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Exponential stabilization of variable coefficient wave equations in a generic tree with small time-delays in the nodal feedbacks☆

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ABSTRACT

In this paper, we study the stability of a general tree network of variable coefficient wave equations with a small delay term in the nodal feedbacks. Using the Lax–Milgram theorem and C_0-semigroup theory, we obtain the well-posedness of the system. By a detailed spectral analysis, we show that the spectrum of the system operator distributes in a strip parallel to the imaginary axis under certain conditions. Furthermore, we prove that there is a sequence of (generalized) eigenfunctions that forms a Riesz basis with parenthesis for the energy state space. As a consequence, we obtain the exponential stabilization of the closed-loop system under certain conditions.

1. Introduction

In the past decades, stabilization problems of 1-d multi-link flexible structures (networks of strings and beams) have been studied by many authors. For instance, [1–9] for networks of strings, [10–15] and more recent papers [16–19] for networks of beams. All works mentioned above were done under constant coefficients. Especially, Nicaise, Pignotti and Valein in [5,20] considered a 1-d wave network of the constant coefficients with the delay term in the nodal and boundary feedbacks. Using the method of observability inequality, they showed that if the coefficient of the delayed damping term is smaller than the one of the undelayed damping term, the observability implies the exponential decay of energy function of the system. However, if the equations have variable coefficients and there exist a small delay term in the nodal feedbacks, the stability analysis of the network becomes very complicated and difficult. The difficulty comes mainly from two aspects: one is differential equation with variable coefficients, and the other is the existence of time-delay terms. This is because there is no explicit solution such as D'Alembert's formula for differential equations of variable coefficients; at the same time the multiplier method fails to be applied for networks and small time-delay can induce some instabilities [21,22]. Even for a single differential equation of variable coefficients, its stability analysis also becomes more complex; for instance, see [23]. Therefore, for a variable coefficient network with time-delays, the stability analysis has been a challenging topic. In the present paper, we shall analyze stability of a generic tree network of variable coefficient wave equations with time delay terms. As a base of our research, we first obtain the well-posedness of the system. To get more detailed property for the system, we choose the spectral analysis method. We mainly adopt two steps: first we discuss the spectral distribution

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and basis properties of eigenfunction and generalized eigenfunctions, and second we analyze stability of the system based on the basis property of eigenfunctions. To get a basic distribution of the spectrum of the system operator, we use the Liouville transform to translate the variable coefficient equations into the equations in which the major terms are of constant coefficients and calculate asymptotic values of the eigenvalues by the asymptotic technique. In short, we overcome these obstacles using the Riesz basis approach and the asymptotic analysis technique of the spectrum of the system operator.

We begin by recalling some notations. Let $G = (V, E)$ be a simply connected graph, as defined by [24], where $V = \{a_0, a_1, a_2, a_3\}$ denotes the vertices set and $E = \{e_1, e_2, e_3\}$ denotes the edges set. The common vertex a_0 named interior node of the graph G and the vertices a_1, a_2 and a_3, each of them receiving only one edge, are called boundary nodes of the graph G. Assume that one of the boundary nodes, say a_3, is fixed and the others are free. Suppose that each of the edges $e_i (i = 1, 2, 3)$ has a finite arc length ℓ_i, which can be parameterized by its arc length by means of the function π_i defined by

$$\pi_i : [0, \ell_i] \longrightarrow e_i, \quad i = 1, 2, 3,$$

so that e_i can be identified as a real interval $[0, \ell_i] (i = 1, 2, 3)$, $\pi_i(0) = a_0$ and $\pi_i(\ell_i) = a_i$.

Suppose that the strings are expanded on G and coincide with G at rest. Denote by $u_i(x, t)$ $(i = 1, 2, 3)$ the displacement function of the ith string departing from the equilibrium position in position $\pi_i(x) \in e_i$ at time t. Thus, the dynamic behavior of the string networks is governed by the following partial differential equations

$$\begin{cases} \rho_i(x)\dfrac{\partial^2 u_i}{\partial t^2}(x,t) - \dfrac{\partial}{\partial x}\left[\sigma_i(x)\dfrac{\partial u_i}{\partial x}(x,t)\right] + q_i(x)u_i(x,t) = 0, \quad x \in (0,\ell_i),\ i = 1,2,3,\ t > 0, \\ u_1(0,t) = u_2(0,t) = u_3(0,t), \\ \displaystyle\sum_{i=1}^{3}\sigma_i(0)\frac{\partial u_i}{\partial x}(0,t) = \alpha_0\frac{\partial u_1}{\partial t}(0,t) + \beta_0\frac{\partial u_1}{\partial t}(0,t-\tau_0), \\ \sigma_1(\ell_1)\dfrac{\partial u_1}{\partial x}(\ell_1,t) = -\alpha_1\dfrac{\partial u_1}{\partial t}(\ell_1,t) - \beta_1\dfrac{\partial u_1}{\partial t}(\ell_1,t-\tau_1), \\ \dfrac{\partial u_2}{\partial x}(\ell_2,t) = 0, \qquad u_3(\ell_3,t) = 0, \\ u_i(x,0) = u_{i0}(x), \qquad \dfrac{\partial u_i}{\partial t}(x,0) = u_{i1}(x), \quad i = 1,2,3, \\ \dfrac{\partial u_1}{\partial t}(0,t-\tau_0) =: f_0(t-\tau_0), \quad 0 < t < \tau_0, \\ \dfrac{\partial u_1}{\partial t}(\ell_1,t-\tau_1) =: f_1(t-\tau_1), \quad 0 < t < \tau_1, \end{cases} \tag{1.1}$$

where $\rho_i(x) > 0$ is the mass density of the ith string, $\sigma_i(x) > 0$ is the elastic modulus of the same string and the rigidity coefficient $q_i(x) \geq 0$; $\alpha_i, \beta_i > 0, i = 0, 1$ are fixed real numbers such that $\beta_i < \alpha_i$, and $\tau_i > 0, i = 0, 1$ are time delays. In the present paper, we shall study the exponential stability of the system (1.1).

The rest of the paper is organized as follows. In Section 2, we discuss the well-posedness of system (1.1). In Section 3, our attention focus on the spectral distribution of the operator $\mathcal{A}$ determined by the system. Under certain conditions we prove that the spectrum of $\mathcal{A}$ distributes in a strip parallel to the imaginary axis. In the final section, we discuss the generation of Riesz basis and the exponential stability of the system.

2. Well-posedness of the system

In this section, we shall devote to the well-posedness of system (1.1) in an appropriate Hilbert space. Let $H^k(0, \ell_i)$ $(i = 1, 2, 3, k = 1, 2)$ be a Sobolev space.

Set $X := \{u = (u_i)_{i=1}^3 \in \prod_{i=1}^3 H^1(0, \ell_i) | u_i(0) = u_j(0), \forall i, j = 1, 2, 3;\ u_3(\ell_3) = 0\}$, equip it with an inner product

$$\langle u, v\rangle_X = \sum_{i=1}^{3}\int_0^{\ell_i}\left(\sigma_i(x)\frac{du_i(x)}{dx}\frac{\overline{dv_i}}{dx}(x) + q_i(x)u_i(x)\overline{v_i(x)}\right)dx, \quad \forall u, v \in X.$$

Assume that $\rho_i(x), \sigma_i(x), q_i(x) \in H^2(0, \ell_i)$, $i = 1, 2, 3$.

We introduce the auxiliary functions $z_i(s, t) = \frac{\partial u_1}{\partial t}(\ell_i, t - \tau_i s)$, $s \in [0, 1]$, $t > 0$, $i = 0, 1$. Thus the function $z_i(s, t)$ satisfy equation $\frac{\partial z_i}{\partial s}(s, t) = -\tau_i \frac{\partial z_i}{\partial t}(s, t)$. System (1.1) can be rewritten in the following form

$$\begin{cases} \rho_i(x)\dfrac{\partial^2 u_i}{\partial t^2}(x,t) - \dfrac{\partial}{\partial x}\left[\sigma_i(x)\dfrac{\partial u_i}{\partial x}(x,t)\right] + q_i(x)u_i(x,t) = 0, \quad x \in (0,\ell_i),\ i = 1,2,3,\ t > 0, \\ \dfrac{\partial z_i}{\partial s}(s,t) + \tau_i \dfrac{\partial z_i}{\partial t}(s,t) = 0, \quad s \in (0,1),\ i = 0,1,\ t > 0, \\ u_1(0,t) = u_2(0,t) = u_3(0,t), \quad t > 0, \\ \displaystyle\sum_{i=1}^{3} \sigma_i(0)\dfrac{\partial u_i}{\partial x}(0,t) = \alpha_0 \dfrac{\partial u_1}{\partial t}(0,t) + \beta_0 z_0(1,t), \quad t > 0, \\ \sigma_1(\ell_1)\dfrac{\partial u_1}{\partial x}(\ell_1,t) = -\alpha_1 \dfrac{\partial u_1}{\partial t}(\ell_1,t) - \beta_1 z_1(1,t), \quad t > 0, \\ u_3(\ell_3,t) = 0, \qquad \dfrac{\partial u_2}{\partial x}(\ell_2,t) = 0, \quad t > 0, \\ z_i(0,t) = \dfrac{\partial u_1}{\partial t}(\ell_i,t), \quad i = 0,1,\ t > 0, \\ u_i(x,0) = u_{i0}(x), \qquad \dfrac{\partial u_i}{\partial t}(x,0) = u_{i1}(x), \quad x \in (0,\ell_i),\ i = 1,2,3, \\ z_i(s,0) = f_i(-\tau_i s), \quad s \in [0,1],\ i = 0,1. \end{cases} \tag{2.1}$$

We define the state space by $\mathcal{H} = X \times \prod_{i=1}^{3} L^2(0,\ell_i) \times [L^2(0,1)]^2$ equipped with the norm

$$\|(u,v,z)\|^2 = \sum_{i=1}^{3}\int_0^{\ell_i}\left[\sigma_i(x)\left|\frac{du_i(x)}{dx}\right|^2 + q_i(x)|u_i(x)|^2 + \rho_i(x)|v_i(x)|^2\right]dx + \sum_{i=0}^{1}\int_0^1 |z_i(s)|^2 ds$$

for any $(u,v,z) \in \mathcal{H}$. Obviously, $(\mathcal{H}, \|\cdot\|)$ is a Hilbert space.

Define the operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} v \\ \left\{\dfrac{1}{\rho_i(x)}\left[\dfrac{d}{dx}\left[\sigma_i(x)\dfrac{du_i(x)}{dx}\right] - q_i(x)u_i(x)\right]\right\}_{i=1}^{3} \\ -\left\{\dfrac{1}{\tau_i}\dfrac{dz_i}{ds}\right\}_{i=0}^{1} \end{pmatrix}, \tag{2.2}$$

where

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u,v,z) \in X \cap \displaystyle\prod_{i=1}^{3} H^2(0,\ell_i) \times X \times [H^1(0,1)]^2 : \\ \quad \displaystyle\sum_{i=1}^{3}\sigma_i(0)\dfrac{du_i}{dx}(0) = \alpha_0 v_1(0) + \beta_0 z_0(1),\ \dfrac{du_2}{dx}(\ell_2) = 0, \\ \quad \sigma_1(\ell_1)\dfrac{du_1}{dx}(\ell_1) = -\alpha_1 v_1(\ell_1) - \beta_1 z_1(1),\ z_0(0) = v_1(0),\ z_1(0) = v_1(\ell_1). \end{array} \right\}. \tag{2.3}$$

Then, we rewrite system (1.1) as an evolutionary equation in $\mathcal{H}$ as follows

$$\begin{cases} \dfrac{dY(t)}{dt} = \mathcal{A}Y(t), \quad t > 0, \\ Y(0) = Y_0, \end{cases} \tag{2.4}$$

where $Y(t) = (u(\cdot,t), \frac{\partial u}{\partial t}(\cdot,t), z)^T$, $Y(0) = (U_0, U_1, \{f_i(-\tau_i\cdot)\}_{i=0}^1)^T \in \mathcal{H}$, $U_0 = \{u_{i0}\}_{i=1}^3$, $U_1 = \{u_{i1}\}_{i=1}^3$ are given.

To obtain the well-posedness of system (1.1), we shall show that $\mathcal{A}$ generates a C_0-semigroup of contractions in $\mathcal{H}$. To this end, notice that $0 < \beta_i < \alpha_i$, $i = 0, 1$ and here similar to the method used in [5,20] we choose positive real scalars b_i, $i = 0, 1$ such that

$$\tau_i\beta_i \le b_i \le \tau_i(2\alpha_i - \beta_i), \quad i = 0,1. \tag{2.5}$$

We now introduce another inner product on $\mathcal{H}$ defined by

$$\begin{aligned} \langle (u,v,z),(f,g,h)\rangle_{\mathcal{H}} := {} & \sum_{i=1}^{3}\int_0^{\ell_i}\left[\sigma_i(x)\frac{du_i(x)}{dx}\overline{\frac{df_i(x)}{dx}} + \rho_i(x)v_i(x)\overline{g_i(x)} + q_i(x)u_i(x)\overline{f_i(x)}\right]dx \\ & + \sum_{i=0}^{1} b_i \int_0^1 z_i(s)\overline{h_i(s)}ds \end{aligned}$$

for any $(u,v,z), (f,g,h) \in \mathcal{H}$. Obviously, this inner product is equivalent to the old one.

