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Generalized inverses and sampling problems

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ABSTRACT

Sampling theory is concerned with the problem of reconstructing a signal f in a Hilbert space from a given a collection of sampled values of f. If a certain decomposition of the Hilbert space is possible (in terms of the sampling and reconstruction subspaces) then a consistent reconstruction can be obtained. In this paper we treat the case in which such a decomposition cannot be found. For this situation, we study the *quasi-consistent reconstructions* which are an extension of the consistent reconstructions. We relate the previous concepts to generalized inverses. We also present some new results and problems regarding consistent sampling.

1. Introduction

Sampling theory is a topic with applications in several fields such as signal and image processing, communication engineering, and information theory, among others. The central idea of this theory is to recover a continuous-time function from a discrete set of samples. One of the first results in this direction was proved by Cauchy in [1]. Nevertheless, the result that has had the most impact in this area is the classical Whittaker–Kotel'nikov–Shannon theorem [2–4] which provides conditions on a function on $\mathbb{R}$ such that it can be reconstructed from its sampled values at integer points. More precisely, for every $f \in L^2(\mathbb{R})$ whose Fourier transform is supported in $[-\frac{1}{2}, \frac{1}{2}]$, it holds that $f(t) = \sum_{n\in\mathbb{Z}} f(n)\,\mathrm{sinc}(t-n)$ with L^2 and uniform convergences and where $\mathrm{sinc}(t) := \frac{\sin(\pi t)}{\pi t}$.

A more general approach to sampling in an arbitrary Hilbert space is to consider the samples of the original signal f as the inner product of f with a set of sampling vectors, which span the sampling subspace $\mathcal{S}$ (see [5–7]). Hence a reconstruction of f, $\tilde{f}$, is obtained as a linear combination of a set of reconstruction vectors that span the reconstruction subspace $\mathcal{W}$. We assume that the coefficients of such a reconstruction are obtained by a bounded linear transformation of the samples. This bounded linear operator will be called a filter. Observe that this framework includes the classical Whittaker–Kotel'nikov–Shannon theorem. Therefore, the sampling problem consists in selecting an appropriate filter such that the reconstruction obtained verifies some optimal criterion.

A common criterion suggested by Unser and Aldroubi in [8] is to design a reconstruction $\tilde{f}$ which is consistent with the samples, i.e., $\tilde{f}$ yields the same samples as f when it is re-injected into the system. The existence of consistent reconstructions in an arbitrary Hilbert space $\mathcal{H}$ was studied by Eldar and Werther in [9]. They proved that there exists a consistent reconstruction for every $f \in \mathcal{H}$ if and only if $\mathcal{H} = \mathcal{W} + \mathcal{S}^\perp$. If in addition $\mathcal{W} \cap \mathcal{S}^\perp = \{0\}$ then the consistent reconstruction is unique. In [10], Corach and Giribet related the consistent sampling condition to oblique projections. In this work, we characterize the consistent filters by means of generalized inverses. Moreover, since the fact that f, $\tilde{f}$ have the same samples does not imply that they are close, we seek among the consistent reconstructions the one which is closest to f in the squared-norm sense. We shall note that this ideal reconstruction can be computed from the sequence of samples if and only if the

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original signal lies in a convenient subspace of $\mathcal{H}$. In this article, we also propose and study a new sampling problem. Namely, we consider the case in which two sequences of samples of the original signal are known. A natural question that arises is that of whether there exist simultaneous consistent reconstructions for the two samples. We provide necessary and sufficient conditions that guarantee the existence. In addition, we present the general form of such recovered signals.

Another goal of this paper is to study a reconstruction-sampling scheme for the case where consistent reconstructions cannot be obtained, i.e., $\mathcal{H} \neq \mathcal{W} + \mathcal{S}^\perp$. We define the *quasi-consistent reconstructions* as those reconstructed signals such that if they are re-injected into the system, then their samples are as close as possible to the original samples. This concept is a generalization of consistent sampling. A first study of reconstructions of this kind can be found in [11]. Here, we characterize the quasi-consistent filters by means of generalized inverses. Furthermore, we obtain conditions for assuring that a quasi-consistent reconstruction minimizes the squared-norm error. Moreover, if there exist infinite quasi-consistent reconstructions, then we provide two criteria for selecting a convenient one. These criteria are motivated by a work of Eldar and Dvorkind [12]. We recommend this article for simulation results in problems of speech and image processing.

The paper is organized as follows: Section 2 contains a survey of results and notation used throughout the article. In Section 3 we relate the notion of consistent reconstructions to generalized inverses. Furthermore, we determine the consistent reconstruction that minimizes the squared-norm error. In Section 4, the problem related to two samples is presented. To conclude, Section 5 is devoted to quasi-consistent reconstructions.

2. Preliminaries

In this section we present some of the results and terminology that we shall need in this paper. Throughout, Hilbert spaces are denoted by $\mathcal{H}$, $\mathcal{F}$, $\mathcal{K}$, $\mathcal{G}$, whereas vectors in these spaces are denoted by lower-case letters. By $L(\mathcal{H}, \mathcal{K})$ we denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and the algebra $L(\mathcal{H}, \mathcal{H})$ is abbreviated as $L(\mathcal{H})$. For any $T \in L(\mathcal{H}, \mathcal{K})$ the range is denoted by $R(T)$, the kernel by $N(T)$ and the adjoint by T^*. In what follows, $\mathcal{S} \dot{+} \mathcal{T}$ denotes the direct sum of the closed subspaces $\mathcal{S}$ and $\mathcal{T}$. In addition, if $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$ then $Q_{\mathcal{S}\|\mathcal{T}}$ denotes the projection with range $\mathcal{S}$ and kernel $\mathcal{T}$. In particular, $P_{\mathcal{S}}$ indicates $Q_{\mathcal{S}\|\mathcal{S}^\perp}$.

