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Second order Riesz transforms associated to the Schrödinger operator for $p \leqslant 1$

Fu Ken Ly

Department of Mathematics, Macquarie University, Sydney, Australia

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ABSTRACT

Let $L = -\Delta + V$ be the Schrödinger operator on $\mathbb{R}^n$, where V belongs to the class of reverse Hölder weights RH_q for some $q > \max\{2, n/2\}$. We show that the second order Riesz transforms $\nabla^2 L^{-1}$ and VL^{-1} are bounded from the Hardy spaces $H_L^p(\mathbb{R}^n)$ associated to L into $L^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$. We show also that the operators $\nabla^2 L^{-1}$ map the classical Hardy spaces $H^p(\mathbb{R}^n)$ into $H^p(\mathbb{R}^n)$ for a restricted range of p.

1. Introduction

Let $n \geqslant 3$ and consider the Schrödinger operator on $\mathbb{R}^n$ with non-negative potential V, given by

$$L = -\Delta + V.$$

We assume that V is locally integrable and belongs to the class of reverse Hölder weights RH_q for some $q > n/2$. That is, there exists $C = C(q, V) > 0$ such that for all balls $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B V(x)^q\,dx\right)^{1/q} \leqslant \frac{C}{|B|}\int_B V(x)\,dx. \tag{1.1}$$

The operator L generates a semigroup e^{-tL} on $L^2(\mathbb{R}^n)$ with integral kernel $p_t(x, y)$ given by

$$e^{-tL} f(x) = \int_{\mathbb{R}^n} p_t(x, y) f(y)\,dy.$$

We shall refer to $p_t(x, y)$ as the heat kernel of L.

In this article we study the second order Riesz transform

$$\nabla^2 L^{-1} = \int_0^\infty \nabla^2 e^{-tL}\,dt \tag{1.2}$$

on Hardy spaces for $p \leqslant 1$.

E-mail address: ken.ly@mq.edu.au.

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For $p \geqslant 1$ the following is known for L^p spaces.

(i) If $V \geqslant 0$ the first order Riesz transforms $\nabla L^{-1/2}$ and $V^{1/2}L^{-1/2}$ are bounded on L^p for all $p \in (1, 2]$. See [19,5].
(ii) If $V \in RH_q$ for $q > 1$ and $n \geqslant 1$, the first order operators are bounded on L^p with $p \in (1, q^*]$ for $\nabla L^{-1/2}$, and with $p \in (1, 2q]$ for $V^{1/2}L^{-1/2}$. Here $q^* = qn/(n-q)$ if $q < n$ and $q^* = \infty$ if $q \geqslant n$. The second order operators $\nabla^2 L^{-1}$ and VL^{-1} are bounded on L^p for all $p \in (1, q]$. See [18,2].

For $p \leqslant 1$ the Hardy spaces H^p continue the L^p scale in the sense that the classical Riesz transforms $\nabla(-\Delta)^{-1/2}$ and $\nabla^2(-\Delta)^{-1}$ map H^p into H^p for all $0 < p \leqslant 1$. See [22].

However when working with differential operators other than the Laplacian it turns out that H^p may not be the most appropriate and, in these situations, it may be more suitable to work with a Hardy space that is adapted to the differential operator. A well developed theory concerning these matters is now available and we refer the reader to [3,6,10,11] for further discussion and historical notes. For Schrödinger operators the class of Hardy spaces H^p_L of relevance can be found in [7–10,13]. See Section 2 of this article for a review of their definitions.

Estimates for the first order Riesz transforms on these spaces are known:

Theorem 1.1. *(See [10,13,5].) Let $V \geqslant 0$ be a locally integrable function on $\mathbb{R}^n$ with $n \geqslant 1$, and $L = -\Delta + V$. Then $\nabla L^{-1/2}$ maps H^p_L to L^p for all $0 < p \leqslant 1$, and maps H^p into H^p for $n/(n+1) < p \leqslant 1$.*

In this article we obtain analogous estimates for the second order Riesz transforms associated to L under the extra condition that V satisfies a reverse Hölder inequality (1.1). The main result of this article is the following.

Theorem 1.2. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geqslant 3$. Assume that $V \in RH_q$ with $q > \max\{2, n/2\}$. Then the following hold:*

(a) *The operators $\nabla^2 L^{-1}$ and VL^{-1} are bounded from $H^p_L(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for each $p \in (0, 1]$.*
(b) *The operator $\nabla^2 L^{-1}$ is bounded from $H^p_L(\mathbb{R}^n)$ into $H^p(\mathbb{R}^n)$ for each $p \in (n/(n+1), 1]$.*

The key to our approach is suitable estimates on the derivatives of the heat kernel $\nabla^2_x p_t(x, y)$, obtained in [15]. See Lemma 3.2 below. This combined with the formula (1.2) allow us to apply the strategy in [10,13] used to study the first order operator $\nabla L^{-1/2}$.

In addition since V is a reverse Hölder potential, the atomic characterization of the Hardy spaces H^p_L given in [8] (see Definition 2.3 below) allow us to state the range of boundedness on the *classical* Hardy spaces.

Corollary 1.3. *Under the assumptions of Theorem 1.2 the operator $\nabla^2 L^{-1}$ is bounded from $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ for each $p \in (n/(n+p_L), 1]$, where $p_L = \min\{1, 2 - n/q\}$.*

We give some remarks about the condition $q > \max\{2, n/2\}$ in our results. The requirement $q > 2$ is required in two instances. The first is in the construction of the H^p_L spaces, which uses L^2-convergence of atomic sums (see Section 2). The other instance is the L^2 boundedness of the operators $\nabla^2 L^{-1}$ and VL^{-1}, which we recall from item (ii) above, is valid when $q > 2$. The reader will observe that our techniques and our heat kernel estimates will still follow through for the range $q < 2$, with suitable modifications, once an alternative construction of H^p_L is available. For the time being however, the range $n = 3$ and $3/2 < q < 2$ remains open.

This result also admits extensions to weighted Hardy spaces introduced in [20] and further studied in [24,23]. Our methods allow extensions for Theorem 1.2 (a) and (b) to weighted variants of H^p_L and H^p. However Corollary 1.3 remains open in this setting. We refer the reader to [16] for the details.