Theorem 2.1. *Let $\mathcal{A}$ be defined as (2.2)–(2.3). Then, $\mathcal{A}$ is a closed and densely defined linear dissipative operator in $\mathcal{H}$.*

Proof. It is easy to check that $\mathcal{A}$ is a densely defined and closed linear operator in $\mathcal{H}$. Here, we only prove that $\mathcal{A}$ is dissipative. For any $(u, v, z) \in D(\mathcal{A})$, we have

$$\begin{aligned}
\Re\langle \mathcal{A}(u,v,z),(u,v,z)\rangle_{\mathcal{H}} &= \frac{1}{2}\left[\langle \mathcal{A}(u,v,z),(u,v,z)\rangle_{\mathcal{H}} + \langle (u,v,z),\mathcal{A}(u,v,z)\rangle_{\mathcal{H}}\right] \\
&= \Re\sum_{i=1}^{3}\int_0^{\ell_i}\left\{\sigma_i(x)\frac{dv_i(x)}{dx}\overline{\frac{du_i(x)}{dx}} + q_i(x)v_i(x)\overline{u_i(x)} + \overline{v_i(x)}\right. \\
&\quad \times \left.\left[\frac{d}{dx}\left(\sigma_i(x)\frac{du_i(x)}{dx}\right) - q_i(x)u_i(x)\right]\right\}dx - \Re\sum_{i=0}^{1}\frac{b_i}{\tau_i}\int_0^1\frac{dz_i(s)}{ds}\overline{z_i(s)}ds \\
&= \Re\sum_{i=1}^{3}\left[\sigma_i(\ell_i)\frac{du_i}{dx}(\ell_i)\overline{v_i(\ell_i)} - \sigma_i(0)\frac{du_i}{dx}(0)\overline{v_i(0)}\right] - \frac{1}{2}\sum_{i=0}^{1}\frac{b_i}{\tau_i}(|z_i(1)|^2 - |z_i(0)|^2) \\
&= -\Re\left\{[\alpha_1 z_1(0) + \beta_1 z_1(1)]\overline{z_1(0)} + [\alpha_0 z_0(0) + \beta_0 z_0(1)]\overline{z_0(0)}\right\} \\
&\quad - \frac{1}{2}\sum_{i=0}^{1}\frac{b_i}{\tau_i}(|z_i(1)|^2 - |z_i(0)|^2) \\
&\le -\sum_{i=0}^{1}\alpha_i|z_i(0)|^2 + \frac{1}{2}\sum_{i=0}^{1}\beta_i[|z_i(1)|^2 + |z_i(0)|^2] - \frac{1}{2}\sum_{i=0}^{1}\frac{b_i}{\tau_i}(|z_i(1)|^2 - |z_i(0)|^2) \\
&\le -\frac{1}{2}\sum_{i=0}^{1}\left\{\left[2\alpha_i - \beta_i - \frac{b_i}{\tau_i}\right]|z_i(0)|^2 + \left[\frac{b_i}{\tau_i} - \beta_i\right]|z_i(1)|^2\right\} \le 0,
\end{aligned}$$

the last inequality is owed to (2.5). The desired result follows. □

The following theorem is necessary when we discuss the spectral distribution, which describes some basic properties of $\mathcal{A}$, whose proof is somewhat similar to the one used in [5,20].

Theorem 2.2. *Let $\mathcal{A}$ be defined by (2.2)–(2.3). Then $0 \in \rho(\mathcal{A})$ and $\mathcal{A}^{-1}$ is compact in $\mathcal{H}$. Hence, $\mathcal{A}$ generates a C_0-semigroup of contractions in $\mathcal{H}$.*

Proof. We first show that $\mathcal{A}^{-1}$ exists. To this end, let $(u, v, z) \in D(\mathcal{A})$ such that $\mathcal{A}(u, v, z) = 0$. Then $v = 0, z = 0$ and $u_i(x)$ satisfies

$$\frac{d}{dx}\left[\sigma_i(x)\frac{du_i(x)}{dx}\right] - q_i(x)u_i(x) = 0, \quad x \in (0,\ell_i),\ i = 1,2,3, \tag{2.6}$$

$$u_1(0) = u_2(0) = u_3(0), \tag{2.7}$$

$$\sum_{i=1}^{3}\sigma_i(0)\frac{du_i}{dx}(0) = 0, \tag{2.8}$$

$$\frac{du_i}{dx}(\ell_i) = 0, \quad i = 1,2, \tag{2.9}$$

$$u_3(\ell_3) = 0. \tag{2.10}$$

Multiplying Eq. (2.6) by $\overline{u_i}$, integrating over the interval $[0, \ell_i]$ and sum them up for $i = 1, 2, 3$, utilizing the conditions (2.7)–(2.10), we deduce

$$\begin{aligned}
0 &= \sum_{i=1}^{3}\left\{\int_0^{\ell_i}\frac{d}{dx}\left[\sigma_i(x)\frac{du_i(x)}{dx}\right]\overline{u_i(x)} - q_i(x)|u_i(x)|^2\right\}dx \\
&= \sum_{i=1}^{3}\sigma_i(x)\frac{du_i(x)}{dx}\overline{u_i(x)}\Big|_0^{\ell_i} - \sum_{i=1}^{3}\int_0^{\ell_i}\left[\sigma_i(x)\left|\frac{du_i(x)}{dx}\right|^2 + q_i(x)|u_i(x)|^2\right]dx \\
&= -\sum_{i=1}^{3}\int_0^{\ell_i}\left[\sigma_i(x)\left|\frac{du_i(x)}{dx}\right|^2 + q_i(x)|u_i(x)|^2\right]dx.
\end{aligned}$$

Since $\sigma_i(x) > 0, q_i(x) \ge 0$, we deduce from above that $u_i(x) \equiv 0$, $i = 1, 2, 3$, which implies $(u, v, z) \equiv 0$. So $\mathcal{A}$ is injective.

Next, we claim that $\mathcal{A}$ is surjective. Indeed, for any $(f, g, h) \in \mathcal{H}$, $\mathcal{A}(u, v, z) = (f, g, h)$ implies that

$$v_i(x) = f_i(x), \quad x \in (0, \ell_i),\ i = 1, 2, 3, \tag{2.11}$$

$$\frac{d}{dx}\left[\sigma_i(x)\frac{du_i(x)}{dx}\right] - q_i(x)u_i(x) = g_i(x)\rho_i(x), \quad x \in (0, \ell_i),\ i = 1, 2, 3, \tag{2.12}$$

$$-\frac{1}{\tau_i}\frac{dz_i(s)}{ds} = h_i(s), \quad s \in (0, 1),\ i = 0, 1, \tag{2.13}$$

$$\sum_{i=1}^{3}\sigma_i(0)\frac{du_i}{dx}(0) = \alpha_0 f_1(0) + \beta_1 z_0(1), \tag{2.14}$$

$$\frac{du_2}{dx}(\ell_2) = 0, \qquad u_3(\ell_3) = 0, \tag{2.15}$$

$$\sigma_1(\ell_1)\frac{du_1}{dx}(\ell_1) = -\alpha_1 f_1(\ell_1) - \beta_1 z_1(1), \tag{2.16}$$

$$z_0(0) = v_1(0) = f_1(0), \qquad z_1(0) = v_1(\ell_1) = f_1(\ell_1). \tag{2.17}$$

From (2.14) and (2.17), we get

$$z_0(s) = f_1(0) - \tau_0\int_0^s h_0(s)ds, \qquad z_1(s) = f_1(\ell_1) - \tau_1\int_0^s h_1(s)ds.$$

In what follows, we shall seek for a solution to (2.12) which meets (2.14)–(2.16).

Multiply (2.12) by test functions $\phi_i(x)$ $(i = 1, 2, 3)$, integrate them on the interval $(0, \ell_i)$ and use integration by parts in x to get

$$\begin{aligned}\sum_{i=1}^{3}\int_0^{\ell_i} g_i(x)\rho_i(x)\phi_i(x)dx &= \sum_{i=1}^{3}\int_0^{\ell_i}\frac{d}{dx}\left[\sigma_i(x)\frac{du_i(x)}{dx}\right]\phi_i(x)dx - \sum_{i=1}^{3}\int_0^{\ell_i} q_i(x)u_i(x)\phi_i(x)dx\\ &= -\sum_{i=1}^{3}\int_0^{\ell_i}\sigma_i(x)\frac{du_i(x)}{dx}\frac{d\phi_i(x)}{dx}dx - \sum_{i=1}^{3}\int_0^{\ell_i} q_i(x)u_i(x)\phi_i(x)dx\\ &\quad + \sum_{i=1}^{3}\sigma_i(x)\frac{du_i(x)}{dx}\phi_i(x)\Big|_0^{\ell_i}.\end{aligned} \tag{2.18}$$

For $(u, v, z) \in D(\mathcal{A})$, we have

$$\begin{aligned}\sum_{i=1}^{3}\sigma_i(x)\frac{du_i(x)}{dx}\phi_i(x)\Big|_0^{\ell_i} &= \sigma_1(\ell_1)\frac{du_1}{dx}(\ell_1)\phi_1(\ell_1) - \phi_1(0)\sum_{i=1}^{3}\sigma_i(0)\frac{du_i}{dx}(0)\\ &= -\phi_1(\ell_1)[\alpha_1 f_1(\ell_1) + \beta_1 z_1(1)] - \phi_1(0)[\alpha_0 f_1(0) + \beta z_0(1)].\end{aligned}$$

Now we define a bilinear function $B(w, z)$ on X by

$$B[w, z] = \sum_{i=1}^{3}\int_0^{\ell_i}\left[\sigma_i(x)\frac{dw_i(x)}{dx}\overline{\frac{dz_i}{dx}}(x) + q_i(x)w_i(x)\overline{z_i(x)}\right]dx, \quad \forall w, z \in X.$$

Obviously,

$$|B[w, z]| \le M\|w\|_X \cdot \|z\|_X, \quad w, z \in X,$$

where M is a positive constant. Moreover, B is coercive since

$$B[w, w] = \sum_{i=1}^{3}\int_0^{\ell_i}\sigma_i(x)\left|\frac{dw_i(x)}{dx}\right|^2 + q_i(x)|w_i(x)|^2 dx = \|w\|_X^2, \quad \forall w \in X.$$

If we choose $\phi(x) = \{\phi_i(x)\}_{i=1}^3 \in [C_0^\infty(0, \ell_i)]^3 \cap X \subset X$, then there exists a unique solution $u \in X$ of Eq. (2.18) by Lax–Milgram's lemma, where $u = \{u_i(x)\}_{i=1}^3$ satisfies

$$\frac{d}{dx}\left[\sigma_i(x)\frac{du_i(x)}{dx}\right] = g_i(x)\rho_i(x) + q_i(x)u_i(x). \tag{2.19}$$

This implies that $u \in \prod_{i=1}^{3} H^2(0, \ell_i)$ and hence $u \in \prod_{i=1}^{3} H^2(0, \ell_i) \cap X$. Again we select special $\phi_i(x)$, insert (2.19) into (2.18) and observe that

$$\sum_{i=1}^{3} \sigma_i(0) \frac{du_i}{dx}(0) = -\alpha_0 f_1(0) - \beta_0 z_0(1),$$

$$\frac{du_1}{dx}(\ell_1) = \alpha_1 f_1(\ell_1) + \beta_1 z_1(1).$$

Up until now, we have found $(u, v, z) = (u, f, z) \in D(\mathcal{A})$ and (u, v, z) satisfies equation $\mathcal{A}(u, v, z) = (f, g, h)$. So, $\mathcal{A}$ is a surjective operator. As a result of the Inverse Operator Theorem, $\mathcal{A}^{-1}$ is continuous, which indicates that $0 \in \rho(\mathcal{A})$. Hence, the Lumer–Phillips Theorem (see [25]) asserts that $\mathcal{A}$ generates a C_0 semigroup of contractions on $\mathcal{H}$. Moreover, in view of $D(\mathcal{A}) \subset \prod_{i=1}^{3} H^2(0, \ell_i) \times X \times H^1(0, 1)^2$, $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$ owing to Sobolev's Embedding Theorem. □

The above two theorems together with semigroup theory yield the following results.

Corollary 2.1. *System (2.4) is well-posed in $\mathcal{H}$.*

Corollary 2.2. *The spectrum of $\mathcal{A}$ consists of all isolated eigenvalues of finite multiplicity* [26].

3. Eigenvalue problem of $\mathcal{A}$

In order to investigate the properties of the semigroup generated by $\mathcal{A}$, we need to find out some spectral properties of $\mathcal{A}$. In this section, we shall calculate the spectrum of $\mathcal{A}$. According to Corollary 2.2, the spectrum of $\mathcal{A}$ consists of all isolated eigenvalues. Therefore, we only need to discuss the eigenvalue problem of $\mathcal{A}$.

Let $\lambda \in \mathbb{C}$, we consider the existence of a nonzero solution to the equation

$$(\lambda I - \mathcal{A})(u, v, z) = 0, \quad (u, v, z) \in \mathcal{D}(\mathcal{A}).$$

From it we have $\lambda z_i + \frac{1}{\tau_i}\frac{dz_i}{ds} = 0$, $i = 0, 1$, which implies $z_i(s) = z_i(0)e^{-\lambda\tau_i s}$.

More precisely,

$$z_0(s) = v_1(0)e^{-\lambda\tau_0 s}, \qquad z_1(s) = v_1(\ell_1)e^{-\lambda\tau_1 s}, \tag{3.1}$$

since $z_0(0) = v_1(0)$ and $z_1(0) = v_1(\ell_1)$.

Again, from $(\lambda I - \mathcal{A})$ $(u, v, z) = 0$ as well as (3.1), we have $v = \lambda u$ and $u = (u_i)_{i=1}^{3}$ satisfies the following boundary eigenvalue problem:

$$\lambda^2 u_i(x) - \frac{1}{\rho_i(x)}\left[\frac{d}{dx}\left(\sigma_i(x)\frac{du_i(x)}{dx}\right) - q_i(x)u_i(x)\right] = 0, \quad x \in (0, \ell_i),\ i = 1, 2, 3, \tag{3.2}$$

$$\sigma_1(\ell_1)\frac{du_1}{dx}(\ell_1) = -\lambda u_1(\ell_1)[\alpha_1 + \beta_1 e^{-\lambda\tau_1}], \tag{3.3}$$

$$\frac{du_2}{dx}(\ell_2) = 0, \tag{3.4}$$

$$u_3(\ell_3) = 0, \tag{3.5}$$

$$u_1(0) = u_2(0) = u_3(0), \tag{3.6}$$

$$\sum_{i=1}^{3} \sigma_i(0)\frac{du_i}{dx}(0) = \lambda u_1(0)[\alpha_0 + \beta_0 e^{-\lambda\tau_0}]. \tag{3.7}$$

Due to the variable coefficients in (3.2), we cannot get an explicit expression of its solution. Indeed, in our analysis we do not need an exact expression of the solution and only need an asymptotic expression of the solution in λ of (3.2)–(3.7).

To obtain an asymptotic expression of the solution, we will introduce some new functions to transform the family of Eqs. (3.2)–(3.7) into a standard form.