Given $A \in L(\mathcal{H}, \mathcal{K})$ with closed range the *Moore–Penrose inverse* of A, denoted by $A^\dagger$, is defined to be the unique operator X satisfying the four Penrose equations:

1. $AXA = A$;
2. $XAX = X$;
3. $(AX)^* = AX$;
4. $(XA)^* = XA$.

Clearly, $AA^\dagger = P_{R(A)}$ and $A^\dagger A = P_{N(A)^\perp}$. An operator X is called a *generalized inverse* of A, denoted by A^-, if it satisfies Eq. (1). In the sequel, $A[i, j, k, l]$ stands for the set of operators that verify conditions i, j, k, l. Furthermore, it holds that $A[1] = \{A^\dagger + T - A^\dagger ATAA^\dagger,\ T \in L(\mathcal{K}, \mathcal{H})\}$. For details of these matters we refer the reader to the books [13,14] among many other sources.

We shall study sampling problems which are expressed as operator equations of the form $AXB = C$ with A, B, C bounded linear operators defined in convenient Hilbert spaces. In what follows the next result, which provides conditions for the solubility of equations of this kind, will play a relevant role (see [15,16]). We shall say that the equation $AXB = C$ is *solvable* if there exists a bounded linear operator $\tilde{X}$ such that $A\tilde{X}B = C$.

Theorem 2.1 (*[15, Theorem 3.1]; [16, Theorem 2]*). *Let $A \in L(\mathcal{H}, \mathcal{K})$, $B \in L(\mathcal{F}, \mathcal{G})$ and $C \in L(\mathcal{F}, \mathcal{K})$. If $R(A)$, $R(B)$ or $R(C)$ is closed then the following conditions are equivalent:*

(1) *The equation $AXB = C$ is solvable.*
(2) *$R(C) \subseteq R(A)$ and $R(C^*) \subseteq R(B^*)$.*

Moreover, if A, B have closed ranges and $AXB = C$ is solvable then the general solution is given by

$$X = A^\dagger CB^\dagger + T - A^\dagger ATBB^\dagger, \tag{1}$$

for arbitrary $T \in L(\mathcal{G}, \mathcal{H})$.

Finally, the concept of frames is a useful tool for studying sampling problems [17–19]. For a complete survey on frame theory and its applications, the reader is referred to [20].

3. Consistent sampling

Now, we are ready to construct a precise formulation of the sampling problem in abstract Hilbert spaces. Let f be the original input signal which is assumed to belong to a Hilbert space $\mathcal{H}$. We consider two closed subspaces of $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{W}$, called the *sampling* and *reconstruction subspaces*, respectively. Given $\mathcal{F}_{\mathcal{S}} = \{s_n\}_{n\in\mathbb{N}}$ a frame of $\mathcal{S}$ with synthesis operator $S \in L(l^2, \mathcal{H})$, $S^*f = \{\langle f, s_n\rangle\}_{n\in\mathbb{N}}$ are the *samples* of f. We point out that we can ensure that the samples have finite energy because we are working with frames. On the other hand, we consider a frame of $\mathcal{W}$, $\mathcal{F}_{\mathcal{W}} = \{w_n\}_{n\in\mathbb{N}}$, with synthesis operator $W \in L(l^2, \mathcal{H})$. Hence, a reconstruction of f has the form

$$\tilde{f} = \sum_n c_n w_n,$$

for some coefficients $\{c_n\}_{n\in\mathbb{N}} \in l^2$ obtained from the samples S^*f under some optimality criterion. Note that the reconstruction is well-defined, i.e. the sum converges, because $\{w_n\}_{n\in\mathbb{N}}$ is a frame of $\mathcal{W}$. That is, sampling problems consist in finding a suitable $X \in L(l^2)$ (called the filter) such that the reconstruction

$$\tilde{f} = WXS^*f, \tag{2}$$

has good (in some sense) approximation properties. We note that both operators, S, W, have closed ranges.

A well-known criterion of reconstruction is to require that the reconstructed signal be *consistent*. Consistency was proposed in [8] as follows: a reconstruction of f, $\tilde{f} \in \mathcal{W}$, is said to be a consistent reconstruction (c.r.) if and only if it yields exactly the same samples if it is re-injected into the system. Using the formulation introduced above this is expressed as

$$S^*\tilde{f} = S^*f.$$

Clearly, the existence of a consistent reconstruction for every $f \in \mathcal{H}$ is equivalent to the solubility of the equation $S^*WXS^* = S^*$. Following the notation used in [10] we define

$$\begin{aligned} \mathcal{CS}(W, S) &:= \{X \in L(l^2) : WXS^*f \text{ is a c.r. for every } f \in \mathcal{H}\} \\ &= \{X \in L(l^2) : S^*WXS^* = S^*\}. \end{aligned} \tag{3}$$

Observe that $\mathcal{CS}(W, S)$ is not empty if and only if $\mathcal{H} = R(W) + N(S^*)$. Indeed, by Theorem 2.1, the equation $S^*WXS^* = S^*$ is solvable if and only if $R(S^*) \subseteq R(S^*W)$, i.e., if and only if $\mathcal{H} = R(W) + N(S^*)$. In the next theorem, we relate the filters in $\mathcal{CS}(W, S)$ to the generalized inverses of S^*W.