This article is organized as follows. In Section 2 we summarize the definitions and properties of the Hardy spaces that we will be working with and then give the proof of Corollary 1.3 is given at the end of the section. Section 3 collects together the estimates on the heat kernel that we shall need. The proof of Theorem 1.2 is given in Section 4.

Throughout this article we write B to mean a ball $B = B(x_B, r_B)$ with a centre x_B and radius r_B that has been fixed. Given a ball B we write λB to mean the dilation $B(x_B, \lambda r_B)$. We use $U_j(B)$ to denote the annulus $2^j B \backslash 2^{j-1} B$ when $j \geqslant 1$ and $U_0(B) = B$. The letter "C" will represent possibly different constants that are independent of the essential variables.

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2. Hardy spaces

In this section we survey the Hardy spaces adapted to the Schrödinger operator $L = -\Delta + V$. Unless otherwise noted, we will assume the potential V is a non-negative and locally integrable function. The material in this section can be found in more complete form in [4,10,13], where more general classes of operators are treated. For a description of the classical Hardy spaces and their properties see [22].

Firstly we set

$$\mathbb{H}^2(\mathbb{R}^n) := \overline{\{Lu \in L^2(\mathbb{R}^n):\ u \in L^2(\mathbb{R}^n)\}}.$$

For each $f \in L^2(\mathbb{R}^n)$, we define area integral function of f associated to L as

$$S_L(f)(x) := \left(\int_0^\infty \int_{|x-y|<t} \left|t^2 Le^{-t^2L} f(y)\right|^2 dy \frac{dt}{t^{n+1}}\right)^{1/2}, \quad x \in \mathbb{R}^n.$$

For each $p \in (0,1]$ we define the *Hardy spaces* $H_L^p(\mathbb{R}^n)$ *associated to* L as the completion of

$$\{f \in \mathbb{H}^2(\mathbb{R}^n):\ \|S_L(f)\|_{L^p(\mathbb{R}^n)} < \infty\}$$

in the quasi-norm $\|f\|_{H_L^p} := \|S_L(f)\|_{L^p}$.

Next we introduce the notion of $(p,2,M)$-atoms for L.

Definition 2.1 *(Atoms for H_L^p).* Let $0 < p \leqslant 1$ and $M \in \mathbb{N}$. A function $a \in L^2(\mathbb{R}^n)$ is called a $(p,2,M)$-atom for L associated to the ball B if for some $b \in \mathcal{D}(L^M)$ we have

(i) $a = L^M b$,
(ii) $\operatorname{supp} L^k b \subseteq B$ for each $k = 0, 1, \ldots, M$,
(iii) $\|(r_B^2 L)^k b\|_2 \leqslant r_B^{2M} |B|^{1/2-1/p}$ for each $k = 0, 1, \ldots, M$.

Let $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$. Then it follows that for each $f \in H_L^p(\mathbb{R}^n)$ there exists a sequence $\{a_B\}_B$ of $(p,2,M)$-atoms for L, and a sequence of scalars $\{\lambda_B\}_B \subset \mathbb{C}$, such that

$$f = \sum_B \lambda_B a_B \quad \text{and} \quad \sum_B |\lambda|^p \leqslant \|f\|_{H_L^p}^p.$$

The convergence is in both $H_L^p(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$.

These atoms allow us to reduce the study of operators on $H_L^p(\mathbb{R}^n)$ to studying their behaviour on single atoms. This is recorded in the following fact, and will be crucial in the proof of Theorem 1.2 (a).

Lemma 2.2. *Let $0 < p \leqslant 1$ and fix an integer $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$. Assume that T is a linear operator (resp. non-negative sublinear) operator that maps $L^2(\mathbb{R}^n)$ continuously into $L^{2,\infty}(\mathbb{R}^n)$ satisfying the following property: there exists $C > 0$ such that for each $(p,2,M)$-atom a,*

$$\|Ta\|_{L^p(\mathbb{R}^n)} \leqslant C.$$

Then T extends to a bounded linear (resp. sublinear) operator from $H_L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Furthermore, there exists $C' > 0$ such that

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leqslant C' \|f\|_{H_L^p(\mathbb{R}^n)}$$

for ever $f \in H_L^p(\mathbb{R}^n)$.

Proof. We refer the reader to [10], Lemma 4.3 or [4], Lemma 3.15. □

Next we specialize to the case that V satisfies a reverse Hölder inequality (1.1). Under this context the Hardy spaces associated to L can be characterized using certain atoms that allow us to directly compare H_L^p with H^p. We will use this fact to give the proof of Corollary 1.3 at the end of this section. The material found here have their origins in the work by Dziubański and Zienkiewicz in the series of papers [7–9].

Firstly we define the *critical radius* associated to V at x by the following expression:

$$\rho(x) = \rho(x,V) := \sup\left\{r > 0:\ \frac{r^2}{|B(x,r)|} \int_{B(x,r)} V \leqslant 1\right\}. \tag{2.1}$$

For each $p \in (0,1]$ we define $\mathcal{H}_L^p(\mathbb{R}^n)$ as the completion of

$$\{f \in L^1_c(\mathbb{R}^n):\ \|\mathcal{M}_L f\|_{L^p} < \infty\}$$

in the quasi-norm $\|f\|_{\mathcal{H}^p_L} = \|\mathcal{M}_L f\|_{L^p}$. Here $L^1_c(\mathbb{R}^n)$ is the space of compactly supported functions on $\mathbb{R}^n$, and the operator $\mathcal{M}_L$ is defined as

$$\mathcal{M}_L f(x) := \sup_{t>0} \left| e^{-tL} f(x) \right|.$$

When $V \in RH_q$ with $q \geqslant n/2$ and $n \geqslant 3$, the authors in [8] give a special atomic characterization of $\mathcal{H}^p_L(\mathbb{R}^n)$.

Definition 2.3 *(Special L-atoms).* A function a is called a *special L-atom* associated to the ball $B = B(x_B, r_B)$ if $r_B \leqslant \rho(x_B)$ and

(i) $\operatorname{supp} a \subseteq B$,
(ii) $\|a\|_{L^\infty} \leqslant |B|^{-1/p}$,
(iii) $\int a(x)\,dx = 0$ whenever $r_B \leqslant \frac{1}{4}\rho(x_B)$.