At first, we expand (3.2) to the following equation

$$\lambda^2 \frac{\rho_i(x)}{\sigma_i(x)}u_i(x) - \frac{1}{\sigma_i(x)}\frac{d\sigma_i}{dx}(x)\frac{du_i(x)}{dx} - \frac{d^2u_i}{dx^2}(x) + \frac{q_i(x)}{\sigma_i(x)}u_i(x) = 0, \quad i = 1, 2, 3. \tag{3.8}$$

Then we define a new independent variable $\xi_i(x) := \int_0^x \sqrt{\frac{\rho_i(\theta)}{\sigma_i(\theta)}}d\theta$, $x \in (0, \ell_i)$ and a new function

$$w_i(\xi_i) := \frac{1}{\sqrt{x'(\xi_i)}}\sqrt{\sigma_i(x(\xi_i))}u_i(x(\xi_i)), \tag{3.9}$$

where $x(\xi_i)$ is the inverse function of $\xi_i(x)$ and the prime denotes the derivative with respect to ξ_i. Furthermore, we set

$$m_i := \int_0^{\ell_i} \sqrt{\frac{\rho_i(\theta)}{\sigma_i(\theta)}} d\theta, \tag{3.10}$$

$$\phi_i(\xi_i) := -\frac{3}{4}(x'(\xi_i))^{-2}(x''(\xi_i))^2 + \frac{1}{2}(x'(\xi_i))^{-1}x'''(\xi_i), \tag{3.11}$$

$$b_i(x) := -\left[\frac{1}{4}\left(\frac{\frac{d\sigma_i}{dx}(x)}{\sigma_i(x)}\right)^2 - \frac{1}{2}\left(\frac{\frac{d^2\sigma_i}{dx^2}}{\sigma_i}\right)(x) - \frac{q_i(x)}{\sigma_i(x)}\right]\frac{\sigma_i(x)}{\rho_i(x)}, \quad i = 1, 2, 3. \tag{3.12}$$

After inserting (3.9), (3.11) and (3.12) into (3.8) as well as (3.3)–(3.7), we obtain the following desired forms

$$w_i''(\xi_i) - \lambda^2 w_i(\xi_i) = [b_i(x(\xi_i)) - \phi_i(\xi_i)]w_i(\xi_i), \quad \xi_i \in (0, m_i),\ i = 1, 2, 3, \tag{3.13}$$

$$\sigma_1(\ell_1)w_1'(m_1) + \left[c_1(\ell_1) + \lambda\left(\frac{\rho_1(\ell_1)}{\sigma_1(\ell_1)}\right)^{-1/2}(\alpha_1 + \beta_1 e^{-\lambda\tau_1})\right]w_1(m_1) = 0, \tag{3.14}$$

$$\sigma_2(\ell_2)w_2'(m_2) + c_2(\ell_2)w_2(m_2) = 0, \tag{3.15}$$

$$w_3(m_3) = 0, \tag{3.16}$$

$$(\rho_1(0)\sigma_1(0))^{-1/4}\, w_1(0) = (\rho_2(0)\sigma_2(0))^{-1/4}\, w_2(0) = (\rho_3(0)\sigma_3(0))^{-1/4}\, w_3(0), \tag{3.17}$$

$$\sum_{i=1}^{3}[\rho_i(0)\sigma_i(0)]^{1/4}w_i'(0) + \sum_{i=1}^{3} d_i(0)w_i(0) = \lambda\,(\rho_1(0)\sigma_1(0))^{-1/4}\,[\alpha_0 + \beta_0 e^{-\lambda\tau_0}]w_1(0), \tag{3.18}$$

where

$$c_i(x) := \frac{1}{4}\left(\frac{\sigma_i(x)}{\rho_i(x)}\right)^{-1}\sigma_i(x)\left(\frac{\sigma_i}{\rho_i}\right)_x(x) - \frac{1}{2}\sigma_{ix}(x)\left(\frac{\rho_i(x)}{\sigma_i(x)}\right)^{-1/2}, \quad i = 1, 2, \tag{3.19}$$

$$d_i(x) := \frac{1}{4}\sigma_i(x)^{1/2}\left(\frac{\sigma_i(x)}{\rho_i(x)}\right)^{-5/4}\left(\frac{\sigma_i}{\rho_i}\right)_x(x) - \frac{1}{2}\sigma_{ix}(x)\,(\rho_i(x)\sigma_i(x))^{-1/4}, \quad i = 1, 2, 3. \tag{3.20}$$

Obviously, the eigenvalue problem of (3.2)–(3.7) is equivalent to that of (3.13)–(3.18).

We are now in a position to determine the eigenvalues of $\mathcal{A}$. In other words, we will determine the eigenvalues of (3.13)–(3.18). According to the theory of ordinary differential equations, there exist two linear independent solutions $F_i(\lambda, \xi_i)$ and $\Psi_i(\lambda, \xi_i)$ to (3.13). The general solutions to (3.13) are of the form

$$w_i(\xi) = a_i^{(1)}F_i(\lambda, \xi_i) + a_i^{(2)}\Psi_i(\lambda, \xi_i), \quad i = 1, 2, 3.$$

Substituting them into (3.14)–(3.18) leads to an algebraic equation

$$\begin{pmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{23} & A_{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & F_3(\lambda, m_3) & \psi_3(\lambda, m_3) \\ A_{41} & A_{42} & A_{43} & A_{44} & 0 & 0 \\ 0 & 0 & A_{53} & A_{54} & A_{55} & A_{56} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{pmatrix}\begin{pmatrix} a_1^{(1)} \\ a_1^{(2)} \\ a_2^{(1)} \\ a_2^{(2)} \\ a_3^{(1)} \\ a_3^{(2)} \end{pmatrix} = 0,$$

where

$$A_{11} = \sigma_1(\ell_1)F_1'(\lambda, m_1) + \left[c_1(\ell_1) + \lambda\left(\frac{\rho_1(\ell_1)}{\sigma_1(\ell_1)}\right)^{-\frac{1}{2}}(\alpha_1 + \beta_1 e^{-\lambda\tau_1})\right]F_1(\lambda, m_1),$$

$$A_{12} = \sigma_1(\ell_1)\psi_1'(\lambda, m_1) + \left[c_1(\ell_1) + \lambda\left(\frac{\rho_1(\ell_1)}{\sigma_1(\ell_1)}\right)^{-\frac{1}{2}}(\alpha_1 + \beta_1 e^{-\lambda\tau_1})\right]\psi_1(\lambda, m_1),$$

$$A_{23} = \sigma_2(\ell_1)F_2'(\lambda, m_2) + c_2(\ell_2)F_2(\lambda, m_2), A_{24} = \sigma_2(\ell_1)\psi_2'(\lambda, m_2) + c_2(\ell_2)\psi_2(\lambda, m_2),$$

$$A_{41} = (\rho_1(0)\sigma_1(0))^{-\frac{1}{4}}F_1(\lambda, 0), \qquad A_{42} = (\rho_1(0)\sigma_1(0))^{-\frac{1}{4}}\psi_1(\lambda, 0),$$

$$A_{43} = -(\rho_2(0)\sigma_2(0))^{-\frac{1}{4}}F_2(\lambda, 0), \qquad A_{44} = -(\rho_2(0)\sigma_2(0))^{-\frac{1}{4}}\psi_2(\lambda, 0),$$

$$
\begin{aligned}
&A_{53} = (\rho_2(0)\sigma_2(0))^{-\frac{1}{4}} F_2(\lambda, 0), \qquad A_{54} = (\rho_2(0)\sigma_2(0))^{-\frac{1}{4}} \psi_2(\lambda, 0),\\
&A_{55} = -(\rho_3(0)\sigma_3(0))^{-\frac{1}{4}} F_3(\lambda, 0), \qquad A_{56} = -(\rho_3(0)\sigma_3(0))^{-\frac{1}{4}} \psi_3(\lambda, 0),\\
&A_{61} = (\rho_1(0)\sigma_1(0))^{\frac{1}{4}} F_1'(\lambda, 0) + \left[d_1(0) - \lambda(\rho_1(0)\sigma_1(0))^{-\frac{1}{4}}(\alpha_0 + \beta_0 e^{-\lambda\tau_0})\right] F_1(\lambda, 0),\\
&A_{62} = (\rho_1(0)\sigma_1(0))^{\frac{1}{4}} \psi_1'(\lambda, 0) + \left[d_1(0) - \lambda(\rho_1(0)\sigma_1(0))^{-\frac{1}{4}}(\alpha_0 + \beta_0 e^{-\lambda\tau_0})\right] \psi_1(\lambda, 0),\\
&A_{63} = (\rho_2(0)\sigma_2(0))^{\frac{1}{4}} F_2'(\lambda, 0) + d_2(0) F_2(\lambda, 0), \qquad A_{64} = (\rho_2(0)\sigma_2(0))^{\frac{1}{4}} \psi_2'(\lambda, 0) + d_2(0)\psi_2(\lambda, 0),\\
&A_{65} = (\rho_3(0)\sigma_3(0))^{\frac{1}{4}} F_3'(\lambda, 0) + d_3(0) F_3(\lambda, 0), \qquad A_{66} = (\rho_3(0)\sigma_3(0))^{\frac{1}{4}} \psi_3'(\lambda, 0) + d_3(0)\psi_3(\lambda, 0).
\end{aligned}
$$

Denoting by $\delta(\lambda)$ the above coefficients matrix, we have the following result.

Proposition 3.1. *$\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}$ if and only if λ is a zero point of the determinant of the coefficient matrix $\delta(\lambda)$.*

To analyze the asymptotic spectrum of $\mathcal{A}$, we shall use the asymptotic analysis technique. For $\lambda \in \mathbb{C}$ with $|\lambda| \geq \delta > 0$, $F_i(\lambda, \xi_i)$ and $\Psi_i(\lambda, \xi_i)$ have the asymptotic expressions (see, [27, Theorem 1, p. 49]):

$$
F_i(\lambda, \xi_i) = e^{\lambda\xi_i}\left[1 + O\left(\frac{1}{\lambda}\right)\right], \qquad \Psi_i(\lambda, \xi_i) = e^{-\lambda\xi_i}\left[1 + O\left(\frac{1}{\lambda}\right)\right], \tag{3.21}
$$

$$
F_i'(\lambda, \xi_i) = \lambda e^{\lambda\xi_i}\left[1 + O\left(\frac{1}{\lambda}\right)\right], \qquad \Psi_i'(\lambda, \xi_i) = \lambda e^{-\lambda\xi_i}\left[-1 + O\left(\frac{1}{\lambda}\right)\right]. \tag{3.22}
$$

Therefore, the general solution of Eq. (3.13) can be asymptotically expressed as

$$
\begin{aligned}
w_i(\lambda, \xi_i) &= A_i(\lambda) F_i(\lambda, \xi_i) + B_i(\lambda)\Psi_i(\lambda, \xi_i)\\
&= A_i(\lambda) e^{\lambda\xi_i}\left[1 + O\left(\frac{1}{\lambda}\right)\right] + B_i(\lambda) e^{-\lambda\xi_i}\left[1 + O\left(\frac{1}{\lambda}\right)\right],
\end{aligned} \tag{3.23}
$$

with derivative

$$
w_i'(\lambda, \xi_i) = \lambda A_i(\lambda) e^{\lambda\xi_i}\left[1 + O\left(\frac{1}{\lambda}\right)\right] - \lambda B_i(\lambda) e^{-\lambda\xi_i}\left[1 + O\left(\frac{1}{\lambda}\right)\right],
$$

where $A_i(\lambda)$ and $B_i(\lambda)$ are coefficients dependent on λ. By using the notations

$$
\begin{aligned}
&[a]_1 = a + O\left(\frac{1}{\lambda}\right), \qquad k := \frac{\sqrt{\sigma_1(\ell_1)}}{\sqrt{\rho_1(\ell_1)}}, \qquad \tilde{k}_i := (\sigma_i(0)\rho_i(0))^{-1/4}, \quad i = 1, 2, 3,\\
&h_i^- := \frac{3\sqrt{\sigma_i(0)\rho_i(0)} - \alpha_0}{3\sqrt[4]{\rho_i(0)\sigma_i(0)}} = \frac{3 - \alpha_0\tilde{k}_i^2}{3\tilde{k}_i},\\
&h_i^+ := \frac{3\sqrt{\sigma_i(0)\rho_i(0)} + \alpha_0}{3\sqrt[4]{\rho_i(0)\sigma_i(0)}} = \frac{3 + \alpha_0\tilde{k}_i^2}{3\tilde{k}_i}, \quad i = 1, 2, 3,
\end{aligned}
$$

we obtain the asymptotic expression $\Delta(\lambda)$ to the coefficient matrix $\delta(\lambda)$ of those equations, which can be resulted from inserting (3.23) into (3.14)–(3.18). In fact, $\Delta(\lambda) =$

$$
\left(\begin{array}{cccccc}
\lambda e^{\lambda(m_1-\tau_1)}[\beta_1 k]_1 + \lambda e^{\lambda m_1}[\alpha_1 k + \sigma_1(\ell_1)]_1 & 0 & 0 & \lambda e^{\lambda(-\tau_1-m_1)}[\beta_1 k]_1 + \lambda e^{-\lambda m_1}[\alpha_1 k - \sigma_1(\ell_1)]_1 & 0 & 0\\
0 & \lambda[\sigma_2(\ell_2)]_1 e^{\lambda m_2} & 0 & 0 & \lambda[-\sigma_2(\ell_2)]_1 e^{-\lambda m_2} & 0\\
0 & 0 & e^{\lambda m_3}[1]_1 & 0 & 0 & e^{-\lambda m_3}[1]_1\\
\left[\tilde{k}_1\right]_1 & -[\tilde{k}_2]_1 & 0 & \left[\tilde{k}_1\right]_1 & -[\tilde{k}_2]_1 & 0\\
\left[\tilde{k}_1\right]_1 & 0 & -[\tilde{k}_3]_1 & \left[\tilde{k}_1\right]_1 & 0 & -[\tilde{k}_3]_1\\
-\lambda e^{-\lambda\tau_0}\left[\frac{\beta_0}{3}\tilde{k}_1\right]_1 + \lambda[h_1^-]_1 & -\lambda e^{-\lambda\tau_0}\left[\frac{\beta_0}{3}\tilde{k}_2\right]_1 + \lambda[h_2^-]_1 & -\lambda e^{-\lambda\tau_0}\left[\frac{\beta_0}{3}\tilde{k}_3\right]_1 + \lambda[h_3^-]_1 & -\lambda e^{-\lambda\tau_0}\left[\frac{\beta_0}{3}\tilde{k}_1\right]_1 - \lambda[h_1^+]_1 & -\lambda e^{-\lambda\tau_0}\left[\frac{\beta_0}{3}\tilde{k}_2\right]_1 - \lambda[h_2^+]_1 & -\lambda e^{-\lambda\tau_0}\left[\frac{\beta_0}{3}\tilde{k}_3\right]_1 - \lambda[h_3^+]_1
\end{array}\right)
$$

$$=: \begin{pmatrix} a_{11}+b_{11} & 0 & 0 & a_{14}+b_{14} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{61}+b_{61} & a_{62}+b_{62} & a_{63}+b_{63} & a_{64}+b_{64} & a_{65}+b_{65} & a_{66}+b_{66} \end{pmatrix}. \tag{3.24}$$

$$\begin{aligned} \text{Thereby, } \det \Delta(\lambda) &= |\Delta(\lambda)| \\ &= \begin{vmatrix} a_{11} & 0 & 0 & a_{14} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 & a_{14} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{vmatrix} \\ &\quad + \begin{vmatrix} b_{11} & 0 & 0 & b_{14} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{vmatrix} + \begin{vmatrix} b_{11} & 0 & 0 & b_{14} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{vmatrix} \\ &=: |A_1| + |A_2| + |B_1| + |B_2|. \end{aligned}$$

