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On the wave-breaking phenomena for the periodic two-component Dullin–Gottwald–Holm system

Min Zhu [a,b,*], Junxiang Xu [a]

[a] Department of Mathematics, Southeast University, Nanjing 211189, PR China
[b] Department of Mathematics, Nanjing Forestry University, Nanjing 210037, PR China

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ABSTRACT

Considered herein is the well-posedness problem of the periodic two-component Dullin–Gottwald–Holm (DGH) system on the circle, which can be derived from Euler's equation with constant vorticity in shallow water waves moving over a linear shear flow. The result of blow-up solutions for certain initial profiles in a manner which corresponds to wave-breaking is established.

1. Introduction

Studied here is the following periodic two-component Dullin–Gottwald–Holm (DGH) system on the circle

$$\begin{cases} m_t - Au_x + um_x + 2u_x m + \gamma u_{xxx} + \rho\rho_x = 0, \quad m = u - u_{xx}, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geqslant 0,\ x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t \geqslant 0,\ x \in \mathbb{R}. \end{cases} \tag{1.1}$$

When $\rho = 0$, (1.1) becomes the DGH equation, that is

$$m_t - Au_x + um_x + 2u_x m + \gamma u_{xxx} = 0. \tag{1.2}$$

This equation was derived using asymptotic expansions directly in the Hamiltonian for Euler's equation in the shallow water regime, and it is completely integrable with a bi-Hamiltonian as well as a Lax pair, see [18].

Using the notation $m = u - \alpha^2 u_{xx}$, (1.2) can be rewritten as

$$u_t - \alpha^2 u_{txx} - Au_x + 3uu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}), \tag{1.3}$$

where A and α are two positive constants. Eq. (1.3) is connected with two separately integrable soliton equations for shallow water waves, which are the Korteweg–de Vries (KdV) equation and the Camassa–Holm (CH) equation [5,22].

* Corresponding author at: Department of Mathematics, Southeast University, Nanjing 211189, PR China.
E-mail addresses: zhumin@njfu.edu.cn (M. Zhu), xujun@seu.edu.cn (J. Xu).

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When $\alpha^2 = 0$, (1.3) becomes the well-known KdV equation

$$u_t - Au_x + 3uu_x + \gamma u_{xxx} = 0,$$

which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity. Bourgain [1] proved that the solutions to the KdV equation are global as long as the initial data is square integrable [31,35]. Another remarkable property is that it is integrable and the solitary waves are nonlinearly stable. It is observed that the KdV equation does not accommodate wave-breaking (which means the wave profile remains bounded while its slope becomes unbounded in finite time [36]).

When $\gamma = 0$, (1.3) turns into the standard CH equation

$$u_t - Au_x - \alpha^2 u_{txx} + 3uu_x = \alpha^2(2u_x u_{xx} + uu_{xxx}), \tag{1.4}$$

modeling the unidirectional propagation of shallow water waves over a flat bottom [5,12,22], where the variable $u(t,x)$ represents the horizontal velocity of the fluid, and the parameter A characterizes a linear underlying shear flow. The CH equation is also a model for the propagation of axially symmetric waves in the hyperelastic rods [17]. Its solitary waves are smooth if $A > 0$ and peaked in the limiting case $A = 0$ [5–7]. Recently, it was claimed in [32] that the CH equation might be relevant to the modeling of tsunami. Some satisfactory results have been obtained for this shallow water equation recently. The CH equation has global strong solutions and also solutions which blow up in finite time, for instance, see [8–10,13,33] and references therein, with a different class of initial profiles in the Sobolev spaces $H^s(\mathbb{R})$, $s > 3/2$. It is shown in [2] and [3] that solution of the CH equation can be uniquely continued after breaking as either global conservative or global dissipative weak solution.

The advantage of the CH equation in comparison with the KdV equation lies in the fact that the CH equation has peaked solitons and models wave-breaking. Wave-breaking is one of the most intriguing long-standing problems of water wave theory [36]. The peaked solitons are the presence of solutions in the form of peaked solitary waves or “peakons” [5–7,19] $u(t,x) = ce^{-|x-ct|}$, $c \neq 0$, which are smooth except at the crests, where they are continuous, but have a jump discontinuity in the first derivative. The peakons replicate a feature that is characteristic for the waves of great height-waves of the largest amplitude that are exact solutions of the governing equations for water waves. These peakons are shown to be stable [14,15].

The interest in the CH equation inspired the search for various generalizations of this equation. The following two-component integrable CH system was first derived in [34] and can be viewed as a model in the context of shallow water theory [11,28],

$$\begin{cases} m_t - Au_x + um_x + 2u_x m + \rho\rho_x = 0, \\ m = u - u_{xx}, \\ \rho_t + (u\rho)_x = 0, \end{cases} \tag{1.5}$$

where $\rho(t,x)$ is related to the free surface elevation from equilibrium(or scalar density), and the parameter A characterizes a linear underlying shear flow. Obviously, if $\rho = 0$, then (1.5) becomes the CH equation. Many recent works are devoted to studying (1.5) (see, for instance, [11,20,23–28,37] and references therein).

In the presence of a linear shear flow and nonzero vorticity, we will follow Ivanov’s approach [28] to derive (1.1). System (1.1) can also be rewritten as the following equivalent form of the system in terms of u and ρ, that is,

$$\begin{cases} u_t - u_{txx} - Au_x + \gamma u_{xxx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \rho\rho_x = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geqslant 0,\ x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t \geqslant 0,\ x \in \mathbb{R}. \end{cases} \tag{1.6}$$

It is very interesting that not only the DGH equation but also the DGH system is completely integrable. The DGH system can be written as a compatibility condition of two linear systems (Lax pair) with a spectral parameter ξ, that is

$$\Psi_{xx} = \left(-\xi^2\rho^2 + \xi\left(m - \frac{A}{2} + \frac{\gamma}{2}\right) + \frac{1}{4}\right)\Psi,$$

$$\Psi_t = \left(\frac{1}{2\xi} - u + \gamma\right)\Psi_x + \frac{1}{2}u_x\Psi.$$

Moreover, this system has the following two Hamiltonians

$$E(u,\rho) = \frac{1}{2}\int \left(u^2 + u_x^2 + (\rho - 1)^2\right) dx$$

and

$$F(u,\rho)=\frac{1}{2}\int\left(u^3+uu_x^2-Au^2-\gamma u_x^2+2u(\rho-1)+u(\rho-1)^2\right)dx.$$

The goal of this paper is to derive the two-component DGH system by the shallow water theory, then to establish a result of blow-up solutions corresponding to only wave-breaking with certain initial profiles for this system. Our main tool to investigate the question of the wave-breaking for this system is due to Constantin and Escher [8,11]. However, since the system has two characteristics (see (3.2)–(3.3) in Section 3), we cannot just follow their approaches. In fact we will make use of the diffeomorphism of the trajectory q_2 defined in (3.3), which captures the maximum/minimum of u_x, therefore the transport equation for ρ can coincide with the equation for u.