Theorem 3.1. *Let $\mathcal{H} = R(W) + N(S^*)$. Then $\mathcal{CS}(W, S) = (S^*W)[1]$.*

Proof. Consider $X \in \mathcal{CS}(W, S)$. Then, $S^*WXS^* = S^*$ and so $S^*WXS^*W = S^*W$, i.e., $X \in (S^*W)[1]$. For the converse, consider $\mathcal{T}$ a closed subspace of $\mathcal{H}$ such that $\mathcal{H} = R(W)\dot{+}\mathcal{T}$ and $\mathcal{T} \subseteq N(S^*)$ (for example, $\mathcal{T} := N(S^*) \cap (R(W) \cap N(S^*))^{\perp}$). By Theorem 2.1, the equation $WX = Q_{R(W)\|\mathcal{T}}$ is solvable. Let W^- be a solution of this equation. Then, $WW^-W = Q_{R(W)\|\mathcal{T}}W = W$, i.e., $W^- \in W[1]$. Therefore, if $X \in (S^*W)[1]$ then $S^*WXS^*WW^- = S^*WW^-$ and so, since $\mathcal{T} \subseteq N(S^*)$, $S^*WXS^* = S^*$, i.e., $X \in \mathcal{CS}(W, S)$. $\square$

Given $f \in \mathcal{H}$ we shall define

$$\mathcal{C}_{W,S}(f) := \{WXS^*f : X \in \mathcal{CS}(W, S)\},$$

i.e., $\mathcal{C}_{W,S}(f)$ is the set of consistent reconstructions in $\mathcal{W}$ of f. Under the assumption that $\mathcal{H} = R(W) + N(S^*)$, $\mathcal{C}_{W,S}(f)$ is not empty for every $f \in \mathcal{H}$. Since in general the consistent reconstruction differs from the original signal, we devote the rest of this section to seeking the element in $\mathcal{C}_{W,S}(f)$ which is closest to f in the squared-norm sense. As we shall see, the solution of this problem can be calculated if extra hypotheses about the original signal are given. For this purpose, the next statement which provides a total description of $\mathcal{C}_{W,S}(f)$ will be useful.

Theorem 3.2 *([21, Theorem 1]). Let $\mathcal{H} = R(W) + N(S^*)$. Then, $\mathcal{C}_{W,S}(f) = F^{\dagger}P_{R(S)}f + N(F)$, where $F = I - P_{N(S^*)}P_{R(W)}$.*

Note that $F^{\dagger}P_{R(S)}f = F^{\dagger}S(S^*S)^{\dagger}(S^*f)$, i.e. $F^{\dagger}P_{R(S)}f$ can be obtained from the samples S^*f. Moreover, since $F^{\dagger}P_{R(S)}f \in N(F)^{\perp}$, then $F^{\dagger}P_{R(S)}f$ is the element in $\mathcal{C}_{W,S}(f)$ with minimal norm. The filter of this minimal reconstruction was described by Corach and Giribet in [10, Theorem 4.2].

Proposition 3.1. *Let $\mathcal{H} = R(W) + N(S^*)$. Then*

$$\arg \min_{\tilde{f}\in\mathcal{C}_{W,S}(f)} \|f - \tilde{f}\|^2 = F^{\dagger}P_{R(S)}f + P_{N(F)}f. \tag{4}$$

Proof. Let us start by noting that $R(F) = R(W)^{\perp} + R(S)$ and $N(F) = R(W) \cap N(S^*)$, i.e., $N(F) = R(F)^{\perp}$. By Theorem 3.2, $\min_{\tilde{f}\in\mathcal{C}_{W,S}(f)} \|f - \tilde{f}\|^2 = \min_{v\in N(F)} \|f - F^{\dagger}P_{R(S)}f - v\|^2$. Now, since $F^{\dagger}P_{R(S)}f \in R(F^{\dagger}) = N(F)^{\perp}$ and $v \in N(F)$ then

$$\min_{v\in N(F)} \|f - F^{\dagger}P_{R(S)}f - v\|^2 = \min_{v\in N(F)} \|P_{N(F)^{\perp}}f - F^{\dagger}P_{R(S)}f\|^2 + \|P_{N(F)}f - v\|^2,$$

and so the assertion follows. $\square$

The main inconvenience of expression (4) is that, in general, $P_{N(F)}f$ cannot be obtained from the samples S^*f by using a bounded linear operator. Indeed, $P_{N(F)} = CS^*$ for some operator $C \in L(l^2, \mathcal{H})$ if and only if $N(F) = R(W) \cap N(S^*) \subseteq R(S)$ (by Theorem 2.1), i.e., $R(W) \cap N(S^*) = \{0\}$ or, equivalently, if the consistent reconstruction is unique. However, as we shall see in the next proposition, if we consider the case in which f is known to lie in an appropriate subspace then the optimal reconstruction (4) can be computed from the samples. The following result can also be found in [10, Theorem 5.1]. The proof presented here differs from that of Corach and Giribet since we do not use the notion of oblique projections.

Proposition 3.2. *Let $\mathcal{T}$ be a closed subspace of $\mathcal{H}$. For every $f \in \mathcal{T}$ the consistent reconstruction (4) can be obtained from the samples S^*f if and only if $P_{\mathcal{T}}(R(W) \cap N(S^*)) \subseteq P_{\mathcal{T}}(R(S))$.*

Proof. Observe that $P_{\mathcal{T}}(R(W) \cap N(S^*)) \subseteq P_{\mathcal{T}}(R(S))$ if and only if, by Theorem 2.1, there exists $Z \in L(l^2, \mathcal{H})$ such that $P_{N(F)}P_{\mathcal{T}} = ZS^*P_{\mathcal{T}}$. Hence, for every $f \in \mathcal{T}$ we have $P_{N(F)}f = ZS^*f$ and so (4) can be obtained from the samples of f. $\square$

4. Consistent reconstructions for two samples

This section is devoted to studying the situation in which two sequences of samples of the original signal are known. We focus our attention on determining conditions for the existence of simultaneous consistent reconstructions for the two samples. In addition, we provide the expression for such recovered signals. We point out that we consider the same reconstruction subspace for both sampling procedures. More precisely, we consider a reconstruction subspace $\mathcal{W}$ with synthesis operator W and two sampling subspaces $\mathcal{S}, \mathcal{S}' \subseteq \mathcal{H}$ with synthesis operators $S, S' \in L(l^2, \mathcal{H})$, respectively. For simplicity of notation, we define $\mathcal{N} = N(S'^*) \cap N(S^*)$.

First, we are interested in finding necessary and sufficient conditions for $\mathcal{C}_{W,S}(f) = \mathcal{C}_{W,S'}(f)$ for every $f \in \mathcal{H}$.

