



Asymptotic behavior of least energy solutions for a 2D nonlinear Neumann problem with large exponent



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ABSTRACT

In this paper, we consider the following elliptic problem with the nonlinear Neumann boundary condition:

$$(E_p) \begin{cases} -\Delta u + u = 0 & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ \frac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in \mathbb{R}^2 , ν is the outer unit normal vector to $\partial\Omega$, and $p > 1$ is any positive number.

We study the asymptotic behavior of least energy solutions to (E_p) when the nonlinear exponent p gets large. Following the arguments of X. Ren and J.C. Wei [13,14], we show that the least energy solutions remain bounded uniformly in p , and it develops one peak on the boundary, the location of which is controlled by the Green function associated to the linear problem.

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1. Introduction

In this paper, we consider the following elliptic problem with the nonlinear Neumann boundary condition:

$$(E_p) \begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in \mathbb{R}^2 , ν is the outer unit normal vector to $\partial\Omega$, and $p > 1$ is any positive number. Let $H^1(\Omega)$ be the usual Sobolev space with the norm $\|u\|_{H^1(\Omega)}^2 = \int_{\Omega} (|\nabla u|^2 + u^2) dx$. Since the trace Sobolev embedding $H^1(\Omega) \hookrightarrow L^{p+1}(\partial\Omega)$ is compact for any $p > 1$, we can obtain at least one solution of (1.1) by a standard variational method. In fact, let us consider the constrained minimization problem

$$C_p^2 = \inf \left\{ \int_{\Omega} (|\nabla u|^2 + u^2) dx \mid u \in H^1(\Omega), \int_{\partial\Omega} |u|^{p+1} ds_x = 1 \right\}. \tag{1.2}$$

Standard variational method implies that C_p^2 is achieved by a positive function $\bar{u}_p \in H^1(\Omega)$ and then $u_p = C_p^{2/(p-1)} \bar{u}_p$ solves (1.1). We call u_p a least energy solution to the problem (1.1).

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In this paper, we prove the followings:

Theorem 1. Let u_p be a least energy solution to (E_p) . Then it holds

$$1 \leq \liminf_{p \rightarrow \infty} \|u_p\|_{L^\infty(\partial\Omega)} \leq \limsup_{p \rightarrow \infty} \|u_p\|_{L^\infty(\partial\Omega)} \leq \sqrt{e}.$$

To state further results, we set

$$v_p = u_p / \left(\int_{\partial\Omega} u_p^p ds_x \right). \tag{1.3}$$

Theorem 2. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Then for any sequence v_{p_n} of v_p defined in (1.3) with $p_n \rightarrow \infty$, there exists a subsequence (still denoted by v_{p_n}) and a point $x_0 \in \partial\Omega$ such that the following statements hold true.

(1)

$$f_n = \frac{u_{p_n}^{p_n}}{\int_{\partial\Omega} u_{p_n}^{p_n} ds_x} \xrightarrow{*} \delta_{x_0}$$

in the sense of Radon measures on $\partial\Omega$.

(2) $v_{p_n} \rightarrow G(\cdot, x_0)$ in $C^1_{loc}(\bar{\Omega} \setminus \{x_0\})$, $L^t(\Omega)$ and $L^t(\partial\Omega)$ respectively for any $1 \leq t < \infty$, where $G(x, y)$ denotes the Green function of $-\Delta$ for the following Neumann problem:

$$\begin{cases} -\Delta_x G(x, y) + G(x, y) = 0 & \text{in } \Omega, \\ \frac{\partial G}{\partial \nu_x}(x, y) = \delta_y(x) & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

(3) x_0 satisfies

$$\nabla_{\tau(x_0)} R(x_0) = \vec{0},$$

where $\tau(x_0)$ denotes a tangent vector at the point $x_0 \in \partial\Omega$ and R is the Robin function defined by $R(x) = H(x, x)$, where

$$H(x, y) := G(x, y) - \frac{1}{\pi} \log |x - y|^{-1}$$

denotes the regular part of G .