The rest of the paper is organized as follows. In Section 2, we will follow the modeling approach in the shallow water theory [28] to derive the DGH system. The local well-posedness result (Theorem 3.1) is presented in Section 3 and the proof of it was enclosed in Appendix A. In Section 4, the wave-breaking phenomena of solutions for the system is analyzed in details.

Notation. Throughout this paper, we identity periodic functions with function spaces over the unit circle $\mathbb{S}$ in $\mathbb{R}^2$, i.e. $\mathbb{S}=\mathbb{R}/\mathbb{Z}$.

2. Derivation of the model

In this section, we will follow Ivanov's approach in [28] to derive the DGH system. Consider the motion of an inviscid incompressible fluid with a constant density ϱ governed by the Euler equations

$$\begin{aligned}&\vec{v}_t+(\vec{v}\cdot\nabla)\vec{v}=-\frac{1}{\varrho}\nabla P+\vec{g},\\&\nabla\cdot\vec{v}=0,\end{aligned}$$

where $\vec{v}(t,x,y,z)$ is the velocity of the fluid, $P(t,x,y,z)$ is the pressure and $\vec{g}=(0,0,-g)$ is the gravity acceleration.

Using the shallow water approximation and non-dimensionalization, the above equations can be written as

$$\begin{aligned}&u_t+\varepsilon(uu_x+wu_z)=-p_x,\\&\delta^2\big(w_t+\varepsilon(uw_x+ww_z)\big)=-p_z,\\&u_x+w_z=0,\\&w=\eta_t+\varepsilon u\eta_x,\qquad p=\eta\quad\text{on } z=1+\varepsilon\eta,\\&w=0\quad\text{on } z=0,\end{aligned}$$

where $\vec{v}=(u,0,w)$, $p(x,z,t)$ is the pressure variable measuring the deviation from the hydrostatic pressure distribution and $\eta(t,x)$ is the deviation from the mean level $z=h$ of the water surface. $\varepsilon=a/h$ and $\delta=h/\lambda$ are the two dimensionless parameters with a being the typical amplitude of the wave and λ being the typical wavelength of the wave.

In the presence of an underlying shear flow, the horizontal velocity of the flow becomes $u+\tilde{U}(z)$. We take the simplest case $\tilde{U}(z)=Az$ in which $A>0$ is a constant. Notice that the Burns condition gives the shallow water limit of the dispersion relation for the waves with vorticity, hence determines the speed of propagation of the linear waves. From Burns condition [4] one has the following expression for the speed c of the traveling waves in linear approximation,

$$c=\frac{1}{2}\big(A\pm\sqrt{4+A^2}\big).\tag{2.1}$$

In the case of the constant vorticity $\omega=A$, we obtain the following equations for u_0 and η by ignoring the terms of $O(\varepsilon^2,\delta^4,\varepsilon\delta^2)$,

$$\left(u_0-\frac{1}{2}\delta^2u_{0,xx}\right)_t+\varepsilon u_0u_{0,x}+\eta_x-\frac{A}{3}\delta^2u_{0,xxx}=0,\tag{2.2}$$

$$\eta_t+A\eta_x+\left((1+\varepsilon\eta)u_0+\frac{A}{2}\varepsilon\eta^2\right)_x-\frac{1}{6}\delta^2u_{0,xxx}=0,\tag{2.3}$$

where u_0 is the leading order approximation for u (see the details in [28]). Let both of the parameters ϵ and δ go to 0. Then by Eqs. (2.2) and (2.3), we have the following system of linear equations

$$\begin{aligned}&u_{0,t}+\eta_x=0,\\&\eta_t+A\eta_x+u_{0,x}=0.\end{aligned}$$

This in turn implies that $\eta_{tt} + A\eta_{tx} - \eta_{xx} = 0$. Introducing a new variable

$$\rho = 1 + \varepsilon\alpha\eta + \varepsilon^2\beta\eta^2 + \varepsilon\delta^2\mu u_{0,xx},$$

for some constants α, β and μ satisfying

$$\frac{\mu}{\alpha} = \frac{1}{6(c-A)},$$
$$\alpha = 1 + \frac{Ac}{2} + \frac{\beta}{\alpha},$$

then Eqs. (2.2) and (2.3) become

$$\begin{cases} m_t + Am_x - Au_{0,x} - \dfrac{1}{6c^2(c-A)}\delta^2 u_{0,xxx} + \varepsilon\left(1 - \dfrac{\alpha^2+2\beta}{\alpha}c^2\right)u_0u_{0,x} + \dfrac{1}{2\varepsilon\alpha}\left(\rho^2\right)_x = 0, \\ \rho_t + A\rho_x + \alpha\varepsilon(\rho u_0)_x = 0, \end{cases} \tag{2.4}$$

where $m = u_0 - \frac{1}{2}\delta^2 u_{0,xx}$. Since

$$3u_0u_{0,x} = 2mu_{0,x} + u_0m_x,$$

(2.4) can be reformulated at the order of $O(\varepsilon, \delta^2)$ as

$$m_t + Am_x - Au_{0,x} - \frac{1}{6c^2(c-A)}\delta^2 u_{0,xxx} + \frac{\varepsilon}{3}\left(1 - \frac{\alpha^2+2\beta}{\alpha}c^2\right)(2mu_{0,x} + u_0m_x) + \frac{1}{2\varepsilon\alpha}\left(\rho^2\right)_x = 0.$$

Using the scaling $u_0 \to \frac{1}{\alpha\varepsilon}u_0$, $x \to \delta x$ and $t \to \delta t$, then (2.4) becomes

$$\begin{cases} m_t + Am_x - Au_{0,x} - \dfrac{1}{6c^2(c-A)}u_{0,xxx} + \dfrac{1}{3\alpha}\left(1 - \dfrac{\alpha^2+2\beta}{\alpha}c^2\right)(2mu_{0,x} + u_0m_x) + \dfrac{1}{2\varepsilon\alpha}\left(\rho^2\right)_x = 0, \\ m = u_0 - u_{0,xx}, \\ \rho_t + A\rho_x + (\rho u_0)_x = 0. \end{cases}$$

Now if we choose

$$\frac{1}{3\alpha}\left(1 - \frac{\alpha^2+2\beta}{\alpha}c^2\right) = 1$$

and denote $\gamma = -\frac{1}{6c^2(c-A)}$, then we arrive at

$$\begin{cases} m_t + Am_x - Au_{0,x} + 2mu_{0,x} + u_0m_x + \gamma u_{0,xxx} + \rho\rho_x = 0, \\ m = u_0 - u_{0,xx}, \\ \rho_t + A\rho_x + (\rho u_0)_x = 0. \end{cases} \tag{2.5}$$

Thus the constants α, β, μ and c satisfy

$$\alpha = \frac{1}{3(c^2+1)} + \frac{c^2}{3},$$
$$\beta = \alpha^2 - \alpha\left(1 + \frac{Ac}{2}\right),$$
$$\mu = \frac{\alpha}{6(c-A)},$$
$$c^2 - Ac - 1 = 0.$$

With a further Galilean transformation $x \to x - ct,\ t \to t$, we can drop the terms $A\rho_x$ and Am_x in (2.5) and obtain (1.1) or (1.6).

