



Stability of solitary waves for the vector nonlinear Schrödinger equation in higher-order Sobolev spaces

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ARTICLE INFO

Article history:

Received 30 August 2012

Available online 1 August 2013

Submitted by B. Straughan

Keywords:

Ground states

Solitary waves

Orbital stability

Nonlinear Schrödinger system

ABSTRACT

In this manuscript, a sharp form of orbital stability for the Schrödinger system, also known as the vector NLS,

$$i \frac{\partial}{\partial t} u_j + \frac{\partial^2}{\partial x^2} u_j + 2 \sum_{i=1}^m |u_i|^2 u_j = 0,$$

where u_j are complex-valued functions of $(x, t) \in \mathbb{R}^2$, $j = 1, 2, \dots, m$, is established in L^2 -based Sobolev classes of arbitrarily high order. The result means practically that not only does the bulk of what emanates from the perturbed solitary wave stay close in shape and propagation and phase speeds to the original solitary wave, but emerging residual oscillations must also be very small and not only in the energy norm.

Published by Elsevier Inc.

1. Introduction

It is well-understood that the nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} \pm 2|u|^2 u = 0 \quad (1.1)$$

where u is a complex valued function of $(x, t) \in \mathbb{R}^2$, arises in a generic situation. The equation describes evolution of small amplitude, slowly varying wave packets in a nonlinear medium [5]. Indeed, it has been derived in such diverse fields as deep water waves [28], plasma physics [29], nonlinear optical fibers [13,14], magneto-static spin waves [30], to name a few. In this manuscript, we consider the following system

$$i \frac{\partial}{\partial t} u_j + \frac{\partial^2}{\partial x^2} u_j + 2 \sum_{i=1}^m |u_i|^2 u_j = 0 \quad (1.2)$$

where u_j is a complex valued function of $(x, t) \in \mathbb{R}^2$, which is usually referred to as the vector nonlinear Schrödinger equation (or VNLS) as it can be rewritten compactly in the vector form as

$$i\mathbf{u}_t + \mathbf{u}_{xx} + 2\|\mathbf{u}\|^2 \mathbf{u} = 0 \quad (1.3)$$

with $\mathbf{u} = (u_1, u_2, \dots, u_m)$ and $\|\mathbf{u}\|^2 = u_1^2 + u_2^2 + \dots + u_m^2$. This system arises physically under conditions similar to those described by (1.1). The VNLS system models physical systems in which the field has more than one components; for example, in optical fibers and waveguides, the propagating electric field has two components that are transverse to the direction of propagation. The VNLS system also arises in the Hartree–Fock theory for a double condensate, i.e., a binary

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mixture of Bose–Einstein condensates in two different hyperfine states. Readers are referred to the works [5,13,14,28,29] for the derivation as well as applications of this system.

The system (1.2) is completely integrable. It shares many interesting features as the cubic, nonlinear Schrödinger equation (1.1). We refer readers to the nice book of Ablowitz et al. [2] for the discussion of those and we para-phase here only the most relevant ones to us.

System (1.2) reduces to the cubic NLS equation (1.1) under the reduction

$$\mathbf{u}(x, t) = \mathbf{p}u(x, t), \quad (1.4)$$

where $\mathbf{p} = (p_1, \dots, p_m)$ is a constant vector such that

$$\|\mathbf{p}\|^2 = |p_1|^2 + |p_2|^2 + \dots + |p_m|^2 = 1. \quad (1.5)$$

Thus, any solution of the scalar cubic NLS equation generates a family of solutions of the system (1.2) parametrized by the choice of $\mathbf{p}$, to which Ablowitz et al. referred to as reduction solutions. The vector $\mathbf{p}$ is usually called the polarization of the reduction solution. The fact that the polarization of a reduction solution is arbitrary (up to the restriction $\|\mathbf{p}\|^2 = 1$) is a manifestation of a symmetry of the vector NLS. The dependent variable transformation

$$\tilde{\mathbf{u}} = \mathbf{u}U, \quad (1.6)$$

where U is any unitary matrix, leaves the vector NLS invariant. Thus, if $\mathbf{u}$ is a reduction solution of the form (1.4), then $\tilde{\mathbf{u}}$ is also a reduction solution with $\tilde{\mathbf{p}} = \mathbf{p}U$ and $\tilde{u} = u$. The transformation (1.6) can be interpreted as a change in the basis of the vector space of the dependent variable. For any reduction solution, the polarization depends entirely on the choice of the basis, and by changing the basis via transformations of the form (1.6) one can arbitrarily change the polarization. Although a single vector NLS soliton is governed by the scalar NLS, the vector nature of the solitons (that is, the associated polarization vector) affects the interaction of the vector solitons with one another. When vector solitons interact, they shift one another's polarization vectors. This polarization shift distinguishes vector soliton interactions from scalar soliton interactions.

Just like the cubic NLS equation, the vector NLS equation (1.2) possesses infinitely many integral invariants $J_i, i = 1, 2, \dots$. Reproduced below are the first six invariants for this equation, where $\mathbf{u}^H$ denotes the complex transpose of $\mathbf{u}$:

$$\begin{aligned} J_1 &= \int_{-\infty}^{\infty} \|\mathbf{u}\| \, dx; \\ J_2 &= \int_{-\infty}^{\infty} \|\mathbf{u}\|^2 \, dx; \\ J_3 &= \int_{-\infty}^{\infty} (\mathbf{u}_x^H \mathbf{u} - \mathbf{u}^H \mathbf{u}_x) \, dx; \\ J_4 &= \int_{-\infty}^{\infty} (\|\mathbf{u}_x\|^2 - \|\mathbf{u}\|^4) \, dx; \\ J_5 &= \int_{-\infty}^{\infty} (\mathbf{u}_{xx}^H \mathbf{u}_x - \mathbf{u}_{xx} \mathbf{u}_x^H + 3\|\mathbf{u}\|^2 \mathbf{u}_x^H \mathbf{u} - 3\|\mathbf{u}\|^2 \mathbf{u}_x \mathbf{u}^H) \, dx; \\ J_6 &= \int_{-\infty}^{\infty} (\|\mathbf{u}_{xx}\|^2 - 4\|\mathbf{u}\|^2 \|\mathbf{u}_x\|^2 + \|\mathbf{u}\|^2 \mathbf{u}^H \mathbf{u}_{xx} + \|\mathbf{u}\|^2 \mathbf{u}_{xx}^H + 2\|\mathbf{u}\|^6) \, dx. \end{aligned}$$

Traveling-wave solutions for (1.2) of interest here have the form

$$\mathbf{u}(x, t) = (u_1, u_2, \dots, u_m)$$

where for $j = 1, 2, \dots, m$,

$$u_j(x, t) = e^{i[(\omega - \frac{1}{4}\theta^2)t + \frac{1}{2}\theta x + m_j]} \varphi_{j,\omega}(x - \theta t) \quad (1.7)$$

for m_j, θ, ω real constants with $\omega - \frac{1}{4}\theta^2 > 0$ and $\varphi_{j,\omega} : \mathbb{R} \rightarrow \mathbb{R}$ are functions of one variable whose values are small when $|\xi| = |x - \theta t|$ is large. An important special case arises when $m_j = \theta = 0$ and $\omega = \Omega > 0$. These special solutions (where, to emphasize the dependence on the parameters, we write $\varphi_{j,\omega}$ as $\phi_{j,\Omega}$)

$$u_j(x, t) = e^{i\Omega t} \phi_{j,\Omega}(x) \quad (1.8)$$

are often referred to as *standing waves*. It is easy to see that, for example, standing waves are solutions of (1.2) if and only if $(u_1, u_2, \dots, u_m)$ is a critical point for the functional $J_4(u_1, u_2, \dots, u_m)$, when the functions $u_j(x)$ are varied subject to the m constraints that $J_2(u_j)$ be held constant. If $(u_1, u_2, \dots, u_m)$ is not only a critical point but in fact a global minimizer then the standing wave is called a *ground-state solution*. In the case of (1.2) as shown in this manuscript, it is possible to show further that the ground-state solutions are solitary waves with the usual sech-profile. (When $m = 2$, this result has been shown, see, for example, [18–20,22,24].)

The mathematically exact theory for the stability of solitary waves began with Benjamin's theory [4] for the Korteweg–de Vries equation (see also [6]). In subsequent works, Benjamin's original conception was refined and extended in many ways to include numerous equations and systems such as Benjamin–Ono equation, intermediate long wave equation, nonlinear Schrödinger equation, Boussinesq systems, nonlinear Schrödinger systems, etc. (see, for example, [3,1,12,18–22,24]). Stability here is referred to the initial-value problem. Consider, for example, the cubic NLS equation (1.1). One imagines being provided with an initial wave profile, say at $t = 0$,

$$u(x, 0) = \psi(x), \quad (1.9)$$

for $x \in \mathbb{R}$, and then inquiring into the subsequent evolution using (1.1). This presumes that the initial-value problem (1.1) is a well-posed problem so that a unique solution $u(x, t)$ departs from ψ under the influence of (1.1). The solitary wave $e^{i\omega t}\varphi_\omega$ is said to be orbitally stable in a Banach space X with norm $\|\cdot\|_X$ if whenever $\epsilon > 0$ is given, there is a corresponding $\delta > 0$ such that $\|\varphi_\omega - \psi\|_X \leq \delta$ implies that

$$\inf_{y, \theta \in \mathbb{R}} \|u(\cdot + y, t) - e^{i\theta}\varphi_\omega(\cdot, t)\|_X \leq \epsilon, \quad (1.10)$$

for all $t \geq 0$. This result is interpreted to say that, if at some time, say $t = 0$, a solution of (1.1) is close to φ_ω relative to X , then it remains close in shape for all subsequent (and previous) times. Of course, it is possible that in some contexts the stipulation $\|\varphi_\omega - \psi\|_X \leq \delta$ might need strengthening, say to $\|\varphi_\omega - \psi\|_Y \leq \delta$, where $Y \subset X$ is a smaller space with a stronger norm, though one would generally prefer that stability in X subsists on the data lying close to φ_ω in X and nothing more. In fact, in many circumstances where stability (1.10) is obtained, one can find smooth functions $p, q : \mathbb{R} \rightarrow \mathbb{R}$ (see, for example, [8,27]) such that

$$\|u(\cdot + y, t) - e^{ip(t)}\varphi_\omega(\cdot - q(t))\|_X \leq \epsilon, \quad (1.11)$$

for all $t \geq 0$. Moreover, p and q are close to ω and θ in the sense that

$$\begin{aligned} p'(t) &= \omega + O(\epsilon), \\ q'(t) &= \theta + O(\epsilon), \end{aligned} \quad (1.12)$$

as $\epsilon \rightarrow 0$, uniformly in t .