Let $p_L := \min\{1, 2 - n/q\}$. The authors show that when $p \in (n/(n+p_L), 1]$, then each $f \in \mathcal{H}^p_L(\mathbb{R}^n)$ has a special atomic decomposition $f = \sum_B \lambda_B a_B$ where the a_B are special L-atoms.

Recall that in the atomic characterization for the classical $H^p(\mathbb{R}^n)$ spaces, the cancellation condition is required for all balls (see [22]). Comparing this with Definition 2.3 (iii) above, we therefore have the following strict inclusion,

$$H^p(\mathbb{R}^n) \subsetneq \mathcal{H}^p_L(\mathbb{R}^n), \quad p \in \left(\frac{n}{n+p_L}, 1 \right]. \tag{2.2}$$

It is also known (see [13], Section 6) that

$$H^p_L(\mathbb{R}^n) = \mathcal{H}^p_L(\mathbb{R}^n), \quad p \in \left(\frac{n}{n+1}, 1 \right]. \tag{2.3}$$

We are now read to give

Proof of Corollary 1.3. We simply observe that $p_L \leqslant 1$ and hence $n/(n+1) \leqslant n/(n+p_L)$. Therefore (2.2) and (2.3) give

$$H^p(\mathbb{R}^n) \subsetneq \mathcal{H}^p_L(\mathbb{R}^n) = H^p_L(\mathbb{R}^n), \quad p \in \left(\frac{n}{n+p_L}, 1 \right].$$

Combining this with Theorem 1.2 (b) we obtain the corollary. □

3. Kernel estimates

In this section we collect here the heat kernel estimates that we will need for the proof of Theorem 1.2.

We first consider the case V is a non-negative and locally integrable function.

Lemma 3.1. *Let $L = -\Delta + V$ on $\mathbb{R}^n$, $n \geqslant 1$ with $0 \leqslant V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then the heat kernel $p_t(x, y)$ of L satisfies the following*:

(a) *For each $x, y \in \mathbb{R}^n$ and $t > 0$,*

$$0 \leqslant p_t(x, y) \leqslant (4\pi t)^{-n/2} e^{-|x-y|^2/4t}. \tag{3.1}$$

(b) *For each $k \in \mathbb{N}$ there exist $C_k > 0$, $c > 0$ satisfying*

$$\left| \frac{\partial^k}{\partial t^k} p_t(x, y) \right| \leqslant \frac{C_k}{t^{n/2+k}} e^{-c\frac{|x-y|^2}{t}} \tag{3.2}$$

for every $x, y \in \mathbb{R}^n$, and $t > 0$.

(c) *There exist $C, c, \alpha > 0$ such that for all $y \in \mathbb{R}^n$, and $t > 0$,*

$$\left(\int \left| \nabla_x p_t(x, y) \right|^2 e^{\alpha \frac{|x-y|^2}{t}}\, dx \right)^{1/2} \leqslant \frac{C}{t^{1/2+n/4}}. \tag{3.3}$$

Proof. Estimate (a) is well known. See page 195 of [17]. For the proof of (b) see [17], Theorem 6.17. The proof of (c) can be found in [1], Lemma 2.5. □

Next we specialize to the case that V satisfies a reverse Hölder inequality (1.1). It is known that under this condition the heat kernel for L satisfies stronger decay properties than the usual Gaussian upper bounds (3.1). Although this extra decay is not needed in the proof of Theorem 1.2, we record it here for completeness. Note that the function ρ has been defined in (2.1).

Lemma 3.2. *Assume that $V \in RH_q$ with $q \geqslant n/2$ for $n \geqslant 3$, or $q > 1$ for $n = 2$. Then the heat kernel $p_t(x, y)$ of $L = -\Delta + V$ satisfies the following estimates.*

(a) *There exist $C_0, c_0, c > 0$ and $\delta \in (0, 1)$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,*

$$p_t(x, y) \leqslant \frac{C_0}{t^{n/2}} e^{-c_0 \frac{|x-y|^2}{t}} e^{-c(1+\frac{t}{\rho(x)^2})^{\delta}}. \tag{3.4}$$

(b) *There exist $c = c(\delta) > 0$ and $c_1 > 0$ such that for each $k \in \mathbb{N}$ there exists $C_k > 0$ satisfying*

$$\left| \frac{\partial^k}{\partial t^k} p_t(x, y) \right| \leqslant \frac{C_k}{t^{n/2+k}} e^{-c_1 \frac{|x-y|^2}{t}} e^{-c(1+\frac{t}{\rho(x)^2})^{\delta}} \tag{3.5}$$

for every $x, y \in \mathbb{R}^n$, and $t > 0$.

(c) *For each $p \in [1, q)$ there exist $\beta_p, C_p, c > 0$ such that for all $y \in \mathbb{R}^n$, and $t > 0$,*

$$\left(\int \left| \nabla_x^2 p_t(x, y) \right|^p e^{\beta_p \frac{|x-y|^2}{t}} \, dx \right)^{1/p} \leqslant \frac{C_p}{t^{1+n/2p'}} e^{-c(1+\frac{t}{\rho(y)^2})^{\delta}}, \tag{3.6}$$

$$\left(\int \left| V(x) p_t(x, y) \right|^p e^{\beta_p \frac{|x-y|^2}{t}} \, dx \right)^{1/p} \leqslant \frac{C_p}{t^{1+n/2p'}} e^{-c(1+\frac{t}{\rho(y)^2})^{\delta}}. \tag{3.7}$$

Proof. The proof of (a) can be found in [14]. Parts (b) and (c) are proved in [15,16]. $\square$

The following is an extension of Lemma 3.2 (c) to time derivatives on the heat kernel.