3. Local well-posedness and preliminaries

To study the wave-breaking problem, we first recall the local existence theory for the periodic DGH system. We will apply Kato's theory to obtain the local well-posedness of (1.6) with initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geqslant 2$.

It is noted that the periodic DGH system can be written as the "transport" type, that is

$$\begin{cases} u_t + (u-\gamma)u_x = -\partial_x G * \left(u^2 + \dfrac{1}{2}u_x^2 + (\gamma - A)u + \dfrac{1}{2}\rho^2\right), & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geqslant 0,\ x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t \geqslant 0,\ x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where

$$G(x) = \frac{\cosh(x - [x] - 1/2)}{2\sinh(1/2)}, \quad x \in \mathbb{S},$$

$(1-\partial_x^2)^{-1} f = G * f$ for all $f \in L^2(\mathbb{S})$, $u = G * m$ and $m = u - u_{xx}$. The system in (3.1) is suitable for applying Kato's theory [29] to obtain the local well-posedness. Therefore we have the following theorem.

Theorem 3.1. *Given an initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geqslant 2$, there exists a maximal $T = T(\|(u_0,\rho_0)\|_{H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})}) > 0$ and a unique solution*

$$(u,\rho) \in C\big([0,T); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})\big) \cap C^1\big([0,T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S})\big)$$

of (1.6)*. Moreover, the solution (u,ρ) depends continuously on the initial value (u_0,ρ_0) and the maximal time of existence $T > 0$ is independent of s.*

Since the proof of this theorem is similar to Theorem 2.2 in [20], we enclose it as Appendix A for completeness.

In order to pursue our goal for wave-breaking solutions, we state here some previously known results which are needed for our proofs.

We consider the following two associated Lagrangian scales of (3.1)

$$\begin{cases} \dfrac{\partial q_1}{\partial t} = u(t,q_1) - \gamma, & 0 < t < T, \\ q_1(0,x) = x, & x \in \mathbb{R}, \end{cases} \tag{3.2}$$

and

$$\begin{cases} \dfrac{\partial q_2}{\partial t} = u(t,q_2), & 0 < t < T, \\ q_2(0,x) = x, & x \in \mathbb{R}, \end{cases} \tag{3.3}$$

where $u \in C^1([0,T), H^{s-1}(\mathbb{S}))$ is the first component of the solution (u,ρ) to (1.6).

Lemma 3.2. *(See [16,11].) Let (u,ρ) be the solution of* (1.6) *with initial data $(u_0,\rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geqslant 2$, and T the maximal time of existence. Then* (3.2) *has a unique solution $q_1 \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$ and* (3.3) *has a unique solution $q_2 \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$. These two solutions satisfy $q_i(t,x+1) = q_i(t,x) + 1$, $i = 1,2$. Moreover, the maps $q_1(t,\cdot)$ and $q_2(t,\cdot)$ are increasing diffeomorphisms of $\mathbb{R}$ with*

$$q_{1x}(t,x) = \exp\left(\int_0^t u_x\big(\tau, q_1(\tau,x)\big)\,d\tau\right) > 0, \quad (t,x) \in [0,T) \times \mathbb{R},$$

$$q_{2x}(t,x) = \exp\left(\int_0^t u_x\big(\tau, q_2(\tau,x)\big)\,d\tau\right) > 0, \quad (t,x) \in [0,T) \times \mathbb{R}.$$

The above lemma indicates that $q_1(t,\cdot): \mathbb{R} \to \mathbb{R}$ and $q_2(t,\cdot): \mathbb{R} \to \mathbb{R}$ are diffeomorphisms of the line for each $t \in [0,T)$. Hence, the L^∞ norm of any function $v(t,\cdot) \in L^\infty(\mathbb{S})$ is preserved under the family of diffeomorphisms $q_1(t,\cdot)$ and $q_2(t,\cdot)$ with $t \in [0,T)$, that is

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{S})} = \big\|v\big(t,q_1(t,\cdot)\big)\big\|_{L^\infty(\mathbb{S})} = \big\|v\big(t,q_2(t,\cdot)\big)\big\|_{L^\infty(\mathbb{S})}, \quad t \in [0,T). \tag{3.4}$$

Similarly, we have

$$\inf_{x\in\mathbb{S}} v(t,x) = \inf_{x\in\mathbb{S}} v\big(t, q_1(t,x)\big) = \inf_{x\in\mathbb{S}} v\big(t, q_2(t,x)\big), \quad t\in[0,T), \tag{3.5}$$

$$\sup_{x\in\mathbb{S}} v(t,x) = \sup_{x\in\mathbb{S}} v\big(t, q_1(t,x)\big) = \sup_{x\in\mathbb{S}} v\big(t, q_2(t,x)\big), \quad t\in[0,T). \tag{3.6}$$

Lemma 3.3. *(See [20].) Let (u,ρ) be the solution of (1.6) with initial data $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$, $s\geqslant 2$, and T the maximal time of existence. Then we have*

$$\rho\big(t, q_2(t,x)\big)q_{2x}(t,x) = \rho_0(x), \quad (t,x)\in[0,T)\times\mathbb{S}. \tag{3.7}$$

Moreover if there exists a $x_0\in\mathbb{S}$ such that $\rho_0(x_0)=0$, then $\rho(t,q_2(t,x_0))=0$ for all $t\in[0,T)$.

Lemma 3.4. *(See [9].) Let $T>0$ and $v\in C^1([0,T);H^2(\mathbb{R}))$. Then for every $t\in[0,T)$, there exists at least one point $\xi(t)\in\mathbb{R}$ with*

$$m(t) := \inf_{x\in\mathbb{R}}\big(v_x(t,x)\big) = v_x\big(t,\xi(t)\big).$$

The function $m(t)$ is absolutely continuous on $(0,T)$ with

$$\frac{dm(t)}{dt} = v_{tx}\big(t,\xi(t)\big) \quad \text{a.e. on } (0,T).$$

Then, we give the useful conservation law of the strong solutions to (1.6).

