such that

$$F^*(gf) = \int_{\mathbb{R}} h(x)\overline{g(x)} f(x)\,dx, \quad \forall g \in \mathcal{S}(\mathbb{R}).$$

Hence F restricted to the linear subspace $\{gf\colon\ g \in \mathcal{S}(\mathbb{R})\}$ acts as a function. This is a contradiction if f (and then ω_f) is non-zero.

Example 4.4 *(Quasi *-algebras of operators).* Let $(\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}), \mathcal{L}^\dagger(\mathcal{D})_b)$ be the locally convex quasi *-algebra of Example 2.3. Let $\xi \in \mathcal{D}$. Then the positive linear functional

$$\omega_\xi(X) = \langle X\xi | \xi \rangle, \quad X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$$

is representable. The corresponding sesquilinear form φ_{ω_ξ} on $\mathcal{L}^\dagger(\mathcal{D})_b \times \mathcal{L}^\dagger(\mathcal{D})_b$ is jointly continuous with respect to the topology τ_{s^*}, so that $D(\overline{\varphi_{\omega_\xi}}) = \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$.

The same is true in the more general case where

$$\omega(X) = \sum_{i=1}^n \langle X\xi_i | \xi_i \rangle, \quad \xi_i \in \mathcal{D},\ i = 1, \ldots, n.$$

Let us now consider $(\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}), \mathcal{L}^\dagger(\mathcal{D})_b)$, where $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is endowed with the weak topology t_w. Then, the following statements hold [3,19]:

(i) Every weakly continuous (or strongly *-continuous) linear functional ω on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ has the form

$$\omega(X)=\sum_{i=1}^{n}\langle X\xi_i|\eta_i\rangle,\quad \xi_i,\eta_i\in\mathcal{D},\ i=1,\ldots,n. \tag{4.1}$$

(ii) Every weakly continuous positive linear functional ω on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ has the form

$$\omega(X)=\sum_{i=1}^{n}\langle X\zeta_i|\zeta_i\rangle,\quad \zeta_i\in\mathcal{D},\ i=1,\ldots,n. \tag{4.2}$$

Note that for both the locally convex quasi *-algebras $(\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})[\mathrm{t}_{s^*}],\mathcal{L}^\dagger(\mathcal{D})_b)$ and $(\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})[\mathrm{t}_w],\mathcal{L}^\dagger(\mathcal{D})_b)$, the equality $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})_{\mathcal{R}}=\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ holds, and therefore both $(\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})[\mathrm{t}_{s^*}],\mathcal{L}^\dagger(\mathcal{D})_b)$ and $(\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})[\mathrm{t}_w],\mathcal{L}^\dagger(\mathcal{D})_b)$ are fully representable.

Indeed, let ω be t_{s^*}-continuous and representable. If $X\in\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, there exists a net $\{X_\delta\}$ of elements of $\mathcal{L}^\dagger(\mathcal{D})_b$ such that $X_\delta\xrightarrow{\mathrm{t}_{s^*}}X$. Then, using the representation (4.2) we have,

$$\begin{aligned}\varphi_\omega(X_\delta-X_\gamma,X_\delta-X_\gamma)&=\omega\big((X_\delta-X_\gamma)^*(X_\delta-X_\gamma)\big)\\&=\sum_{i=1}^{n}\big\langle(X_\delta-X_\gamma)\zeta_i|(X_\delta-X_\gamma)\zeta_i\big\rangle\\&=\sum_{i=1}^{n}\big\|(X_\delta-X_\gamma)\zeta_i\big\|^2\to 0.\end{aligned}$$

This proves that every $X\in\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is in the domain of the closure of φ_ω with respect to the topology t_{s^*}. The statement for the weak topology follows by observing that if ω is weakly continuous, then it is automatically t_{s^*}-continuous.