Concerning related results, X. Ren and J.C. Wei [13,14] first studied the asymptotic behavior of least energy solutions to the semilinear problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

as $p \rightarrow \infty$, where Ω is a bounded smooth domain in \mathbb{R}^2 . They proved that the least energy solutions remain bounded and bounded away from zero in L^∞ -norm uniformly in p . As for the shape of solutions, they showed that the least energy solutions must develop one ‘‘peak’’ in the interior of Ω , which must be a critical point of the Robin function associated with the Green function subject to the Dirichlet boundary condition. Later, Adimurthi and Grossi [1] improved their results by showing that, after some scaling, the limit profile of solutions is governed by the Liouville equation

$$-\Delta U = e^U \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^U dx < \infty,$$

and obtained that $\lim_{p \rightarrow \infty} \|u_p\|_{L^\infty(\Omega)} = \sqrt{e}$ for least energy solutions u_p . Actual existence of concentrating solutions to (1.1) is recently obtained by H. Castro [4] by a variational reduction procedure, along the line of [8] and [6]. As for construction of concentrating solutions to related problems, see also [7,9], and [11].

Also in our case, we may conjecture that the limit problem of (1.1) is

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^2_+, \\ \frac{\partial U}{\partial \nu} = e^U & \text{on } \partial\mathbb{R}^2_+, \\ \int_{\partial\mathbb{R}^2_+} e^U ds < \infty, \end{cases}$$

and $\lim_{p \rightarrow \infty} \|u_p\|_{L^\infty(\partial\Omega)} = \sqrt{e}$ holds true at least for least energy solutions u_p . Verification of these conjectures remains as the future work.

2. Some estimates for C_p^2

In this section, we provide some estimates for C_p^2 in (1.2) as $p \rightarrow \infty$.

Lemma 3. For any $s \geq 2$, there exists $\tilde{D}_s > 0$ such that for any $u \in H^1(\Omega)$,

$$\|u\|_{L^s(\partial\Omega)} \leq \tilde{D}_s s^{\frac{1}{2}} \|u\|_{H^1(\Omega)}$$

holds true. Furthermore, we have

$$\lim_{s \rightarrow \infty} \tilde{D}_s = (2\pi e)^{-\frac{1}{2}}.$$

Proof. Let $u \in H^1(\Omega)$. By Trudinger–Moser trace inequality, see [5] and the references therein, we have

$$\int_{\partial\Omega} \exp\left(\frac{\pi |u(x) - u_{\partial\Omega}|^2}{\|\nabla u\|_{L^2(\Omega)}^2}\right) ds_x \leq C(\Omega)$$

for any $u \in H^1(\Omega)$, where $u_{\partial\Omega} = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u ds_x$. Thus, by an elementary inequality $\frac{x^s}{\Gamma(s+1)} \leq e^x$ for any $x \geq 0$ and $s \geq 0$, where $\Gamma(s)$ is the Gamma function, we see

$$\begin{aligned} \frac{1}{\Gamma((s/2) + 1)} \int_{\partial\Omega} |u - u_{\partial\Omega}|^s ds_x &= \frac{1}{\Gamma((s/2) + 1)} \int_{\partial\Omega} \left(\pi \frac{|u(x) - u_{\partial\Omega}|^2}{\|\nabla u\|_{L^2(\Omega)}^2}\right)^{s/2} ds_x \pi^{-s/2} \|\nabla u\|_{L^2(\Omega)}^s \\ &\leq \int_{\partial\Omega} \exp\left(\pi \frac{|u(x) - u_{\partial\Omega}|^2}{\|\nabla u\|_{L^2(\Omega)}^2}\right) ds_x \pi^{-s/2} \|\nabla u\|_{L^2(\Omega)}^s \\ &\leq C(\Omega) \pi^{-s/2} \|\nabla u\|_{L^2(\Omega)}^s. \end{aligned}$$