Lemma 3.5. *Let (u,ρ) be the solution of (1.6) with initial data $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$, $s\geqslant 2$, and T the maximal time of existence. Then for all $t\in[0,T)$, we have*

$$\int_{\mathbb{S}}\big(u^2(t,x)+u_x^2(t,x)+\rho^2(t,x)\big)\,dx = \int_{\mathbb{S}}\big(u_0^2(0,x)+u_{0x}^2(0,x)+\rho_0^2(0,x)\big)\,dx.$$

Proof. Multiplying the first equation of (1.6) by $2u$ and integrating by parts, we have

$$\frac{d}{dt}\int_{\mathbb{S}}\big(u^2(t,x)+u_x^2(t,x)\big)\,dx = \frac{d}{dt}\int_{\mathbb{S}} u_x(t,x)\rho^2(t,x)\,dx.$$

Multiplying the second equation of (1.6) by 2ρ and integrating by parts, we get

$$\frac{d}{dt}\int_{\mathbb{S}}\rho^2(t,x) = -\frac{d}{dt}\int_{\mathbb{S}} u_x(t,x)\rho^2(t,x)\,dx.$$

Adding the above two equalities, we obtain

$$\frac{d}{dt}\int_{\mathbb{S}}\big(u^2(t,x)+u_x^2(t,x)+\rho^2(t,x)\big)\,dx = 0.$$

This implies the desired result in this lemma. □

Lemma 3.6. *(See [21].) For all $u\in H^1(\mathbb{S})$, the following inequality holds*

$$G*\left(u^2+\frac{1}{2}u_x^2\right)\geqslant \kappa u^2(x),$$

with

$$\kappa = \frac{1}{2}+\frac{\arctan(\sinh(1/2))}{2\sinh(1/2)+2\arctan(\sinh(1/2))\sinh^2(1/2)}\approx 0.869.$$

Moreover,

$$\kappa = \min_{u\in H^1, u\neq 0}\frac{G*(u^2+\frac{1}{2}u_x^2)}{u^2(x)},$$

which is the optimal constant obtained by the function

$$u_1 = \frac{1 + \arctan(\sinh(x - [x] - 1/2))\sinh(x - [x] - 1/2)}{1 + \arctan(\sinh(1/2))\sinh(1/2)}.$$

Lemma 3.7. *(See [9].) Let g be a monotone function on $[a, b]$, and f be a real continuous function on $[a, b]$. Then there exists a $\xi \in [a, b]$ such that*

$$\int_a^b f(s)g(s)\,ds = g(a)\int_a^\xi f(s)\,ds + g(b)\int_\xi^b f(s)\,ds.$$

4. Wave-breaking

In this section, we establish a result of a wave-breaking solution (i.e. the wave profile remains bounded while its slope becomes unbounded in finite time) with certain initial profiles for the periodic DGH system (1.6).

Firstly, in order to obtain the precise blow-up mechanism of the DGH system, we recall the following lemma derived in [26].

Lemma 4.1. *Let $0 < s < 1$. Suppose that $f_0 \in H^s(\mathbb{S})$, $g \in L^1([0, T]; H^s(\mathbb{S}))$, $v, v_x \in L^1([0, T]; L^\infty(\mathbb{S}))$ and that $f \in L^\infty([0, T]; H^s(\mathbb{S})) \cap C([0, T]; S'(\mathbb{S}))$ solves the one-dimensional linear transport equation*

$$\begin{cases} f_t + v f_x = g, \\ f(0, x) = f_0(x), \end{cases}$$

then $f \in C([0, T]; H^s(\mathbb{R}))$. More precisely, there exists a constant C depending only on s such that the following estimate holds,

$$\|f(t)\|_{H^s} \leqslant \|f_0\|_{H^s} + C\left(\int_0^t \|g(\tau)\|_{H^s}\,d\tau + \int_0^t \|f(\tau)\|_{H^s} V'(\tau)\,d\tau\right).$$

Hence,

$$\|f(t)\|_{H^s} \leqslant e^{CV(t)}\left(\|f_0\|_{H^s} + C\int_0^t \|g(\tau)\|_{H^s}\,d\tau\right),$$

where

$$V(t) = \int_0^t \left(\|v(\tau)\|_{L^\infty} + \|v_x(\tau)\|_{L^\infty}\right)d\tau.$$

The above lemma was proved using the Littlewood–Paley analysis for the transport equation and the Moser-type estimates. Using this result and performing the same argument as in [26], we can obtain the following blow-up criterion (up to a slight modification, the proof is omitted).

Lemma 4.2. *Let (u, ρ) be the solution of (4.1) with initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geqslant 2$, and T the maximal time of existence. Then*

$$T < \infty \quad \Rightarrow \quad \int_0^T \|u_x(\tau)\|_{L^\infty(\mathbb{S})}\,d\tau = \infty.$$

Based on the above results, let us state the following precise blow-up mechanism of (1.6).

Proposition 4.3 *(Wave-breaking criterion). (See [38].) Let (u, ρ) be the solution of (1.6) with initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geqslant 2$, and T the maximal time of existence. Then the solution blows up in finite time if and only if*

$$\liminf_{t \to T_0^-}\left\{\inf_{x \in \mathbb{S}} u_x(t, x)\right\} = -\infty.$$

Using Proposition 4.3, we study the wave-breaking phenomena for (1.6). We set

$$m_1(t) := \min_{x\in\mathbb{S}}\big(u_x(t,x)\big), \qquad m_2(t) := \max_{x\in\mathbb{S}}\big(u_x(t,x)\big).$$

Theorem 4.4. *Let $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$, $s\geqslant 2$ satisfy*

$$m_1(0)+m_2(0) < -4|\gamma - A|\frac{\cosh(1/2)-1}{\sinh(1/2)} - 2\sqrt{2}C_1, \tag{4.1}$$

where

$$C_0 = \int_{\mathbb{S}}\big(u_0^2(0,x)+u_{0x}^2(0,x)+\rho_0^2(0,x)\big)\,dx, \qquad C_1 = \left(\frac{(1-\kappa)\cosh(1/2)}{2\kappa\sinh(1/2)}C_0\right)^{\frac{1}{2}},$$

and $\kappa = \frac{1}{2}+\frac{\arctan(\sinh(1/2))}{2\sinh(1/2)+2\arctan(\sinh(1/2))\sinh^2(1/2)}$. If there are some $x_1,x_2\in\mathbb{S}$ such that

$$\rho_0(x_1)=0, \qquad u_{0,x}(x_1)=\inf_{x\in\mathbb{S}}u_{0,x}(x), \tag{4.2}$$

and

$$\rho_0(x_2)=0, \qquad u_{0,x}(x_2)=\sup_{x\in\mathbb{S}}u_{0,x}(x), \tag{4.3}$$

then the solution of (1.6) *blows up in finite time.*

Proof. Let $T>0$ be the maximal time of existence of the corresponding solution (u,ρ) to (1.6). By Theorem 3.1, we need only to prove this theorem for $s\geqslant 3$. According to Lemma 3.4, we can define $\xi(t)\in\mathbb{S}$ as

$$m_1(t)=u_x\big(t,\xi(t)\big)=\inf_{x\in\mathbb{S}}u_x(t,x), \quad t\in[0,T). \tag{4.4}$$