Note that in Examples 4.2 and 4.4 the condition (P) is satisfied, while for Example 4.3 it is meaningless. We do not know whether (P) holds in any case when $\mathcal{R}_c(\mathfrak{A},\mathfrak{A}_0)$ is sufficient.

5. Bounded elements

The concept of a bounded element in a locally convex algebra was first introduced by G.R. Allan (1965), in [1], for building a spectral theory for this kind of algebras. Bounded elements in the context of quasi *-algebras and partial *-algebras have been considered by the second named author and jointly with some of his collaborators, for studying the structure of these algebras (see, for instance, [3,15,17]). K. Schmüdgen has considered Allan's bounded elements in his research on the unbounded operator algebras called O^*-algebras and recently (2005), the same author considered bounded elements in a purely algebraic sense (see also [21]) and studied the structure of the set of the introduced bounded elements, in order to use them for proving "A strict Positivstellensatz for the Weyl algebra" [13]. Motivated from this, and having in hands the results of Section 3, we introduce the concept of "order boundedness" and what is interesting, is that this concept coincides under some conditions with the usual notion of boundedness (see, e.g., [3]), which one gets when the *-algebra under consideration admits *-representations (see, for instance, Proposition 5.4, Theorem 5.5, and Corollary 5.8). Furthermore, for suitable fully representable locally convex quasi *-algebras $(\mathfrak{A}[\tau],\mathfrak{A}_0)$, considering the set $\mathfrak{A}_b$ of all order bounded elements of $\mathfrak{A}[\tau]$, we prove that $\mathfrak{A}_b$ becomes either a partial C^*-algebra or a C^*-algebra (Theorem 5.16).

Let $(\mathfrak{A}[\tau],\mathfrak{A}_0)$ be an arbitrary locally convex quasi *-algebra. As we have seen in Section 3, $(\mathfrak{A}[\tau],\mathfrak{A}_0)$ has a natural order related to the topology τ. This order can be used to define *bounded* elements. In what follows, we will assume that $(\mathfrak{A},\mathfrak{A}_0)$ has a unit e.

Let $x\in\mathfrak{A}$; put $\Re(x)=\frac{1}{2}(x+x^*)$, $\Im(x)=\frac{1}{2i}(x-x^*)$. Then $\Re(x),\Im(x)\in\mathfrak{A}_h$ and $x=\Re(x)+i\Im(x)$.

Definition 5.1. An element $x\in\mathfrak{A}$ is called *order bounded* if there exists $\gamma\geqslant 0$ such that

$$\pm\Re(x)\leqslant\gamma e,\qquad \pm\Im(x)\leqslant\gamma e.$$

We denote by $\mathfrak{A}_b$ the set of all order bounded elements of $\mathfrak{A}[\tau]$.

In this regard, we have

Proposition 5.2. *The following statements hold*:

(1) $\alpha x + \beta y \in \mathfrak{A}_b$, $\forall x, y \in \mathfrak{A}_b$, $\alpha, \beta \in \mathbb{C}$.
(2) $x \in \mathfrak{A}_b \Leftrightarrow x^* \in \mathfrak{A}_b$.
(3) $x \in \mathfrak{A}_b$, $a \in \mathfrak{A}_b \cap \mathfrak{A}_0 \Rightarrow xa \in \mathfrak{A}_b$.
(4) $a \in \mathfrak{A}_b \cap \mathfrak{A}_0 \Leftrightarrow aa^* \in \mathfrak{A}_b \cap \mathfrak{A}_0$.

Hence, $(\mathfrak{A}_b, \mathfrak{A}_b \cap \mathfrak{A}_0)$ *is a quasi* *-*algebra.*

Proof. The proof is similar to that of [13, Lemma 2.1]. □

For $x \in (\mathfrak{A}_b)_h$, put

$$\|x\|_b := \inf\{\gamma > 0\colon -\gamma e \leqslant x \leqslant \gamma e\}.$$

Then $\|\cdot\|_b$ is a seminorm on the real vector space $(\mathfrak{A}_b)_h$.