A direct calculation gives

$$|A_1| = -\lambda^3 e^{-\lambda(\tau_0+\tau_1)} \left[k\sigma_2(\ell_2)\beta_0\beta_1 \prod_{i=1}^{3} \tilde{k}_i \right]_1 \left(e^{\lambda \sum_{i=1}^3 m_i} + e^{-\lambda \sum_{i=1}^3 m_i} \right) + o\left(\lambda^2 e^{-\lambda(\tau_0+\tau_1)} \cosh\left(\lambda \sum_{i=1}^{3} m_i \right) \right),$$

$$\begin{aligned} |A_2| &= -\lambda^3 e^{-\lambda\tau_1} [k\sigma_2(\ell_2)\beta_1]_1 \left\{ e^{\lambda \sum_{i=1}^3 m_i} \left[\sum_{i=1}^{3} \prod_{j=1, j\neq i}^{3} \tilde{k}_j h_i^+ \right]_1 + e^{-\lambda \sum_{i=1}^3 m_i} \left[\sum_{i=1}^{3} \prod_{j=1, j\neq i}^{3} \tilde{k}_j h_i^- \right]_1 \right\} \\ &\quad + o\left(\lambda^2 e^{-\lambda\tau_1} \cosh\left(\lambda \sum_{i=1}^{3} m_i \right) \right), \end{aligned}$$

$$\begin{aligned} |B_1| &= -\lambda^3 e^{-\lambda\tau_0} \left[\beta_0 \prod_{i=1}^{3} \tilde{k}_i \right]_1 \left\{ e^{\lambda \sum_{i=1}^3 m_i} [(\alpha_1 k + \sigma_1(\ell_1))\sigma_2(\ell_2)]_1 + e^{-\lambda \sum_{i=1}^3 m_i} [(\alpha_1 k - \sigma_1(\ell_1))\sigma_2(\ell_2)]_1 \right\} \\ &\quad + o\left(\lambda^2 e^{-\lambda\tau_0} \cosh\left(\lambda \sum_{i=1}^{3} m_i \right) \right), \end{aligned}$$

$$\begin{aligned} |B_2| &= -\lambda^3 e^{\lambda \sum_{i=1}^3 m_i} [(\alpha_1 k + \sigma_1(\ell_1))\sigma_2(\ell_2)]_1 \left[\sum_{i=1}^{3} \prod_{j=1, j\neq i}^{3} \tilde{k}_j h_i^+ \right]_1 - \lambda^3 e^{-\lambda \sum_{i=1}^3 m_i} [(\alpha_1 k - \sigma_1(\ell_1))\sigma_2(\ell_2)]_1 \\ &\quad \times \left[\sum_{i=1}^{3} \prod_{j=1, j\neq i}^{3} \tilde{k}_j h_i^- \right]_1 + o\left(\lambda^2 \cosh\left(\lambda \sum_{i=1}^{3} m_i \right) \right), \end{aligned}$$

where the expression in the form $o(\alpha(\lambda))$ means some function satisfying $\lim_{|\lambda|\to+\infty} \frac{o(\alpha(\lambda))}{\alpha(\lambda)} = 0$. Hence, we have

$$\lim_{\Re\lambda\to-\infty} \frac{\det \Delta(\lambda)}{\lambda^3 e^{-\lambda\left(\sum_{i=1}^{3} m_i + \tau_0 + \tau_1\right)}} = -k\sigma_2(\ell_2)\beta_0\beta_1 \prod_{i=1}^{3} \tilde{k}_i \neq 0, \tag{3.25}$$

$$\lim_{\Re\lambda\to+\infty} \frac{\det \Delta(\lambda)}{\lambda^3 e^{\lambda \sum_{i=1}^{3} m_i}} = -[\alpha_1 k + \sigma_1(\ell_1)]\sigma_2(\ell_2) \left(\sum_{i=1}^{3} \prod_{j=1, j\neq i}^{3} \tilde{k}_j h_i^+ \right) \neq 0. \tag{3.26}$$

Theorem 3.1. *Let $\mathcal{A}$ be defined by (2.2)–(2.3). Then the spectrum of $\mathcal{A}$ distributes in a strip parallel to the imaginary axis. Moreover, $\sigma(\mathcal{A})$ is a union of finitely many separable sets.*

Proof. The spectrum of $\mathcal{A}$ distributes in a strip parallel to the imaginary axis is a direct result of (3.25) and (3.26), which implies that $|\Delta(\lambda)|$ is a sine-type function in λ. Then the conclusion follows from the Levin lemma [28]. □

4. The basis property and the exponential stability of the system

In this section, we shall establish the Riesz basis property and exponential stability of the system (2.4). To show the Riesz basis property of system (2.4), verification of the completeness about generalized eigenvectors of $\mathcal{A}$ in $\mathcal{H}$ is necessary. The following proposition gives a sufficient condition, which comes from [6].

Proposition 4.1. *Let $\mathcal{A}$ be the generator of a C_0-semigroup in a Hilbert space $\mathcal{H}$. Assume that $\mathcal{A}$ is discrete and for $\lambda \in \rho(\mathcal{A}^*)$ ($\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$), $R(\lambda, \mathcal{A}^*)$ is of the form*

$$R(\lambda, \mathcal{A}^*)x = \frac{G(\lambda)x}{F(\lambda)}, \quad \forall x \in \mathcal{H},$$

where for each $x \in \mathcal{H}$, $G(\lambda)x$ is a $\mathcal{H}$-valued entire function with order less than or equal to ρ_1 and $F(\lambda)$ is a scalar entire function of order ρ_2. Let $\rho = \max\{\rho_1, \rho_2\} < \infty$ and $n \in \mathbb{N}$ such that $n-1 \le \rho < n$. If there are $n+1$ rays $\gamma_j, j = 0, 1, \ldots, n$ on the complex plane satisfying

$$\arg \gamma_0 = \frac{\pi}{2} < \arg \gamma_1 \le \arg \gamma_2 \le \cdots \le \arg \gamma_n = \frac{3\pi}{2}$$

and

$$\arg \gamma_{j+1} - \arg \gamma_j \le \frac{\pi}{n}, \quad 0 \le j \le n-1$$

such that $R(\lambda, \mathcal{A}^*)x$ is bounded on each ray γ_j, $0 \le j \le n$ as $|\lambda| \to \infty$, $\forall x \in \mathcal{H}$, then $Sp(\mathcal{A}) = Sp(\mathcal{A}^*) = \mathcal{H}$, where $Sp(\mathcal{A})$ is the closed subspace spanned by all generalized eigenvectors of $\mathcal{A}$.

Using Proposition 4.1, we can prove the following result.

Theorem 4.1. *All (generalized) eigenvectors of $\mathcal{A}$ are complete in $\mathcal{H}$.*

Proof. We verify the assertion by three steps.

Step 1. Given arbitrary $(f, g, p) \in \mathcal{H}$, $\|R(\lambda, \mathcal{A}^*)(f, g, p)\|$ is bounded as $\lambda \to -\infty$.

We define an auxiliary operator $\mathcal{A}_0$ by $\mathcal{A}_0(u, v, z) := \mathcal{A}(u, v, z)$, $(u, v, z) \in D(\mathcal{A}_0)$ with domain

$$D(\mathcal{A}_0) = \left\{ \begin{array}{l} (u, v, z) \in X \cap \prod_{i=1}^{3} H^2(0, \ell_i) \times X \times H^1(0, 1)^2 : \\ \displaystyle\sum_{i=1}^{3} \sigma_i(0) \frac{du_i}{dx}(0) = 0;\ \frac{du_i}{dx}(\ell_i) = 0;\ i = 1, 2;\ z_0(0) = v_1(0),\ z_1(0) = v_1(\ell_1). \end{array} \right\}.$$

Then, $\mathcal{A}_0$ is a skew-adjoint operator in $\mathcal{H}$ and hence $\|R(\lambda, \mathcal{A}_0^*)\| \le \frac{1}{|\lambda|}$, $\forall \lambda \in \mathbb{R} \setminus \{0\}$.

Let $\lambda \in \rho(A^*) \cap \rho(A_0^*) \cap \mathbb{R}^-$ and $(f, g, p) \in \mathcal{H}$, we write

$$(u, v, z) := \mathcal{R}(\lambda, \mathcal{A}_0^*)(f, g, p), \qquad (w, \eta, r) := \mathcal{R}(\lambda, \mathcal{A}^*)(f, g, p) - (u, v, z), \tag{4.1}$$

where $\mathcal{A}^*$ is given by

$$\mathcal{A}^* \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} -v \\ \left\{ -\dfrac{1}{\rho_i(x)} \left[\dfrac{d}{dx} \left[\sigma_i(x) \dfrac{du_i(x)}{dx} \right] - q_i(x) u_i(x) \right] \right\}_{i=1}^{3} \\ \left\{ \dfrac{1}{\tau_i} \dfrac{dz_i}{ds} \right\}_{i=0}^{1} \end{pmatrix}$$

and

$$D(\mathcal{A}^*) = \left\{ \begin{array}{l} (u, v, z) \in X \cap \prod_{i=1}^{3} H^2(0, \ell_i) \times X \times H^2(0, 1)^2 : \\ \displaystyle\sum_{i=1}^{3} \sigma_i(0) \frac{du_i}{dx}(0) = -\alpha_0 v_1(0) - \beta_0 z_0(1),\ \frac{du_2}{dx}(\ell_2) = 0, \\ \sigma_1(\ell_1) \dfrac{du_1}{dx}(\ell_1) = \alpha_1 v_1(\ell_1) + \beta_1 z_1(1),\ z_0(0) = v_1(0), z_1(0) = v_1(\ell_1) \end{array} \right\}.$$

Thus, we have

$$\|\mathcal{R}(\lambda, \mathcal{A}^*)(f, g, p)\| = \|(w, \eta, r) + (u, v, z)\| \le \|(w, \eta, r)\| + \|(u, v, z)\| \le \|(w, \eta, r)\| + \frac{1}{|\lambda|} \|(f, g, p)\|,$$

which implies that the conclusion is true provided that $\|(w, \eta, r)\|$ is bounded as $\lambda \to -\infty$.

A direct calculation shows that (w, η, r) satisfies the following differential equations

$$\begin{cases} -\eta = \lambda w, \\ \left\{ -\dfrac{1}{\rho_i(x)} \left[\dfrac{d}{dx} \left[\sigma_i(x) \dfrac{dw_i(x)}{dx} \right] - q_i(x) w_i(x) \right] \right\}_{i=1}^{3} = \lambda \eta, \\ \left\{ \dfrac{1}{\tau_i} \dfrac{dr_i}{ds} \right\}_{i=0}^{1} = \lambda r(s), \end{cases}$$

with appropriate boundary conditions. Then,

$$
\begin{aligned}
&\Re(\lambda\|(w,\eta,r)\|^2) = \Re\,\langle \lambda(w,\eta,r),(w,\eta,r)\rangle\\
&= \left\langle \left(-\eta,\left\{-\frac{1}{\rho_i(x)}\left[\frac{d}{dx}\left(\sigma_i(x)\frac{dw_i(x)}{dx}\right)-q_i(x)w_i(x)\right]\right\}_{i=1}^{3},\left\{\frac{1}{\tau_i}\frac{dr_i(s)}{ds}\right\}_{i=0}^{1}\right),(w,\eta,r)\right\rangle\\
&= -\sum_{i=1}^{3}\int_0^{\ell_i}\left\{\sigma_i(x)\frac{d\eta_i(x)}{dx}\overline{\frac{dw_i(x)}{dx}}+q_i(x)\eta_i(x)\overline{w_i(x)}+\overline{\eta_i(x)}\left[\frac{d}{dx}\left(\sigma_i(x)\frac{dw_i(x)}{dx}\right)-q_i(x)w_i(x)\right]\right\}dx\\
&\quad+\sum_{i=0}^{1}\frac{b_i}{\tau_i}\int_0^1\frac{dr_i(s)}{ds}\overline{r_i(s)}ds\\
&= -\sum_{i=1}^{3}\left\{\int_0^{\ell_i}\sigma_i(x)\frac{d\eta_i}{dx}\overline{\frac{dw_i}{dx}}dx+\sigma_i(x)\frac{dw_i}{dx}\overline{\eta_i(x)}\Big|_0^{\ell_i}-\int_0^{\ell_i}\sigma_i(x)\frac{dw_i}{dx}\overline{\frac{d\eta_i}{dx}}dx\right\}+\frac{1}{2}\sum_{i=0}^{1}\frac{b_i}{\tau_i}|r_i(s)|^2\Big|_0^1\\
&= -\sum_{i=1}^{3}\left[\sigma_i(\ell_i)\frac{dw_i}{dx}(\ell_i)\overline{\eta_i(\ell_i)}-\sigma_i(0)\frac{dw_i}{dx}(0)\overline{\eta_i(0)}\right]+\frac{1}{2}\sum_{i=0}^{1}\frac{b_i}{\tau_i}(|r_i(1)|^2-|r_i(0)|^2)\\
&=: -I_1+\frac{1}{2}I_2. \qquad (4.2)
\end{aligned}
$$