Set

$$D_s := (\Gamma(s/2 + 1))^{1/s} C(\Omega)^{1/s} \pi^{-1/2} s^{-1/2}.$$

Then we have

$$\|u - u_{\partial\Omega}\|_{L^s(\partial\Omega)} \leq D_s s^{1/2} \|\nabla u\|_{L^2(\Omega)}.$$

Stirling’s formula says that $(\Gamma(\frac{s}{2} + 1))^{\frac{1}{s}} \sim (\frac{s}{2e})^{1/2}$ as $s \rightarrow \infty$, so we have

$$\lim_{s \rightarrow \infty} D_s = \left(\frac{1}{2\pi e}\right)^{1/2}.$$

On the other hand, by the embedding $\|u\|_{L^2(\partial\Omega)} \leq C(\Omega) \|u\|_{H^1(\Omega)}$ for any $u \in H^1(\Omega)$, we see

$$|u_{\partial\Omega}| \leq \frac{1}{|\partial\Omega|^{1/2}} \left(\int_{\partial\Omega} |u|^2 ds_x\right)^{1/2} \leq \frac{C(\Omega)}{|\partial\Omega|^{1/2}} \|u\|_{H^1(\Omega)}.$$

Thus,

$$\begin{aligned} \|u\|_{L^s(\partial\Omega)} &\leq \|u - u_{\partial\Omega}\|_{L^s(\partial\Omega)} + \|u_{\partial\Omega}\|_{L^s(\partial\Omega)} \leq \|u - u_{\partial\Omega}\|_{L^s(\partial\Omega)} + |u_{\partial\Omega}| |\partial\Omega|^{1/s} \\ &\leq s^{1/2} \|u\|_{H^1(\Omega)} \left(D(s) + \frac{C(\Omega) |\partial\Omega|^{1/s-1/2}}{s^{1/2}}\right). \end{aligned}$$

Put

$$\tilde{D}(s) = D(s) + \frac{C(\Omega) |\partial\Omega|^{1/s-1/2}}{s^{1/2}}.$$

Then, we have $\lim_{s \rightarrow \infty} \tilde{D}(s) = \lim_{s \rightarrow \infty} D(s) = \frac{1}{\sqrt{2\pi e}}$ and

$$\|u\|_{L^s(\partial\Omega)} \leq \tilde{D}_s s^{\frac{1}{2}} \|u\|_{H^1(\Omega)}$$

holds. \square

Lemma 4. Let Ω be a smooth bounded domain in \mathbb{R}^2 . Then we have

$$\lim_{p \rightarrow \infty} pC_p^2 = 2\pi e.$$

Proof. For the estimate from below, we use Lemma 3. By Lemma 3, we have

$$\|u\|_{L^{p+1}(\partial\Omega)}^2 \leq \tilde{D}_{p+1}^2(p+1)\|u\|_{H^1(\Omega)}^2$$

for any $u \in H^1(\Omega)$, which leads to $\tilde{D}_{p+1}^{-2}(\frac{p}{p+1}) \leq pC_p^2$. Thus, we have $2\pi e \leq \liminf_{p \rightarrow \infty} pC_p^2$, since $\lim_{p \rightarrow \infty} \tilde{D}_{p+1} = (2\pi e)^{-1/2}$.

For the estimate from above, we use the Moser function. Let $0 < l < L$. First, we assume $\Omega \cap B_L(0) = \Omega \cap B_L^+$ where $B_L^+ = B_L(0) \cap \{y = (y_1, y_2) \mid y_2 > 0\}$. Define

$$m_l(y) = \frac{1}{\sqrt{\pi}} \begin{cases} (\log L/l)^{1/2}, & 0 \leq |y| \leq l, y \in B_L^+, \\ \frac{(\log L/|y|)}{(\log L/l)^{1/2}}, & l \leq |y| \leq L, y \in B_L^+, \\ 0, & L \leq |y|, y \in B_L^+. \end{cases}$$