Lemma 5.3. *If* $\mathfrak{A} \cap (-\mathfrak{A}^+) = \{0\}$, $\|\cdot\|_b$ *is a norm on* $(\mathfrak{A}_b)_h$.

Proof. Put $E = \{\gamma > 0\colon -\gamma e \leqslant x \leqslant \gamma e\}$. If $\inf E = 0$, then for every $\varepsilon > 0$ there exists $\gamma_\varepsilon \in E$ such that $\gamma_\varepsilon < \varepsilon$. This implies that $-\varepsilon e \leqslant x \leqslant \varepsilon e$. If $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, we get $-\varepsilon\omega(e) \leqslant \omega(x) \leqslant \varepsilon\omega(e)$ (we may suppose $\omega(e) > 0$, for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, since from (L3) of Proposition 2.5, if $\omega(e) = 0$, then $\omega \equiv 0$). Hence, $\omega(x) = 0$. By the sufficiency of $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ (Definition 3.7), it follows that $x = 0$. □

Proposition 5.4. *If* $x \in \mathfrak{A}_b$, *then* $\pi(x)$ *is a bounded operator, for every* (τ, t_w)-*continuous* *-*representation* π *of* $(\mathfrak{A}, \mathfrak{A}_0)$. *Moreover, if* $x = x^*$, *then* $\|\pi(x)\| \leqslant \|x\|_b$.

Proof. It follows easily from Proposition 3.1 and the very definitions. □

Theorem 5.5. *Let* $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ *be fully representable and assume that condition* (P) *holds. Then for* $x \in \mathfrak{A}$, *the following statements are equivalent*:

(i) *x is order bounded.*
(ii) *There exists* $\gamma_x > 0$ *such that*

$$\bigl|\omega\bigl(a^*xa\bigr)\bigr| \leqslant \gamma_x\omega\bigl(a^*a\bigr), \quad \forall\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0),\ a \in \mathfrak{A}_0.$$

(iii) *There exists* $\gamma_x > 0$ *such that*

$$\bigl|\omega\bigl(b^*xa\bigr)\bigr| \leqslant \gamma_x\omega\bigl(a^*a\bigr)^{1/2}\omega\bigl(b^*b\bigr)^{1/2}, \quad \forall\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0),\ a, b \in \mathfrak{A}_0.$$

Proof. It is sufficient to consider the case $x = x^*$. Also, as in the proof of Lemma 5.3, we suppose $\omega(e) > 0$, for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$.

(i) $\Rightarrow$ (ii) If $x = x^*$ is bounded, there exists $\gamma > 0$ such that $-\gamma e \leqslant x \leqslant \gamma e$. Hence from Proposition 3.9, for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, $\omega(\gamma e - x) \geqslant 0$. It follows that $\omega(a^*(\gamma e - x)a) \geqslant 0$. Thus, $\omega(a^*xa) \leqslant \gamma\omega(a^*a)$, for every $a \in \mathfrak{A}_0$. Similarly we can show that $-\gamma\omega(a^*a) \leqslant \omega(a^*xa)$, for every $a \in \mathfrak{A}_0$.

(ii) $\Rightarrow$ (i) Assume now that there exists $\gamma_x > 0$ such that

$$\bigl|\omega\bigl(a^*xa\bigr)\bigr| \leqslant \gamma_x\omega\bigl(a^*a\bigr), \quad \forall\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0),\ a \in \mathfrak{A}_0.$$

Define

$$\tilde{\gamma} := \sup\bigl\{\bigl|\omega\bigl(a^*xa\bigr)\bigr|\colon \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0),\ a \in \mathfrak{A}_0,\ \omega\bigl(a^*a\bigr) = 1\bigr\}.$$

Then (see Proposition 3.9 and Remark 3.10(1)), for an arbitrary $\omega' \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, we get,

$$\omega'(\tilde{\gamma} e \pm x) = \tilde{\gamma}\omega'(e) \pm \omega'(x) = \omega'(e)\bigl(\tilde{\gamma} \pm \omega'\bigl(u^*xu\bigr)\bigr) \geqslant 0,$$

where $u = \frac{e}{\omega'(e)^{1/2}}$.