Proposition 3.3. *Assume $V \in RH_q$ with $q \geqslant n/2$ for $n \geqslant 3$ or $q > 1$ for $n = 2$. Let δ be the constant from (3.4). Then for each $p \in [1, q)$ and $k \in \mathbb{Z}_+$ there exist $\xi = \xi(k, p) > 0$ and $C_{k,p} > 0$ such that*

$$\left(\int_{\mathbb{R}^n} \left| \nabla_x^2 \frac{\partial^k}{\partial t^k} p_t(x, y) \right|^p e^{\xi \frac{|x-y|^2}{t}} \, dx \right)^{1/p} \leqslant \frac{C_{k,p}}{t^{1+n/2p'+k}} e^{-c(1+\frac{t}{\rho(x)^2})^{\delta}}, \tag{3.8}$$

$$\left(\int_{\mathbb{R}^n} \left| V(x) \frac{\partial^k}{\partial t^k} p_t(x, y) \right|^p e^{\xi \frac{|x-y|^2}{t}} \, dx \right)^{1/p} \leqslant \frac{C_{k,p}}{t^{1+n/2p'+k}} e^{-c(1+\frac{t}{\rho(x)^2})^{\delta}} \tag{3.9}$$

for every $y \in \mathbb{R}^n$ and $t > 0$.

Proof. We shall make use of the commutativity property of the semigroup e^{-tL} to see that for each $k \geqslant 1$,

$$\frac{\partial^k}{\partial t^k} e^{-2tL} = (-2L)^k e^{-2tL} = 2^k e^{-tL} \frac{\partial^k}{\partial t^k} e^{-tL}.$$

In particular this implies

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\partial^k}{\partial t^k} p_{2t}(x, y) f(y) \, dy &= \frac{\partial^k}{\partial t^k} e^{-2tL} f(x) = 2^k e^{-tL} \frac{\partial^k}{\partial t^k} e^{-tL} f(x) \\
&= 2^k \int_{\mathbb{R}^n} p_t(x, w) \frac{\partial^k}{\partial t^k} e^{-tL} f(w) \, dw \\
&= 2^k \int_{\mathbb{R}^n} p_t(x, w) \int_{\mathbb{R}^n} \frac{\partial^k}{\partial t^k} p_t(w, y) f(y) \, dy \, dw \\
&= 2^k \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} p_t(x, w) \frac{\partial^k}{\partial t^k} p_t(w, y) \, dw \right) f(y) \, dy,
\end{aligned}$$

giving the identity

$$\frac{\partial^k}{\partial t^k} p_{2t}(x,y) = \int_{\mathbb{R}^n} p_t(x,w) \frac{\partial^k}{\partial t^k} p_t(w,y)\, dw \tag{3.10}$$

for each $x, y \in \mathbb{R}^n$.

Now fix $k \geqslant 1$ and $p \in [1, q)$. We first estimate (3.8). Let ξ be a constant such that $0 < \xi < \min\{\beta_p/2, pc_1/4\}$ where c_1 is the constant in the time derivative bounds of Proposition 3.2 (b) and β_p is the constant in Lemma 3.2 (c). Then using (3.10) we have for each $y \in \mathbb{R}^n$ and $t > 0$,

$$\int_{\mathbb{R}^n} \left| \nabla_x^2 \frac{\partial^k}{\partial t^k} p_{2t}(x,y) \right|^p e^{\xi \frac{|x-y|^2}{t}}\, dx = 2^k \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \nabla_x^2 p_t(x,w) \frac{\partial^k}{\partial t^k} p_t(w,y)\, e^{\xi \frac{|x-y|^2}{pt}}\, dw \right|^p dx.$$

Now for each $w \in \mathbb{R}^n$ the triangle inequality gives

$$e^{\xi \frac{|x-y|^2}{pt}} \leqslant e^{2\xi \frac{|x-w|^2}{pt}} e^{2\xi \frac{|w-y|^2}{pt}} = e^{2\xi \frac{|x-w|^2}{pt}} e^{-2\xi \frac{|w-y|^2}{pt}} e^{4\xi \frac{|w-y|^2}{pt}}.$$

Therefore for each $x, y \in \mathbb{R}^n$, by Hölder's inequality with exponent p and p',

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \nabla_x^2 p_t(x,w) \frac{\partial^k}{\partial t^k} p_t(w,y)\, e^{\xi \frac{|x-y|^2}{pt}}\, dw \right|^p \\ &\quad \leqslant \left(\int_{\mathbb{R}^n} \left|\nabla_x^2 p_t(x,w)\right|^p e^{2\xi \frac{|x-w|^2}{t}} e^{-2\xi \frac{|w-y|^2}{t}}\, dw \right) \left(\int_{\mathbb{R}^n} \left| \frac{\partial^k}{\partial t^k} p_t(w,y) \right|^{p'} e^{4p'\xi \frac{|w-y|^2}{pt}}\, dw \right)^{p/p'}. \end{aligned}$$

Using that $\xi < pc_1/4$ the time derivative bounds of Lemma 3.2 (b) give

$$\int_{\mathbb{R}^n} \left| \frac{\partial^k}{\partial t^k} p_t(w,y) \right|^{p'} e^{4p'\xi \frac{|w-y|^2}{pt}}\, dw \leqslant \frac{C_{k,p}}{t^{np'/2+kp'}} \int_{\mathbb{R}^n} e^{-p'(c_1 - 4\xi/p) \frac{|w-y|^2}{t}}\, dw \leqslant \frac{C_{k,p}}{t^{np'/2p+kp'}}$$

since $p' - 1 = p'/p$. Note that the constant $C_{k,p}$ is independent of y. We therefore obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \nabla_x^2 \frac{\partial^k}{\partial t^k} p_{2t}(x,y) \right|^p e^{\xi \frac{|x-y|^2}{t}}\, dx &\leqslant \frac{C_{k,p}}{t^{n/2+kp}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left|\nabla_x^2 p_t(x,w)\right|^p e^{2\xi \frac{|x-w|^2}{t}}\, dx \right) e^{-2\xi \frac{|w-y|^2}{t}}\, dw \\ &\leqslant C_{k,p} \frac{e^{-cp(1+\frac{t}{\rho(x)^2})^{\delta}}}{t^{n/2+kp+np/2p'}} \int_{\mathbb{R}^n} e^{-2\xi \frac{|w-y|^2}{t}}\, dw \\ &\leqslant C_{k,p} \frac{e^{-cp(1+\frac{t}{\rho(x)^2})^{\delta}}}{t^{p+kp+np/2p'}}, \end{aligned}$$

where we have applied (3.6) in the second inequality because $2\xi < \beta_p$. This concludes the proof of estimate (3.8).

We can obtain (3.9) in the same way, but we use (3.7) in place of (3.6). □

These estimates allow us to obtain the following decay estimates, which will be crucial in the subsequent sections.