We will calculate I_1 and I_2. According to (4.1), $(w+u,\eta+v,r+z)\in D(\mathcal{A}^*)$ and $(u,v,z)\in D(\mathcal{A}_0)$. Thus,

$$
\begin{aligned}
&\eta=-\lambda w,\qquad v=-\lambda u+f,\\
&z_0(s)=e^{\lambda\tau_0\eta}\left[v_1(0)-\tau_0\int_0^s e^{-\lambda\tau_0 h}p_0(h)dh\right],\quad s\in(0,1);\\
&z_1(s)=e^{\lambda\tau_1 s}\left[v_1(\ell_1)-\tau_1\int_0^s e^{-\lambda\tau_1 h}p_1(h)dh\right],\quad s\in(0,1);\\
&r_0(s)=\eta(0)e^{\tau_0\lambda s}=-\lambda w(0)e^{\tau_0\lambda s},\quad s\in(0,1);\\
&r_1(s)=\eta_1(\ell_1)e^{\tau_1\lambda s}=-\lambda w_1(\ell_1)e^{\tau_1\lambda s},\quad s\in(0,1),
\end{aligned}
$$

where $w=(w_i(x))_{i=1}^3$ satisfies the following equations

$$-\lambda^2 w_i(x)+\frac{1}{\rho_i(x)}\left[\frac{d}{dx}\left(\sigma_i(x)\frac{dw_i(x)}{dx}\right)-q_i(x)w_i(x)\right]=0,\quad x\in(0,\ell_i),\ i=1,2,3, \qquad (4.3)$$

$$\sigma_1(\ell_1)\frac{dw_1}{dx}(\ell_1)=(\alpha_1+\beta_1e^{\lambda\tau_1})\{-\lambda[w_1(\ell_1)+u_1(\ell_1)]+f_1(\ell_1)\}-\tau_1\beta_1e^{\lambda\tau_1}\int_0^1 e^{-\lambda\tau_1 h}p_1(h)dh, \qquad (4.4)$$

$$\frac{dw_2}{dx}(\ell_2)=0, \qquad (4.5)$$

$$w_3(\ell_3)=0, \qquad (4.6)$$

$$w_1(0)=w_2(0)=w_3(0), \qquad (4.7)$$

$$\sum_{i=1}^{3}\sigma_i(0)\frac{dw_i}{dx}(0)=(\alpha_0+\beta_0e^{\lambda\tau_0})\{\lambda[w_1(0)+u_1(0)]-f_1(0)\}+\tau_0\beta_0e^{\lambda\tau_0}\int_0^1 e^{-\lambda\tau_0 h}p_0(h)dh. \qquad (4.8)$$

Consequently,

$$
\begin{aligned}
I_1 &= \sum_{i=1}^{3}\left[\sigma_i(\ell_i)\frac{dw_i}{dx}(\ell_i)\overline{\eta_i(\ell_i)}-\sigma_i(0)\frac{dw_i}{dx}(0)\overline{\eta_i(0)}\right]\\
&= \lambda^2(\alpha_1+\beta_1e^{\lambda\tau_1})|w_1(\ell_1)|^2+\lambda(\alpha_1+\beta_1e^{\lambda\tau_1})[\lambda u_1(\ell_1)-f_1(\ell_1)]\overline{w_1(\ell_1)}\\
&\quad+\lambda\beta_1\tau_1e^{\lambda\tau_1}\int_0^1 e^{-\lambda\tau_1 h}p_1(h)dh\overline{w_1(\ell_1)}+\lambda^2(\alpha_0+\beta_0e^{\tau_0\lambda})|w_1(0)|^2\\
&\quad+\lambda(\alpha_0+\beta_0e^{\tau_0\lambda})[\lambda u_1(0)-f_1(0)]\overline{w_1(0)}+\lambda\beta_0\tau_0e^{\lambda\tau_0}\int_0^1 e^{-\lambda\tau_0 h}p_0(h)dh\overline{w_1(0)}\\
&\le \left[(\alpha_1+\beta_1e^{\lambda\tau_1})+(\alpha_1+\beta_1e^{\tau_1\lambda})+\frac{\beta_1\tau_1}{2}e^{\lambda\tau_1}\right]\lambda^2|w_1(\ell_1)|^2+\left[(\alpha_1+\beta_1e^{\tau_1\lambda})+\tau_1\beta_1e^{\lambda\tau_1}\right]\|(f,g,p)\|^2\\
&\quad+\left[(\alpha_0+\beta_0e^{\lambda\tau_0})+(\alpha_0+\beta_0e^{\tau_0\lambda})+\frac{\beta_0\tau_0}{2}e^{\lambda\tau_0}\right]\lambda^2|w_1(0)|^2+\left[(\alpha_0+\beta_0e^{\tau_0\lambda})+\tau_0\beta_0e^{\lambda\tau_0}\right]\|(f,g,p)\|^2,
\end{aligned}
$$

$$\begin{aligned}I_2 &= \sum_{i=0}^{1}\frac{b_i}{\tau_i}(|r_i(1)|^2-|r_i(0)|^2) = \frac{b_0}{\tau_0}|\eta_1(0)|^2(e^{2\lambda\tau_0}-1)+\frac{b_1}{\tau_1}|\eta_1(\ell_1)|^2(e^{2\lambda\tau_1}-1)\\ &= \frac{b_0\lambda^2}{\tau_0}|w_1(0)|^2(e^{2\lambda\tau_0}-1)+\frac{b_1\lambda^2}{\tau_1}|w_1(\ell_1)|^2(e^{2\lambda\tau_1}-1).\end{aligned}$$

Therefore,

$$\begin{aligned}\|(w,\eta,r)\|^2 &= \frac{1}{\lambda}\left(-I_1+\frac{1}{2}I_2\right) \le -\left[2(\alpha_1+\beta_1e^{\lambda\tau_1})+\frac{\beta_1\tau_1}{2}e^{\lambda\tau_1}\right]\lambda|w_1(\ell_1)|^2\\ &\quad -\frac{1}{\lambda}\left[(\alpha_1+\beta_1e^{\lambda\tau_1})+\tau_1\beta_1e^{\lambda\tau_1}\right]\|(f,g,p)\|^2\\ &\quad -\left[2(\alpha_0+\beta_0e^{\lambda\tau_0})+\frac{\beta_0\tau_0}{2}e^{\lambda\tau_0}\right]\lambda|w_1(0)|^2\\ &\quad -\frac{1}{\lambda}\left[(\alpha_0+\beta_0e^{\lambda\tau_0})+\tau_0\beta_0e^{\lambda\tau_0}\right]\|(f,g,p)\|^2\\ &\quad +\frac{\lambda}{2\tau_0}|w_1(0)|^2(e^{2\lambda\tau_0}-1)+\frac{\lambda}{2\tau_1}|w_1(\ell_1)|^2(e^{2\lambda\tau_1}-1).\end{aligned} \tag{4.9}$$

Inequality (4.9) indicates that the assertion will be verified if $|w_1(0)|^2$ and $|w_1(\ell_1)|^2$ can be estimated appropriately. Define

$$h_i(\xi_i) := \sqrt[4]{\rho_i(x(\xi_i))\sigma_i(x(\xi_i))}\,w_i(x(\xi_i)), \tag{4.10}$$

where $x(\xi_i)$ is the inverse function of $\xi_i(x)$ and $\xi_i(x) := \int_0^x\sqrt{\frac{\rho_i(t)}{\sigma_i(t)}}dt$. Then we only need to estimate the values of $h_1(\xi_1)$ at the two endpoints since $\rho_1(x)$ and $\sigma_1(x)$ are known functions. To this end, the main idea is to obtain the asymptotic expression of $h_1(\xi_1)$ by applying the asymptotic analysis technique that used in Section 3 and then utilize the Cramer's rule.

Inserting (4.10) into (4.3)–(4.8), we get

$$h_i''(\xi_i)-\lambda^2h_i(\xi_i) = [b_i(\xi_i)-\phi_i(\xi_i)]h_i(\xi_i), \quad \xi_i\in(0,m_i),\ i=1,2,3, \tag{4.11}$$

$$\begin{aligned}&h_1'(m_1)+\left[\hat{c}_1(\ell_1)+\lambda(\alpha_1+\beta_1e^{\lambda\tau_1})[\rho_1(\ell_1)\sigma_1(\ell_1)]^{-1/2}\right]h_1(m_1)\\ &\quad = [\rho_1(\ell_1)\sigma_1(\ell_1)]^{-1/4}\left[(\alpha_1+\beta_1e^{\lambda\tau_1})[f_1(\ell_1)-\lambda u_1(\ell_1)]-\beta_1\tau_1e^{\lambda\tau_1}\int_0^1e^{-\lambda\tau_1h}p_1(h)dh\right],\end{aligned} \tag{4.12}$$

$$h_2'(m_2)+\hat{c}_2(\ell_2)h_2(m_2) = 0, \tag{4.13}$$

$$h_3(m_3) = 0, \tag{4.14}$$

$$[\rho_1(0)\sigma_1(0)]^{-1/4}h_1(0) = [\rho_2(0)\sigma_2(0)]^{-1/4}h_2(0) = [\rho_3(0)\sigma_3(0)]^{-1/4}h_3(0), \tag{4.15}$$

$$\begin{aligned}&\sum_{i=1}^{3}[\rho_i(0)\sigma_i(0)]^{1/4}h_i'(0)+\sum_{i=1}^{3}\left\{\hat{d}_i(0)-\lambda(\alpha_0+\beta_0e^{\lambda\tau_0})[\rho_i(0)\sigma_i(0)]^{-1/4}\right\}h_i(0)\\ &\quad = -(\alpha_0+\beta_0e^{\lambda\tau_0})[-\lambda u_1(0)+f_1(0)]+\tau_0\beta_0e^{\lambda\tau_0}\int_0^1e^{-\lambda\tau_0h}p_0(h)dh,\end{aligned} \tag{4.16}$$

where $b_i(x(\xi_i))$, $\phi_i(\xi_i)$ and m_i are given by (3.12), (3.11) and (3.10) respectively, and

$$\hat{c}_i(x) = -\frac{1}{2}[\sigma_i(x)]_x[\sigma_i(x)\rho_i(x)]^{-1/2}+\frac{(\sigma_i)_x\rho_i-\rho_i(\sigma_i)_x}{4\sigma_i(x)\rho_i(x)}, \quad i=1,2,$$

$$\hat{d}_i(x) = -\frac{1}{2}[\sigma_i(x)]_x[\sigma_i(x)\rho_i(x)]^{-1/4}+\frac{(\sigma_i)_x\rho_i-\rho_i(\sigma_i)_x}{4[\sigma_i(x)\rho_i(x)]^{3/4}}, \quad i=1,2,3.$$

Again applying the theorem in [27, p. 49] to Eq. (4.11), we get the asymptotical expression of the general solution $h_i(\lambda,\xi_i)$ to (4.11)

$$\begin{aligned}h_i(\lambda,\xi_i) &= A_i(\lambda)e^{\lambda\xi_i}\left[1+O\left(\frac{1}{\lambda}\right)\right]+B_i(\lambda)e^{-\lambda\xi_i}\left[1+O\left(\frac{1}{\lambda}\right)\right]\\ &=: A_i(\lambda)e^{\lambda\xi_i}[1]_1+B_i(\lambda)e^{-\lambda\xi_i}[1]_1, \quad \xi_i\in(0,m_i),\ i=1,2,3,\end{aligned} \tag{4.17}$$

where $[1]_1 = 1+O\left(\frac{1}{\lambda}\right)$.

Substitute (4.17) into (4.12)–(4.16), then apply the Cramer's rule to $A_1(\lambda)$ and $B_1(\lambda)$. A complicated calculation gives

$$A_1(\lambda) = -\frac{\alpha_0\tilde{k}_2\tilde{k}_3u_1(0)}{\alpha_0\prod_{i=1}^{3}\tilde{k}_i - \sum_{i=1}^{3}\tilde{k}_i\prod_{j=1,j\neq i}^{3}\tilde{k}_j} + o(1),$$

$$B_1(\lambda) = \frac{\hat{k}\alpha_1e^{\lambda m_1}u_1(\ell_1)}{-1+\alpha_1\hat{k}} + o(e^{\lambda m_1}).$$

Consequently,

$$\begin{aligned}
h_1(0) &= A_1(\lambda)[1]_1 + B_1(\lambda)[1]_1 \\
&= -\left\{\frac{\alpha_0\tilde{k}_2\tilde{k}_3u_1(0)}{\alpha_0\prod_{i=1}^{3}\tilde{k}_i - \sum_{i=1}^{3}\tilde{k}_i\prod_{j=1,j\neq i}^{3}\tilde{k}_j} + o(1)\right\}[1]_1 + \left\{\frac{\hat{k}\alpha_1e^{\lambda m_1}u_1(\ell_1)}{-1+\alpha_1\hat{k}} + o(e^{\lambda m_1})\right\}[1]_1 \\
&= -\frac{\alpha_0\tilde{k}_2\tilde{k}_3u_1(0)}{\alpha_0\prod_{i=1}^{3}\tilde{k}_i - \sum_{i=1}^{3}\tilde{k}_i\prod_{j=1,j\neq i}^{3}\tilde{k}_j} + \frac{\hat{k}\alpha_1e^{\lambda m_1}u_1(\ell_1)}{-1+\alpha_1\hat{k}} + [u_1(0)+e^{\lambda m_1}u_1(\ell_1)]O\left(\frac{1}{\lambda}\right) + o(e^{\lambda m_1}), \\
h_1(m_1) &= A_1(\lambda)e^{\lambda m_1}[1]_1 + B_1(\lambda)e^{-\lambda m_1}[1]_1 \\
&= -\left\{\frac{\alpha_0\tilde{k}_2\tilde{k}_3u_1(0)}{\alpha_0\prod_{i=1}^{3}\tilde{k}_i - \sum_{i=1}^{3}\tilde{k}_i\prod_{j=1,j\neq i}^{3}\tilde{k}_j} + o(1)\right\}e^{\lambda m_1}[1]_1 + \left\{\frac{\hat{k}\alpha_1e^{\lambda m_1}u_1(\ell_1)}{-1+\alpha_1\hat{k}} + o(e^{\lambda m_1})\right\}e^{-\lambda m_1}[1]_1 \\
&= -\frac{\alpha_0\tilde{k}_2\tilde{k}_3e^{\lambda m_1}u_1(0)}{\alpha_0\prod_{i=1}^{3}\tilde{k}_i - \sum_{i=1}^{3}\tilde{k}_i\prod_{j=1,j\neq i}^{3}\tilde{k}_j} + \frac{\hat{k}\alpha_1u_1(\ell_1)}{-1+\alpha_1\hat{k}} + [u_1(0)+u_1(\ell_1)]O\left(\frac{1}{\lambda}\right) + o\left(\frac{1}{\lambda}\right) + o(1).
\end{aligned}$$