Proposition 4.1. *If $\mathcal{C}_{W,S}(f)$ and $\mathcal{C}_{W,S'}(f)$ are not empty then $\mathcal{C}_{W,S}(f) = \mathcal{C}_{W,S'}(f)$ for every $f \in \mathcal{H}$ if and only if $N(S^*) = N(S'^*)$.*

Proof. Suppose $\mathcal{C}_{W,S}(f) = \mathcal{C}_{W,S'}(f)$ for all $f \in \mathcal{H}$ and let $f \in N(S^*)$. Then, $\tilde{f} = 0 \in \mathcal{C}_{W,S}(f)$. Thus, $\tilde{f} = 0 \in \mathcal{C}_{W,S'}(f)$ and so $S'^*f = S'^*\tilde{f} = 0$, i.e., $f \in N(S'^*)$. Conversely, suppose that $N(S^*) = N(S'^*)$ and let $\tilde{f} \in \mathcal{C}_{W,S}(f)$. Then, $S^*f = S^*\tilde{f}$, i.e., $\tilde{f} - f \in N(S^*) = N(S'^*)$. Hence, $S'^*\tilde{f} = S'^*f$ and so $\tilde{f} \in \mathcal{C}_{W,S'}(f)$. $\square$

The following theorem provides different criteria for $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$ to be not empty. Moreover, we present a full description of this set.

Theorem 4.1. *The following conditions are equivalent:*

(1) *$\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$ is not empty for every $f \in \mathcal{H}$;*
(2) *$\mathcal{H} = R(W) + \mathcal{N}$;*
(3) *for every $f \in \mathcal{H}$, $W^\dagger(\tilde{f}_S - \tilde{f}_{S'}) \in N(S^*W) + N(S'^*W)$ where $\tilde{f}_S \in \mathcal{C}_{W,S}(f)$; and $\tilde{f}_{S'} \in \mathcal{C}_{W,S'}(f)$.*

Moreover, if one of the previous conditions holds then

$$\tilde{f}_{S,S'} = \tilde{f}_S + WP_{N(S^*W)}G^\dagger S'^*(\tilde{f}_{S'} - \tilde{f}_S) \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f), \tag{5}$$

where $\tilde{f}_S \in \mathcal{C}_{W,S}(f)$, $\tilde{f}_{S'} \in \mathcal{C}_{W,S'}(f)$ and $G = S'^*WP_{N(S^*W)}$. Furthermore,

$$\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f) = \{\tilde{f}_{S,S'} + WP_{N(S^*W)}(I - G^-G)h,\ h \in \mathcal{H}\}. \tag{6}$$

Proof. $1 \Leftrightarrow 2$ Suppose that $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$ is not empty for every $f \in \mathcal{H}$ and consider $f \in \mathcal{H}$. Then, there exists $\tilde{f} \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$. Clearly, $\tilde{f} \in R(W)$. Moreover, since $S^*f = S^*\tilde{f}$ and $S'^*f = S'^*\tilde{f}$, then $z := f - \tilde{f} \in \mathcal{N}$. Then, $f = \tilde{f} + z \in R(W) + \mathcal{N}$.

Conversely, suppose that $\mathcal{H} = R(W) + \mathcal{N}$. Note that this implies that $\mathcal{C}_{W,S}(f)$ and $\mathcal{C}_{W,S'}(f)$ are not empty for every f. Now, given $f \in \mathcal{H}$ let $f = \tilde{f} + w$ with $\tilde{f} \in R(W)$ and $w \in \mathcal{N}$. Hence, $S^*f = S^*\tilde{f}$ and $S'^*f = S'^*\tilde{f}$. Therefore, $\tilde{f} \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$.

$1 \Leftrightarrow 3$ Let $f \in \mathcal{H}$, $\tilde{f}_S \in \mathcal{C}_{W,S}(f)$ and $\tilde{f}_{S'} \in \mathcal{C}_{W,S'}(f)$. Observe that $W^\dagger\tilde{f}_S$ is a solution of $S^*Wx = S^*f$. Now, let $\tilde{f} = W\xi \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$ for some $\xi \in l^2$. Note that ξ is also a solution of $S^*Wx = S^*f$. Then, $W^\dagger\tilde{f}_S - \xi \in N(S^*W)$. Analogously, $W^\dagger\tilde{f}_{S'} - \xi \in N(S'^*W)$. So, $W^\dagger(\tilde{f}_S - \tilde{f}_{S'}) = (W^\dagger\tilde{f}_S - \xi) - (W^\dagger\tilde{f}_{S'} - \xi) \in N(S^*W) + N(S'^*W)$.

Conversely, suppose that $W^\dagger(\tilde{f}_S - \tilde{f}_{S'}) = \mu - \nu$ with $\mu \in N(S'^*W)$, $\nu \in N(S^*W)$. Then, $W^\dagger\tilde{f}_S + \nu = W^\dagger\tilde{f}_{S'} + \mu$ and, from this, $\tilde{f}_S + W\nu = \tilde{f}_{S'} + W\mu$. Let $\tilde{f} := \tilde{f}_S + W\nu \in R(W)$. Now, since $\mu \in N(S'^*W)$, $\nu \in N(S^*W)$, then $S^*\tilde{f} = S^*f$ and $S'^*\tilde{f} = S'^*f$, i.e., $\tilde{f} \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$.