It is our purpose here to extend the existing theory in another direction, namely in L^2 -based Sobolev classes of arbitrarily high order whose norms are much stronger. What this means practically is that not only does the bulk of what emanates from the perturbed solitary wave stay close in shape and propagation and phase speeds to the original solitary wave, but emerging residual oscillations must also be very small and not only in the energy norm. The approach used here follows the one previously introduced in [7]. An example of what our theory denies can easily be displayed. (For $k \geq 0$, $H^k(\mathbb{R})$ is the usual Sobolev space consisting of measurable functions such that f and all the derivatives of f up to the order k are in L^2 .) Suppose initial data ψ lies near to a solitary-wave solution φ_ω in the strong sense of a small space like $H^k(\mathbb{R})$ for k large. The energy space in this case is $H^1(\mathbb{R})$. The currently existing theory would not preclude, for some positive time t , that the solution u corresponding to initial data ψ might have the form

$$u(x, t) = e^{i\omega t}(\varphi_\omega + \rho(x))$$

where ρ has the form

$$\rho(x) = \epsilon \mu(x) \cos(\epsilon^\alpha x)$$

with $\alpha = -1/k$. For $\epsilon \ll 1$, ρ is indeed small in $H^1(\mathbb{R})$, but features very significant high frequency oscillations that, for example, never appear in numerical simulations when ψ is as described above. In particular, the $H^k(\mathbb{R})$ -norm is not generally small, no matter how small is ϵ . The present theory precludes such behavior if ψ does not initially feature these sorts of oscillations.

It should be pointed out that asymptotic stability which may be deduced by now classical methods, can provide stability in higher-order Sobolev norms, even for non-integrable systems, as done in the paper of Rodnianski et al. [23], on asymptotic stability of multi-solitons of NLS.

Notation. For $1 \leq p \leq \infty$, we denote by $L^p = L^p(\mathbb{R})$ the space of all measurable functions f on $\mathbb{R}^N$ for which the norm $\|f\|_p = (\int_{\mathbb{R}} |f|^p dx)^{1/p}$ is finite for $1 \leq p < \infty$ and $\|f\|_\infty$ is the essential supremum of $|f|$ on $\mathbb{R}$. Whether we intend the functions in L^p to be real-valued or complex-valued will be clear from the context. For $s \geq 0$, the Sobolev space $H^s(\mathbb{R})$ is the subspace of measurable functions $f \in L^2$ such that

$$\|f\|_{H^s}^2 = \int_{-\infty}^{\infty} (1 + \xi^2)^s |\widehat{f}(\xi)|^2 d\xi < \infty.$$

The cases $s = 1$ and $s = k$ were featured in our previous commentary. We define the spaces $\mathcal{L}^p$, $\mathcal{H}^1$ and $\mathcal{H}^s$ to be the m -cartesian products $L^p(\mathbb{R}) \times \cdots \times L^p(\mathbb{R})$, $H^1(\mathbb{R}) \times \cdots \times H^1(\mathbb{R})$ and $H^s(\mathbb{R}) \times \cdots \times H^s(\mathbb{R})$ respectively. If $T > 0$ and Y is any Banach space, we denote by $C([0, T], Y)$ the Banach space of continuous maps $f : [0, T] \rightarrow Y$, with norms given by $\|f\|_{C([0, T], Y)} = \sup_{t \in [0, T]} \|f(t)\|_Y$.

This manuscript is organized as follows. In Section 2, known results for the system (1.2) are first summarized, then the main contributions of this manuscript are presented. In Section 3, the stability for ground-state solutions in $\mathcal{H}^1$ in the sense of (1.10) is first established. This result is then improved in the sense of (1.11) by picking unique trajectory and phase shifts that the perturbed ground states must follow. Properties of these maps are also studied in detail. These stability results in $\mathcal{H}^1$ are extended in Section 4 to include solitary-wave solutions as well. Section 5 gives the proof for the stability of solitary waves for the vector NLS equation (1.2) in higher-order Sobolev spaces $\mathcal{H}^n$ for arbitrary positive integer n .

2. Statement of results

2.1. Review

Logically, prior to a discussion of stability in terms of perturbations of the initial data should be a theory for the initial valued problem itself. It follows from [9,10] that for any initial data $(u_1(x, 0), \dots, u_m(x, 0)) \in \mathcal{H}^1$, there exists a unique solution $(u_1(x, t), \dots, u_m(x, t))$ of (1.2) in $\mathcal{C}(\mathbb{R}; \mathcal{H}^1)$ emanating from $(u_1(x, 0), \dots, u_m(x, 0))$, and $(u_1(x, t), \dots, u_m(x, t))$ satisfies for $j = 1, 2, \dots, m$

$$\begin{aligned} Q(u_j(x, t)) &= Q(u_j(x, 0)), \\ E(u_1(x, t), \dots, u_m(x, t)) &= E(u_1(x, 0), \dots, u_m(x, 0)). \end{aligned}$$

Indeed, we also have the following results [9,10,15].

Theorem 2.1. *Let $s \geq 1$. For each $\mathbf{u}_0 \in \mathcal{H}^s$, there exist a $T > 0$ depending only on $\|\mathbf{u}_0\|_{\mathcal{H}^s}$ and a unique solution $\mathbf{u}$ to the VNLS equation (1.2) such that $\mathbf{u} \in C([0, T]; \mathcal{H}^s)$. For any fixed $T > 0$, let U_T be the mapping which associates to $\mathbf{u}_0$ the solution $\mathbf{u}$ on the interval $[0, T]$. Then U_T is continuous from $\mathcal{H}^s$ into $C([0, T]; \mathcal{H}^s)$.*

The following generalized system

$$\begin{cases} iu_t + u_{xx} + (b_{11}|u|^2 + b_{12}|v|^2)u = 0, \\ iv_t + v_{xx} + (b_{21}|u|^2 + b_{22}|v|^2)v = 0, \end{cases} \quad (2.1)$$

where $b_{12} = b_{21}$ includes the system (1.2) when $m = 2$. In [26], Wei and Yao have established the uniqueness of positive solutions to this case for a range of b_{11} , b_{12} and b_{22} , for any fixed $\Omega > 0$. In particular, they proved that for $0 \leq b_{12} \neq \{\min(b_{11}, b_{22}), \max(b_{11}, b_{22})\}$, all solutions are given by $(\alpha_1 \phi_\Omega, \alpha_2 \phi_\Omega)$ where $\alpha_1 = \sqrt{\frac{b_{12}-b_{22}}{b_{12}^2-b_{11}b_{22}}}$, $\alpha_2 = \sqrt{\frac{b_{12}-b_{11}}{b_{12}^2-b_{11}b_{22}}}$ and

$$\phi_\Omega(\xi) = \sqrt{2\Omega} \operatorname{sech}(\sqrt{\Omega}\xi).$$

In fact, when $b_{12} > 0$, all possible ground-state solutions are classified and the non-degeneracy is also proven in [26].

In [20] (see also [22]), it was shown that the standing waves for (2.1) are indeed *ground states* and that they are $\mathcal{H}^1$ -orbitally stable in the sense of (1.10) for any fixed $\Omega > 0$ provided that either

$$(A1) \quad 0 < b_{12} < \min\{b_{11}, b_{22}\};$$

or

$$(A2) \quad b_{12} > 0 \text{ with } b_{12} > \max\{b_{11}, b_{22}\} \text{ and } b_{12}^2 > b_{11}b_{22}.$$

In [19], the above result was further improved. The improvement was made by picking unique trajectory and phase shifts that the perturbed ground states must follow. The results were then extended to include solitary-wave solutions as well.

In [12] a different variational setting than the one in [20] is used to prove the stability of solitary waves for (2.1), namely using the sum of the L^2 -norms of the two components. The two variational problems can have different solitary-wave solutions. In fact, the last two pages of [12] show that in the case when $b_{11} = b_{22} = \alpha$ and $b_{12} < \alpha$, the solitary waves which solve the variational problem in [20] are not the same as those in [12].

In [24,25], Song proves $\mathcal{H}^1$ -stability of standing waves to a system of Schrödinger equations with combined power-type nonlinearities, which includes (2.1) when the dimension $n = 1$. However, due to the nature of the problem being posed on higher dimension, the uniqueness of the ground-state solutions is not studied in [24,25]; moreover, the range of stability for the coefficients b_{ij} is smaller compared to the one obtained in [20] (namely, in [24], b_{ij} are strictly positive while b_{11}, b_{22} are allowed to be negative as well in [20]).

In [21], stability of ground-state solutions to the 3-coupled nonlinear Schrödinger system

$$i \frac{\partial}{\partial t} u_j + \frac{\partial^2}{\partial x^2} u_j + \sum_{i=1}^3 b_{ij} |u_i|^2 u_j = 0, \quad (2.2)$$

where u_j are complex-valued functions of $(x, t) \in \mathbb{R}^2$, $j = 1, 2, 3$, and b_{ij} are positive constants satisfying $b_{ij} = b_{ji}$ was studied. The following assumptions on the matrix $B = (b_{ij})$ were made.

- (A) B is invertible and the linear system $B\vec{\alpha} = \vec{1}$ is solvable for $\vec{\alpha} = (\alpha_1^2, \alpha_2^2, \alpha_3^2) \in \mathbb{R}_+^3$ where $\vec{1} = (1, 1, 1)$.
 (B1) For all pairs $i \neq j$, $b_{ij} < \min\{b_{ii}, b_{jj}\}$ and $\det B_{ij}$ has the sign of $(-1)^{i+j+1}$ for $i \neq j$.
 (B2) For all pairs $i \neq j$, $b_{ij} > \max\{b_{ii}, b_{jj}\}$ and $\det B_{ij}$ has the sign of $(-1)^{i+j}$ for $i \neq j$.

It was proved that if the matrix $B = (b_{ij})$ satisfies (A) and (B1) or (B2), then for any $\Omega > 0$, the ground-state solutions $(e^{i\Omega t}\alpha_1\phi_\Omega(x), e^{i\Omega t}\alpha_2\phi_\Omega(x), e^{i\Omega t}\alpha_3\phi_\Omega(x))$ of (2.2) are orbitally stable in $\mathcal{H}^1$ in the sense of (1.11).