Up until now, for $\lambda < 0$, we obtain

$$\begin{aligned}
|h_1(0)| &\le \frac{\alpha_0\tilde{k}_2\tilde{k}_3|u_1(0)|}{\left|\alpha_0\prod_{i=1}^{3}\tilde{k}_i - \sum_{i=1}^{3}\tilde{k}_i\prod_{j=1,j\neq i}^{3}\tilde{k}_j\right|} + \frac{\hat{k}\alpha_1|u_1(\ell_1)|}{|-1+\alpha_1\hat{k}|} \\
&\quad + [|u_1(0)|+|u_1(\ell_1)|]O\left(\frac{1}{\lambda}\right) + o\left(\frac{1}{\lambda}\right) + o(1) \\
&\le \frac{\alpha_0\tilde{k}_2\tilde{k}_3}{\left|\alpha_0\prod_{i=1}^{3}\tilde{k}_i - \sum_{i=1}^{3}\tilde{k}_i\prod_{j=1,j\neq i}^{3}\tilde{k}_j\right|}\cdot\frac{1}{|\lambda|}\|(f,g,p)\| + \frac{\hat{k}\alpha_1}{|-1+\alpha_1\hat{k}|}\cdot\frac{1}{|\lambda|}\|(f,g,p)\| \\
&\quad + o\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{\lambda^2}\right)\|(f,g,p)\| + o(1) \\
&= \left(1+\frac{1}{|\lambda|}\right)O\left(\frac{1}{\lambda}\right)\|(f,g,p)\| = O\left(\frac{1}{\lambda}\right)\|(f,g,p)\|, \qquad (4.18)
\end{aligned}$$

$$\begin{aligned}
|h_1(m_1)| &\le \frac{\alpha_0\tilde{k}_2\tilde{k}_3e^{\lambda m_1}}{\left|\alpha_0\prod_{i=1}^{3}\tilde{k}_i - \sum_{i=1}^{3}\tilde{k}_i\prod_{j=1,j\neq i}^{3}\tilde{k}_j\right|}\cdot\frac{1}{|\lambda|}\|(f,g,p)\| + \frac{\hat{k}\alpha_1}{|-1+\alpha_1\hat{k}|}\cdot\frac{1}{|\lambda|}\|(f,g,p)\| \\
&\quad + o\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{\lambda^2}\right)\|(f,g,p)\| + o(1) \\
&= \left(1+e^{\lambda m_1}+\frac{1}{|\lambda|}\right)O\left(\frac{1}{\lambda}\right)\|(f,g,p)\| \\
&= O\left(\frac{1}{\lambda}\right)\|(f,g,p)\|. \qquad (4.19)
\end{aligned}$$

By inserting Eqs. (4.18)–(4.19) into Eq. (4.9), we get

$$\begin{aligned}\|(w,\eta,r)\|^2 &\le \left[-2(\alpha_1+\beta_1 e^{\lambda\tau_1}) - \frac{\beta_1\tau_1}{2}e^{\lambda\tau_1} + \frac{\lambda}{2\tau_1}(e^{2\lambda\tau_1}-1)\right]\lambda\hat{k}|h_1(m_1)|^2 - \frac{1}{\lambda}\left[(\alpha_1+\beta_1 e^{\lambda\tau_1}) + \tau_1\beta_1 e^{\lambda\tau_1}\right] \\ &\quad \times \|(f,g,p)\|^2 + \left[-2(\alpha_0+\beta_0 e^{\lambda\tau_0}) - \frac{\beta_0\tau_0}{2}e^{\lambda\tau_0} + \frac{\lambda}{2\tau_0}(e^{2\lambda\tau_0}-1)\right]\lambda\tilde{k}^2|h_1(0)|^2 \\ &\quad - \frac{1}{\lambda}\left[(\alpha_0+\beta_0) + \tau_0\beta_0 e^{\lambda\tau_0}\right] \\ &= \left[-2\alpha_1 + O(e^{\lambda\tau_1}) - \frac{\lambda}{2\tau_1}\right]\lambda O\left(\frac{1}{\lambda^2}\right)\|(f,g,p)\|^2 - \frac{1}{\lambda}\left[\alpha_1 + O(e^{\lambda\tau_1})\right]\|(f,g,p)\|^2 \\ &\quad + \left[-2\alpha_0 + O(e^{\lambda\tau_0}) - \frac{\lambda}{2\tau_0}\right]\lambda O\left(\frac{1}{\lambda^2}\right)\|(f,g,p)\|^2 - \frac{1}{\lambda}\left[\alpha_0 + O(e^{\lambda\tau_0})\right]\|(f,g,p)\|^2 \\ &= \left[C + O\left(\frac{1}{\lambda}\right)\right]\|(f,g,p)\|^2,\end{aligned}$$

where C is a positive constant. Therefore,

$$\begin{aligned}\|R(\lambda,\mathcal{A}^*)(f,g,p)\| &\le \|(w,\eta,r)\| + \|R(\lambda,\mathcal{A}_0^*)(f,g,p)\| \\ &\le \left\{\left[C + O\left(\frac{1}{\lambda}\right)\right]^{1/2} + \frac{1}{|\lambda|}\right\}\|(f,g,p)\|,\end{aligned}$$

which implies $\|R(\lambda,\mathcal{A}^*)\ (f,g,p)\|$ is bounded as $\lambda\to-\infty$.

Step 2.

$$R(\lambda,\mathcal{A}^*)(f,g,p) = \frac{H(\lambda;f,g,p)}{M(\lambda)},$$

where $H(\lambda;f,g,p)$ is a $\mathcal{H}$-value entire function of exponential type of finite order at most 1 and $M(\lambda)$ is a scalar entire function of order 1.

In fact, let $\lambda\in\rho(\mathcal{A}^*), \lambda\neq 0$. For any $(f,g,p)\in\mathcal{H}$, denote $(\tilde{u},\tilde{v},\tilde{z}) := (\lambda I-\mathcal{A}^*)^{-1}(f,g,p)$, which indicates $(\tilde{u},\tilde{v},\tilde{z})\in D(\mathcal{A}^*)$ and

$$\begin{cases}\tilde{v}_i(x) = f_i(x) - \lambda\tilde{u}_i(x), \quad x\in(0,\ell_i),\ i=1,2,3,\\ \lambda\tilde{v}_i(x) + \dfrac{1}{\rho_i(x)}\left[\dfrac{d}{dx}\left(\sigma_i(x)\dfrac{d\tilde{u}_i(x)}{dx}\right) - q_i(x)\tilde{u}_i(x)\right] = g_i(x), \quad x\in(0,\ell_i),\ i=1,2,3,\\ \lambda\tilde{z}_i(s) - \dfrac{1}{\tau_i}\dfrac{d\tilde{z}_i(s)}{ds} = p_i, \quad s\in(0,1),\ i=0,1.\end{cases}$$

Setting

$$\tilde{h}_i(\xi_i) := (\rho_i\sigma_i)^{1/4}(x(\xi_i))\tilde{u}_i(x(\xi_i)),$$

where $\xi_i, i=1,2,3$ is defined as before, we can change above equations into the following forms in a more precise way.

$$\tilde{h}_i''(\xi_i) - \lambda^2\tilde{h}_i(\xi_i) + \tilde{b}_i(x(\xi_i))\tilde{h}_i(\xi_i) = [\rho_i(x(\xi_i))\sigma_i(x(\xi_i))]^{1/4}[g_i(x(\xi_i)) - \lambda f_i(x(\xi_i))], \quad \xi_i\in(0,m_i),\ i=1,2,3, \tag{4.20}$$

$$\begin{aligned}&\tilde{h}_1'(m_1) + [\tilde{c}_1(\ell_1) + \lambda(\rho_1\sigma_1)^{-1/2}(\ell_1)(\alpha_1+\beta_1 e^{\lambda\tau_1})]\tilde{h}_1(m_1) \\ &\quad = (\rho_1\sigma_1)^{-1/4}(\ell_1)\left[(\alpha_1+\beta_1 e^{\lambda\tau_1})f_1(\ell_1) + \beta_1 e^{\lambda\tau_1}\int_0^1 p_1(\eta)e^{-\lambda\tau_1\eta}d\eta\right],\end{aligned} \tag{4.21}$$

$$\tilde{h}_2'(m_2) + \tilde{c}_2(\ell_2)\tilde{h}_2(m_2) = 0, \tag{4.22}$$

$$\tilde{h}_3(m_3) = 0, \tag{4.23}$$

$$(\rho_1(0)\sigma_1(0))^{-1/4}\tilde{h}_1(0) = (\rho_2(0)\sigma_2(0))^{-1/4}\tilde{h}_2(0) = (\rho_3(0)\sigma_3(0))^{-1/4}\tilde{h}_3(0), \tag{4.24}$$

$$\begin{aligned}&\sum_{i=1}^3\left(\frac{\rho_i(0)}{\sigma_i(0)}\right)^{1/4}\tilde{h}_i'(0) + \sum_{i=1}^3\tilde{d}_i(0)\tilde{h}_i(0) + \frac{\lambda}{3}(\alpha_0+\beta_0)\sum_{i=1}^3\left(\frac{\rho_i(0)}{\sigma_i(0)}\right)^{-1/4}\tilde{h}_i(0) \\ &\quad = -(\alpha_0+\beta_0)f_1(0) - \tau\beta_0\int_0^1 p_1(\eta)e^{-\lambda\tau(1-\eta)}d\eta,\end{aligned} \tag{4.25}$$

where

$$\tilde{b}_i(x) = \frac{5}{16}(\rho_i\sigma_i)^{-9/4}(x)(\rho_i\sigma_i)_x^2(x) - \frac{1}{4}(\rho_i\sigma_i)^{-5/4}(x)(\rho_i\sigma_i)_{xx}(x) + q_i(x)(\rho_i\sigma_i)^{-1/4}(x)\sigma_i^{-1}(x), \quad i = 1, 2, 3,$$

$$\tilde{c}_i(x) = -\frac{1}{4}(\rho_i\sigma_i)^{-3/2}(x)(\rho_i\sigma_i)_x(x)\sigma_i(x), \quad i = 1, 2,$$

$$\tilde{d}_i(x) = \frac{1}{4}\left(\frac{\rho_i}{\sigma_i}\right)^{-5/4}(x)\sigma_i(x)^{1/2}\left(\frac{\rho_i}{\sigma_i}\right)_x(x) - \frac{1}{2}(\rho_i\sigma_i)^{-1/4}(x)\sigma_{ix}(x), \quad i = 1, 2, 3.$$

Two linearly independent solutions $\tilde{F}_i(\lambda, \xi_i)$, $\tilde{\Psi}_i(\lambda, \xi_i)$ of Eq. (4.20) satisfy the Volterra integral equations:

$$\begin{aligned}\tilde{F}_i(\lambda, \xi_i) &= \cosh(\lambda\xi_i) + \frac{1}{\lambda}\int_0^{\xi_i}\sinh(\lambda(\xi_i-\eta))[\rho_i(\eta)\sigma_i(\eta)]^{1/4}[g_i(\eta) - \lambda f_i(\eta)]d\eta \\ &\quad - \frac{1}{\lambda}\int_0^{\xi_i}\sinh(\lambda(\xi_i-\eta))\tilde{b}_i(\eta)\tilde{F}_i(\lambda,\eta)d\eta \\ &=: \cosh(\lambda\xi_i) + \frac{1}{\lambda}Q(\lambda,\xi_i) + T(\lambda,\xi_i) - \frac{1}{\lambda}\int_0^{\xi_i}\sinh(\lambda(\xi_i-\eta))\tilde{b}_i(\eta)\tilde{F}_i(\lambda,\eta)d\eta,\end{aligned} \tag{4.26}$$

$$\begin{aligned}\tilde{\Psi}_i(\lambda, \xi_i) &= \sinh(\lambda\xi_i) + \frac{1}{\lambda}\int_0^{\xi_i}\sinh(\lambda(\xi_i-\eta))[\rho_i(\eta)\sigma_i(\eta)]^{1/4}[g_i(\eta) - \lambda f_i(\eta)]d\eta \\ &\quad + \frac{1}{\lambda}\int_0^{\xi_i}\sinh(\lambda(\xi_i-\eta))\tilde{b}_i(\eta)\tilde{\Psi}_i(\lambda,\eta)d\eta \\ &=: \sinh(\lambda\xi_i) + \frac{1}{\lambda}Q(\lambda,\xi_i) + T(\lambda,\xi_i) + \frac{1}{\lambda}\int_0^{\xi_i}\sinh(\lambda(\xi_i-\eta))\tilde{b}_i(\eta)\tilde{\Psi}_i(\lambda,\eta)d\eta,\end{aligned} \tag{4.27}$$

where

$$Q(\lambda, \xi_i) := \int_0^{\xi_i}\sinh(\lambda(\xi_i-\eta))[\rho_i(\eta)\sigma_i(\eta)]^{1/4}g_i(\eta)d\eta,$$

$$T(\lambda, \xi_i) := -\int_0^{\xi_i}\sinh(\lambda(\xi_i-\eta))[\rho_i(\eta)\sigma_i(\eta)]^{1/4}f_i(\eta)d\eta.$$

Furthermore, $\tilde{F}_i$, $\tilde{\Psi}_i$ are determined uniquely by (4.26) and (4.27), respectively, using [29, Theorem 3.10, p. 36].

Since the kernel $\tilde{K}_i(\eta)$ of integral equations (4.26) and (4.27) are entire functions in λ, the solutions $\tilde{F}_i$, $\tilde{\Psi}_i$ obtained by the method of successive approximation are also entire functions with respect to λ. Thereby, the general solution of Eq. (4.20) has the form

$$\tilde{u}_i(\xi_i) = \tilde{A}_i(\lambda)\tilde{F}_i(\lambda, \xi_i) + \tilde{B}_i(\lambda)\tilde{\Psi}_i(\lambda, \xi_i) \tag{4.28}$$

for any constants $\tilde{A}_i(\lambda)$, $\tilde{B}_i(\lambda)$ dependent on λ and is an entire function with respect to λ.

By combining (4.21)–(4.26), a straightforward but complicated calculation gives

$$R(\lambda, \mathcal{A}^*)(f, g, p) = \frac{H(\lambda; f, g, p)}{\det\tilde{\Delta}(\lambda)},$$

where $\tilde{\Delta}(\lambda)$ is the coefficient matrix obtained by inserting (4.26) into (4.21)–(4.25), which is a scalar entire function of finite exponential type. $H(\lambda; f, g, p)$ is a $\mathcal{H}$-valued entire function of finite exponential type, because $H(\lambda; f, g)$ is the linear combination of $\sinh(\lambda\xi_i)$, $\cosh(\lambda\xi_i)$, some constants dependent on the values of $\sigma_i(x)$, $\rho_i(x)$ and $q_i(x)$ at endpoint 0 or ℓ_i as well as $\mathcal{H}$-valued functions $f(x)$, $g(x)$.

Step 3. The sequence of generalized eigenvectors of $\mathcal{A}$ is complete in the state Hilbert space $\mathcal{H}$.

Obviously, $\rho_2 = \rho_1 = \rho = 1$. We take $\gamma_0 = -N + iy$, $\gamma_1 = -N - y$, $\gamma_2 = -N - iy$ for $y \in (0, \infty)$ and sufficiently large positive real number N in Proposition 4.1. Steps 1 and 2 show that all the conditions in Proposition 4.1 are fulfilled. The desired result follows from Proposition 4.1. □

To prove the Riesz basis property of system (2.4), we need the notion of a Riesz basis with parentheses and another proposition.