Then $\|\nabla m_l\|_{L^2(B_L^+)} = 1$ and since $m_l \equiv 0$ on $\partial B_L^+ \cap \{y_2 > 0\}$, we have

$$\begin{aligned} \|m_l\|_{L^{p+1}(\partial B_L^+)}^{p+1} &= 2 \int_0^l |m_l(y_1)|^{p+1} dy_1 + 2 \int_l^L |m_l(y_1)|^{p+1} dy_1 \\ &\geq 2 \int_0^l \left(\frac{1}{\sqrt{\pi}} \sqrt{\log(L/l)} \right)^{p+1} dy_1 = 2l \left(\sqrt{\frac{1}{\pi} \log(L/l)} \right)^{p+1}. \end{aligned}$$

Thus $\|m_l\|_{L^{p+1}(\partial B_L^+)}^2 \geq (2l)^{\frac{2}{p+1}} \frac{1}{\pi} \log(L/l)$. Also,

$$\|m_l\|_{L^2(B_L^+)}^2 = \int_0^\pi \int_0^L |m_l|^2 r dr d\theta = \int_0^\pi \int_0^l |m_l|^2 r dr d\theta + \int_0^\pi \int_l^L |m_l|^2 r dr d\theta =: I_1 + I_2.$$

We calculate

$$I_1 = \frac{l^2}{2} \log(L/l), \quad I_2 = \frac{1}{\log(L/l)} \int_l^L (\log L/r)^2 r dr = -\frac{l^2}{2} - \frac{l^2}{2} \log(L/l) + \frac{1}{\log(L/l)} \frac{L^2 - l^2}{4}.$$

Thus we have $\|m_l\|_{L^2(B_L^+)}^2 = -\frac{l^2}{2} + \frac{1}{\log(L/l)} \frac{L^2 - l^2}{4}$.

Now, put $l = Le^{-\frac{p+1}{2}}$ and extend m_l by 0 outside B_L^+ and consider it as a function in $H^1(\Omega)$. Then

$$pC_p^2 \leq p \frac{\|m_l\|_{H^1(B_L^+)}^2}{\|m_l\|_{L^{p+1}(\partial B_L^+)}^2} = \frac{p}{\|m_l\|_{L^{p+1}(\partial B_L^+)}^2} + \frac{p\|m_l\|_{L^2(B_L^+)}^2}{\|m_l\|_{L^{p+1}(\partial B_L^+)}^2}.$$

We estimate

$$\frac{p}{\|m_l\|_{L^{p+1}(\partial B_L^+)}^2} \leq \frac{p}{(2l)^{\frac{2}{p+1}} \frac{1}{\pi} \log(L/l)} = \left(\frac{p}{p+1} \right) 2\pi e \frac{1}{(2L)^{\frac{2}{p+1}}} \rightarrow 2\pi e,$$

and

$$\frac{p\|m_l\|_{L^2(B_L^+)}^2}{\|m_l\|_{L^{p+1}(\partial B_L^+)}^2} \leq \frac{p \left(-\frac{l^2}{2} + \frac{1}{\log(L/l)} \frac{L^2 - l^2}{4} \right)}{(2l)^{\frac{2}{p+1}} \frac{1}{\pi} \log(L/l)} = \frac{2\pi e}{(2L)^{\frac{2}{p+1}}} \left(\frac{p}{p+1} \right) \left\{ -\frac{L^2}{2} e^{-(p+1)} + \frac{2}{p+1} \frac{L^2(1 - e^{-(p+1)})}{4} \right\} \rightarrow 0$$

as $p \rightarrow \infty$. Therefore, we have obtained $\limsup_{p \rightarrow \infty} pC_p^2 \leq 2\pi e$ in this case.