2.2. Main results

Our first result in this work regards the $\mathcal{H}^1$ -stability for ground-state solutions of (1.2). We define the orbit of the ground-state solutions $(e^{i\Omega t}\phi_\Omega, \dots, e^{i\Omega t}\phi_\Omega)$ to be the collection of all translations in the spatial variable, together with the m phase shifts $\{e^{ij\theta}f_j(x+r)\}_{r \in \mathbb{R}}$.

Theorem 2.2. *The ground-state solution $(e^{i\Omega t}\phi_\Omega, \dots, e^{i\Omega t}\phi_\Omega)$ is orbitally stable in the sense that for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if*

$$\inf_{y_j, y \in \mathbb{R}} \left\{ \sum_{j=1}^m \|u_{j0} - e^{iy_j}\phi_\Omega(\cdot + y)\|_{H^1} \right\} < \delta$$

then there are C^1 -functions $\theta_j, \eta : \mathbb{R} \rightarrow \mathbb{R}$ for which the solution $(u_1(x, t), \dots, u_m(x, t))$ emanating from the initial data $(u_1(\cdot, 0), \dots, u_m(\cdot, 0)) = (u_{10}, \dots, u_{m0})$ satisfies

$$\sum_{j=1}^m \|u_j(\cdot, t) - e^{i\theta_j(t)}\phi_\Omega(\cdot - \eta(t))\|_{H^1} < \epsilon$$

for all $t \geq 0$. Moreover,

$$\begin{aligned} \eta'(t) &= O(\epsilon), \\ \theta_j'(t) &= \Omega + O(\epsilon) \end{aligned}$$

for $j = 1, \dots, m$ as $\epsilon \rightarrow 0$, uniformly in t .

This $\mathcal{H}^1$ -stability result is then extended to include solitary-wave solutions as well.

Theorem 2.3. *The solitary-wave solutions $(e^{i\omega t}\varphi_\omega, \dots, e^{i\omega t}\varphi_\omega)$ of (1.2) are orbitally stable in the sense that for any $\epsilon > 0$ given, there exists $\delta = \delta(\epsilon) > 0$ such that if*

$$\inf_{y_j, y \in \mathbb{R}} \left\{ \sum_{j=1}^m \|u_{j0} - e^{iy_j}\varphi_\omega(\cdot + y)\|_{H^1} \right\} \leq \delta$$

then there are C^1 -mappings $p_1, \dots, p_m, q : \mathbb{R} \rightarrow \mathbb{R}$ for which the solution $(u_1, \dots, u_m)$ of (1.2) emanating from the initial data $(u_{10}, \dots, u_{m0})$ satisfies

$$\sum_{j=1}^m \|u_j(\cdot, t) - e^{ip_j(t)}\varphi_\omega(\cdot - q(t))\|_{H^1} \leq \epsilon$$

for all $t \geq 0$. Moreover, $p_1, \dots, p_m$ and q are close to ω and θ in the sense that

$$\begin{aligned} p_j'(t) &= \omega + O(\epsilon), \\ q'(t) &= \theta + O(\epsilon), \end{aligned}$$

as $\epsilon \rightarrow 0$, uniformly in t .

Finally, the stability of solitary-wave solutions for (1.2) in higher-order Sobolev spaces is presented.

Theorem 2.4. *The solitary-wave solutions $(e^{i\omega t}\varphi_\omega, \dots, e^{i\omega t}\varphi_\omega)$ of (1.2) are orbitally stable in $\mathcal{H}^n$, for any positive integer n . That is, for any $\epsilon_n > 0$, there exists $\delta_n = \delta_n(\epsilon_n) > 0$ such that if*

$$\inf_{y_j, y \in \mathbb{R}} \left\{ \sum_{j=1}^m \|u_{j0} - e^{iy_j}\varphi_\omega(\cdot + y)\|_{H^n} \right\} < \delta_n,$$

then the solution $(u_1(x, t), \dots, u_m(x, t))$ emanating from the initial data $(u_1(\cdot, 0), \dots, u_m(\cdot, 0)) = (u_{10}, \dots, u_{m0})$ satisfies

$$\sum_{j=1}^m \|u_j(\cdot, t) - e^{ip_j(t)} \varphi_\omega(\cdot - q(t))\|_{H^n} < \epsilon_n$$

for all $t \geq 0$.

Remark. (1) The above result means that one can pick unique trajectory and phase shifts such that the stability theory holds throughout the evolution. Of course, it is more desirable to have an orbital stability theory where all the components are in phase, i.e., $p_j(t) = p(t)$ for all j and t ; in another words, the orbit comprises just the spatial translation along with one common phase shift. However, in lieu of the polarization mentioned above, [Theorem 2.4](#) appears to be the best possible.

(2) Notice that we prove something a little stronger than just orbital stability in $\mathcal{H}^n$. It is deduced in fact that if

$$\inf_{\gamma_j, y \in \mathbb{R}} \left\{ \sum_{j=1}^m \|u_{j0} - e^{i\gamma_j} \varphi_\omega(\cdot + y)\|_{H^n} \right\} \leq \delta_n$$

then

$$\sum_{j=1}^m \|u_j - e^{ip_j(t)} \varphi_\omega(\cdot - q(t))\|_{H^n} \leq \epsilon_n$$

for all $t \geq 0$, where $p_j(t)$ and $q(t)$ are the same smooth functions obtained in the $\mathcal{H}^1$ -stability result.

3. Stability of ground-state solutions

3.1. Variational problem

Let $u_1, \dots, u_m \in H^1(\mathbb{R})$ and consider the following functional associated with conserved quantity of (1.2):

$$J_4(\mathbf{u}) = \int_{-\infty}^{\infty} (\|\mathbf{u}_x\|^2 - \|\mathbf{u}\|^4) dx = \int_{-\infty}^{\infty} \sum_{i=1}^m \left[|\partial_x u_i(x)|^2 - \sum_{j=1}^m |u_i(x)|^2 |u_j(x)|^2 \right] dx. \quad (3.1)$$

For $i = 1, 2, \dots, m$ and $u \in H^1(\mathbb{R})$, define

$$E(u) = \int_{-\infty}^{\infty} (|u_x(x, t)|^2 - m|u(x, t)|^4) dx.$$

For some fixed $\Omega > 0$, let $\phi_\Omega(\xi) = \sqrt{\frac{\Omega}{m}} \operatorname{sech}(\sqrt{\Omega} \xi)$ and set

$$\lambda(\Omega) = \frac{2}{m} \sqrt{\Omega}. \quad (3.2)$$

For fixed $\Omega > 0$ (hence $\lambda > 0$ is also fixed), define the real numbers I and I_1 as follows:

$$I(\lambda, \dots, \lambda) = \inf \{J_4(\mathbf{u}) : \mathbf{u} = (u_1, \dots, u_m) \in \mathcal{H}^1(\mathbb{R}), \|u_1\|_{L^2}^2 = \dots = \|u_m\|_{L^2}^2 = \lambda\},$$

and

$$I_1(\lambda) = \inf \{E(u) : u \in H^1(\mathbb{R}), \|u\|_{L^2}^2 = \lambda\}.$$

The sets of minimizers for $I(\lambda, \dots, \lambda)$ and $I_1(\lambda)$ are, respectively,

$$G(\lambda, \dots, \lambda) = \{\mathbf{u} = (u_1, \dots, u_m) \in \mathcal{H}^1 : I(\lambda, \dots, \lambda) = J_4(\mathbf{u}), \|u_1\|_{L^2}^2 = \dots = \|u_m\|_{L^2}^2 = \lambda\},$$

and

$$G_1(\lambda) = \{u \in H^1(\mathbb{R}) : I_1(\lambda) = E(u), \|u\|_{L^2}^2 = \lambda\}.$$

From now on, we always assume that $\Omega > 0$ is fixed (hence so is the positive number λ). It is easy to see that for all $\lambda > 0$, one has

$$-\infty < I(\lambda, \dots, \lambda) < 0,$$

and that any minimizing sequence $\{(u_{1n}, \dots, u_{mn})\}$ is bounded uniformly in $\mathcal{H}^1$.

Let $\{(u_{in})\} \in \mathcal{H}^1$ be a minimizing sequence for J_4 and consider a sequence of nondecreasing functions $M_n : [0, \infty) \rightarrow [0, m\lambda]$ as follows

$$M_n(s) = \sup_{y \in \mathbb{R}} \int_{y-s}^{y+s} \sum_{i=1}^m |u_{in}(x)|^2 dx.$$

As $M_n(s)$ is a uniformly bounded sequence of nondecreasing functions in s , one can show that it has a subsequence, which is still denoted as M_n , that converges point-wisely to a nondecreasing limit function $M(s) : [0, \infty) \rightarrow [0, m\lambda]$. Let

$$\rho = \lim_{s \rightarrow \infty} M(s) \equiv \lim_{s \rightarrow \infty} \lim_{n \rightarrow \infty} M_n(s) = \lim_{s \rightarrow \infty} \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-s}^{y+s} \sum_{i=1}^m |u_{in}(x)|^2 dx.$$

Then $0 \leq \rho \leq m\lambda$.

Lions' Concentration Compactness Lemma [16,17] shows that there are three possibilities for the value of ρ :

- Case 1: (*Vanishing*) $\rho = 0$. Since $M(s)$ is non-negative and nondecreasing, this is equivalent to saying $M(s) = \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-s}^{y+s} \sum_{i=1}^m |u_{in}(x)|^2 dx = 0$, for all $s < \infty$, or
- Case 2: (*Dichotomy*) $\rho \in (0, m\lambda)$, or
- Case 3: (*Compactness*) $\rho = m\lambda$, which implies that there exists $\{y_n\}_{n=1} \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists $s < \infty$ such that $\int_{y_n-s}^{y_n+s} \sum_{i=1}^m |u_{in}(x)|^2 dx \geq m\lambda - \epsilon$.

Straightforward calculations give the following.

Lemma 3.1. For any fixed Ω , $\mu_i > 0$, one has

- (1) $I(\mu_1, \dots, \mu_m) \geq \sum_{i=1}^m I_1(\mu_i)$.
- (2) $I_1(\lambda(\Omega)) = -\frac{m^2 \lambda^3}{96} = -\frac{2\Omega^{3/2}}{3m}$.
- (3) $I(\lambda, \dots, \lambda) = mI_1(\lambda)$.

Consequently, we know the following.