A sequence $\{f_i\}_{i=1}^{\infty}$ is called a Riesz basis with parentheses if there is a method in parentheses; for instance, there is a sequence of integers $n_0 = 1 \le n_1 \le \cdots \le n_k \le \cdots$ such that the following conditions hold:

(1) $\{f_i\}_{i=1}^{\infty}$ is complete in $\mathcal{H}$,
(2) there exist two positive constants c and C such that for any integer p and any finite scalars $\eta_1, \ldots, \eta_{n_1}, \eta_{n_1+1}, \ldots, \eta_{n_p}$,

$$c\sum_{k=1}^{p}\left|\sum_{i=n_{k-1}}^{n_k}\eta_i\right|^2 \le \sum_{k=1}^{p}\left\|\sum_{i=n_{k-1}}^{n_k}\eta_i f_i\right\|^2 \le C\sum_{k=1}^{p}\left|\sum_{i=n_{k-1}}^{n_k}\eta_i\right|^2, \tag{4.29}$$

(3) $f = \sum_{k=1}^{\infty}\left(\sum_{i=n_{k-1}}^{n_k}\alpha_i f_i\right)$ converges in $\mathcal{H}$ (see [30]).

Proposition 4.2 (*[31]*)**.** *Let $\mathcal{A}$ be the generator of a C_0-semigroup $T(t)$ $(t \ge 0)$ on a separable Hilbert space $\mathcal{H}$. Suppose that the following conditions are satisfied:*

(1) *the spectrum of $\mathcal{A}$ has a decomposition $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \bigcup \sigma_2(\mathcal{A})$;*
(2) *there exists a real number $\alpha \in \mathbb{R}$ such that*

$$\sup\{\Re\lambda|\lambda \in \sigma_1(\mathcal{A})\} \le \alpha \le \inf\{\Re\lambda|\lambda \in \sigma_2(\mathcal{A})\};$$

(3) *the set $\sigma_2(\mathcal{A}) = \{\lambda_k\}_{k\in\mathbb{N}}$ consists of isolated eigenvalues of $\mathcal{A}$ and is a union of finitely separated sets.*

Then there exist two $T(t)$-invariant closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with

$$\mathcal{H}_1 = \{f \in \mathcal{H}|E(\lambda, \mathcal{A})f = 0, \forall\lambda \in \sigma_2(\mathcal{A})\},$$
$$\mathcal{H}_2 = \overline{\operatorname{span}\left\{\sum_{k=1}^{m} E(\lambda_k, \mathcal{A})f : \lambda_k \in \sigma(\mathcal{A}), \forall m \in \mathbb{N},\ \forall f \in \mathcal{H}\right\}}$$

such that $\mathcal{H}_1 \bigcap \mathcal{H}_2 = \{0\}$ with property that $\sigma(\mathcal{A}|_{\mathcal{H}_1}) = \sigma_1(\mathcal{A})$ and $\sigma(\mathcal{A}|_{\mathcal{H}_2}) = \sigma_2(\mathcal{A})$. Moreover, there exists a finite collection Ω_k of elements in $\sigma_2(\mathcal{A})$ such that $\{E(\Omega_k, \mathcal{A})\mathcal{H}_2\}_{k\in\mathbb{N}}$ forms a subspace Riesz basis for $\mathcal{H}_2$, where $E(\Omega_k, \mathcal{A}) = \sum_{\lambda\in\Omega_k} E(\lambda, \mathcal{A})$ is the Riesz projector corresponding to Ω_k.

Theorem 4.2. *Let $\mathcal{A}$ be defined by (2.2)–(2.3). There exists a sequence of generalized eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}$.*

Proof. We take $\sigma_1(\mathcal{A}) = \{-\infty\}$, $\sigma_2(\mathcal{A}) = \sigma_p(\mathcal{A})$ in Proposition 4.2, then $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A})$. Condition (1) is satisfied. Theorem 3.1 ensures that the spectrum of $\mathcal{A}$ distributes in a strip parallel to the imaginary axis and $\sigma(\mathcal{A})$ is a union of finitely separable sets, which together with Corollary 2.2 imply that Conditions (2) and (3) are fulfilled. Hence, there exists $T(t)$-invariant closed subspace $\mathcal{H}_2$ such that the sequence of generalized eigenvectors of $\mathcal{A}$ forms a subspace Riesz basis (i.e. the Riesz basis with parentheses) for $\mathcal{H}_2$ by Proposition 4.2. Furthermore, Theorem 4.1 shows that the sequence of generalized eigenvectors of the system operator $\mathcal{A}$ is complete in $\mathcal{H}$, which implies that $\mathcal{H}_2 = \mathcal{H}$. Hence, we complete the proof. □

Now, we are in a position to discuss the stability of the closed loop system (2.4). First we consider the following two eigenvalue problems of ordinary differential equations

$$\begin{cases} -\dfrac{d}{dx}\left(\sigma_2(x)\dfrac{du_2(x)}{dx}\right) + q_2(x)u_2(x) = \mu\rho_2(x)u_2(x), \quad x \in (0, \ell_2), \\ u_2(0) = 0, \qquad u_2'(\ell_2) = 0 \end{cases} \tag{4.30}$$

and

$$\begin{cases} -\dfrac{d}{dx}\left(\sigma_3(x)\dfrac{du_3(x)}{dx}\right) + q_3(x)u_3(x) = \mu\rho_3(x)u_3(x), \quad x \in (0, \ell_3), \\ u_3(0) = 0, \qquad u_3(\ell_3) = 0. \end{cases} \tag{4.31}$$

Let $\psi_j(x, \mu)$ be a solution to the following equation

$$\begin{cases} -\dfrac{d}{dx}\left(\sigma_j(x)\dfrac{d\psi_j(x)}{dx}\right) + q_j(x)\psi_j(x) = \mu\rho_j(x)\psi_j(x), \quad x \in (0, \ell_j), \\ \psi_j(0) = 0, \qquad \psi_j'(0) = 1. \end{cases}$$

Then the eigenvalues of (4.30) and (4.31) are given respectively by

$$\Sigma_2 = \{\mu_n > 0|\psi_2'(\ell_2, \mu_n) = 0\}, \qquad \Sigma_3 = \{\nu_n > 0|\psi_3(\ell_3, \nu_n) = 0\}. \tag{4.32}$$

Theorem 4.3. *If $\Sigma_2 \cap \Sigma_3 \ne \emptyset$, then system (2.4) (or system (1.1)) is unstable.*

Proof. If $\Sigma_2 \cap \Sigma_3 \neq \emptyset$, we take $\mu \in \Sigma_2 \cap \Sigma_3$ and define functions as follows

$$u_1(x,t) \equiv 0, \qquad u_2(x,t) = \frac{e^{i\sqrt{\mu}t}}{\sigma_2(0)}\psi_2(x,\mu), \qquad u_3(x,t) = -\frac{e^{i\sqrt{\mu}t}}{\sigma_3(0)}\psi_3(x,\mu).$$

A direct verification shows that $\{u_j(x,t)\}_{j=1}^3$ satisfy (1.1). Therefore the system is unstable. □

Note that the eigenvalue problems (4.30) and (4.31) correspond to different physical systems, from the physical point of view, different physics systems have different eigenvalues. Therefore, we can assume that $\Sigma_2 \cap \Sigma_3 = \emptyset$. In the case of constant coefficients, for instance, $\sigma_2(x) = \sigma_3(x) \equiv 1$, $\rho_2(x) = \rho_3(x) \equiv 1$ and $q_2(x) = q_3(x) \equiv 0$, the condition $\Sigma_2 \cap \Sigma_3 = \emptyset$ is equivalent to request that

$$\frac{\ell_2}{\ell_3} \neq \frac{2m+1}{2k}, \quad \forall k,\ m \in \mathbb{N}.$$

In what follows, we discuss the asymptotic stability and the exponential stability of system (2.4) under the condition $\Sigma_2 \cap \Sigma_3 = \emptyset$.

Theorem 4.4. *Let Σ_2 and Σ_3 be defined by (4.32). Suppose that $\Sigma_2 \cap \Sigma_3 = \emptyset$, then the following statements are true.*

(1) *System (2.4) is asymptotically stable.*
(2) *Let m_2 and m_3 be defined by (3.17), then system (2.4) is exponentially stable if and only if m_2 and m_3 satisfy the condition*

$$\frac{m_2}{m_3} \in \left\{\frac{2n_1+1}{2n_2+1}, n_1, n_2 \in \mathbb{N}\right\} \quad \text{or} \quad \frac{m_2}{m_3} \in \left\{\frac{2n_1}{2n_2+1}, n_1, n_2 \in \mathbb{N}\right\}.$$

Proof. As a direct result of Theorems 4.2 and 3.1, we have that the spectrum-determined growth condition $\omega(\mathcal{A}) = \sup\{\Re\lambda \mid \lambda \in \sigma(\mathcal{A})\}$ holds. Obviously, if

$$\sup\{\Re\lambda \mid \lambda \in \sigma(\mathcal{A})\} < 0, \tag{4.33}$$

then system (2.4) is stable exponentially.

First, we show that the system is asymptotically stable. Since the operator $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$ by Theorem 2.1 and $\sigma(\mathcal{A}) \subseteq \{\lambda \in \mathbb{C} \mid \Re\lambda \le 0\}$, we only need to show that there is no eigenvalue on the imaginary axis by Corollary 2.2. In fact, if $\sigma(\mathcal{A}) \cap i\mathbb{R} \neq \emptyset$, we take $\lambda \in \sigma(\mathcal{A}) \cap i\mathbb{R}$, $\lambda \neq 0$ such that $\mathcal{A}(u,v,z) = \lambda(u,v,z)$, where $(u,v,z) \in D(\mathcal{A})$ and is a nonzero vector. From the relation

$$\Re\lambda\|(u,v,z)\|^2 = \Re\langle \mathcal{A}(u,v,z),(u,v,z)\rangle \le -\frac{1}{2}\sum_{i=0}^{1}\left\{\left[2\alpha_i - \beta_i - \frac{b_i}{\tau_i}\right]|z_i(0)|^2 + \left[\frac{b_i}{\tau_i} - \beta_i\right]|z_i(1)|^2\right\} \le 0,$$

we get $z_i(0) = z_i(1) = 0$, $i = 0, 1$. Using $\mathcal{A}(u,v,z) = \lambda(u,v,z)$, we have $z_i(x) = z_i(0)e^{-\lambda\tau_i x} = 0$, $i = 0, 1$, $v = \lambda u$ and $u = \{u_i\}$ satisfy the equations

$$\frac{d}{dx}\left[\sigma_i(x)\frac{du_i(x)}{dx}\right] - q_i(x)u_i(x) = \lambda^2\rho_i(x)u_i(x), \quad x \in (0,\ell_i),\ i = 1,2,3, \tag{4.34}$$

$$u_1(0) = u_2(0) = u_3(0) \tag{4.35}$$

$$\sum_{i=1}^{3}\sigma_i(0)\frac{du_i}{dx}(0) = \alpha_0\lambda u_1(0) + \beta_0 z_0(1) = 0, \tag{4.36}$$

$$\frac{du_2}{dx}(\ell_2) = 0, \qquad u_3(\ell_3) = 0 \tag{4.37}$$

$$\sigma_1(\ell_1)\frac{du_1}{dx}(\ell_1) = -\alpha_1\lambda u_1(\ell_1) - \beta_1 z_1(1) = 0, \tag{4.38}$$

$$z_0(0) = \lambda u_1(0) = 0, \qquad z_1(0) = \lambda u_1(\ell_1) = 0. \tag{4.39}$$

Eqs. (4.39) and (4.35) show that $u_1(0) = u_2(0) = u_3(0) = 0$ and $u_1(\ell_1) = 0$, which together with (4.38) indicate that $u_1(x) \equiv 0$. Hence, $u_2(x)$ and $u_3(x)$ satisfy (4.30) and (4.31) for $\mu = \lambda^2$, respectively. We assert that $u_2(x) = u_3(x) \equiv 0$ by the assumption $\Sigma_2 \cap \Sigma_3 = \emptyset$. Therefore, $(u,v,z) = 0$; this is a contradiction. So, there is no eigenvalue on the imaginary axis.

Next, we prove that the imaginary axis is not an asymptote of $\sigma(\mathcal{A})$. Indeed, we only need to show that the imaginary axis is not an asymptote of the zeros of $\det\Delta(\lambda)$. From (3.25) and (3.26), we know that $\frac{\det\Delta(\lambda)}{\lambda^3}$ is an entire function of sine type, so is $[\frac{\det\Delta(\lambda)}{\lambda^3}]'$, which implies that $[\frac{\det\Delta(\lambda)}{\lambda^3}]'$ is bounded in the strip parallel to the imaginary axis (see, [32]).