Since $q_2(t,\cdot)$ defined by (3.3) is a diffeomorphism of the circle for any $t\in[0,T)$, we obtain there exists a $x_1(t)\in\mathbb{S}$ such that

$$q_2\big(t,x_1(t)\big)=\xi(t), \quad t\in[0,T). \tag{4.5}$$

Then (4.2) and (4.4) imply that

$$m_1(0)=u_x\big(0,\xi(0)\big)=\inf_{x\in\mathbb{S}}u_{0,x}(x)=u_{0,x}(x_1).$$

Therefore we can choose $\xi(0)=x_1$ and

$$\rho_0\big(\xi(0)\big)=\rho_0(x_1)=0.$$

Using Lemma 3.3, we have

$$\rho\big(t,q_2\big(t,x_1(t)\big)\big)=\rho\big(t,\xi(t)\big)=0, \quad \forall t\in[0,T). \tag{4.6}$$

On the other hand, since $\sup_{x\in\mathbb{S}}(v_x(t,x))=-\inf_{x\in\mathbb{S}}(-v_x(t,x))$, we similarly define

$$m_2(t)=u_x\big(t,\eta(t)\big)=\sup_{x\in\mathbb{S}}u_x(t,x), \quad t\in[0,T), \tag{4.7}$$

then there exists a $x_2(t)\in\mathbb{S}$ such that $q_2(t,x_2(t))=\eta(t)$, $t\in[0,T)$. Moreover, we have

$$\rho\big(t,q_2\big(t,x_2(t)\big)\big)=\rho\big(t,\xi(t)\big)=0, \quad \forall t\in[0,T). \tag{4.8}$$

Now, differentiating the first equation in (3.1) with respect to the x, we have

$$u_{tx}+(u-\gamma)u_{xx}=-\frac{1}{2}u_x^2+u^2+\frac{1}{2}\rho^2-(\gamma-A)\partial_x^2G*u-G*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\right).$$

In view of the definitions of $m_i(t)$ $(i=1,2)$ in (4.4) and (4.7), let $x=x_i(t)$, $(t=1,2)$, we obtain that

$$\begin{aligned}\frac{dm_1}{dt}&=-\frac{1}{2}m_1^2+u^2-(\gamma-A)\partial_x^2G*u-G*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\right)\\&=-\frac{1}{2}m_1^2+u^2-(\gamma-A)\int_0^1G(y)u_{xx}\big(t,\xi(t)-y\big)\,dy-\int_0^1G\big(\xi(t)-y\big)\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\right)dy\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\frac{dm_2}{dt} &= -\frac{1}{2}m_2^2 + u^2 - (\gamma - A)\partial_x^2 G * u - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right) \\ &= -\frac{1}{2}m_2^2 + u^2 - (\gamma - A)\int_0^1 G(y)u_{xx}\big(t,\eta(t)-y\big)\,dy - \int_0^1 G\big(\eta(t)-y\big)\left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right)dy. \end{aligned} \tag{4.10}$$

In view of $\frac{1}{2\sinh(1/2)} \leqslant G(y) \leqslant \frac{\cosh(1/2)}{2\sinh(1/2)}$ and thanks to Lemma 3.6, we have

$$\begin{aligned} u^2 - \int_0^1 G(x-y)\left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right)dy &\leqslant \frac{1}{\kappa}\int_0^1 G(x-y)\left(u^2 + \frac{1}{2}u_x^2\right)dy - \int_0^1 G(x-y)\left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right)dy \\ &\leqslant \frac{1-\kappa}{\kappa}\frac{\cosh(1/2)}{2\sinh(1/2)}\int_0^1\left(u^2 + \frac{1}{2}u_x^2\right)dy - \frac{1}{2}\int_0^1 G(x-y)\rho^2\,dy \\ &\leqslant \frac{(1-\kappa)\cosh(1/2)}{2\kappa\sinh(1/2)}\int_0^1\big(u^2 + u_x^2 + \rho^2\big)\,dy = C_1^2. \end{aligned} \tag{4.11}$$

The function $G(y)$ is continuous, decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$, with $G(\frac{1}{2}) = \frac{1}{2\sinh(1/2)}$ and $G(0) = G(1) = \frac{\cosh(1/2)}{2\sinh(1/2)}$. So that we choose the function

$$g(y) = G(y) - \frac{1}{2\sinh(1/2)}, \quad y \in \mathbb{S},$$

which is continuous, decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$, with $g(\frac{1}{2}) = 0$ and $g(0) = g(1) = \frac{\cosh(1/2)-1}{2\sinh(1/2)}$. Note the periodicity of u_{xx}, we find for $i = 1, 2$, that

$$\begin{aligned} \left|\int_0^1 G(y)u_{xx}(t, x_i - y)\,dy\right| &= \left|\int_0^1 g(y)u_{xx}(t, x_i - y)\,dy\right| \\ &\leqslant \left|\int_0^{\frac{1}{2}} g(y)u_{xx}(t, x_i - y)\,dy\right| + \left|\int_{\frac{1}{2}}^1 g(y)u_{xx}(t, x_i - y)\,dy\right|. \end{aligned} \tag{4.12}$$

Using Lemma 3.7, we have

$$\begin{aligned} \left|\int_0^{\frac{1}{2}} g(y)u_{xx}(t, x_i - y)\,dy\right| &= \left|g(0)\int_0^{\varphi} u_{xx}(t, x_i - y)\,dy + g\left(\frac{1}{2}\right)\int_{\varphi}^{\frac{1}{2}} u_{xx}(t, x_i - y)\,dy\right| \\ &= \left|\frac{\cosh(1/2)-1}{2\sinh(1/2)}\big(u_x(t, x_i) - u_x(t, x_i - \varphi)\big)\right| \\ &\leqslant \frac{\cosh(1/2)-1}{2\sinh(1/2)}\big(m_2(t) - m_1(t)\big). \end{aligned} \tag{4.13}$$

In the same way, we obtain

$$\left|\int_{\frac{1}{2}}^1 g(y)u_{xx}(t, x_i - y)\,dy\right| \leqslant \frac{\cosh(1/2)-1}{2\sinh(1/2)}\big(m_2(t) - m_1(t)\big). \tag{4.14}$$