Corollary 3.2. For any $\Omega > 0$ fixed,

$$\{(e^{i\alpha_1} \phi_\Omega(\cdot + y), \dots, e^{i\alpha_m} \phi_\Omega(\cdot + y)) \mid \alpha_i, y \in \mathbb{R}\} \subset G(\lambda(\Omega), \dots, \lambda(\Omega)).$$

The following lemma provides *strict sub-additivity* of the function I needed to rule out the dichotomy of minimizing sequences. Notice that, for example, when we write $I(0, \lambda, \dots, \lambda)$, we intend for the first component of the minimizing sequence to vanish in the sense of concentration compactness introduced above.

Lemma 3.3. For any $\beta_j \in [0, \lambda]$ satisfying $0 < \sum_{j=1}^m \beta_j < m\lambda$, we have

$$I(\lambda, \dots, \lambda) < I(\beta_1, \dots, \beta_m) + I(\lambda - \beta_1, \dots, \lambda - \beta_m).$$

Proof. We consider separately the following cases.

Case 1: $\beta_j \in (0, \lambda)$ for $j = 1, \dots, m$. From (3) in Lemma 3.1, we have

$$I(\lambda, \dots, \lambda) = mI_1(\lambda);$$

and from Theorem II.1 in [11] we have, for any $\beta_j \in (0, \lambda)$

$$I_1(\lambda) < I_1(\beta_j) + I_1(\lambda - \beta_j).$$

Consequently, we obtain that

$$\begin{aligned} I(\lambda, \dots, \lambda) &= mI_1(\lambda) < \sum_{j=1}^m I_1(\beta_j) + \sum_{j=1}^m I_1(\lambda - \beta_j) \\ &\leq I(\beta_1, \dots, \beta_m) + I(\lambda - \beta_1, \dots, \lambda - \beta_m) \end{aligned}$$

where (1) in Lemma 3.1 has been used. Thus case 1 is proved.

Case 2: Exactly k of $\{\beta_1, \beta_2, \dots, \beta_m\}$ vanish, $k = 1, \dots, m-1$, and without loss of generality, we may assume that

$$\beta_{m-k+1} = \dots = \beta_m = 0; \quad \beta_j \in (0, \lambda], \quad \text{for } j = 1, 2, \dots, m-k.$$

The variational problem then becomes

$$\inf \left\{ \int_{-\infty}^{\infty} \sum_{j=1}^{m-k} \left[|\partial_x u_j|^2 - \sum_{i=1}^{m-k} |u_i|^2 |u_j|^2 \right] dx : \|u_j\|_{L^2}^2 = \beta_j \right\} \geq \sum_{j=1}^{m-k} I_1(\beta_j)$$

where an application of (1) in Lemma 3.1 has been used. Thus,

$$\begin{aligned} I(\lambda, \dots, \lambda) &= mI_1(\lambda) \leq kI_1(\lambda) + \sum_{j=1}^{m-k} (I_1(\beta_j) + I_1(\lambda - \beta_j)) \\ &\leq I(\lambda - \beta_1, \dots, \lambda - \beta_{m-k}, \lambda, \dots, \lambda) + \sum_{j=1}^{m-k} I_1(\beta_j) \\ &< I(\beta_1, \dots, \beta_{m-k}, 0, \dots, 0) + I(\lambda - \beta_1, \dots, \lambda - \beta_{m-k}, \lambda, \dots, \lambda) \end{aligned}$$

proving case 2. Thus the lemma is proved. $\square$

With the above lemmas in hand, it is straightforward to show that any minimizing sequence can neither vanish nor dichotomize. (The arguments follow, for example, the same lines as in [20–22].) Consequently, Lions' Concentration Compactness Lemma guarantees that any minimizing sequence must be compact. Moreover, the following holds true.

Proposition 3.4. *If $\{u_{1n}, \dots, u_{mn}\}$ is any minimizing sequence for J_4 , then*

1. $\lim_{n \rightarrow \infty} \inf_{(\phi_1, \dots, \phi_m) \in G, y \in \mathbb{R}} \|(u_{1n}(\cdot + y), \dots, u_{mn}(\cdot + y)) - (\phi_1, \dots, \phi_m)\|_{\mathcal{H}^1} = 0.$
2. $\lim_{n \rightarrow \infty} \inf_{(\phi_1, \dots, \phi_m) \in G} \|(u_{1n}, \dots, u_{mn}) - (\phi_1, \dots, \phi_m)\|_{\mathcal{H}^1} = 0.$

An immediate consequence of the above results is that the set of minimizers $G(\lambda, \dots, \lambda)$ is stable. Precisely, we have the following. Let $\Phi = (\Phi_1, \dots, \Phi_m)$.

Proposition 3.5. *For every $\epsilon > 0$ given, there exists $\delta > 0$ such that if*

$$\inf_{\Phi \in G} \|(u_{10}, \dots, u_{m0}) - (\Phi_1, \dots, \Phi_m)\|_{\mathcal{H}^1} < \delta,$$

then the solution $(u_1(x, t), \dots, u_m(x, t))$ of (1.2) with $(u_1(x, 0), \dots, u_m(x, 0)) = (u_{10}, \dots, u_{m0})$ satisfies

$$\inf_{\Phi \in G} \|(u_1(\cdot, t), \dots, u_m(\cdot, t)) - (\Phi_1, \dots, \Phi_m)\|_{\mathcal{H}^1} < \epsilon$$

for all $t \in \mathbb{R}$.

Proof. The proof is standard (see, for example, [20–22]) and hence is omitted. $\square$

We next show that the set of minimizers $G(\lambda, \dots, \lambda)$ contains just a single m -tuple of functions (modulo translations and phase shifts), and that this m -tuple of functions is indeed a ground-state solution of (1.2) given by

$$(\Phi_1(x, t), \dots, \Phi_m(x, t)) = (e^{i\Omega t} \phi_\Omega(x), \dots, e^{i\Omega t} \phi_\Omega(x)).$$

We start first with the following lemma that relates the functions $\Phi_1, \dots, \Phi_m$ whenever $\Phi = (\Phi_1, \dots, \Phi_m) \in G(\lambda, \dots, \lambda)$.

Lemma 3.6. *Let $\Phi \in G(\lambda, \dots, \lambda)$. Then for any $x \in \mathbb{R}$,*

$$|\Phi_1(x)| = |\Phi_2(x)| = \dots = |\Phi_m(x)|.$$

Proof. It follows from Lemma 3.1 that for any $(\Phi_1, \dots, \Phi_m) \in G(\lambda, \dots, \lambda)$

$$I(\lambda, \dots, \lambda) = J_4(\Phi_1, \dots, \Phi_m) \geq \sum_{j=1}^m E(\Phi_j) \geq mI_1(\lambda) = I(\lambda, \dots, \lambda).$$

Thus,

$$\int_{-\infty}^{\infty} \sum_{i=1}^m \sum_{j=1}^m |\Phi_i|^2 |\Phi_j|^2 dx = m \sum_{j=1}^m \|\Phi_j\|_{L^4}^4$$

which implies that

$$\int_{-\infty}^{\infty} \sum_{i,j=1; i \neq j}^m \left| |\Phi_i(x)|^2 - |\Phi_j(x)|^2 \right|^2 dx = 0,$$

from which the statement of the lemma immediately follows. $\square$

We proceed to show the following.

Proposition 3.7. For any $\Omega > 0$ fixed,

$$\left\{ (e^{i\alpha_1} \phi_\Omega(\cdot + y), \dots, e^{i\alpha_m} \phi_\Omega(\cdot + y)) \right\} = G(\lambda(\Omega), \dots, \lambda(\Omega))$$

where $y, \alpha_j \in \mathbb{R}, j = 1, 2, \dots, m$.

Proof. It has been established (Corollary 3.2) that for any $\Omega > 0$ fixed and for $y, \alpha_j \in \mathbb{R}, j = 1, 2, \dots, m$,

$$(e^{i\alpha_1} \phi_\Omega(\cdot + y), \dots, e^{i\alpha_m} \phi_\Omega(\cdot + y)) \in G(\lambda(\Omega), \dots, \lambda(\Omega)).$$

Hence, the proposition is proved if we can show that any minimizer in $G(\lambda(\Omega), \dots, \lambda(\Omega))$ must be of the form given above. Now, since the constrained minimizer for the variational problem exists, there are Lagrange multipliers $\Omega_1, \dots, \Omega_m \in \mathbb{R}$ such that for $j = 1, 2, \dots, m$

$$-\partial_{xx} \Phi_j + \Omega_j \Phi_j = 2 \sum_{i=1}^m |\Phi_i|^2 \Phi_j. \quad (3.3)$$

Using Lemma 3.6, we can rewrite this system as m -uncoupled equations

$$-\partial_{xx} \Phi_j + \Omega_j \Phi_j = 2m |\Phi_j|^2 \Phi_j. \quad (3.4)$$

A bootstrap argument shows that any m -tuple L^2 -distribution solution of (3.4) must indeed be smooth and given by (see, for example, [10])

$$\Phi_j(x) = e^{i\alpha_j} \phi_\Omega(x + y_j),$$

where $\phi_\Omega(\xi) = \sqrt{\Omega/m} \operatorname{sech}(\sqrt{\Omega} \xi)$, and $\alpha_j, y_j \in \mathbb{R}$ for $j = 1, 2, \dots, m$. Now recall that for any $x \in \mathbb{R}$, we must have

$$|\Phi_1(x)| = |\Phi_2(x)| = \dots = |\Phi_m(x)|;$$

and

$$\|\Phi_1\|_{L^2}^2 = \|\Phi_2\|_{L^2}^2 = \dots = \|\Phi_m\|_{L^2}^2 = \lambda = \frac{2}{m} \sqrt{\Omega}.$$

It is easy to see then that

$$y_1 = y_2 = \dots = y_m,$$

and

$$\Omega = \Omega_1 = \Omega_2 = \dots = \Omega_m.$$

The proposition is thus proved. $\square$

We arrive at the stability in the sense of (1.10) for ground-state solutions of (1.2).

Theorem 3.8. For any $\Omega > 0$, the ground-state solutions $(e^{i\Omega t} \phi_\Omega(x), \dots, e^{i\Omega t} \phi_\Omega(x))$ of (1.2) are orbitally stable in the following sense: for any $\epsilon > 0$, there exists $\delta > 0$ such that if $(u_{10}, \dots, u_{m0}) \in \mathcal{H}^1$ satisfies

$$\inf_{y_j, y \in \mathbb{R}} \left\{ \sum_{j=1}^m \|u_{j0} - e^{iy_j} \phi_\Omega(\cdot + y)\|_{H^1} \right\} < \delta$$

then the solution $(u_1(x, t), \dots, u_m(x, t))$ with $(u_1(\cdot, 0), \dots, u_m(\cdot, 0)) = (u_{10}, \dots, u_{m0})$ satisfies

$$\inf_{\theta_j, y \in \mathbb{R}} \left\{ \sum_{j=1}^m \|u_j(\cdot, t) - e^{i\theta_j} \phi_\Omega(\cdot + y, t)\|_{H^1} \right\} < \epsilon$$

uniformly for all $t \geq 0$.