Lemma 3.4. *Assume $V \in RH_q$ with $q > \max\{2, n/2\}$ and $n \geqslant 3$. Then for each $k \in \mathbb{N} \cup \{0\}$, there exist $C_k, c > 0$ such that*

$$\left(\int_{|x-y| \geqslant \sqrt{s}} \left| \nabla_x^2 \frac{\partial^k}{\partial t^k} p_t(x,y) \right|^2 dx \right)^{1/2} \leqslant \frac{C_k}{t^{1+n/4+k}} e^{-cs/t}, \tag{3.11}$$

$$\left(\int_{|x-y| \geqslant \sqrt{s}} \left| V(x) \frac{\partial^k}{\partial t^k} p_t(x,y) \right|^2 dx \right)^{1/2} \leqslant \frac{C_k}{t^{1+n/4+k}} e^{-cs/t}, \tag{3.12}$$

for each $y \in \mathbb{R}^n$ and $s, t > 0$.

Proof. Since $q > 2$ we may apply Proposition 3.3 with $p = 2$. Let ξ be the constant in Proposition 3.3. Then by (3.8),

$$\begin{aligned}\left(\int_{|x-y|\geqslant\sqrt{s}}\left|\nabla_x^2\frac{\partial^k}{\partial t^k}p_t(x,y)\right|^2dx\right)^{1/2} &= \left(\int_{|x-y|\geqslant\sqrt{s}}\left|\nabla_x^2\frac{\partial^k}{\partial t^k}p_t(x,y)\right|^2 e^{\xi\frac{|x-y|^2}{t}}e^{-\xi\frac{|x-y|^2}{t}}dx\right)^{1/2}\\ &\leqslant \sup_{|x-y|\geqslant\sqrt{s}} e^{-\xi\frac{|x-y|^2}{t}}\left(\int_{\mathbb{R}^n}\left|\nabla_x^2\frac{\partial^k}{\partial t^k}p_t(x,y)\right|^2 e^{\xi\frac{|x-y|^2}{t}}dx\right)^{1/2}\\ &\leqslant \frac{C}{t^{1+n/4+k}}e^{-\xi s/t}.\end{aligned}$$

Estimate (3.12) can be obtained similarly but with (3.9) in place of (3.8). □

We also record corresponding estimates for the first spatial derivative. These are needed in the proof of Theorem 1.2 (b).

Lemma 3.5. *Assume $n \geqslant 1$ and $0 \leqslant V \in L^1_{loc}(\mathbb{R}^n)$. Then for each $k \in \mathbb{N}\cup\{0\}$, there exist $C_k, c > 0$ such that*

$$\int_{|x-y|\geqslant\sqrt{s}}\left|\nabla_x\frac{\partial^k}{\partial t^k}p_t(x,y)\right|dx \leqslant \frac{C_k}{t^{1/2+k}}e^{-cs/t}, \tag{3.13}$$

for each $y \in \mathbb{R}^n$ and $s, t > 0$.

Proof. We first observe that a similar argument to the proof of Proposition 3.3, but with (3.3) in place of (3.6), and with the time derivative bounds in (3.2) in place of (3.5), give the following estimates: for each $k \in \mathbb{N}\cup 0$, there exist $\xi_k > 0$ and $C_k > 0$ such that

$$\left(\int_{\mathbb{R}^n}\left|\nabla_x\frac{\partial^k}{\partial t^k}p_t(x,y)\right|^2 e^{\xi\frac{|x-y|^2}{t}}dx\right)^{1/2} \leqslant \frac{C_k}{t^{1/2+n/4+k}}. \tag{3.14}$$

Note that the case $k = 0$ is simply the estimate in (3.3).

Now we combine (3.14) with the Cauchy–Schwarz inequality to obtain

$$\int_{|x-y|\geqslant\sqrt{s}}\left|\nabla_x\frac{\partial^k}{\partial t^k}p_t(x,y)\right|dx \leqslant \left(\int_{\mathbb{R}^n}\left|\nabla_x\frac{\partial^k}{\partial t^k}p_t(x,y)\right|^2 e^{\xi\frac{|x-y|^2}{t}}dx\right)^{1/2}\left(\int_{|x-y|\geqslant\sqrt{s}}e^{-\xi\frac{|x-y|^2}{t}}dx\right)^{1/2} \leqslant \frac{C_k}{t^{1/2+k}}e^{-cs/t}$$

as desired. □

4. Proof of the main result

In this section we prove Theorem 1.2.

Proof of Theorem 1.2 (a). We show that Lemma 2.2 holds for each of the operators ∇^2L^{-1} and VL^{-1}, for all $0 < p \leqslant 1$. More precisely let $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$ be an integer and a_B be a $(p, 2, M)$-atom for L associated to the ball $B = B(x_B, r_B)$.

We first consider the operator ∇^2L^{-1}. By Lemma 2.2 it suffices to show that

$$\|\nabla^2L^{-1}a_B\|_{L^p} \leqslant C \tag{4.1}$$

with C independent of a_B.

Since $0 < p \leqslant 1$ we may apply Hölder's inequality with exponents $2/p$ and $2/(2-p)$ to obtain

$$\begin{aligned}\|\nabla^2L^{-1}a_B\|_{L^p}^p &= \sum_{j=0}^{\infty}\left\||\nabla^2L^{-1}a_B|^p\right\|_{L^1(U_j(B))}\\ &\leqslant \sum_{j=0}^{\infty}\left|2^jB\right|^{1-p/2}\|\nabla^2L^{-1}a_B\|_{L^2(U_j(B))}^p\\ &\leqslant |B|^{1-p/2}\sum_{j=0}^{\infty}2^{jn(1-p/2)}\|\nabla^2L^{-1}a_B\|_{L^2(U_j(B))}^p.\end{aligned} \tag{4.2}$$

Since $q > 2$ the operator $\nabla^2 L^{-1}$ is bounded on $L^2(\mathbb{R}^n)$ (recall item (ii) from Section 1), and hence for $j = 0, 1, 2$,

$$\left\|\nabla^2 L^{-1} a_B\right\|_{L^2(U_j(B))} \leqslant C\|a_B\|_{L^2} \leqslant C|B|^{1/2-1/p}. \tag{4.3}$$