Substituting (4.13) and (4.14) into (4.12), we deduce that

$$\left|\int_0^1 G(y)u_{xx}(t, x_i - y)\,dy\right| \leqslant \frac{\cosh(1/2)-1}{\sinh(1/2)}\big(m_2(t) - m_1(t)\big). \tag{4.15}$$

In view of (4.9), (4.10), (4.11) and (4.15), we obtain for a.e. $t \in (0, T)$ that

$$\frac{dm_1}{dt} \leqslant -\frac{1}{2}m_1^2 + |\gamma - A|\frac{\cosh(1/2) - 1}{\sinh(1/2)}(m_2 - m_1) + C_1^2, \tag{4.16}$$

$$\frac{dm_2}{dt} \leqslant -\frac{1}{2}m_2^2 + |\gamma - A|\frac{\cosh(1/2) - 1}{\sinh(1/2)}(m_2 - m_1) + C_1^2. \tag{4.17}$$

Summing up the above two equations gives

$$\begin{aligned}\frac{d(m_1 + m_2)}{dt} &\leqslant -\frac{1}{2}\left(m_1^2 + m_2^2\right) + 2|\gamma - A|\frac{\cosh(1/2) - 1}{\sinh(1/2)}(m_2 - m_1) + 2C_1^2 \\ &= -\frac{1}{2}\left(m_1^2 + m_2^2\right) + 2|\gamma - A|\frac{\cosh(1/2) - 1}{\sinh(1/2)}(m_2 + m_1) - 4|\gamma - A|\frac{\cosh(1/2) - 1}{\sinh(1/2)}m_1 + 2C_1^2.\end{aligned} \tag{4.18}$$

Let

$$C_2 = |\gamma - A|\frac{\cosh(1/2) - 1}{\sinh(1/2)}. \tag{4.19}$$

Then (4.16) and (4.18) become the following equations

$$\frac{dm_1}{dt} \leqslant -\frac{1}{2}m_1^2 + C_2(m_2 - m_1) + C_1^2, \tag{4.20}$$

and

$$\frac{d(m_1 + m_2)}{dt} \leqslant -\frac{1}{2}\left(m_1^2 + m_2^2\right) + 2C_2(m_2 + m_1) - 4C_2 m_1 + 2C_1^2. \tag{4.21}$$

Since $(m_1 + m_2)(0) < -4|\gamma - A|\frac{\cosh(1/2)-1}{\sinh(1/2)} - 2\sqrt{2}C_1 = -4C_2 - 2\sqrt{2}C_1$, there is $\delta \in (0, \frac{1}{2}]$ such that $(m_1 + m_2)(0) \leqslant -\alpha - 2\sqrt{2}(1 + \delta)C_1$ with $\alpha = 4C_2 + \delta$, $\alpha > 4C_2$.

We first claim that there holds for all $t \in (0, T]$,

$$(m_1 + m_2)(t) \leqslant -\alpha - 2\sqrt{2}(1 + \delta)C_1. \tag{4.22}$$

Let $\overline{m} =: (m_1 + m_2)(t) + \alpha + 2\sqrt{2}(1 + \delta)C_1$. Then we claim that $\overline{m}(t) \leqslant 0$. It is observed that $\overline{m}$ is continuous on $[0, T)$. If (4.22) does not hold, we can find a $t_0 \in (0, T)$ such that $\overline{m}(t) > 0$. Denote

$$t_1 = \max\left(t < t_0\colon\ \overline{m}(t_0) = 0\right).$$

Then

$$\overline{m}(t_1) = 0, \qquad \overline{m}'(t_1) \geqslant 0. \tag{4.23}$$

Thanks to

$$m_1(t_1) \leqslant \frac{1}{2}(m_1 + m_2)(t_1) = -\frac{1}{2}\alpha - \sqrt{2}(1 + \delta)C_1$$

and

$$m_2(t_1) = -\alpha - 2\sqrt{2}(1 + \delta)C_1 - m_1(t_1),$$

using (4.21) and (4.23), we get

$$\begin{aligned}\overline{m}'(t_1) &= (m_1 + m_2)'(t_1) \\ &\leqslant -\frac{1}{2}m_1^2(t_1) - \frac{1}{2}m_2^2(t_1) + 2C_2(m_2 + m_1)(t_1) - 4C_2 m_1(t_1) + 2C_1^2 \\ &= -\frac{1}{2}m_1^2(t_1) - \frac{1}{2}\left(-\alpha - 2\sqrt{2}(1 + \delta)C_1 - m_1(t_1)\right)^2 + 2C_2\left(-\alpha - 2\sqrt{2}(1 + \delta)C_1\right) - 4C_2 m_1(t_1) + 2C_1^2 \\ &= -m_1^2(t_1) - m_1(t_1)\left(\alpha + 2\sqrt{2}(1 + \delta)C_1 + 4C_2\right) - \frac{1}{2}\left(\alpha + 2\sqrt{2}(1 + \delta)C_1\right)^2 \\ &\quad - 2C_2\left(\alpha + 2\sqrt{2}(1 + \delta)C_1\right) + 2C_1^2 \\ &= -\left(m_1(t_1) + \frac{1}{2}\left(\alpha + 2\sqrt{2}(1 + \delta)C_1 + 4C_2\right)\right)^2 - \frac{1}{4}\left(\alpha + 2\sqrt{2}(1 + \delta)C_1\right)^2 + 4C_2^2 + 2C_1^2,\end{aligned} \tag{4.24}$$

which together with the fact $\alpha > 4C_2$ implies

$$\overline{m}'(t_1) \leqslant -\frac{1}{4}\big(\alpha + 2\sqrt{2}(1+\delta)C_1\big)^2 + 4C_2^2 + 2C_1^2 < 0.$$

This yields a contradiction with (4.23).