While this result is new, it still leaves open the question of the speeds with which the solution $(u_1, u_2, \dots, u_m)$ propagates. For example, for the single cubic NLS equation (1.1), it is known that the bulk of the disturbance flowing out of the perturbation travels at a speed near to the speed of the unperturbed ground state. Another interesting question is in regard to the phase with which the solution propagates. Because the system (1.2) is invariant under the Galilean transformation

$$(u_1(x, t), \dots, u_m(x, t)) \rightarrow (e^{-i\frac{1}{4}\theta^2 t + i\frac{1}{2}\theta x} u_1(x - \theta t, t), \dots, e^{-i\frac{1}{4}\theta^2 t + i\frac{1}{2}\theta x} u_m(x - \theta t, t)),$$

for $\theta \in \mathbb{R}$ and the phase transformation

$$(u_1(x, t), \dots, u_m(x, t)) \rightarrow (e^{ik_1} u_1(x, t), \dots, e^{ik_m} u_m(x, t)), \quad k_j \in \mathbb{R}, j = 1, \dots, m,$$

one can always obtain the solitary-wave solutions as in (1.7) from ground states, but then it is not clear how the phase shifts will be affected in the previously stated stability results. This question is addressed next. It will be shown that instead of allowing the ground-state solutions to wander around at random, one can pick unique trajectory and phase shifts that the ground-state solutions must follow. Properties of these functions are then studied.

As noted in [8,27], the crucial argument for choosing the functions θ_j and η , $j = 1, \dots, m$ is deduced from appreciating that the first-order conditions corresponding to minimizing the function

$$R = R(\theta_1, \dots, \theta_m, \eta) : \mathbb{R}^{m+1} \rightarrow \mathbb{R},$$

$$R = \sum_{j=1}^m (\Omega \|e^{-i\theta_j} u_j(\cdot + \eta) - \phi_\Omega\|_{L^2}^2 + \|e^{-i\theta_j} u'_j(\cdot + \eta) - \phi'_\Omega\|_{L^2}^2)$$

produce the orthogonality relations, holding for all $j = 1, 2, \dots, m$,

$$\mathcal{R}e \left\{ \int_{-\infty}^{\infty} \sum_{j=1}^m \phi_\Omega^2(x) \phi'_\Omega(x) (e^{-i\theta_j} u_j(x + \eta, t)) dx \right\} = 0;$$

$$\mathcal{I}m \left\{ \int_{-\infty}^{\infty} \phi_\Omega^3(x) (e^{-i\theta_j} u_j(x + \eta, t)) dx \right\} = 0.$$

Following this idea, we consider the vector-valued function $Q : \mathcal{H}^1 \times \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{m+1}$ defined by

$$Q((\psi_1, \dots, \psi_m), \theta_1, \dots, \theta_m, \eta) = (F_1, F_2, \dots, F_{m+1})$$

where F_1 is given by

$$F_1((\psi_1, \dots, \psi_m), \theta_1, \dots, \theta_m, \eta) = \mathcal{R}e \left\{ \int_{-\infty}^{\infty} \sum_{j=1}^m \phi_\Omega^2(x) \phi'_\Omega(x) (e^{-i\theta_j} \psi_j(x + \eta)) dx \right\};$$

and for $k = 2, 3, \dots, m + 1$,

$$F_k = \mathcal{I}m \left\{ \int_{-\infty}^{\infty} \phi_\Omega^3(x) (e^{-i\theta_k} \psi_k(x + \eta)) dx \right\}. \quad (3.5)$$

The following lemma is straightforward, hence stated without proof.

Lemma 3.9. When evaluating at $\mathbf{r} = ((\phi_\Omega, \dots, \phi_\Omega), 0, \dots, 0)$, one has

$$(a) \quad Q(\mathbf{r}) = (0, \dots, 0);$$

$$(b) \quad \frac{\partial Q}{\partial(\theta_1, \dots, \theta_m, \eta)}(\mathbf{r}) = (-1)^{m+1} m \left(\int_{-\infty}^{\infty} \phi_\Omega^4(x) dx \right)^m \left(\int_{-\infty}^{\infty} \phi_\Omega^2(x) (\phi'_\Omega(x))^2 dx \right) \neq 0. \quad (3.6)$$

For $\beta > 0$, define an $\mathcal{H}^1$ -neighborhood of the trajectory of $(e^{i\theta_1} \phi_\Omega, \dots, e^{i\theta_m} \phi_\Omega)$ by

$$\mathcal{U}_\beta = \left\{ (\psi_1, \dots, \psi_m) \in \mathcal{H}^1 : \inf_{\theta_j, \eta \in \mathbb{R}} \left\{ \sum_{j=1}^m \|e^{-i\theta_j} \psi_j(\cdot + \eta) - \phi_\Omega\|_{H^1} \right\} < \beta \right\}.$$

The next lemma provides a choice of functions $\theta_1, \dots, \theta_m, \eta$ by demanding the satisfaction of the orthogonality condition (3.5).

Lemma 3.10. Fix $\Omega > 0$. There exist $\beta > 0$ and C^1 -functions:

$$\theta_1, \dots, \theta_m, \eta : \mathcal{U}_\beta \rightarrow \mathbb{R}$$

such that for all $(\psi_1, \dots, \psi_m) \in \mathcal{U}_\beta$ the following statements hold

$$(1) \quad \mathcal{R}e \left\{ \sum_{j=1}^m \int_{-\infty}^{\infty} \phi_\Omega^2(x) \phi'_\Omega(x) (e^{-i\theta_j(\psi_1, \dots, \psi_m)} \psi_j(x + \eta(\psi_1, \dots, \psi_m))) dx \right\} = 0;$$

$$(2) \quad \mathcal{I}m \left\{ \int_{-\infty}^{\infty} \phi_\Omega^3(x) (e^{-i\theta_j(\psi_1, \dots, \psi_m)} \psi_j(x + \eta(\psi_1, \dots, \psi_m))) dx \right\} = 0;$$

with $\theta_j(\phi_\Omega, \dots, \phi_\Omega) = 0$, for $i = 1, \dots, m$; $\eta(\phi_\Omega, \dots, \phi_\Omega) = 0$ and

$$Q((\psi_1, \dots, \psi_m), \theta_1(\psi_1, \dots, \psi_m), \dots, \theta_m(\psi_1, \dots, \psi_m), \eta(\psi_1, \dots, \psi_m)) = (0, \dots, 0).$$

Proof. It is clear that Q is C^1 on $\mathcal{U}_\beta \times \mathbb{R}^{m+1}$. In fact, the derivatives of Q with respect to components η and θ_j exist up to any order. Now, for any $\rho > 0$, denote the ball centered at $(e^{i\theta_1}\phi_\Omega, \dots, e^{i\theta_m}\phi_\Omega) \in \mathcal{H}^1$ with radius ρ as

$$\mathcal{B}_\rho(e^{i\theta_1}\phi_\Omega, \dots, e^{i\theta_m}\phi_\Omega) = \left\{ (\psi_1, \dots, \psi_m) \in \mathcal{H}^1 : \sum_{j=1}^m \|e^{-i\theta_j}\psi_j - \phi_\Omega\|_{H^1} < \rho \right\}.$$

The implicit function theorem together with Lemma 3.9 implies the existence of positive numbers β, r and unique C^1 -functions

$$\theta_1, \dots, \theta_m, \eta : \mathcal{B}_\beta(e^{i\theta_1}\phi_\Omega, \dots, e^{i\theta_m}\phi_\Omega) \longrightarrow (-r, r)$$

such that for all $(\psi_1, \dots, \psi_m) \in \mathcal{B}_\beta(e^{i\theta_1}\phi_\Omega, \dots, e^{i\theta_m}\phi_\Omega)$,

$$\theta_j(\phi_\Omega, \dots, \phi_\Omega) = 0, \quad j = 1, \dots, m;$$

$$\eta(\phi_\Omega, \dots, \phi_\Omega) = 0; \quad \text{and}$$

$$Q((\psi_1, \dots, \psi_m), \theta_1(\psi_1, \dots, \psi_m), \dots, \theta_m(\psi_1, \dots, \psi_m), \eta(\psi_1, \dots, \psi_m)) = (0, \dots, 0).$$

The lemma follows immediately. $\square$

Because of the stability result stated in Theorem 3.8, $(u_1(\cdot, t), \dots, u_m(\cdot, t)) \in \mathcal{U}_\beta$ hence the corresponding functions $\theta_1, \dots, \theta_m$ and η are defined on $(u_1(\cdot, t), \dots, u_m(\cdot, t))$; one can now consider the functions $\theta_1, \dots, \theta_m$ and η from $\mathbb{R} \rightarrow \mathbb{R}$ as

$$\eta(t) = \eta(u_1(\cdot, t), \dots, u_m(\cdot, t)),$$

and for $j = 1, \dots, m$,

$$\theta_j(t) = \theta_j(u_1(\cdot, t), \dots, u_m(\cdot, t)).$$

Notice that since $(u_1, \dots, u_m)$ is a solution of (1.2), it must satisfy

$$u_{jt} = i[u_{jxx} + 2(|u_1|^2 + \dots + |u_m|^2)u_j]. \quad (3.7)$$

Proposition 3.11. The function Q as defined in (3.5) is continuously differentiable with respect to t .