Now, let us see that $\tilde{f}_{S,S'} = \tilde{f}_S + WP_{N(S^*W)}G^\dagger S'^*(\tilde{f}_{S'} - \tilde{f}_S) \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$. Clearly, $\tilde{f}_{S,S'} \in R(W)$, so it only remains to show that $S^*\tilde{f}_{S,S'} = S^*f$ and $S'^*\tilde{f}_{S,S'} = S'^*f$. Now,

$$S^*\tilde{f}_{S,S'} = S^*\tilde{f}_S + S^*WP_{N(S^*W)}G^\dagger S'^*(\tilde{f}_{S'} - \tilde{f}_S) = S^*\tilde{f}_S = S^*f,$$

because $\tilde{f}_S \in \mathcal{C}_{W,S}(f)$. On the other hand, $S'^*\tilde{f}_{S,S'} = S'^*\tilde{f}_S + GG^\dagger S'^*(\tilde{f}_{S'} - \tilde{f}_S)$. Thus, if we show that $S'^*(\tilde{f}_{S'} - \tilde{f}_S) \in R(G)$ then the result is obtained. Now, by item 3, there exist $\nu \in N(S^*W)$, $\mu \in N(S'^*W)$ such that $W^\dagger(\tilde{f}_{S'} - \tilde{f}_S) = \nu + \mu$. Then, $S'^*(\tilde{f}_{S'} - \tilde{f}_S) = S'^*W(W^\dagger(\tilde{f}_{S'} - \tilde{f}_S)) = S'^*W\nu \in R(G)$.

Finally, let us prove that equality (6) holds. First, note that $\tilde{f} := \tilde{f}_{S,S'} + WP_{N(S^*W)}(I - G^-G)h \in R(W)$, $S^*\tilde{f} = S^*\tilde{f}_{S,S'} = S^*f$ and $S'^*\tilde{f} = S'^*\tilde{f}_{S,S'} + S'^*WP_{N(S^*W)}(I - G^-G)h = S'^*\tilde{f}_{S,S'} + G(I - G^-G)h = S'^*\tilde{f}_{S,S'} = S'^*f$. Hence, $\tilde{f} \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$.

On the other hand, let $\tilde{f} \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$. Then, $\tilde{f} - \tilde{f}_{S,S'} = Wh$ for some $h \in \mathcal{H}$. Moreover, since $h \in N(S^*W) \cap N(S'^*W)$ then it follows that $h \in N(G)$. So, $h = P_{N(S^*W)}(I - G^-G)h$ and the result follows. $\square$

Remark 4.2. We highlight that a characterization of the set $\mathcal{CS}(W, S) \cap \mathcal{CS}(W', S')$ is equivalent to studying simultaneous solutions of a system of operator equations. We recommend [22] for a treatment on this topic for matrix equations. Moreover, we suggest [10] for a relationship between $\mathcal{CS}(W, S)$ and $\mathcal{CS}(W', S')$ under some range hypotheses.

Proposition 4.3. *The set $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$ has a unique reconstruction for every $f \in \mathcal{H}$ if and only if $\mathcal{H} = R(W) \dot{+} \mathcal{N}$.*

Proof. Let us suppose that $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f) = \{\tilde{f}_{S,S'}\}$. Then, by (6), $WP_{N(S^*W)}(I - G^-G)h = 0$, for every $h \in \mathcal{H}$, where $G = S'^*WP_{N(S^*W)}$. Now, let $v \in R(W) \cap \mathcal{N}$. Thus, $v = Wz$ for some $z \in N(S^*W)$. So, $v = WP_{N(S^*W)}z$. On the other hand, $0 = S'^*v = S'^*WP_{N(S^*W)}z = Gz$, i.e., $z \in N(G)$ and so $z = (I - G^-G)z$. Summarizing, $v = WP_{N(S^*W)}(I - G^-G)z = 0$.

Conversely, suppose that $\mathcal{H} = R(W) \dot{+} \mathcal{N}$. Now, consider $\tilde{f}_{S,S'}, \tilde{\tilde{f}}_{S,S'} \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$. Then, $\tilde{f}_{S,S'} - \tilde{\tilde{f}}_{S,S'} \in R(W)$. Furthermore, $0 = S^*(\tilde{f}_{S,S'} - \tilde{\tilde{f}}_{S,S'}) = S'^*(\tilde{f}_{S,S'} - \tilde{\tilde{f}}_{S,S'})$. Hence, $\tilde{f}_{S,S'} - \tilde{\tilde{f}}_{S,S'} \in R(W) \cap \mathcal{N} = \{0\}$. $\square$

Motivated by Theorem 1 in [21], we obtain two new descriptions of the consistent reconstructions for the two samples.

Proposition 4.4. *Let $\mathcal{H} = R(W) + \mathcal{N}$. Then,*

(1) $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f) = \{Q_{L\|\mathcal{N}}f : L \subseteq R(W)\}$.
(2) $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f) = \{J^\dagger P_{\mathcal{N}^\perp}f + \nu : \nu \in R(W) \cap \mathcal{N}\}$, *where $J = I - P_{\mathcal{N}}P_{R(W)}$.*

Furthermore, $J^\dagger P_{\mathcal{N}^\perp}f$ is the reconstruction in $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$ with minimal norm.

Proof. Observe that, since $\mathcal{H} = R(W) + \mathcal{N}$, then $\mathcal{C}_{W,E}(f)$ is not empty for every $f \in \mathcal{H}$ where $E \in L(\mathcal{H}, l^2)$ is such that $N(E^*) = \mathcal{N}$. Then, it is straightforward that $\mathcal{C}_{W,E}(f) = \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$ for every $f \in \mathcal{H}$.

(1) The proof follows from the fact that $\tilde{f} \in \mathcal{C}_{W,E}(f)$ if and only if $\tilde{f} = Q_{L\|N(E^*)}f$ with $L \subseteq R(W)$.
(2) The proof follows by Theorem 1 in [21].

Finally, since $N(J) = R(W) \cap \mathcal{N}$, it is clear that $J^\dagger P_{\mathcal{N}^\perp}f$ is the reconstruction with minimal norm. $\square$

In the previous proposition we obtained the simultaneous consistent reconstruction with minimal norm, namely $J^\dagger P_{\mathcal{N}^\perp}f$. What is still lacking is an explicit description of this optimal recovered signal in terms of the samples.

5. Quasi-consistent reconstructions

In this section we treat the case in which $\mathcal{H} \neq R(W) + N(S^*)$, i.e., when it is not possible to find a consistent reconstruction for every $f \in \mathcal{H}$. Hence, we are interested in finding a reconstruction $\tilde{f} \in \mathcal{W}$ such that if it is re-injected into the system then the samples obtained are as close as possible to the original samples.