Now for each $j \geqslant 3$ we note that

$$\operatorname{dist}\big(U_j(B), B\big) \geqslant 2^{j-1} r_B - r_B \geqslant 2^{j-2} r_B.$$

Then using the identity

$$L^{-1} = \int_0^\infty e^{-tL}\, dt,$$

we obtain

$$\left\|\nabla^2 L^{-1} a_B\right\|_{L^2(U_j(B))} \leqslant \left\| \int_0^{r_B^2} \nabla^2 e^{-tL} a_B\, dt \right\|_{L^2(U_j(B))} + \left\| \int_{r_B^2}^\infty \nabla^2 e^{-tL} a_B\, dt \right\|_{L^2(U_j(B))} =: I_j + II_j.$$

We first estimate term I_j. Using estimate (3.11) with $k = 0$ we have

$$\begin{aligned}
\left\|\nabla^2 e^{-tL} a_B\right\|_{L^2(U_j(B))} &= \left(\int_{U_j(B)} \left| \int_B \nabla_x^2 p_t(x,y)\, a_B(y)\, dy \right|^2 dx \right)^{1/2} \\
&\leqslant \int_B \left|a_B(y)\right| \left(\int_{|x-y| \geqslant 2^{j-2} r_B} \left|\nabla_x^2 p_t(x,y)\right|^2 dx \right)^{1/2} dy \\
&\leqslant C\|a_B\|_{L^1} \frac{e^{-c4^j r_B^2/t}}{t^{1+n/4}}.
\end{aligned} \tag{4.4}$$

In the following let α be a number satisfying $\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) < \alpha < M$. Then (4.4) gives

$$\begin{aligned}
I_j &\leqslant \int_0^{r_B^2} \left\|\nabla^2 e^{-tL} a_B\right\|_{L^2(U_j(B))} dt \leqslant C\|a_B\|_{L^1} \int_0^{r_B^2} e^{-c4^j r_B^2/t} \frac{dt}{t^{n/4+1}} \\
&\leqslant C|B|^{1-1/p} \int_0^{r_B^2} \left(\frac{t}{4^j r_B^2} \right)^\alpha \frac{dt}{t^{n/4+1}} \leqslant C 2^{-2j\alpha} |B|^{1/2-1/p}.
\end{aligned} \tag{4.5}$$

In the last line we used that $\alpha > n/4$, because $p \leqslant 1$ implies that $\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) \geqslant \frac{n}{4}$.

We turn to the term II_j. For this estimate we apply L-cancellation to transfer powers of L to powers of t^{-1} increasing the decay as $t \to \infty$. More precisely we write $a_B = L^M b_B$ for some $b_B \in \mathcal{D}(L^M)$, and obtain

$$e^{-tL} a_B = e^{-tL} L^M b_B = L^M e^{-tL} b_B = (-1)^M \frac{\partial^M}{\partial t^M} e^{-tL} b_B.$$

Now we apply (3.11) with $k = M$ to obtain the extra powers of t^{-1}. This gives

$$\begin{aligned}
\left\|\nabla^2 \frac{\partial^M}{\partial t^M} e^{-tL} b_B\right\|_{L^2(U_j(B))} &= \left(\int_{U_j(B)} \left| \int_B \nabla_x^2 \frac{\partial^M}{\partial t^M} p_t(x,y)\, b_B(y)\, dy \right|^2 dx \right)^{1/2} \\
&\leqslant \int_B \left|b_B(y)\right| \left(\int_{|x-y| \geqslant 2^{j-2} r_B} \left|\nabla_x^2 \frac{\partial^M}{\partial t^M} p_t(x,y)\right|^2 dx \right)^{1/2} dy \\
&\leqslant C\|b_B\|_{L^1} \frac{e^{-c4^j r_B^2/t}}{t^{M+n/4+1}}.
\end{aligned} \tag{4.6}$$

Then, with α as before, we use (4.6) to get

$$II_j \leqslant \int_{r_B^2}^{\infty} \left\| \nabla^2 \frac{\partial^M}{\partial t^M} e^{-tL} b_B \right\|_{L^2(U_j(B))} dt \leqslant C \|b_B\|_{L^1} \int_{r_B^2}^{\infty} e^{-c4^j r_B^2/t} \frac{dt}{t^{M+n/4+1}}$$

$$\leqslant C r_B^{2M} |B|^{1-1/p} \int_{r_B^2}^{\infty} \left(\frac{t}{4^j r_B^2} \right)^{\alpha} \frac{dt}{t^{M+n/4+1}} \leqslant C 2^{-2j\alpha} |B|^{1/2-1/p}. \tag{4.7}$$

In the last line we used that $\alpha < M + n/4$.

Collecting estimates (4.3), (4.5) and (4.7) into (4.2) we obtain

$$\left\| \nabla^2 L^{-1} a_B \right\|_{L^p}^p \leqslant C + |B|^{1-p/2} \sum_{j=3}^{\infty} 2^{jn(1-p/2)} \{I_j + II_j\}^p \leqslant C + C \sum_{j=3}^{\infty} 2^{-j(2\alpha p - n(1-p/2))} \leqslant C$$

with the sum converging because $\alpha > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$. Therefore (4.1) holds.

Turning to the operator VL^{-1} we observe that we can repeat the proof to obtain

$$\left\| V L^{-1} a_B \right\|_{L^p} \leqslant C$$

using (3.12) in place of (3.11). $\square$

Proof of Theorem 1.2 (b). The proof we give here follows the same strategy as in [12] Proposition 5.6. We utilize a certain characterization of $H^p(\mathbb{R}^n)$ for $p \leqslant 1$ given there on page 38: for each $p \in (0, 1]$, $\varepsilon > 0$, and $N \in \mathbb{N} \cup \{0\}$ with $N \geqslant [n(\frac{1}{p} - 1)]$, we call $m \in L^2(\mathbb{R}^n)$ a $(p, 2, N, \varepsilon)$-molecule for $H^p(\mathbb{R}^n)$ associated to a ball B if

(a) $\int_{\mathbb{R}^n} x^\alpha m(x)\, dx = 0$ for all multi-indices $0 \leqslant |\alpha| \leqslant N$,
(b) $\|m\|_{L^2(U_j(B))} \leqslant 2^{-j\varepsilon} |2^j B|^{1/2-1/p}$ for all $j = 0, 1, \ldots.$

Then one may characterize the classical $H^p(\mathbb{R}^n)$ as follows

$$H^p(\mathbb{R}^n) = \left\{ \sum_j \lambda_j m_j \colon \{\lambda_j\} \in l^p,\ m_j \text{ are } (p, 2, N, \varepsilon)\text{-molecules} \right\}$$

with

$$\|f\|_{H^p} \approx \inf\left\{ \left(\sum_j |\lambda_j|^p \right)^{1/p} \right\}, \tag{4.8}$$

where the infimum being taken over all decompositions $f = \sum_j \lambda_j m_j$ and the sum converging the space of tempered distributions $\mathcal{S}'$.