Proof. It has been proved [9,10] that for any $\mathbf{u}(x, 0) = (u_1(x, 0), \dots, u_m(x, 0)) \in \mathcal{H}^1$, there exists a unique solution $\mathbf{u}(x, t) = (u_1(x, t), \dots, u_m(x, t))$ of (1.2) in $C(\mathbb{R}; \mathcal{H}^1)$ emanating from the initial data $\mathbf{u}(x, 0)$, and $\mathbf{u}(x, t)$ satisfies

$$J_2(\mathbf{u}(x, t)) = J_2(\mathbf{u}(x, 0)),$$

$$J_4(\mathbf{u}(x, t)) = J_4(\mathbf{u}(x, 0)).$$

Thus $\mathbf{u}(x, t)$ is differentiable as a distribution-valued function of t with $(u_{1t}, \dots, u_{mt}) \in C(\mathbb{R}, H^{-1} \times \dots \times H^{-1})$. Hence for any functions $\chi_1, \dots, \chi_m$ in the Schwarz class $\mathcal{S}$, the action

$$\langle (u_1, \dots, u_m), (\chi_1, \dots, \chi_m) \rangle = \int_{-\infty}^{\infty} [u_1(x, t)\chi_1(x) + \dots + u_m(x, t)\chi_m(x)]dx$$

of $u_1(\cdot, t), \dots$ and $u_m(\cdot, t)$ on $\chi_1, \dots$ and χ_m will be a differentiable function of t , with derivative

$$\langle (u_{1t}, \dots, u_{mt}), (\chi_1, \dots, \chi_m) \rangle = \left\langle \left(i \left[u_{1xx} + 2 \sum_{j=1}^m |u_j|^2 u_1 \right], \chi_1 \right), \dots, \left(i \left[u_{mxx} + 2 \sum_{j=1}^m |u_j|^2 u_m \right], \chi_m \right) \right\rangle$$

which is a continuous function of t , where an application of (3.7) is used. It follows that Q is continuously differentiable with respect to t . $\square$

We now proceed to show that the orbital stability result obtained in Theorem 3.8 can be improved in the sense explained above.

Theorem 3.12. The ground-state solution $(e^{i\Omega t}\phi_\Omega, \dots, e^{i\Omega t}\phi_\Omega)$ is orbitally stable in the sense that for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if

$$\inf_{y_j, y \in \mathbb{R}} \left\{ \sum_{j=1}^m \|u_{j0} - e^{iy_j}\phi_\Omega(\cdot + y)\|_{H^1} \right\} < \delta$$

then there are C^1 -functions $\theta_j, \eta : \mathbb{R} \rightarrow \mathbb{R}$ for which the solution $(u_1(x, t), \dots, u_m(x, t))$ emanating from the initial data $(u_1(\cdot, 0), \dots, u_m(\cdot, 0)) = (u_{10}, \dots, u_{m0})$ satisfies

$$\sum_{j=1}^m \|u_j(\cdot, t) - e^{i\theta_j(t)} \phi_\Omega(\cdot - \eta(t))\|_{H^1} < \epsilon$$

for all $t \geq 0$. Moreover,

$$\begin{aligned} \eta'(t) &= O(\epsilon), \\ \theta_j'(t) &= \Omega + O(\epsilon) \end{aligned}$$

for $j = 1, \dots, m$ as $\epsilon \rightarrow 0$, uniformly in t .

Proof. Proposition 3.11 guarantees that the function Q is continuously differentiable with respect to t , hence we can differentiate $Q(u_1(\cdot, t), \dots, u_m(\cdot, t), \theta_1(t), \dots, \theta_m, \eta(t)) = (0, \dots, 0)$ with respect to the variable t to obtain the following $(m+1)$ equations

$$\begin{aligned} (1) \quad \mathcal{R}e \left\{ \int_{-\infty}^{\infty} \sum_{j=1}^m (u_{jt} e^{-i\theta_j(t)} \phi_\Omega^2 \phi_\Omega' - i u_j e^{-i\theta_j(t)} \theta_j'(t) \phi_\Omega^2 \phi_\Omega' - \eta'(t) u_j e^{-i\theta_j(t)} \phi_\Omega^2 \phi_\Omega'') dx \right\} \\ + \int_{-\infty}^{\infty} \phi_\Omega (\phi_\Omega')^2 \left(\sum_{j=1}^m -2 u_j e^{-i\theta_j(t)} \eta'(t) \right) dx; \\ (2) \quad \mathcal{I}m \left\{ \int_{-\infty}^{\infty} (u_{jt} e^{-i\theta_j(t)} \phi_\Omega^3 - i u_j e^{-i\theta_j(t)} \theta_j'(t) \phi_\Omega^3 - 3 \eta'(t) u_j e^{-i\theta_j(t)} \phi_\Omega^2 \phi_\Omega') dx \right\} = 0, \end{aligned} \quad (3.8)$$

where $u_j = u_j(x + \eta(t), t)$ and $\phi_\Omega = \phi_\Omega(x)$. Define the function h_j , by

$$h_j(x, t) = e^{-i\theta_j(t)} u_j(x + \eta(t), t) - \phi_\Omega(x) = h_{1j} + i h_{2j}.$$

Then

$$e^{-i\theta_j(t)} u_j = \phi_\Omega + h_{1j} + i h_{2j}.$$

The orbital stability result obtained in Theorem 3.8 coupled with Lemma 3.10 implies the existence of $\delta > 0$ such that if

$$\inf_{y_j, y \in \mathbb{R}} \left\{ \sum_{j=1}^m \|u_{j0} - e^{iy_j} \phi_\Omega(\cdot + y)\|_{H^1} \right\} < \delta$$

then the solution $(u_1(x, t), \dots, u_m(x, t))$ with $(u_1(\cdot, 0), \dots, u_m(\cdot, 0)) = (u_{10}, \dots, u_{m0})$ satisfies

$$\sum_{j=1}^m \|h_j(x, t)\|_{H^1} < \epsilon. \quad (3.9)$$

Attention is now turned to the functions η and θ_j . Using the definitions of h_j and the fact that $(u_1, \dots, u_m)$ must solve (3.7), the first equation in (3.8) can be expressed in terms of h_j rather than u_j as follows (where, for convenience, we denote $\frac{\partial h_{1j}}{\partial x}, \frac{\partial h_{2j}}{\partial x}$ as h'_{1j} and h'_{2j} respectively)

$$\begin{aligned} \int_{-\infty}^{\infty} \sum_{j=1}^m \left(h'_{2j} (\phi_\Omega^2 \phi_\Omega')' - 2 \sum_{j=1}^m |u_j|^2 h_{2j} \phi_\Omega^2 \phi_\Omega' + \theta_j'(t) h_{2j} \phi_\Omega^2 \phi_\Omega' \right. \\ \left. - \eta'(t) h_{1j} \phi_\Omega^2 \phi_\Omega'' - 2 \eta'(t) \phi_\Omega^2 (\phi_\Omega')^2 - 2 \eta'(t) h_{1j} \phi_\Omega (\phi_\Omega')^2 - \eta'(t) \phi_\Omega^3 \phi_\Omega'' \right) dx = 0. \end{aligned}$$

Because of (3.9) and the fact that $\sum_{j=1}^m \|u_j(\cdot, t)\|_{L^2} \leq C$ for some constant $C > 0$ independent of t , one can readily justify that

$$\eta'(t) = O(\epsilon) + O(\epsilon) \sum_{j=1}^m \theta_j'(t), \quad (3.10)$$

as $\epsilon \rightarrow 0$, uniformly in t . Similarly, the second equation in (3.8) can be rewritten as

$$\int_{-\infty}^{\infty} \left(\phi_\Omega'' \phi_\Omega^3 + h_{1j} (\phi_\Omega^3)'' + 2 \sum_{j=1}^m |u_j|^2 (\phi_\Omega^4 + h_{1j} \phi_\Omega^3) - \theta_j'(t) \phi_\Omega^4 - \theta_j'(t) h_{1j} \phi_\Omega^3 - 3 \eta'(t) h_{2j} \phi_\Omega^2 \phi_\Omega' \right) dx = 0,$$

and using (3.9) and the fact that the ground-state solutions satisfy

$$-\phi''_{\Omega} + \Omega \phi_{\Omega} = 2m\phi_{\Omega}^3, \quad (3.11)$$

one can deduce that

$$\theta'_j(t) \int_{-\infty}^{\infty} \phi_{\Omega}^4 = \int_{-\infty}^{\infty} \phi_{\Omega}^3 \phi''_{\Omega} + 2\phi_{\Omega}^4 \sum_{j=1}^m |u_j|^2 + O(\epsilon) + O(\epsilon)\eta'(t), \quad (3.12)$$

as $\epsilon \rightarrow 0$, uniformly in t . It is straightforward to see that

$$\int_{-\infty}^{\infty} \phi_{\Omega}^4 \sum_{j=1}^m |u_j|^2 = m \int_{-\infty}^{\infty} \phi_{\Omega}^6 + O(\epsilon), \quad (3.13)$$

as $\epsilon \rightarrow 0$, uniformly in t . It is then concluded using (3.11)–(3.13) that

$$\theta'_j(t) = \Omega + O(\epsilon) + O(\epsilon)\eta'(t), \quad (3.14)$$

as $\epsilon \rightarrow 0$, uniformly in t . We can now deduce from (3.10) and (3.14) that

$$\begin{aligned} \eta'(t) &= O(\epsilon), \\ \theta'_j(t) &= \Omega + O(\epsilon) \end{aligned}$$

for $j = 1, \dots, m$ as $\epsilon \rightarrow 0$, uniformly in t . $\square$

4. Orbital stability of solitary-wave solutions

The result obtained in Section 3 is now broadened to include solitary-wave solutions and improved by providing a more detailed view of the connection between the functions η and θ_j . Using the same approach employed in [8], a relation between ground states and solitary-wave solutions is first exhibited as follows. For $\theta \in \mathbb{R}$, define the operator $T_{\theta} : H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$ by

$$(T_{\theta}u)(x) = \exp\left(\frac{i\theta x}{2}\right)u(x)$$

for $u \in H^1(\mathbb{R})$. Then for any $\theta \in \mathbb{R}$,

$$(1 + |\theta|)^{-1} \|u\|_{H^1} \leq \|T_{\theta}u\|_{H^1} \leq (1 + |\theta|) \|u\|_{H^1}. \quad (4.1)$$

For any pair $(\omega, \theta) \in \mathbb{R} \times \mathbb{R}$ such that $\Omega = \omega - \frac{1}{4}\theta^2 > 0$, let

$$\varphi_{\omega} = T_{\theta}\phi_{\Omega}. \quad (4.2)$$

A straightforward calculation reveals the following lemma whose proof is omitted.

Lemma 4.1. *If $(e^{i\Omega t}\phi_{\Omega}, \dots, e^{i\Omega t}\phi_{\Omega})$ is a ground state solution of (1.2), then $(e^{i\omega t}\varphi_{\omega}, \dots, e^{i\omega t}\varphi_{\omega})$ is a solitary-wave solution of (1.2).*

The $\mathcal{H}^1$ -stability theory for solitary-wave solutions in the sense of (1.11) is as follows.