Therefore, we shall say that $\tilde{f} = WXS^*f$ is a *quasi-consistent reconstruction* (q-c.r.) of f if

$$\|S^*\tilde{f} - S^*f\| \leq \|S^*\hat{f} - S^*f\|, \tag{7}$$

for every reconstruction $\hat{f} \in \mathcal{W}$ of f. In the sequel we shall use the notation

$$\mathcal{QC}(W, S) := \{X \in L(l^2) : WXS^*f \text{ is q-c.r. for every } f \in \mathcal{H}\}.$$

Clearly, if $\mathcal{H} = R(W) + N(S^*)$ then $\mathcal{CS}(W, S) = \mathcal{QC}(W, S)$. From now on we make the following assumptions: $R(W) + N(S^*)$ is a closed subspace and $S^*W \neq 0$. The first condition is equivalent to S^*W having closed range (see [23, Theorem 22]). We note that if $S^*W = 0$ then $\mathcal{QC}(W, S) = L(l^2)$.

The following theorem is an analogue of the characterization of $\mathcal{CS}(W, S)$ given in (3).

Theorem 5.1.

$$\mathcal{QC}(W, S) = \{X \in L(l^2) : S^*WXS^* = P_{R(S^*W)}S^*\}.$$

The resulting quasi-consistent reconstructions are

$$\tilde{f} = [W(S^*W)^\dagger + WP_{N(S^*W)}T]S^*f, \tag{8}$$

with $T \in L(l^2)$. Moreover, there exists a unique q-c.r for every $f \in \mathcal{H}$ if and only if $R(W) \cap N(S^*) = \{0\}$.

Proof. Let $\hat{f} = WXS^*f$ be a reconstruction of f. Then,

$$\begin{aligned}\|S^*\hat{f} - S^*f\|^2 &= \|S^*WXS^*f - S^*f\|^2 \\ &= \|S^*WXS^*f - P_{R(S^*W)}S^*f - (S^*f - P_{R(S^*W)}S^*f)\|^2 \\ &= \|S^*WXS^*f - P_{R(S^*W)}S^*f - P_{R(S^*W)^\perp}S^*f\|^2 \\ &= \|S^*WXS^*f - P_{R(S^*W)}S^*f\|^2 + \|P_{R(S^*W)^\perp}S^*f\|^2 \geq \|P_{R(S^*W)^\perp}S^*f\|^2. \end{aligned} \tag{9}$$

Now, since the equation $S^*WXS^* = P_{R(S^*W)}S^*$ is solvable then the minimum in (9) is achieved. Moreover, this minimum will be attained in those reconstructions $\tilde{f} = WXS^*f$ such that $S^*\tilde{f} = S^*WXS^*f = P_{R(S^*W)}S^*f$. Hence, in order to prove the equality (8) we shall prove that $S^*WXS^* = P_{R(S^*W)}S^*$ if and only if $WXS^* = W(S^*W)^\dagger S^* + WP_{N(S^*W)}TS^*$ for some $T \in L(l^2)$. Thus, let us suppose that $S^*WXS^* = P_{R(S^*W)}S^*$. Then, by Theorem 2.1,

$$X = (S^*W)^\dagger S^*(S^*)^\dagger + T - (S^*W)^\dagger S^*WTS^*(S^*)^\dagger, \tag{10}$$

for some $T \in L(l^2)$. Therefore, $WXS^* = W(S^*W)^\dagger S^* + WTS^* - W(S^*W)^\dagger S^*WTS^* = W(S^*W)^\dagger S^* + W(I - (S^*W)^\dagger S^*W)TS^* = W(S^*W)^\dagger S^* + WP_{N(S^*W)}TS^*$. The converse is trivial.

The unicity of the q-c.r follows from the fact that $\tilde{f}$ is a quasi-consistent reconstruction of f if and only if $S^*\tilde{f} = P_{R(S^*W)}S^*f$. □

The fact that a q-c.r. of f, $\tilde{f}$, yields the samples closest to the original ones does not necessarily imply that $\tilde{f}$ is close to f. In the next proposition, we study this problem for the case where there exists a unique q-c.r. The first part of the next result can also be found in [11]. We include the proof for completeness.

Proposition 5.1. *Let $R(W) \cap N(S^*) = \{0\}$. Then, the unique q-c.r. of f is given by Qf where $Q := W(S^*W)^\dagger S^*$ is a projection with $R(Q) = R(W)$. Moreover, $Q = P_{R(W)}$ if and only if $N(S^*) \subseteq R(W)^\perp \subseteq N(P_{R(S^*W)}S^*)$.*

Proof. Assume that $R(W) \cap N(S^*) = \{0\}$. Thus, as a consequence of Theorem 5.1, the unique q-c.r. of f is given by Qf where $Q := W(S^*W)^\dagger S^*$. Now, it is clear that $Q^2 = Q$. We claim that $R(Q) = R(W)$. Indeed, given $Wz \in R(W)$ we get $QWz = W(S^*W)^\dagger S^*Wz = WP_{N(S^*W)^\perp}z$. Now, since $N(S^*W) = N(W)$, we get $QWz = WP_{N(W)^\perp}z = Wz$, so $R(W) = R(Q)$.