By contradictory, assume that there is a sequence $\lambda_n \subset \sigma(\mathcal{A})$ with $\Re\lambda_n \to 0$ and $|\lambda_n| \to \infty$ such that $\det \Delta(\lambda_n) = 0$, $\forall n \in \mathbb{N}$. Let $\lambda_n = \alpha_n + i\beta_n$, then there exist $\theta_n \in (\alpha_n, 0)$ such that

$$\frac{\det \Delta(\lambda_n)}{\lambda_n^3} - \frac{\det \Delta(i\beta_n)}{(i\beta)_n^3} = \left[\frac{\det \Delta'(\theta_n + i\beta_n)}{(\theta_n + i\beta_n)^3}\right]' \alpha_n, \quad \theta_n \in (\alpha_n, 0).$$

By the boundedness of $[\frac{\det \Delta(\lambda)}{\lambda^3}]'$ and $\alpha_n \to 0$, we get

$$\lim_{n\to\infty} \frac{\det \Delta(\lambda_n)}{\lambda_n^3} = \lim_{n\to\infty} \frac{\det \Delta(i\beta_n)}{(i\beta_n)^3} = 0. \tag{4.40}$$

On the other hand, denote the main part of $\Delta(\lambda)$ by $\Delta_0(\lambda)$, i.e. $\Delta_0(\lambda) =$

$$\left(\begin{matrix} \lambda e^{\lambda(m_1-\tau_1)}\beta_1 k + \lambda e^{\lambda m_1}[\alpha_1 k + \sigma_1(\ell_1)] & 0 & 0 \\ 0 & \lambda\sigma_2(\ell_2)e^{\lambda m_2} & 0 \\ 0 & 0 & e^{\lambda m_3} \\ \tilde{k}_1 & -\tilde{k}_2 & 0 \\ \tilde{k}_1 & 0 & -\tilde{k}_3 \\ -\lambda e^{-\lambda\tau_0}\left[\frac{\beta_0}{3}\tilde{k}_1\right] + \lambda h_1^- & -\lambda e^{-\lambda\tau_0}\left[\frac{\beta_0}{3}\tilde{k}_2\right] + \lambda h_2^- & -\lambda e^{-\lambda\tau_0}\left[\frac{\beta_0}{3}\tilde{k}_3\right] + \lambda h_3^- \end{matrix}\right.$$

$$\left.\begin{matrix} \lambda\beta_1 k e^{\lambda(-\tau_1-m_1)} + \lambda e^{-\lambda m_1}[\alpha_1 k - \sigma_1(\ell_1)] & 0 & 0 \\ 0 & -\lambda\sigma_2(\ell_2)e^{-\lambda m_2} & 0 \\ 0 & 0 & e^{-\lambda m_3} \\ \tilde{k}_1 & -\tilde{k}_2 & 0 \\ \tilde{k}_1 & 0 & -\tilde{k}_3 \\ -\lambda e^{-\lambda\tau_0}\left[\frac{\beta_0}{3}\tilde{k}_1\right] - \lambda h_1^+ & -\lambda e^{-\lambda\tau_0}\left[\frac{\beta_0}{3}\tilde{k}_2\right] - \lambda h_2^+ & -\lambda e^{-\lambda\tau_0}\left[\frac{\beta_0}{3}\tilde{k}_3\right] - \lambda h_3^+ \end{matrix}\right). \tag{4.41}$$

A direct calculation gives

$$\begin{aligned} S(\lambda) &= \frac{\det \Delta_0(\lambda)}{\lambda^3} \\ &= \left(\frac{\rho_3(0)\sigma_3(0)}{\rho_1(0)\sigma_1(0)\rho_2(0)\sigma_2(0)}\right)^{1/4} F_1(\lambda)F_2(\lambda)\cosh\lambda m_3 \\ &\quad + \frac{\sinh\lambda m_3}{(\rho_3(0)\sigma_3(0))^{1/4}}\left[\left(\frac{\rho_1(0)\sigma_1(0)}{\rho_2(0)\sigma_2(0)}\right)^{1/4} F_2(\lambda)G_1(\lambda) + \left(\frac{\rho_2(0)\sigma_2(0)}{\rho_1(0)\sigma_1(0)}\right)^{1/4} F_1(\lambda)G_2(\lambda)\right. \\ &\quad \left. + \left(\frac{\rho_1(0)\sigma_1(0)}{\rho_2(0)\sigma_2(0)}\right)^{1/4} C_0 F_1(\lambda)F_2(\lambda)\right], \end{aligned}$$

where

$$\begin{aligned} F_1(\lambda) &= \sigma_1(\ell_1)\cosh\lambda m_1 + \left(\frac{\rho_1(\ell_1)}{\sigma_1(\ell_1)}\right)^{-1/2} C_1 \sinh\lambda m_1, \\ G_1(\lambda) &= \sigma_1(\ell_1)\sinh\lambda m_1 + \left(\frac{\rho_1(\ell_1)}{\sigma_1(\ell_1)}\right)^{-1/2} C_1 \cosh\lambda m_1, \\ F_2(\lambda) &= \cosh\lambda m_2, \qquad G_2(\lambda) = \sinh\lambda m_2 \end{aligned}$$

and

$$C_0 = \beta_0 e^{-\lambda\tau_0} + \alpha_0, \qquad C_1 = \beta_1 e^{-\lambda\tau_1} + \alpha_1.$$

Since $\alpha_i > \beta_i$, $i = 0, 1$, for any $\lambda = ix \in i\mathbb{R}$, $|C_i| > \alpha_i - \beta_i > 0$, it is easy to check that

$$\inf_{x\in\mathbb{R}} |F_1(ix)| > 0 \quad \text{and} \quad \inf_{x\in\mathbb{R}} |G_1(ix)| > 0.$$

Thus we have

$$\frac{S(ix)}{F_1(ix)} = \left(\frac{\rho_3(0)\sigma_3(0)}{\rho_1(0)\sigma_1(0)\rho_2(0)\sigma_2(0)}\right)^{1/4} \cos xm_2 \cos xm_3 - \left(\frac{\rho_2(0)\sigma_2(0)}{\rho_1(0)\sigma_1(0)\rho_3(0)\sigma_3(0)}\right)^{1/4} \sin xm_3 \sin xm_2$$

$$-\left(\frac{\rho_1(0)\sigma_1(0)}{\rho_2(0)\sigma_2(0)\rho_3(0)\sigma_3(0)}\right)^{1/4}\left(\sigma_1^2(\ell_1)-\frac{\sigma_1(\ell_1)}{\rho_1(\ell_1)}C_1^2\right)\frac{1}{|F_1(ix)|^2}\sin xm_3\cos xm_2\sin xm_1\cos xm_1$$
$$-i\left(\frac{\rho_1(0)\sigma_1(0)}{\rho_2(0)\sigma_2(0)\rho_3(0)\sigma_3(0)}\right)^{1/4}\left[\frac{1}{|F_1(ix)|^2}\sigma_1(\ell_1)\left(\frac{\rho_1(\ell_1)}{\sigma_1(\ell_1)}\right)^{-1/2}C_1+C_0\right]\sin xm_3\cos xm_2.$$

Obviously, if there is an $x\in\mathbb{R}$ such that $\sin xm_3=\cos xm_2=0$, then $S(ix)=0$. In this case, there are infinitely many points $\{x_n\}$ on $\mathbb{R}$ such that $S(ix_n)=0$.

If $\cos xm_2=0$ and $\sin xm_3\neq 0$, then $x=\frac{(2n+1)\pi}{2m_2}$, $n\in\mathbb{N}$, we have

$$\frac{S(ix)}{F_1(ix)}=\pm\left(\frac{\rho_2(0)\sigma_2(0)}{\rho_1(0)\sigma_1(0)\rho_3(0)\sigma_3(0)}\right)^{1/4}\sin\frac{(2n+1)\pi m_3}{2m_2}.$$

If $\cos xm_2\neq 0$ and $\sin xm_3=0$, then $x=\frac{n\pi}{m_3}$, $n\in\mathbb{N}$, we have

$$\frac{S(ix)}{F_1(ix)}=\pm\left(\frac{\rho_3(0)\sigma_3(0)}{\rho_1(0)\sigma_1(0)\rho_2(0)\sigma_2(0)}\right)^{1/4}\cos\frac{n\pi m_2}{m_3}.$$

If $\cos xm_2\neq 0$ and $\sin xm_3\neq 0$, we write

$$\frac{S(ix)}{F_1(ix)}=\Re\frac{S(ix)}{F_1(ix)}+i\Im\frac{S(ix)}{F_1(ix)}$$
$$\Im\frac{S(ix)}{F_1(ix)}=-\left(\frac{\rho_1(0)\sigma_1(0)}{\rho_2(0)\sigma_2(0)\rho_3(0)\sigma_3(0)}\right)^{1/4}\left\{\frac{(\beta_1\cos x\tau_1+\alpha_1)}{|F_1(ix)|^2}\right.$$
$$\left.\times\left[\left(\frac{\sigma_1(\ell_1)}{\rho_1(\ell_1)}\right)\beta_1\sin x\tau_1\sin 2xm_1+\sigma_1(\ell_1)\left(\frac{\sigma_1(\ell_1)}{\rho_1(\ell_1)}\right)^{1/2}\right]+\beta_0\cos x\tau_0+\alpha_0\right\}\sin xm_3\cos xm_2.$$

Note that

$$\left[\left(\frac{\sigma_1(\ell_1)}{\rho_1(\ell_1)}\right)\beta_1\sin x\tau_1\sin 2xm_1+\sigma_1(\ell_1)\left(\frac{\sigma_1(\ell_1)}{\rho_1(\ell_1)}\right)^{1/2}\right]\geq 0,$$

if $\beta_1<\alpha_1$, $\beta_0<\alpha_0$ and $\beta_1^2\leq\sigma_1(\ell_1)\rho_1(\ell_1)$.

Therefore, when

$$\frac{m_2}{m_3}\in\left\{\frac{2n_1+1}{2n_2+1},n_1,n_2\in\mathbb{N}\right\}\quad\text{or}\quad\frac{m_2}{m_3}\in\left\{\frac{2n_1}{2n_2+1},n_1,n_2\in\mathbb{N}\right\},\tag{4.42}$$

we always have $\inf_{x\in\mathbb{R}}\left|\frac{S(ix)}{F_1(ix)}\right|>0$ and in the other case it holds that $\inf_{x\in\mathbb{R}}\left|\frac{S(ix)}{F_1(ix)}\right|=0$. So, if the condition (4.42) holds we get

$$\inf_{x\in\mathbb{R}}\left|\frac{\det\Delta(ix)}{(ix)^3}\right|>0.$$

This contradicts to (4.40). Thus, we can conclude that $\sup\{\Re\lambda|\lambda\in\sigma(\mathcal{A})\}<0$. The desired result follows. □

5. Concluding remarks

With the help of the Riesz basis approach and the asymptotic analysis technique, the exponential stability problem for variable coefficient wave network with small time-delay in nodal feedbacks has been solved under certain conditions in this paper. The key point in stability analysis is to replace checking uniform boundedness of the resolvent $R(\lambda,\mathcal{A})$ on the imaginary axis by verifying $\inf_{x\in\mathbb{R}}\frac{|\det\Delta(ix)|}{|ix^3|}>0$. Hence we get the exponential stability of the network system. Although this method has a tedious computation, it can be used to discuss the variable coefficient differential equation.

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References

[1] K. Ammari, M. Jellouli, Stabilization of star-shaped networks of strings, Differential Integral Equations 17 (2004) 1395–1410.
[2] K. Ammari, M. Jellouli, M. Khenissi, Stabilization of generic trees of strings, J. Dyn. Control Syst. 11 (2) (2005) 177–193.
[3] R. Dáger, E. Zuazua, Wave Propagation, Observation and Control in 1-D Flexible Multi-Structures, in: Series: Mathématiques et Applications, vol. 50, Springer-Verlag, Berlin, New York, 2006.
[4] K.S. Liu, F.L. Huang, G. Chen, Exponential stability analysis of a long chain of coupled vibrating strings with dissipative linkage, SIAM J. Appl. Math. 49 (1989) 1694–1707.
[5] S. Nicaise, J. Valein, Stabilization of the wave equation on 1-D networks with a delay term in the nodal feedbacks, Netw. Heterog. Media 2 (2007) 425–479.
[6] G.Q. Xu, B.Z. Guo, Riesz basis property of evolution equations in Hilbert space and application to a coupled string equation, SIAM J. Control Optim. 42 (3) (2003) 966–984.
[7] E. Ait Benhassi, K. Ammari, S. Boulite, L. Maniar, Feedback stabilization of a class of evolution equations with delay, J. Evol. Equ. 9 (2009) 103–121.
[8] S. Nicaise, J. Valein, E. Fridman, Stability of the heat and wave equations with boundary time-varying delays, Discrete Contin. Dyn. Syst. 2 (2009) 559–581.
[9] J. Valein, E. Zuazua, Stabilization of the wave equation on 1-D networks, SIAM J. Control Optim. 48 (4) (2009) 2771–2797.
[10] G. Chen, M.C. Delfour, A.M. Krall, G. Payre, Modeling, stabilization and control of serially connected beams, SIAM J. Control Optim. 25 (3) (1987) 526–546.
[11] B. Dekoninck, S. Nicaise, Control of networks of Euler–Bernoulli beams, ESAIM Control Optim. Calc. Var. 4 (1999) 57–81.
[12] B. Dekoninck, S. Nicaise, The eigenvalue problem for networks of beams, Linear Algebra Appl. 314 (2000) 165–189.
[13] J.E. Lagnese, G. Leugering, E.J.P.G. Schmidt, Modeling, Analysis and Control of Dynamic Elastic Multi-Link Structures, Birkhäuser, Boston, Basel, Berlin, 1994.
[14] J.E. Lagnese, G. Leugering, E.J.P.G. Schmidt, Modeling, Analysis and Control of Multilink Flexible Structures, in: System and Control: Foundamentions and Applications, Birkhäuser, Basel, 1994.
[15] G. Leugering, E. Zuazua, Exact controllability of generic trees, in: Control of Systems Governed by Partial Differential Equations, ESAIM Proceeding, Nancy, France, 1999.
[16] Z.J. Han, G.Q. Xu, Analysis of stability for n-connected Timoshenko beams with both ends fixed and feedback controller at intermediate nodes, in: Proceedings of the 25th Chinese Control Conference, 7–11 August, 2006, Harbin, Heilongjiang.
[17] J.M. Wang, B.Z. Guo, Riesz basis and stabilization for the flexible structure of a symmetric tree-shaped beam networks, Math. Methods Appl. Sci. 31 (2008) 289–314.
[18] G.Q. Xu, Z.J. Han, S.P. Yung, Riesz basis property of serially connected Timoshenko beams, Internat. J. Control 80 (3) (2007) 470–485.
[19] G.Q. Xu, S.P. Yung, Stability and Riesz basis property of a star-shaped network of Euler–Bernoulli beams with joint damping, Netw. Heterog. Media 3 (4) (2008) 723–747.
[20] S. Nicaise, C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, SIAM J. Control Optim. 45 (2006) 1561–1585.
[21] R. Datko, J. Lagnese, M.P. Polis, An example on the effect of time delays in boundary feedbacks stabilization of wave equations, SIAM J. Control Optim. 24 (1) (1986) 152–156.
[22] G.Q. Xu, S.P. Yung, L.K. Li, Stabilization of wave systems with input delay in the boundary control, ESAIM Control Optim. Calc. Var. 12 (2006) 770–785.
[23] M.A. Shubov, Basis property of eigenfunctions of nonselfadjoint operator pencils generated by the equations of nonhomogeneous damped string, Integral Equations Operator Theory 25 (1996) 289–328.
[24] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, London, 1976.
[25] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
[26] N. Dunford, J.T. Schwartz, Linear Operators, Part III, Spectral Operators, New York, 1971.
[27] M.A. Naimark, Linear Differential Operators, Frederick Ungar, New York, 1967.
[28] S.A. Avdonin, S.A. Ivanov, Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems, Cambridge University Press, Cambridge, 1995.
[29] R. Kress, Linear Integral Equations, in: Applied Mathematical Sciences, vol. 82, 1999.
[30] A.A. Shkalikov, Boundary problems for ordinary differential equations with parameter in the boundary conditions, J. Math. Sci. 33 (6) (1986) 1311–1342.
[31] G.Q. Xu, D.Y. Liu, Y.Q. Liu, Abstract second order hyperbolic system and applications, SIAM J. Control Optim. 47 (4) (2008) 1762–1784.
[32] Robert M. Young, An Introduction to Nonharmonic Fourier Series, 2001.