In the general case, we introduce a diffeomorphism which flattens the boundary $\partial\Omega$, see Ni and Takagi [12]. We may assume $0 \in \partial\Omega$ and in a neighborhood U of 0 , the boundary $\partial\Omega$ can be written by the graph of function $\psi: \partial\Omega \cap U = \{x = (x_1, x_2) \mid x_2 = \psi(x_1)\}$, with $\psi(0) = 0$ and $\frac{\partial\psi}{\partial x_1}(0) = 0$. Define $x = \Phi(y) = (\Phi_1(y), \Phi_2(y))$ for $y = (y_1, y_2)$, where

$$x_1 = \Phi_1(y) = y_1 - y_2 \frac{\partial\psi}{\partial x_1}(y_1), \quad x_2 = \Phi_2(y) = y_2 + \psi(y_1),$$

and put $D_L = \Phi(B_L^+)$. Note that $\partial D_L \cap \partial\Omega = \Phi(\partial B_L^+ \cap \{(y_1, 0)\})$. Since $D\Phi(0) = Id$, we obtain there exists $\Psi = \Phi^{-1}$ in a neighborhood of 0 . Finally, define $\tilde{m}_l \in H^1(\Omega)$ as $\tilde{m}_l(x) = m_l(\Psi(x))$ for $x \in U \cap \Omega$. Then, Lemma A.1 in [12] implies the estimates

$$\begin{aligned} \|\nabla \tilde{m}_l\|_{L^2(D_L)}^2 &= \|\nabla m_l\|_{L^2(B_L^+)}^2 + O\left(\frac{1}{p}\right), & \|\tilde{m}_l\|_{L^2(D_L)}^2 &\leq (1 + O(L)) \|m_l\|_{L^2(B_L^+)}^2, \\ \|\tilde{m}_l\|_{L^{p+1}(\partial D_L \cap \partial\Omega)}^2 &\geq \|m_l\|_{L^{p+1}(\partial B_L^+ \cap \{(y_1, 0)\})}^2. \end{aligned}$$

The last inequality comes from that, if we put $I = \{(y_1, 0) \mid -L \leq y_1 \leq L\} \subset \partial B_L^+$ and $J = \Phi(I) \subset \partial\Omega$, then $ds_x = \sqrt{1 + (\psi'(x_1))^2} dx_1$ and $J = \{(x_1, x_2) \mid x_1 = y_1, x_2 = \psi(y_1)\}$. Thus

$$\int_J |\tilde{m}_l(x)|^{p+1} ds_x = \int_I |m_l(y)|^{p+1} \sqrt{1 + (\psi'(y_1))^2} dy_1 \geq \int_I |m_l(y)|^{p+1} dy_1.$$

By testing C_p^2 with \tilde{m}_l , again we obtain $\limsup_{p \rightarrow \infty} pC_p^2 \leq 2\pi e$. \square

Corollary 5. Let u_p be a least energy solution to (E_p) . Then we have

$$\lim_{p \rightarrow \infty} p \int_{\partial\Omega} u_p^{p+1} ds_x = 2\pi e, \quad \lim_{p \rightarrow \infty} p \int_{\Omega} (|\nabla u_p|^2 + u_p^2) dx = 2\pi e.$$

Proof. Since u_p satisfies

$$\int_{\Omega} (|\nabla u_p|^2 + u_p^2) dx = \int_{\partial\Omega} u_p^{p+1} ds_x$$

and

$$pC_p^2 = p \frac{\int_{\Omega} (|\nabla u_p|^2 + u_p^2) dx}{\left(\int_{\partial\Omega} u_p^{p+1} ds_x\right)^{\frac{2}{p+1}}} = \left(p \int_{\partial\Omega} u_p^{p+1} ds_x\right)^{\frac{p-1}{p+1}} p^{\frac{2}{p+1}},$$

the results follow from Lemma 4. \square

3. Proof of Theorem 1

The uniform estimate of $\|u\|_{L^\infty(\partial\Omega)}$ from below holds true for any solution u of (E_p) , as in [13].

Lemma 6. There exists $C_1 > 0$ independent of p such that

$$\|u\|_{L^\infty(\partial\Omega)} \geq C_1$$

holds true for any solution u to (E_p) .