Hence, $\omega'(\tilde{\gamma} e \pm x) \geqslant 0$, for every $\omega' \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$. Then, by Remark 3.10(1), $-\tilde{\gamma} e \leqslant x \leqslant \tilde{\gamma} e$; i.e., x is order bounded.

(i) $\Rightarrow$ (iii) The GNS representation π_ω is (τ, t_w)-continuous, hence if $x = x^* \in \mathfrak{A}$, by Proposition 5.4, $\pi_\omega(x)$ is bounded. Thus,

$$\begin{aligned} \left|\omega\left(b^* x a\right)\right| &= \left|\left\langle \pi_\omega(x)\lambda_\omega(a) | \lambda_\omega(b)\right\rangle\right| \leqslant \left\|\pi_\omega(x)\right\| \left\|\lambda_\omega(a)\right\| \left\|\lambda_\omega(b)\right\| \\ &\leqslant \|x\|_b \omega\left(a^*a\right)^{1/2} \omega\left(b^*b\right)^{1/2}, \quad \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0),\ a, b \in \mathfrak{A}_0. \end{aligned}$$

(iii) $\Rightarrow$ (ii) is obvious. □

Remark 5.6. The proof above shows that for $x = x^*$,

$$\|x\|_b \leqslant \sup\left\{\left|\omega\left(a^* x a\right)\right|:\ \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0),\ a \in \mathfrak{A}_0,\ \omega\left(a^*a\right) = 1\right\}.$$

Corollary 5.7. *Let $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ be fully representable. If x is order bounded, there exists $\gamma_x > 0$ such that*

$$\left|\overline{\varphi_\omega}(xa, z)\right| \leqslant \gamma_x \omega\left(a^*a\right)^{1/2} \overline{\varphi_\omega}(z, z)^{1/2}, \quad \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0),\ a \in \mathfrak{A}_0,\ z \in \mathfrak{A}.$$

Proof. Let $x \in \mathfrak{A}$ be order bounded. Since $\bigcap_{\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)} D(\overline{\varphi_\omega}) = \mathfrak{A}$, for every $z \in \mathfrak{A}$ there exists a net $\{z_\delta\} \subset \mathfrak{A}_0$ such that $z_\delta \xrightarrow{\tau} z$ and $\overline{\varphi_\omega}(z - z_\delta, z - z_\delta) \to 0$. Then, by Theorem 5.5(iii), we get

$$\begin{aligned} \left|\overline{\varphi_\omega}(xa, z)\right| &= \lim_\delta \left|\overline{\varphi_\omega}(xa, z_\delta)\right| \leqslant \gamma_x \omega\left(a^*a\right)^{1/2} \lim_\delta \overline{\varphi_\omega}(z_\delta, z_\delta)^{1/2} \\ &= \gamma_x \omega\left(a^*a\right)^{1/2} \overline{\varphi_\omega}(z, z)^{1/2}, \quad \forall a \in \mathfrak{A}_0. \qquad \square \end{aligned}$$

Corollary 5.8. *Let $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ be fully representable and $x \in \mathfrak{A}$. Then, x is order bounded if, and only if, there exists $\gamma_x > 0$ such that*

$$\overline{\varphi_\omega}(xa, xa) \leqslant \gamma_x^2 \omega\left(a^*a\right), \quad \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0),\ a \in \mathfrak{A}_0.$$

Proof. The necessity follows by putting $z = xa$ in the inequality of Corollary 5.7. The sufficiency is clear. □