Theorem 4.2. *The solitary-wave solutions $(e^{i\omega t}\varphi_{\omega}, \dots, e^{i\omega t}\varphi_{\omega})$ of (1.2) are orbitally stable in the sense that for any $\epsilon > 0$ given, there exists $\delta = \delta(\epsilon) > 0$ such that if*

$$\inf_{y_j, y \in \mathbb{R}} \left\{ \sum_{j=1}^m \|u_{j0} - e^{iy_j}\varphi_{\omega}(\cdot + y)\|_{H^1} \right\} \leq \delta$$

then there are C^1 -mappings $p_1, \dots, p_m, q : \mathbb{R} \rightarrow \mathbb{R}$ for which the solution $(u_1, \dots, u_m)$ of (1.2) emanating from the initial data $(u_{10}, \dots, u_{m0})$ satisfies

$$\sum_{j=1}^m \|u_j(\cdot, t) - e^{ip_j(t)}\varphi_{\omega}(\cdot - q(t))\|_{H^1} \leq \epsilon$$

for all $t \geq 0$. Moreover, $p_1, \dots, p_m$ and q are close to ω and θ in the sense that

$$\begin{aligned} p'_j(t) &= \omega + O(\epsilon), \\ q'(t) &= \theta + O(\epsilon), \end{aligned}$$

as $\epsilon \rightarrow 0$, uniformly in t .

We need the following calculations before we present the proof of [Theorem 4.2](#). For $(u_1, \dots, u_m)$ a solution of the initial-value problem for (1.2) with initial data $(u_{10}, \dots, u_{m0})$, define

$$f_j(x, t) = e^{i\left(\frac{1}{2}\theta x - \frac{1}{4}\theta^2 t\right)} u_j(x - \theta t, t). \quad (4.3)$$

The proof of the following lemma is straightforward, hence omitted.

Lemma 4.3. *Let $(u_1, \dots, u_m)$ be a solution of the initial-value problem for (1.2) with initial data $(u_{10}, \dots, u_{m0})$. Then $(f_1, \dots, f_m)$ defined as in (4.3) is a solution of the initial-value problem*

$$if_{jt} + f_{jxx} + 2 \sum_{j=1}^m |f_j|^2 f_j = 0,$$

with initial data $(f_1(x, 0), \dots, f_m(x, 0)) = (f_{10}, \dots, f_{m0}) = (T_\theta u_{10}, \dots, T_\theta u_{m0})$.

Consider $(f_{10}, \dots, f_{m0})$ which lies close to a solitary wave $(e^{i\omega t} \varphi_\omega, \dots, e^{i\omega t} \varphi_\omega)$ in $\mathcal{H}^1$ and define

$$(u_{10}, \dots, u_{m0}) = (T_{-\theta} f_{10}, \dots, T_{-\theta} f_{m0}).$$

Let $(f_1, \dots, f_m)$ and $(u_1, \dots, u_m)$ be the solutions of (1.2) with initial data $(f_{10}, \dots, f_{m0})$ and $(u_{10}, \dots, u_{m0})$ respectively. Notice that from (4.1),

$$\sum_{j=1}^m \|u_{j0} - e^{i\gamma_j} \phi_\Omega\|_{H^1} = \sum_{j=1}^m \|T_{-\theta} f_{j0} - e^{i\gamma_j} T_{-\theta} \varphi_\omega\|_{H^1} \leq (1 + |\theta|) \sum_{j=1}^m \|f_{j0} - e^{i\gamma_j} \varphi_\omega\|_{H^1}. \quad (4.4)$$

Proof of Theorem 4.2. Let $\epsilon > 0$ be fixed, let $\tilde{\epsilon} = \epsilon/(1 + |\theta|)$, and let δ be such that the stability result established in [Theorem 3.12](#) holds relative to $\tilde{\epsilon}$ for the ground state $(e^{i\Omega t} \phi_\Omega(x, t), \dots, e^{i\Omega t} \phi_\Omega(x, t))$, that is,

$$\sum_{j=1}^m \|u_j - e^{i\theta_j(t)} \phi_\Omega(\cdot - \eta(t))\|_{H^1} < \tilde{\epsilon} \quad (4.5)$$

for all $t \geq 0$, provided

$$\inf_{\gamma_j, y \in \mathbb{R}} \left\{ \sum_{j=1}^m \|u_{j0} - e^{i\gamma_j} \phi_\Omega(\cdot + y)\|_{H^1} \right\} < \delta.$$

But then according to (4.4),

$$\inf_{\gamma_j, y \in \mathbb{R}} \left\{ \sum_{j=1}^m \|u_{j0} - e^{i\gamma_j} \phi_\Omega(\cdot + y)\|_{H^1} \right\} \leq \delta,$$

provided $\sum_{j=1}^m \|f_{j0} - e^{i\gamma_j} \varphi_\omega\|_{H^1} \leq \delta/(1 + \theta)$. Now, because of [Lemma 4.3](#), it follows that

$$u_j(x, t) = e^{-i\left(\frac{1}{2}\theta x + \frac{1}{4}\theta^2 t\right)} f_j(x + \theta t, t).$$

Hence (4.5) can be rewritten as

$$\sum_{j=1}^m \left\| e^{-\frac{i}{2}\theta x} \left\{ e^{-\frac{i}{4}\theta^2 t} f_j(\cdot + \theta t, t) - e^{i\left[\theta_j(t) + \frac{1}{2}\theta\eta(t)\right]} \varphi_\omega(\cdot - \eta(t)) \right\} \right\|_{H^1} \leq \tilde{\epsilon},$$

for all $t \geq 0$. Using the above inequality and (4.1), it is concluded that

$$\sum_{j=1}^m \left\| f_j(\cdot, t) - e^{i\left(\theta_j(t) + \frac{1}{4}\theta^2 t + \frac{1}{2}\theta\eta(t)\right)} \varphi_\omega(\cdot - \eta(t) - \theta t) \right\|_{H^1} \leq (1 + |\theta|) \tilde{\epsilon} = \epsilon,$$

for all $t \geq 0$. Thus $(e^{i\omega t} \varphi_\omega, \dots, e^{i\omega t} \varphi_\omega)$ is seen to be orbitally stable in the sense specified in the statement of [Theorem 4.2](#) if one chooses

$$p_j(t) = \theta_j(t) + \frac{1}{4}\theta^2 t + \frac{1}{2}\theta\eta(t);$$

$$q(t) = \eta(t) + \theta t. \quad \square$$

5. Higher-order stability of solitary-wave solutions

Recall that the system (1.2) is completely integrable with the first six conserved quantities being listed in Section 1. The general form for calculating the infinitely many conservation laws for (1.2) can be found in [2]. As appears immediately from

an examination of the generating function for these conservation laws, the even numbered ones are potentially useful as regards stability of solitary waves. Indeed, there is a sequence $\{P_{2j}\}_{j \geq 1}$ of polynomials with P_{2j} a function of j variables, i.e.

$$P_{2j} = P_{2j}(y_{(0)}, \dots, y_{(j-1)}),$$

such that if $\mathbf{u} = (u_1, \dots, u_m)$ is a solution of (1.2) that lies at least in $C(0, T; \mathcal{H}^1)$, then

$$J_{2j} = \int_{-\infty}^{\infty} P_{2j}(\mathbf{u}, \mathbf{u}_x, \dots, \partial_x^{j-1} \mathbf{u}) dx = \text{Constant}. \quad (5.1)$$

(This result relies upon the well-posedness theory for (1.2) in the Sobolev spaces $\mathcal{H}^s$, see, for example, [9,10].)

As follows from an analysis of the structure of these polynomials, P_{2j} is exactly a linear combination of monomials $z_0^{r_0} \dots z_n^{r_n}$ where z_k is $\mathbf{y}_k$ or $\mathbf{y}_k^H$ and

$$\sum_{i=0}^n (1+i)r_i = 2j. \quad (5.2)$$

Thus, for $j \geq 1$, the invariant J_{2j} has the form

$$J_{2j}(u_1, \dots, u_m) = \int_{-\infty}^{\infty} c_{2j} \|\partial_x^{j-1} \mathbf{u}\|^2 dx + \int_{-\infty}^{\infty} d_{2j} \|\mathbf{u}\|^{2j} dx + \sum_{n=1}^{2j-1} \int_{-\infty}^{\infty} P_{2j-n,n}(\mathbf{u}) dx \quad (5.3)$$

where c_{2j} , d_{2j} are constants and $P_{j,k}(\mathbf{u})$ denotes the sum of all terms which are homogeneous of degree j in $\mathbf{u}$ and which involve exactly k derivatives in x . In particular, it is known that $c_{2j} \neq 0$, and hence we may take it to be positive.

Theorem 5.1. *The solitary-wave solutions $(e^{i\omega t} \varphi_\omega, \dots, e^{i\omega t} \varphi_\omega)$ of (1.2) are stable in $\mathcal{H}^n$, for any positive integer n .*

Proof. Orbital stability in $\mathcal{H}^1$ is already proved in Section 4. Attention is given next to stability in $\mathcal{H}^2$. Let

$$h_j(x, t) = u_j(x, t) - e^{ip_j(t)} \varphi_\omega(x - q(t)),$$

where p_j and q are the C^1 -functions that provide $\mathcal{H}^1$ -stability as outlined above. Stability in $\mathcal{H}^2$ follows if we can show that, given $\epsilon_2 > 0$, there is a $\delta_2 > 0$ such that

$$\sum_{j=1}^m \|h_j(\cdot, t)\|_{H^2} \leq \epsilon_2$$

for all t , provided

$$\sum_{j=1}^m \|h_j(\cdot, 0)\|_{H^2} \leq \delta_2.$$

Write φ_j for $e^{ip_j(t)} \varphi_\omega(x - q(t))$ for ease of reading. Because of the $\mathcal{H}^1$ -stability result, for any $\epsilon_1 > 0$, there is a $\delta_1 > 0$ such that if $\sum_{j=1}^m \|h_j(\cdot, 0)\|_{H^2} \leq \delta_1$, then $\sum_{j=1}^m \|h_j(\cdot, t)\|_{H^2} \leq \epsilon_1$, for all t . Let $\Psi = (\varphi_1, \dots, \varphi_m)$ and $\mathbf{h}(x, t) = (h_1(x, t), \dots, h_m(x, t))$.