Finally, suppose that $Q = P_{R(W)}$. Then, $N(S^*) \subseteq N(Q) = N(P_{R(W)}) = R(W)^\perp$. On the other hand, let $x \in R(W)^\perp = N(Q)$. Then, $0 = Qx = W(S^*W)^\dagger S^*x$ and so $0 = S^*Qx = S^*W(S^*W)^\dagger S^*x = P_{R(S^*W)}S^*x$, i.e., $x \in N(P_{R(S^*W)}S^*)$. Conversely, in order to prove that $Q = P_{R(W)}$ we shall prove that $N(P_{R(W)}) = R(W)^\perp \subseteq N(Q)$. For this, it is sufficient to show that $N(P_{R(S^*W)}S^*) \subseteq N(Q)$. Hence, given $y \in N(P_{R(S^*W)}S^*)$ we have that $0 = P_{R(S^*W)}S^*y = S^*W(S^*W)^\dagger S^*y = S^*Qy$. Therefore, $Qy = W(S^*W)^\dagger S^*y \in R(W) \cap N(S^*) = \{0\}$ and the result is proved. □

Remark 5.2. Note that if $N(S^*) \subseteq R(W)^\perp$ then there exists $X \in L(l^2)$ such that $P_{R(W)}f = WXS^*f$ for every $f \in \mathcal{H}$. Thus, $P_{R(W)}f$ is the reconstruction of f that minimizes the squared error $\|f - \tilde{f}\|^2$. Now, by the preceding proposition, the additional condition $R(W)^\perp \subseteq N(P_{R(S^*W)}S^*)$ guarantees that the optimal reconstruction $P_{R(W)}f$ is also quasi-consistent.

By means of Theorem 5.1, we establish now how the notion of quasi-consistent reconstruction is related to generalized inverses.

Theorem 5.2. *The following inclusions hold:*

$$(S^*W)[1, 3] \subseteq \mathcal{QC}(W, S) \subseteq (S^*W)[1].$$

Moreover,

(1) $\mathcal{QC}(W, S) = (S^*W)[1]$ *if and only if* $\mathcal{H} = R(W) + N(S^*)$.
(2) $\mathcal{QC}(W, S) = (S^*W)[1, 3]$ *if and only if* S^* *is surjective.*

Proof. If $X \in (S^*W)[1, 3]$ then $S^*WXS^* = P_{R(S^*W)}S^*$. So, $X \in \mathcal{QC}(W, S)$. On the other hand, if $X \in \mathcal{QC}(W, S)$ then $S^*WXS^* = P_{R(S^*W)}S^*$. Thus, $S^*WXS^*W = P_{R(S^*W)}S^*W = S^*W$, i.e., $X \in (S^*W)[1]$.

(1) If $\mathcal{QC}(W, S) = (S^*W)[1]$ then $X = (S^*W)^\dagger + T - (S^*W)^\dagger S^*WTS^*W(S^*W)^\dagger \in \mathcal{QC}(W, S)$ for every $T \in L(l^2)$. Therefore, by Theorem 5.1,

$$\begin{aligned} P_{R(S^*W)}S^* &= S^*WXS^* = S^*W(S^*W)^\dagger S^* + S^*WTS^* - S^*WTS^*W(S^*W)^\dagger S^* \\ &= P_{R(S^*W)}S^* + S^*WT(I - P_{R(S^*W)})S^*. \end{aligned}$$

Hence, for every $T \in L(l^2)$, $S^*WTP_{R(S^*W)^\perp}S^* = 0$. So, we get that $P_{R(S^*W)^\perp}S^* = 0$ or, what is equivalent, $\mathcal{H} = R(W) + N(S^*)$.
Conversely, if $\mathcal{H} = R(W) + N(S^*)$ then $\mathcal{QC}(W, S) = \mathcal{CS}(W, S)$ and the assertion follows by Theorem 3.1.

(2) Suppose $\mathcal{QC}(W, S) = (S^*W)[1, 3]$. Then by (10), for every $T \in L(l^2)$,

$$X = (S^*W)^\dagger S^*(S^*)^\dagger + T - (S^*W)^\dagger S^*WTS^*(S^*)^\dagger \in (S^*W)[1, 3].$$

Then

$$\begin{aligned} P_{R(S^*W)} &= S^*WX = S^*W(S^*W)^\dagger S^*(S^*)^\dagger + S^*WT - S^*WTS^*(S^*)^\dagger \\ &= P_{R(S^*W)}P_{R(S^*)} - S^*WTP_{R(S^*)^\perp}. \end{aligned}$$

Therefore, for every $T \in L(l^2)$, $S^*W(T - (S^*W)^\dagger)P_{R(S^*)^\perp} = 0$. Hence, we obtain that $P_{R(S^*)^\perp} = 0$ and so S^* is surjective. The converse is immediate. □

As we have mentioned, if $R(W) \cap N(S^*) \neq \{0\}$ then there exist infinite quasi-consistent reconstructions for every $f \in \mathcal{H}$. For this situation, we present two criteria for selecting a convenient quasi-consistent reconstruction. These criteria are motivated by the work of Eldar and Dvorkind in [12].

The first method consists in minimizing the worst error between the quasi-consistent reconstructions. That is,

$$\min_{X \in \mathcal{QC}(W,S)} \max_{\|f'\| \le \alpha} \|WXS^*f' - f'\|^2 = \alpha^2 \min_{X \in \mathcal{QC}(W,S)} \|WXS^* - I\|^2.$$

In the second method we seek the quasi-consistent reconstruction that minimizes the worst regret, i.e.,

$$\min_{X \in \mathcal{QC}(W,S)} \max_{\|f'\| \le \alpha} \|WXS^*f' - P_{R(W)}f'\|^2 = \alpha^2 \min_{X \in \mathcal{QC}(W,S)} \|WXS^* - P_{R(W)}\|^2.$$

In order to solve the previous problems we shall use the following results.

Lemma 5.3 (*[24, Corollary 7]*). *If $A, B \in B(\mathcal{H})$ such that $R(A) \perp R(B)$, then*

$$\max\{\|A\|^2, \|B\|^2\} \le \|A + B\|^2 \le \max\{\|A\|^2, \|B\|^2\} + \|AB^*\|.$$

If in addition, $R(A^*) \perp R(B^*)$ then $\max\{\|A\|, \|B\|\} = \|A + B\|$.