We shall show that for each $p \in (n/(n+1), 1]$ and $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$, the operator $\nabla^2 L^{-1}$ maps $(p, 2, M)$-atoms for H_L^p to multiples of $(p, 2, 0, \varepsilon)$-molecules for H^p with some $\varepsilon > 0$. Fix a $(p, 2, M)$-atom a_B for L associated to a ball $B = B(x_B, r_B)$. Set $m_B := \nabla^2 L^{-1}$. Since $p > n/(n+1)$ then we may take $N = 0$ in the above cancellation condition (a). Then we aim to show that there exist $C > 0$ and $\varepsilon > 0$ such that

$$\|m_B\|_{L^2(U_j(B))} \leqslant C 2^{-j\varepsilon} \left| 2^j B \right|^{1/2-1/p}, \tag{4.9}$$

$$\int_{\mathbb{R}^n} m_B(x)\, dx = 0, \tag{4.10}$$

for all $j \geqslant 0$.

Before we prove (4.9) and (4.10) we explain how these imply the estimate

$$\left\| \nabla^2 L^{-1} f \right\|_{H^p} \leqslant C \|f\|_{H_L^p}.$$

Since $f \in H_L^p(\mathbb{R}^n)$ there is a sequence of $(p, 2, M)$-atoms $\{a_B\}_B$ for L and constants $\{\lambda_B\}_B$ such that $f = \sum_B \lambda_B a_B$ in $L^2(\mathbb{R}^n)$ and

$$\|f\|_{H_L^p} \approx \left(\sum_B |\lambda_B|^p \right)^{1/p}. \tag{4.11}$$

Now since the sum converges in $L^2(\mathbb{R}^n)$ we have

$$\nabla^2 L^{-1} f = \sum_B \lambda_B \left(\nabla^2 L^{-1} a_B\right) =: \sum_B \lambda_B m_B.$$

By (4.9) and (4.10) each m_B is a $(p, 2, 0, \varepsilon)$-molecule and hence this last sum converges in $L^2(\mathbb{R}^n)$, and hence also in $\mathcal{S}'$. Therefore $\sum_B \lambda_B m_B \in H^p(\mathbb{R}^n)$ and furthermore

$$\left\|\nabla^2 L^{-1} f\right\|_{H^p} = \left\|\sum_B \lambda_B m_B\right\|_{H^p} \leqslant \left(\sum_B |\lambda_B|^p\right)^{1/p} \approx \|f\|_{H^p_L}$$

from (4.8) and (4.11).

Having these facts in hand we now proceed to estimate (4.9). We recall from the proof of Theorem 1.2 (a) that for any $\frac{n}{2}(\frac{1}{2} - \frac{1}{2}) < \alpha < M$ we have from estimates (4.3), (4.5), and (4.7) that there exists $C > 0$ with

$$\begin{aligned}\|m_B\|_{L^2(U_j(B))} &= \left\|\nabla^2 L^{-1} a_B\right\|_{L^2(U_j(B))} \leqslant C 2^{-2j\alpha} |B|^{1/2-1/p} \\ &= C 2^{-j(2\alpha+n/2-n/p)} \left|2^j B\right|^{1/2-1/p}.\end{aligned} \tag{4.12}$$

Since $\alpha > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$ then $2\alpha + n/2 - n/p > 0$ and we obtain (4.9) with $\varepsilon = 2\alpha + n/2 - n/p$.

We now prove the moment condition (4.10). To do so we shall need the following result. It is implicit in [13] but we give a proof here for completeness.

Lemma 4.1. *Assume that $f \in L^1(\mathbb{R}^n)$ and $\partial_k f \in L^1(\mathbb{R}^n)$ for some $k \in \{1, \ldots, n\}$. Then*

$$\int_{\mathbb{R}^n} \partial_k f(x)\, dx = 0. \tag{4.13}$$

Here the derivative $\partial_k f$ is taken in the sense of distributions.

Proof of Lemma 4.1. Pick $\theta \in C_0^\infty(\mathbb{R}^n)$ such that $\theta = 1$ on $B(0, 1)$ and $\theta = 0$ outside $B(0, 2)$. For each $j \in \mathbb{N}$ set $\varphi_j(x) := \theta(x/j)$ for $x \in \mathbb{R}^n$. Then we have

$$\int_{\mathbb{R}^n} \partial_k f(x)\, dx = \lim_{j\to\infty} \int_{\mathbb{R}^n} \varphi_j(x) \partial_k f(x)\, dx = -\lim_{j\to\infty} \int_{\mathbb{R}^n} (\partial_k \varphi_j)(x) f(x)\, dx = 0$$

where the first equality is by Lebesgue's Dominated Convergence Theorem (LDCT), the second one uses the definition of the distributional derivative $\partial_k f$, and the third is based on the fact that $|\partial_k \varphi_j| \leqslant j^{-1} \|\partial_k \theta\|_{L^\infty}$ and again by LDCT. □