Let x be order bounded. Define

$$\mathsf{q}(x) := \sup\left\{\left|\omega\left(b^* x a\right)\right|:\ \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0),\ a, b \in \mathfrak{A}_0;\ \omega\left(a^*a\right) = \omega\left(b^*b\right) = 1\right\}.$$

Then we have

Lemma 5.9. $\mathsf{q}(x) = \|x\|_b$, *for every $x = x^* \in \mathfrak{A}_b$.*

Proof. The inequality $\|x\|_b \leqslant \mathsf{q}(x)$ follows from Remark 5.6. Let $\gamma > 0$ such that $-\gamma e \leqslant x \leqslant \gamma e$. Then by the proof of Theorem 5.5, we have, $\mathsf{q}(x) \leqslant \gamma$; whence the statement follows. □

Lemma 5.9 shows that q extends $\|\cdot\|_b$. For this reason, we will use the symbol $\|\cdot\|_b$ for q too. It is easily seen that $\|\cdot\|_b$ is a norm on $\mathfrak{A}_b$.

An easy consequence of the above statements is now the following

Proposition 5.10. *For every $x \in \mathfrak{A}_b$,*

$$\|x\|_b = \sup\left\{\overline{\varphi_\omega}(xa, xa):\ \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0):\ a \in \mathfrak{A}_0,\ \omega\left(a^*a\right) = 1\right\}.$$

5.1. Partial multiplication

If $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ is fully representable we can introduce on $\mathfrak{A}$ a partial multiplication which makes it into a partial *-algebra.

Definition 5.11. Let $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ be fully representable. The weak product $x \,\Box\, y$ of two elements $x, y \in \mathfrak{A}$ is well defined if there exists $z \in \mathfrak{A}$ such that

$$\overline{\varphi_\omega}\left(ya, x^*b\right) = \overline{\varphi_\omega}(za, b), \quad \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0),\ \forall a, b \in \mathfrak{A}_0.$$

In this case we put $x \,\Box\, y := z$.

Since $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ is sufficient, the element z is unique.

Proposition 5.12. $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ *endowed with the weak multiplication $\Box$ is a partial *-algebra with $\mathfrak{A}_0 \subset R\mathfrak{A}$.*

Proposition 5.13. *Let x, y be order bounded elements of $\mathfrak{A}$. The following statements hold*:

(i) *x^* is order bounded too, and $\|x^*\|_b = \|x\|_b$;*
(ii) *If $x \,\square\, y$ is well defined, then $x \,\square\, y$ is order bounded and*

$$\|x \,\square\, y\|_b \leqslant \|x\|_b \|y\|_b.$$

Proof. (i) The first part of (i) is given by Proposition 5.2(2). The second part follows from the property (L2) (Proposition 2.5) of every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, by Corollary 5.8 and the definition of $\|\cdot\|_b$.

(ii) If $x \,\square\, y$, $x, y \in \mathfrak{A}$, is well defined, then for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, Corollary 5.7 implies

$$\begin{aligned}\left|\overline{\varphi_\omega}\big((x \,\square\, y)a, b\big)\right| &= \left|\overline{\varphi_\omega}\big(ya, x^*b\big)\right| \leqslant \overline{\varphi_\omega}(ya, ya)^{1/2}\overline{\varphi_\omega}\big(x^*b, x^*b\big)^{1/2} \\ &\leqslant \|x\|_b, \|y\|_b \overline{\varphi_\omega}(a, a)^{1/2}\overline{\varphi_\omega}(b, b)^{1/2}, \quad \forall a, b \in \mathfrak{A}_0.\end{aligned}$$

Taking now sup on the left-hand side (see Proposition 5.10), we get the desired inequality. □

We recall that an *unbounded C^*-seminorm p* on a partial *-algebra $\mathfrak{A}$ is a seminorm defined on a partial *-subalgebra $D(p) \subseteq \mathfrak{A}$, the domain of p, with the properties:

- $p(x \cdot y) \leqslant p(x)p(y)$ whenever $x \cdot y$ is well defined;
- $p(x^* \cdot x) = p(x)^2$, whenever $x^* \cdot x$ is well defined

(see, e.g., [7,16,2]).