Putting (4.22) and $m_1(t) \leqslant \frac{1}{2}(m_1 + m_2)(t) < -2C_2 - \sqrt{2}(1+\delta)C_1$ back to (4.20), we have

$$\begin{aligned}
\frac{d(m_1(t)+2C_2)}{dt} = \frac{dm_1}{dt} &\leqslant -\frac{1}{2}m_1^2(t) + C_2(m_2 - m_1)(t) + C_1^2 \\
&= -\frac{1}{2}m_1^2(t) + C_2(m_2 + m_1)(t) - 2C_2 m_1(t) + C_1^2(t) \\
&\leqslant -\frac{1}{2}m_1^2(t) + C_2\big(-\alpha - 2\sqrt{2}(1+\delta)C_1\big) - 2C_2 m_1(t) + C_1^2 \\
&= -\frac{1}{2}\big(m_1(t) + 2C_2\big)^2 - C_2\alpha - 2\sqrt{2}(1+\delta)C_1C_2 + 2C_2^2 + C_1^2 \\
&= -\frac{1}{2}\big(m_1(t) + 2C_2\big)^2 + C_1^2 - C_2\big(\alpha - 2C_2 + 2\sqrt{2}(1+\delta)C_1\big) \\
&< -\frac{\delta(\delta+2)}{2(1+\delta)^2}\big(m_1(t) + 2C_2\big)^2.
\end{aligned} \tag{4.25}$$

Since $m_1(t)$ is locally Lipshitz on $(0, T)$, we have that $\frac{1}{m_1(t)+2C_2}$ is also locally Lipshitz on $(0, T)$. Being locally Lipshitz, $\frac{1}{m_1(t)+2C_2}$ is absolutely continuous on $(0, T)$, it is then inferred from (4.22) that

$$\frac{d}{dt}\left(\frac{1}{m_1(t) + 2C_2}\right) > \frac{\delta(\delta+2)}{2(1+\delta)^2}, \quad t \in (0, T).$$

Consequently,

$$m_1(t) < \frac{2(1+\delta)^2(m_1(0) + 2C_2)}{(m_1(0) + 2C_2)\delta(\delta+2)t + 2(1+\delta)^2} - 2C_2, \quad t \in (0, T).$$

Using Proposition 4.3, the above equation implies that $T < -\frac{2(1+\delta)^2}{(m_1(0)+2C_2)\delta(\delta+2)}$. Therefore, the proof of the theorem is complete. □

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Appendix A

In this appendix, we will apply Kato's semigroup theory to establish the local well-posedness for the periodic initial value problem to (3.1). For convenience, we present here Kato's theorem in a form suitable for our purpose. Consider the abstract quasilinear evolution equation

$$\begin{cases} \dfrac{dv}{dt} + A(v)v = f(v), & t \geqslant 0, \\ v(0) = v_0. \end{cases} \tag{A.1}$$

Let X and Y be two Hilbert spaces such that Y is continuously and densely embedded in X, let $Q : Y \to X$ be a topological isomorphism, and let $\|\cdot\|_X$ and $\|\cdot\|_Y$ be the norms of the Banach spaces X and Y, respectively. Let $L(Y, X)$ denote the space of all bounded linear operators from Y to X. In particular, it is denoted by $L(X)$ if $X = Y$. If A is an unbounded operator, we denote the domain of A by $D(A)$. $[A, B]$ denotes the commutator of two linear operators A and B. The linear operator A belongs to $G(X, 1, \beta)$ where β is a real number, if $-A$ generates a C_0-semigroup such that $\|e^{-sA}\|_{L(X)} \leqslant e^{\beta s}$. The inner product in H^s is denoted by $\langle\cdot,\cdot\rangle_s$, particularly the L^2 inner product is $\langle\cdot,\cdot\rangle$.

We make the following assumptions, where μ_i $(i = 1, 2, 3, 4)$ are constants depending only on $\max\{\|y\|_Y, \|z\|_Y\}$:

(i) $A(y) \in L(Y, X)$ for $y \in X$ with

$$\big\|\big(A(y) - A(z)\big)w\big\|_X \leqslant \mu_1\|y - z\|_X\|w\|_Y, \quad y, z, w \in Y$$

and $A(y) \in G(X, 1, \beta)$ (i.e., $A(y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(ii) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y) \in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\left\|\big(B(y) - B(z)\big)w\right\|_X \leqslant \mu_2 \|y - z\|_Y \|w\|_X, \quad y, z \in Y,\ w \in X.$$

(iii) $f : Y \to Y$ extends to a map from X into X, is bounded on bounded sets in Y, and satisfies

$$\left\|f(y) - f(z)\right\|_Y \leqslant \mu_3 \|y - z\|_Y, \quad y, z \in Y$$

and

$$\left\|f(y) - f(z)\right\|_X \leqslant \mu_4 \|y - z\|_X, \quad y, z \in Y.$$

Lemma A.1. *(See [29].) Assume the conditions* (i), (ii) *and* (iii) *hold. Given $v_0 \in Y$, there is a maximal $T > 0$ depending only on $\|v_0\|_Y$ and a unique solution v to* (A.1) *such that*

$$v = v(\cdot, v_0) \in C\big([0, T); Y\big) \cap C^1\big([0, T); X\big).$$

Moreover, the map $v_0 \mapsto v(\cdot, v_0)$ is a continuous map from Y to $C([0, T); Y) \cap C^1([0, T); X)$.

To prove Theorem 3.1, we will apply Lemma A.1 with

$$U = \begin{pmatrix} u \\ \rho \end{pmatrix},$$

$$A(U) = \begin{pmatrix} (u - \gamma)\partial_x & 0 \\ 0 & u\partial_x \end{pmatrix}, \tag{A.2}$$

$$f(U) = \begin{pmatrix} -\partial_x(1 - \partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2 + (\gamma - A)u + \frac{1}{2}\rho^2) \\ -u_x\rho \end{pmatrix}, \tag{A.3}$$

$Y = H^s \times H^{s-1}$, $X = H^{s-1} \times H^{s-2}$, $\Lambda = (1 - \partial_x^2)^{1/2}$ and

$$Q = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}. \tag{A.4}$$

Obviously, Q is an isomorphism of $H^s \times H^{s-1}$ onto $H^{s-1} \times H^{s-2}$. Thus, to derive Theorem 3.1, we only need to check that $A(U)$ and $f(U)$ satisfy the conditions (i)–(iii), and this can be formulated through several lemmas.

The following lemmas in [29] (Lemma A1) and [30] (Lemma 2.6) are useful in our proof.

Lemma A.2. *Let r, t be two real numbers such that $-r < t \leqslant r$. Then,*

$$\|fg\|_t \leqslant c\|f\|_r\|g\|_t, \quad \text{if } r > \frac{1}{2} \tag{A.5}$$

and

$$\|fg\|_{r+t-\frac{1}{2}} \leqslant c\|f\|_r\|g\|_t, \quad \text{if } r < \frac{1}{2}, \tag{A.6}$$

where c is a positive constant depending on r and t.

Lemma A.3. *Let $f \in H^r$ for some $r > \frac{3}{2}$. Then*

$$\left\|\Lambda^{-\bar{s}}\big[\Lambda^{\bar{s}+\bar{t}+1}, M_f\big]\Lambda^{-\bar{t}}\right\|_{L(L^2)} \leqslant c\|\partial_x f\|_{r-1}, \quad |\bar{s}|, |\bar{t}| \leqslant r - 1, \tag{A.7}$$

where M_f is the operator of multiplication by f and c is a constant depending only on $\bar{s}$ and $\bar{t}$.