Define $\Delta J_6(\mathbf{u}) = J_6(\mathbf{u}(\cdot, t)) - J_6(\Psi)$. A calculation reveals that

$$\begin{aligned} \Delta J_6(\mathbf{u}) = & \int_{-\infty}^{\infty} \|(\Psi_{xx} + \mathbf{h}_{xx})\|^2 - 4 \int_{-\infty}^{\infty} \|\Psi + \mathbf{h}\|^2 \|\Psi_x + \mathbf{h}_x\|^2 + \|\Psi + \mathbf{h}\|^2 (\bar{\Psi} + \bar{\mathbf{h}})(\Psi_{xx} + \mathbf{h}_{xx}) \\ & + \|\Psi + \mathbf{h}\|^2 (\Psi + \mathbf{h})(\bar{\Psi}_{xx} + \bar{\mathbf{h}}_{xx}) + 2 \int_{-\infty}^{\infty} \|\Psi + \mathbf{h}\|^6 - \int_{-\infty}^{\infty} \|\Psi_{xx}\|^2 + 4 \int_{-\infty}^{\infty} \|\Psi\|^2 \|\Psi_x\|^2 \\ & - \|\Psi\|^2 \bar{\Psi} \Psi_{xx} - \|\Psi\|^2 \Psi \bar{\Psi}_{xx} - 2 \int_{-\infty}^{\infty} \|\Psi\|^6. \end{aligned} \quad (5.4)$$

The following inequalities come to our aid in analyzing ΔJ_6 :

$$\begin{aligned} \text{(a)} \quad & \int_{-\infty}^{\infty} (\Psi_{xx} + \mathbf{h}_{xx})(\bar{\Psi}_{xx} + \bar{\mathbf{h}}_{xx}) - \int_{-\infty}^{\infty} \Psi_{xx} \bar{\Psi}_{xx} \leq \|\mathbf{h}_{xx}\|_{L^2}^2 + a_1 \|\mathbf{h}_x\|_{L^2}^2. \\ \text{(b)} \quad & \left| -4 \int_{-\infty}^{\infty} (\Psi + \mathbf{h})(\bar{\Psi} + \bar{\mathbf{h}})(\Psi_x + \mathbf{h}_x)(\bar{\Psi}_x + \bar{\mathbf{h}}_x) + 4 \int_{-\infty}^{\infty} \Psi \bar{\Psi} \Psi_x \bar{\Psi}_x \right| \\ & \leq b_1 \|\mathbf{h}\|_{L^2}^2 + b_2 \|\mathbf{h}_x\|_{L^2}^2 + b_3 \|\mathbf{h}\|_{L^2}^2 + b_4 \|\mathbf{h}_x\|_{L^2}^2 + b_5 \|\mathbf{h}_2\|_{L^2}^3 + b_6 \|\mathbf{h}_x\|_{L^2}^4. \end{aligned}$$

$$\begin{aligned}
 (c) \quad & \left| 2 \int_{-\infty}^{\infty} (\Psi + \mathbf{h})^3 (\bar{\Psi} + \bar{\mathbf{h}})^3 - 2 \int_{-\infty}^{\infty} \Psi^3 \bar{\Psi}^3 \right| \\
 & \leq c_1 \|\mathbf{h}\|_{\mathcal{L}^2}^2 + c_2 \|\mathbf{h}_x\|_{\mathcal{L}^2}^2 + c_3 \|\mathbf{h}_x\|_{\mathcal{L}^2}^4 + c_4 \|\mathbf{h}_x\|_{\mathcal{L}^2}^6 + c_5 \|\mathbf{h}_x\|_{\mathcal{L}^2}^8 + c_6 \|\mathbf{h}_x\|_{\mathcal{L}^2}^{10}. \\
 (d) \quad & \left| \int_{-\infty}^{\infty} (\Psi + \mathbf{h})(\bar{\Psi} + \bar{\mathbf{h}})^2 (\Psi_{xx} + \mathbf{h}_{xx}) - \Psi \bar{\Psi}^2 \Psi_{xx} \right| \\
 & \leq d_1 \|\mathbf{h}\|_{\mathcal{L}^2}^2 + d_2 \|\mathbf{h}_x\|_{\mathcal{L}^2}^2 + d_3 \|\mathbf{h}_x\|_{\mathcal{L}^2}^4 + d_4 \|\mathbf{h}_x\|_{\mathcal{L}^2}^6 + d_5 \|\mathbf{h}_x\|_{\mathcal{L}^2}^8 + d_6 \|\mathbf{h}_x\|_{\mathcal{L}^2}^{10} + d_7 \|\mathbf{h}_x\|_{\mathcal{L}^2}^{12}.
 \end{aligned}$$

Here a_j, b_j, c_j, d_j denote various constants that depend only on Ψ . Combining (a)–(d) with (5.4), there obtains

$$\Delta J_6(\mathbf{u}) \leq C_0 \delta_2 + C_1 \delta_2^6 \quad (5.5)$$

where δ_2 is any upper bound for $\|\mathbf{u}_0 - \Psi\|_{\mathcal{H}^2}$ and C_0, C_1 are constants depending only on norms of Ψ , and hence only on ω and θ .

Next, an effective lower bound for $\Delta J_6(\mathbf{u})$ is exhibited. To this end, notice that

$$\int_{-\infty}^{\infty} (\Psi_{xx} + \mathbf{h}_{xx})(\bar{\Psi}_{xx} + \bar{\mathbf{h}}_{xx}) - \int_{-\infty}^{\infty} \|\Psi_{xx}\|^2 \geq \int_{-\infty}^{\infty} \|\mathbf{h}_{xx}\|^2 - c_1 \int_{-\infty}^{\infty} \|\mathbf{h}_x\|^2 \geq \int_{-\infty}^{\infty} \|\mathbf{h}_{xx}\|^2 - c_1 \epsilon_1^2 \quad (5.6)$$

according to (5.3). Combining (5.5) and (5.6) and (a)–(d), there obtains the lower bound

$$\Delta J_6(\mathbf{u}) \geq \|\mathbf{h}_{xx}\|_{\mathcal{L}^2}^2 - \epsilon_1 D_0 - \epsilon_1^6 D_1 \quad (5.7)$$

where D_0 and D_1 are constants depending only on ω and θ . Using (5.5) and (5.7), one can conclude that

$$\|\mathbf{h}_{xx}(\cdot, t)\|_{\mathcal{L}^2}^2 \leq \epsilon_1 D_0 + \epsilon_1^6 D_1 + \delta_2 C_0 + \delta_2^6 C_1$$

holding for all t . In consequence, it is deduced that

$$\|\mathbf{h}(\cdot, t)\|_{\mathcal{H}^2}^2 \leq \epsilon_1 M_0 + \delta_2 M_1$$

where M_0 and M_1 are smooth functions of $\epsilon_1, \delta_2, \omega, \theta$, and in particular are bounded on bounded sets.

It remains simply to choose ϵ_1 so that $\epsilon_1 M_0 < \epsilon_2^2/2$. This implies the existence of a $\delta_1 > 0$ for which $\|\mathbf{u}(\cdot, t) - \Psi\|_{\mathcal{H}^1} = \|\mathbf{h}(\cdot, t)\|_{\mathcal{H}^1} \leq \epsilon_1$, provided $\|\Psi - \mathbf{u}_0\|_{\mathcal{H}^1} \leq \delta_1$. Then choose $\delta_2 \leq \delta_1$ small enough that $\delta_2 M_1 \leq \epsilon_2^2/2$ also. The $\mathcal{H}^2$ -stability conclusion then follows.

We now proceed inductively, supposing that for all $j < k$, stability holds in $\mathcal{H}^j$ in the stronger sense that, given an $\epsilon_j > 0$, there is a $\delta_j > 0$ such that if $\|\mathbf{u}_0 - \Psi\|_{\mathcal{H}^j} \leq \delta_j$ then $\|\mathbf{h}(\cdot, t)\|_{\mathcal{H}^j} \leq \epsilon_j$ for all t . Presuming that $\mathbf{u}_0 \in \mathcal{H}^k$, the stability in $\mathcal{H}^k$ is established by using the invariant functional J_{2k+2} .

Fix an $\epsilon_k > 0$. As in the case $k = 2$, define

$$\Delta J_{2k+2}(\mathbf{u}) = J_{2k+2}(\mathbf{u}(\cdot, t)) - J_{2k+2}(\Psi) \quad (5.8)$$

where $q(t)$ is as before, a C^1 -function that provides stability in $\mathcal{H}^1$. The upper bound for $\Delta J_{2k+2}(\mathbf{u})$ is calculated in terms of an upper bound δ_k for $\|\mathbf{u}_0 - \Psi\|_{\mathcal{H}^k}$ by evaluating (5.8) at $t = 0$, i.e.

$$\Delta J_{2k+2}(\mathbf{u}) = \Delta J_{2k+2}(\mathbf{u}_0) \leq c_k \delta_k^2 + c'_k \delta_k^{2k+2}. \quad (5.9)$$

For any positive value ϵ_{k-1} , there is a $\delta_{k-1} > 0$ for which $\|\mathbf{u}_0 - \Psi\|_{\mathcal{H}^{k-1}} \leq \delta_{k-1}$ implies

$$\|\mathbf{u}(\cdot, t) - \Psi\|_{\mathcal{H}^{k-1}} \leq \epsilon_{k-1} \quad (5.10)$$

for all t . Using (5.10), a direct calculation of $\Delta J_{2k+2}(\mathbf{u})$ in terms of $\mathbf{h}$ and Ψ yields a lower bound of the form

$$\Delta J_{2k+2}(\mathbf{u}) \geq c_k \|\partial_x^k \mathbf{h}\|_{\mathcal{L}^2}^2 - \epsilon_{k-1} N_k - \epsilon_{k-1}^{k+2} N'_k \quad (5.11)$$

where $c_k > 0$, and N_k, N'_k depend only on ω and θ . Stability in $\mathcal{H}^k$ follows from (5.9) and (5.11) just as in the $\mathcal{H}^2$ case. $\square$

Remark. Notice that we proved something a little stronger than just orbital stability in $\mathcal{H}^n$. It is deduced in fact that if

$$\inf_{\gamma_j, y \in \mathbb{R}} \left\{ \sum_{j=1}^m \|u_{j0} - e^{i\gamma_j} \varphi_\omega(\cdot + y)\|_{H^n} \right\} \leq \delta_n$$

then the solution $(u_1, \dots, u_m)$ of (1.2) emanating from the initial data $(u_{10}, \dots, u_{m0})$ satisfies

$$\sum_{j=1}^m \|u_j(\cdot, t) - e^{ip_j(t)} \varphi_\omega(\cdot - q(t))\|_{H^n} \leq \epsilon_n$$

for all $t \geq 0$, where $p_j(t)$ and $q(t)$ are the same smooth functions obtained in the $\mathcal{H}^1$ -stability result.

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