Proposition 5.14. *$\|\cdot\|_b$ is an unbounded C^*-norm on $\mathfrak{A}$ with domain $\mathfrak{A}_b$.*

Proof. This can be deduced from [20, Proposition 2.6]. □

It is worth of mentioning here that certain unbounded C^*-seminorms give rise to “well-behaved” (unbounded) *-representations (for more details, see [2, Chapter 8] and [7,16]).

Now having $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ to be fully representable, we can endow $\mathfrak{A}$ with the strong and strong* topology, where both are defined in a natural way through the elements of $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$. Indeed:

- The *strong* topology τ_s, is defined by the family of seminorms

$$x \in \mathfrak{A} \to \overline{\varphi_\omega}(x, x)^{1/2}, \quad \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0).$$

- The *strong** topology τ_{s^*}, is defined by the family of seminorms

$$x \in \mathfrak{A} \to \max\left\{\overline{\varphi_\omega}(x, x)^{1/2}, \ \overline{\varphi_\omega}\big(x^*, x^*\big)^{1/2}\right\}, \quad \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0).$$

The sufficiency of $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ guarantees that these topologies are Hausdorff. Thus, $\mathfrak{A}[\tau_s]$, $\mathfrak{A}[\tau_{s^*}]$ are locally convex spaces.

Definition 5.15. Let $\mathfrak{A}$ be a partial *-algebra. We say that $\mathfrak{A}$ is a partial C^*-algebra if $\mathfrak{A}$ is a Banach space under a norm $\|\cdot\|$ satisfying the following properties:

(i) $\|x^*\| = \|x\|$, $\forall x \in \mathfrak{A}$;
(ii) $\|x \cdot y\| \leqslant \|x\|\|y\|$, whenever $x \cdot y$ is well defined;
(iii) $\|x^* \cdot x\| = \|x\|^2$, whenever $x^* \cdot x$ is well defined.

The theorem that follows, shows that the quasi *-algebra $(\mathfrak{A}_b, \mathfrak{A}_b \cap \mathfrak{A}_0)$ (see Proposition 5.2), under certain conditions achieves a very rich structure.

Theorem 5.16. *Let $(\mathfrak{A}[\tau], \mathfrak{A}_0)$ be a fully representable locally convex quasi *-algebra with unit e. Assume that $\mathfrak{A}$ is τ_{s^*}-complete. Then $\mathfrak{A}_b$ is a partial C^*-algebra with the weak multiplication $\square$ and the norm $\|\cdot\|_b$.*

Assume, in addition, that

(R) *If $x, y \in \mathfrak{A}$ and $\pi_\omega(x) \,\square\, \pi_\omega(y)$ is well defined for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, then there exists $z \in \mathfrak{A}$ such that $\pi_\omega(x) \,\square\, \pi_\omega(y) = \pi_\omega(z)$, for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$.*

Then $\mathfrak{A}_b$ is a C^*-algebra with the weak multiplication $\square$ and the norm $\|\cdot\|_b$.

Proof. Since $\|\cdot\|_b$ satisfies (i)–(iii) of Definition 5.15 on $\mathfrak{A}_b$ (see e.g., Proposition 5.13), we need only to prove the completeness of $\mathfrak{A}_b$.