By Lemma 4.1, to show that

$$\int_{\mathbb{R}^n} \partial_k \partial_l L^{-1} a_B(x)\, dx = 0$$

for each $1 \leqslant k, l \leqslant n$, it suffices to show that the functions $\partial_k L^{-1} a_B$ and $\partial_k \partial_l L^{-1} a_B$ are integrable. We note that $\partial_k \partial_l L^{-1} a_B \in L^1(\mathbb{R}^n)$ follows from (4.12). Indeed,

$$\begin{aligned}\left\|\partial_k \partial_l L^{-1} a_B\right\|_{L^1} &\leqslant \sum_{j=0}^\infty \left\|\nabla^2 L^{-1} a_B\right\|_{L^1(U_j(B))} \\ &\leqslant \sum_{j=0}^\infty |B|^{1/2} \left\|\nabla^2 L^{-1} a_B\right\|_{L^2(U_j(B))} \\ &\leqslant C|B|^{1/2} \sum_{j=0}^\infty 2^{-j\varepsilon} \left|2^j B\right|^{1/2-1/p} \\ &= C|B|^{1-1/p} \sum_{j=0}^\infty 2^{-j(\varepsilon+n/p-n/2)} \\ &\leqslant C|B|^{1-1/p},\end{aligned}$$

with the sum being convergent since $\varepsilon + n/p - n/2 = 2\alpha > 0$. To check $\partial_k L^{-1} a_B \in L^1(\mathbb{R}^n)$ we write

$$\|\partial_k L^{-1} a_B\|_{L^1} \leqslant \|\nabla L^{-1} a_B\|_{L^1} = \sum_{j=0}^{\infty} \|\nabla L^{-1} a_B\|_{L^1(U_j(B))}.$$

For $j \geqslant 3$,

$$\|\nabla L^{-1} a_B\|_{L^1(U_j(B))} \leqslant \left\| \int_0^{r_B^2} \nabla e^{-tL} a_B\, dt \right\|_{L^1(U_j(B))} + \left\| \int_{r_B^2}^{\infty} \nabla e^{-tL} a_B\, dt \right\|_{L^1(U_j(B))} =: I_j + II_j.$$

Let β be a number satisfying $0 < \beta < M - \frac{1}{2}$. Then using (3.13) with $k = 0$, we have

$$\begin{aligned}
I_j &\leqslant \int_0^{r_B^2} \|\nabla e^{-tL} a_B\|_{L^1(U_j(B))}\, dt \\
&= \int_0^{r_B^2} \int_{U_j(B)} \left| \int_B \nabla_x p_t(x, y) a_B(y)\, dy \right| dx\, dt \\
&\leqslant \int_0^{r_B^2} \int_B |a_B(y)| \int_{|x-y| \geqslant 2^{j-2} r_B} |\nabla_x p_t(x, y)|\, dx\, dy\, dt \\
&\leqslant C\|a_B\|_{L^1} \int_0^{r_B^2} e^{-c4^j r_B^2/t} \frac{dt}{\sqrt{t}} \\
&\leqslant C|B|^{1-1/p} \int_0^{r_B^2} \left(\frac{t}{4^j r_B^2} \right)^{\beta} \frac{dt}{\sqrt{t}} \\
&\leqslant 4^{-j\beta} |B|^{1-1/p+1/n}.
\end{aligned} \tag{4.14}$$

For the second term we use L-cancellation and estimate (3.13) with $k = M$ to obtain

$$\begin{aligned}
II_j &\leqslant \int_{r_B^2}^{\infty} \left\| \nabla \frac{\partial^M}{\partial t^M} e^{-tL} b_B \right\|_{L^1(U_j(B))} dt \\
&\leqslant \int_{r_B^2}^{\infty} \int_B |b_B(y)| \int_{|x-y| \geqslant 2^{j-2} r_B} \left| \nabla_x \frac{\partial^M}{\partial t^M} p_t(x, y) \right| dx\, dy\, dt \\
&\leqslant C\|b_B\|_{L^1} \int_{r_B^2}^{\infty} e^{-c4^j r_B^2/t} \frac{dt}{t^{M+1/2}} \\
&\leqslant C r_B^{2M} |B|^{1-1/p} \int_{r_B^2}^{\infty} \left(\frac{t}{4^j r_B^2} \right)^{\beta} \frac{dt}{t^{M+1/2}} \\
&\leqslant C 4^{-j\beta} |B|^{1-1/p+1/n}.
\end{aligned} \tag{4.15}$$

The last line holds because $0 < \beta < M - \frac{1}{2}$. For $j = 0, 1, 2$ we use that the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^2(\mathbb{R}^n)$, and that the fractional power $L^{-1/2}$ maps $L^{2n/(n+2)}(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. The latter fact is a consequence of the following.

Lemma 4.2. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geqslant 1$ and $0 \leqslant V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let $\alpha \in (0, n)$ and $p \in (1, n/\alpha)$, with q satisfying $1/p - 1/q = \alpha/n$. Then $L^{-\alpha/2}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Proof. From the heat kernel bounds of (3.1), we have

$$\left|L^{-\alpha/2} f(x)\right| \lesssim (-\Delta)^{-\alpha/2}|f|(x)$$

and hence the mapping properties of $L^{-\alpha/2}$ follow from that of $(-\Delta)^{-\alpha/2}$ (which can be found in [21] for example). □

More precisely we have

$$\left\|\nabla L^{-1} a_B\right\|_{L^2(8B)} = \left\|\nabla L^{-1/2} L^{-1/2} a_B\right\|_{L^2(8B)} \leqslant C\left\|L^{-1/2} a_B\right\|_{L^2} \leqslant C\|a_B\|_{L^{2n/(n+2)}}.$$

Now we apply Hölder's inequality with exponents $s := (n+2)/n$ and $s' := (n+2)/2$ to obtain

$$\|a_B\|_{L^{2/s}}^{2/s} \leqslant \|a_B\|_{L^2}^{2/s} |B|^{1/s'} \leqslant |B|^{1-2/ps},$$

and therefore

$$\left\|\nabla L^{-1} a_B\right\|_{L^1(8B)} \leqslant C|B|^{1/2}\left\|\nabla L^{-1} a_B\right\|_{L^2(8B)} \leqslant C|B|^{1-1/p+1/n}.$$

Collecting these estimates for $j \geqslant 0$ we obtain for some $0 < \beta < M - \frac{1}{2}$,

$$\left\|\nabla L^{-1} a_B\right\|_{L^1} \leqslant C + C|B|^{1-1/p+1/n} \sum_{j=3}^{\infty} 4^{-j\beta} \leqslant C|B|^{1-1/p+1/n}.$$

We have shown that $\partial_k L^{-1} a_B \in L^1(\mathbb{R}^n)$ for each $1 \leqslant k \leqslant n$, and hence by Lemma 4.1, estimate (4.10) holds.

The proof of Theorem 1.2 (b) is therefore complete. □

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