Lemma A.4. *With $U \in H^s \times H^{s-1} (s \geqslant 2)$, the operator $A(U) \in G(H^{s-1} \times H^{s-2}, 1, \beta)$.*

Proof. Taking the $H^{s-1} \times H^{s-2}$ inner product with $W = \binom{w_1}{w_2}$ on both sides of the equation $\frac{dW}{dt} + A(U)W = 0$ gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|W\|^2_{H^{s-1}\times H^{s-2}} &= -\big\langle W, A(U)W\big\rangle_{(s-1)\times(s-2)} \\ &= -\left\langle \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} (u - \gamma)\partial_x w_1 \\ u\partial_x w_2 \end{pmatrix} \right\rangle_{(s-1)\times(s-2)} \end{aligned}$$

$$
\begin{aligned}
&= -\langle w_1, (u-\gamma)\partial_x w_1\rangle_{s-1} - \langle w_2, u\partial_x w_2\rangle_{s-2} \\
&= -\langle \Lambda^{s-1} w_1, \Lambda^{s-1}\big((u-\gamma)\partial_x w_1\big)\rangle - \langle \Lambda^{s-2} w_2, \Lambda^{s-2}(u\partial_x w_2)\rangle \\
&= -\langle \Lambda^{s-1} w_1, [\Lambda^{s-1}, u-\gamma]\partial_x w_1\rangle - \langle \Lambda^{s-1} w_1, (u-\gamma)\partial_x \Lambda^{s-1} w_1\rangle \\
&\quad - \langle \Lambda^{s-2} w_2, [\Lambda^{s-2}, u]\partial_x w_2\rangle - \langle \Lambda^{s-2} w_2, u\partial_x \Lambda^{s-2} w_2\rangle \\
&= -\langle \Lambda^{s-1} w_1, [\Lambda^{s-1}, u-\gamma]\partial_x w_1\rangle - \frac{1}{2}\langle \Lambda^{s-1} w_1, \partial_x u \Lambda^{s-1} w_1\rangle \\
&\quad - \langle \Lambda^{s-2} w_2, [\Lambda^{s-2}, u]\partial_x w_2\rangle - \frac{1}{2}\langle \Lambda^{s-2} w_2, \partial_x u \Lambda^{s-2} w_2\rangle \\
&\leqslant \|\Lambda^{s-1} w_1\|_{L^2}^2 \|[\Lambda^{s-1}, u-\gamma]\Lambda^{2-s}\|_{L(L^2)} + \frac{1}{2}\|u_x\|_{L^\infty} \|\Lambda^{s-1} w_1\|_{L^2} \\
&\quad + \|\Lambda^{s-2} w_2\|_{L^2}^2 \|[\Lambda^{s-2}, u]\Lambda^{3-s}\|_{L(L^2)} + \frac{1}{2}\|u_x\|_{L^\infty} \|\Lambda^{s-2} w_2\|_{L^2} \\
&\leqslant c\big(\|U\|_{H^s} + |\gamma|\big)\big(\|w_1\|_{H^{s-1}}^2 + \|w_2\|_{H^{s-2}}^2\big) \\
&= c\big(\|U\|_{H^s} + |\gamma|\big)\|W\|_{H^{s-1}\times H^{s-2}}^2.
\end{aligned}
$$

By integrating both of sides in the above the estimate, it follows that $A(U) \in G(H^{s-1} \times H^{s-2}, 1, c(\|U\|_{H^s} + \gamma))$. □

Lemma A.5. *The operator $A(U)$ defined by* (A.2) *belongs to $L(H^s \times H^{s-1}, H^{s-1} \times H^{s-2})$. Moreover*

$$
\big\|\big(A(U) - A(V)\big)W\big\|_{H^{s-1}\times H^{s-2}} \leqslant \mu_1 \|U - V\|_{H^s\times H^{s-1}} \|W\|_{H^s\times H^{s-1}}, \quad U, V, W \in H^s \times H^{s-1}. \tag{A.8}
$$

Proof. In view of (A.2), we have

$$
\begin{aligned}
\big(A(U) - A(V)\big)W &= \begin{pmatrix} (u-\gamma)\partial_x - (v_1-\gamma)\partial_x & 0 \\ 0 & u\partial_x - v_1\partial_x \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\
&= \begin{pmatrix} (u-v_1)\partial_x w_1 \\ (u-v_1)\partial_x w_2 \end{pmatrix}.
\end{aligned}
$$

Since H^{s-1} $(s \geqslant 2)$ is a Banach algebra, taking $r = s-1$, $t = s-2$ in Lemma A.2, we have

$$
\begin{aligned}
\big\|\big(A(U) - A(V)\big)W\big\|_{H^{s-1}\times H^{s-2}} &\leqslant \big\|(u-v_1)\partial_x w_1\big\|_{H^{s-1}} + \big\|(u-v_1)\partial_x w_2\big\|_{H^{s-2}} \\
&\leqslant c\|u - v_1\|_{H^{s-1}}\big(\|\partial_x w_1\|_{H^{s-1}} + \|\partial_x w_2\|_{H^{s-2}}\big) \\
&\leqslant c\|U - V\|_{H^{s-1}\times H^{s-2}} \|W\|_{H^{s-1}\times H^{s-2}}.
\end{aligned}
$$

Taking $V = 0$ in (A.8), we deduce that $A(U) \in L(H^s \times H^{s-1}, H^{s-1} \times H^{s-2})$. □

Lemma A.6. *(See [20].) Let $B(U) = QA(U)Q^{-1} - A(U)$, for $U \in H^s \times H^{s-1}$ $(s \geqslant 2)$. Then $B(U) \in L(H^{s-1} \times H^{s-2})$ and*

$$
\big\|\big(B(U) - B(V)\big)W\big\|_{H^{s-1}\times H^{s-2}} \leqslant \mu_2 \|U - V\|_{H^s\times H^{s-1}} \|W\|_{H^{s-1}\times H^{s-2}}, \quad U, V \in H^s \times H^{s-1},\ W \in H^{s-1} \times H^{s-2}.
$$

Lemma A.7. *(See [20].) Let $U \in H^s \times H^{s-1}$ $(s \geqslant 2)$. Then the operator defined by* (A.3) *is bounded on bounded sets in $H^s \times H^{s-1}$, and satisfies*

(a) $\|f(U) - f(V)\|_{H^s\times H^{s-1}} \leqslant \mu_3 \|U - V\|_{H^s\times H^{s-1}}$, $U, V \in H^s \times H^{s-1}$,
(b) $\|f(U) - f(V)\|_{H^{s-1}\times H^{s-2}} \leqslant \mu_4 \|U - V\|_{H^{s-1}\times H^{s-2}}$, $U, V \in H^s \times H^{s-1}$.

Proof of Theorem 3.1. The result follows from the combination of Lemmas A.4–A.7. □

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