Let $\{x_n\}$ be a Cauchy sequence with respect to the norm $\|\cdot\|_b$. Then $\{x_n^*\}$ is Cauchy too. Hence, for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ and $a \in \mathfrak{A}_0$ we have

$$\overline{\varphi_\omega}\big((x_n - x_m)a, (x_n - x_m)a\big) \to 0, \quad \text{as } n, m \to \infty$$

and

$$\overline{\varphi_\omega}\big(\big(x_n^* - x_m^*\big)a, \big(x_n^* - x_m^*\big)a\big) \to 0, \quad \text{as } n, m \to \infty.$$

Therefore, $\{x_n\}$ is also Cauchy with respect to τ_{s^*}. Then, since $\mathfrak{A}$ is τ_{s^*}-complete, there exists $x \in \mathfrak{A}$ such that $x_n \xrightarrow{\tau_{s^*}} x$. Moreover,

$$\overline{\varphi_\omega}(xa, xa) = \lim_{n\to\infty} \overline{\varphi_\omega}(x_n a, x_n a) \leqslant \limsup_{n\to\infty} \|x_n\|_b^2 \overline{\varphi_\omega}(a, a), \quad \forall a \in \mathfrak{A}_0,$$

with $\limsup_{n\to\infty} \|x_n\|_b^2 < \infty$ (by the boundedness of the sequence $\{\|x_n\|_b\}$), so by Corollary 5.8, we conclude that x is order bounded. Finally, by the Cauchy condition, for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that, for every $n, m > n_\varepsilon$, $\|x_n - x_m\|_b < \varepsilon$. This implies that

$$\overline{\varphi_\omega}\big((x_n - x_m)a, (x_n - x_m)a\big) < \varepsilon \overline{\varphi_\omega}(a, a), \quad \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0),\ a \in \mathfrak{A}_0.$$

Then if we fix $n > n_\varepsilon$ and let $m \to \infty$, we obtain

$$\overline{\varphi_\omega}\big((x_n - x)a, (x_n - x)a\big) \leqslant \varepsilon \overline{\varphi_\omega}(a, a), \quad \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0),\ a \in \mathfrak{A}_0.$$

This, in turn, implies that $\|x_n - x\|_b \leqslant \varepsilon$, for $n \geqslant n_\varepsilon$. So completeness of $\mathfrak{A}_b[\|\cdot\|_b]$ is proved.

Now, assume that condition (R) holds. By Proposition 5.4 it follows that if $x, y \in \mathfrak{A}_b$, then the operators $\pi_\omega(x)$, $\pi_\omega(y)$ are bounded, therefore, $\pi_\omega(x) \,\square\, \pi_\omega(y)$ is well defined, hence (Proposition 5.13) bounded. Thus, by (R), there exists $z \in \mathfrak{A}$ such that $\pi_\omega(x) \,\square\, \pi_\omega(y) = \pi_\omega(z)$, for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$. Now, for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ and $a, b \in \mathfrak{A}_0$, we have

$$\begin{aligned}
\overline{\varphi_\omega}\big(ya, x^*b\big) &= \big\langle \pi_\omega(y)\lambda_\omega(a) | \pi_\varphi\big(x^*\big)\lambda_\omega(b)\big\rangle \\
&= \big\langle \pi_\omega(x) \,\square\, \pi_\omega(y)\lambda_\omega(a) | \lambda_\omega(b)\big\rangle \\
&= \big\langle \pi_\omega(z)\lambda_\omega(a) | \lambda_\omega(b)\big\rangle \\
&= \overline{\varphi_\omega}(za, b).
\end{aligned}$$

Hence $x \,\square\, y$ is well defined (Definition 5.11). Thus (see also Proposition 5.14) $\mathfrak{A}_b$ is a C^*-algebra. □

Example 5.17. Let us consider again the locally convex quasi *-algebra $(\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}), \mathcal{L}^\dagger(\mathcal{D})_b)$ of Example 4.4. As proved there, this quasi *-algebra is fully representable and from (4.2) it follows that the topology τ_{s^*} defined before Definition 5.15 coincides with the strong* topology t_{s^*} of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. One can prove easily that an element $X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is order bounded if and only if $\overline{X} \in \mathcal{B}(\mathcal{H})$. So that

$$\big(\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})\big)_b = \big\{X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}):\ X \text{ is a bounded operator}\big\}.$$

This is clearly a C^*-algebra as expected by Theorem 5.16.

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