Theorem 5.3. *Let $A \in L(\mathcal{H}, \mathcal{K})$, $B \in L(\mathcal{F}, \mathcal{G})$ with closed ranges and $C \in L(\mathcal{F}, \mathcal{K})$. If $R((B^\dagger B - I)C^*AA^\dagger) \perp R(C^*(AA^\dagger - I))$ then, for every $X \in L(\mathcal{G}, \mathcal{H})$,*

$$\|AA^\dagger CB^\dagger B - C\| \le \|AXB - C\|,$$

with equality if $X = A^\dagger CB^\dagger + T - A^\dagger ATBB^\dagger$ for all $T \in L(\mathcal{G}, \mathcal{H})$.

Proof. Observe that $AXB - C = (AXB - AA^\dagger C) + (AA^\dagger C - C)$. Now, since $R(AXB - AA^\dagger C) \perp R(AA^\dagger C - C)$ then by the previous lemma,

$$\|AXB - C\| \ge \max\{\|AXB - AA^\dagger C\|, \|AA^\dagger C - C\|\}.$$

On the other hand, $AXB - AA^\dagger C = (AXB - AA^\dagger CB^\dagger B) + (AA^\dagger CB^\dagger B - AA^\dagger C)$. As $R((AXB - AA^\dagger CB^\dagger B)^*) \perp R((AA^\dagger CB^\dagger B - AA^\dagger C)^*)$ then

$$\|AXB - AA^\dagger C\| \ge \max\{\|AXB - AA^\dagger CB^\dagger B\|, \|AA^\dagger CB^\dagger B - AA^\dagger C\|\}.$$

Summarizing, $\|AXB - C\| \ge \max\{\|AXB - AA^\dagger CB^\dagger B\|, \|AA^\dagger CB^\dagger B - AA^\dagger C\|, \|AA^\dagger C - C\|\}$. From $R(AA^\dagger CB^\dagger B - AA^\dagger C) \perp R(AA^\dagger C - C)$ and the hypothesis we have

$$\|AA^\dagger CB^\dagger B - C\| = \max\{\|AA^\dagger CB^\dagger B - AA^\dagger C\|, \|AA^\dagger C - C\|\}.$$

Finally, $\|AXB - C\| \ge \max\{\|AXB - AA^\dagger CB^\dagger B\|, \|AA^\dagger CB^\dagger B - C\|\}$ and this concludes the proof. □

Theorem 5.4. *Consider the problems*

$$\min_{X \in \mathcal{QC}(W,S)} \|WXS^* - I\|^2, \tag{11}$$

and

$$\min_{X \in \mathcal{QC}(W,S)} \|WXS^* - P_{R(W)}\|^2. \tag{12}$$

The resulting reconstruction are

$$\tilde{f} = [P_{\mathcal{M}^\perp}W(S^*W)^\dagger + P_{\mathcal{M}}(S^*)^\dagger]S^*f, \tag{13}$$

and

$$\tilde{f} = [P_{\mathcal{M}^\perp}W(S^*W)^\dagger + P_{\mathcal{M}}P_{R(W)}(S^*)^\dagger]S^*f, \tag{14}$$

respectively, where $\mathcal{M}$ is the closed subspace $R(WP_{N(S^*W)})$.

Proof. First, note that $\mathcal{M} = R(WP_{N(S^*W)})$ is a closed subspace since $N(W)+N(S^*W) = N(S^*W)$ is closed (see [23, Theorem 22]). Now, by Theorem 5.1,

$$\min_{X\in\mathcal{QC}(W,S)} \|WXS^* - I\|^2 = \min_{L\in L(\mathcal{H})} \|W((S^*W)^\dagger + P_{N(S^*W)}L)S^* - I\|^2$$
$$= \min_{L\in L(\mathcal{H})} \|WP_{N(S^*W)}LS^* - (I - W(S^*W)^\dagger S^*)\|^2.$$

In order to apply Theorem 5.3, we note that

$$P_{\mathcal{M}^\perp}(I - W(S^*W)^\dagger S^*)P_{N(S^*)}(I - S(W^*S)^\dagger W^*)P_{\mathcal{M}} = P_{\mathcal{M}^\perp}P_{N(S^*)}P_{\mathcal{M}} = 0$$

where the last equality follows from the fact that $\mathcal{M} \subseteq N(S^*)$.

Now, applying Theorems 5.1 and 5.3, we get that for all $T \in L(l^2)$,

$$X = (S^*W)^\dagger + P_{N(S^*W)}[(WP_{N(S^*W)})^\dagger[I - W(S^*W)^\dagger S^*] + T - (WP_{N(S^*W)})^\dagger WP_{N(S^*W)}TS^*](S^*)^\dagger,$$

are solutions of (11). Therefore the optimal reconstruction is

$$\tilde{f} = WXS^*f = W\left((S^*W)^\dagger + P_{N(S^*W)}(WP_{N(S^*W)})^\dagger[I - W(S^*W)^\dagger S^*](S^*)^\dagger\right)S^*f$$
$$= W(S^*W)^\dagger S^*f + WP_{N(S^*W)}(WP_{N(S^*W)})^\dagger[I - W(S^*W)^\dagger S^*](S^*)^\dagger S^*f$$
$$= W(S^*W)^\dagger S^*f + P_{\mathcal{M}}[(S^*)^\dagger S^*f - W(S^*W)^\dagger S^*f]$$
$$= [P_{\mathcal{M}^\perp}W(S^*W)^\dagger + P_{\mathcal{M}}(S^*)^\dagger]S^*f.$$

Finally, problem (12) can be solved in a similar manner. □

Remarks 5.5. (1) The problems of minimizing the worst error and worst regret among all possible reconstructions were studied by Eldar and Dvorkind in [12].

(2) If $(S^*W)^\dagger = W^\dagger(S^*)^\dagger$ then $W(S^*W)^\dagger S^* = WW^\dagger(S^*)^\dagger S^* = P_{R(W)}P_{R(S)}$. Thus, replacing in (14) we have $\tilde{f} = P_{R(W)}P_{R(S)}f$ which coincides with the solution obtained in Theorem 2 of [12]. For equivalent conditions for $(S^*W)^\dagger = W^\dagger(S^*)^\dagger$ see [25].

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