Proof. Let $\lambda_1 > 0$ be the first eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta\varphi + \varphi = 0 & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = \lambda\varphi & \text{on } \partial\Omega \end{cases}$$

and let φ_1 be the corresponding eigenfunction. It is known that λ_1 is simple, isolated, and φ_1 can be chosen positive on $\bar{\Omega}$. (See [15].) Then by integration by parts, we have

$$0 = \int_{\Omega} \{(-\Delta u + u)\varphi_1 - (-\Delta\varphi_1 + \varphi_1)u\} dx = \int_{\partial\Omega} \left(\frac{\partial\varphi_1}{\partial\nu}u - \frac{\partial u}{\partial\nu}\varphi_1\right) ds_x = \int_{\partial\Omega} \varphi_1 u (\lambda_1 - u^{p-1}) ds_x.$$

Since $\varphi_1 u > 0$ on $\partial\Omega$, this implies $\|u\|_{L^\infty(\partial\Omega)}^{p-1} \geq \lambda_1$. \square

Lemma 7. Let u_p be a least energy solution to (E_p) . Then it holds

$$\limsup_{p \rightarrow \infty} \|u_p\|_{L^\infty(\partial\Omega)} \leq \sqrt{e}.$$

Proof. We follow the argument of [14], which in turn originates from [10], and use Moser’s iteration procedure. Let u be a solution to (E_p) . For $s \geq 1$, multiplying $u^{2s-1} \in H^1(\Omega)$ to the equation of (E_p) and integrating, we get

$$\left(\frac{2s-1}{s^2}\right)^2 \int_{\Omega} |\nabla(u^s)|^2 dx + \int_{\Omega} u^{2s} dx = \int_{\partial\Omega} u^{2s-1+p} ds_x.$$

Since $\frac{2s-1}{s^2} \leq 1$ for $s \geq 1$, we have

$$\left(\frac{2s-1}{s^2}\right) \|u^s\|_{H^1(\Omega)}^2 \leq \int_{\partial\Omega} u^{2s-1+p} ds_x. \tag{3.1}$$

Also by Lemma 3 applied to $u^s \in H^1(\Omega)$, we have

$$\left(\int_{\partial\Omega} u^{vs} ds_x\right)^{1/v} \leq \tilde{D}_v v^{\frac{1}{2}} \|u^s\|_{H^1(\Omega)}$$

for any $v \geq 2$. Thus by (3.1), we see

$$\left(\int_{\partial\Omega} u^{vs} ds_x\right)^{1/v} \leq \tilde{D}_v v^{\frac{1}{2}} \left(\frac{s^2}{2s-1}\right)^{1/2} \left(\int_{\partial\Omega} u^{2s-1+p} ds_x\right)^{1/2}.$$

Since $\tilde{D}_v^2 \left(\frac{s}{2s-1}\right) \leq C_1$ for some $C_1 > 0$ independent of $s \geq 1$ and $v \geq 2$, we obtain

$$\left(\int_{\partial\Omega} u^{vs} ds_x\right)^{2/v} \leq C_1 v s \int_{\partial\Omega} u^{2s-1+p} ds_x. \tag{3.2}$$

Once the iteration scheme (3.2) is obtained, the rest of the argument is exactly the same as one in [14]. Indeed, by Lemma 3, we have

$$\left(\int_{\partial\Omega} u^v ds_x\right)^{1/v} \leq (2\pi e)^{-\frac{1}{2}} (1 + o(1)) v^{1/2} \|u\|_{H^1(\Omega)}, \tag{3.3}$$

here $o(1) \rightarrow 0$ as $v \rightarrow \infty$. Now, we fix $\alpha > 0$ and $\varepsilon > 0$ which will be chosen small later and put $v = (1 + \alpha)(p + 1) > 2$ in (3.3). By Corollary 5, $p^{1/2} (2\pi e)^{-1/2} \|u_p\|_{H^1(\Omega)} \rightarrow 1$ as $p \rightarrow \infty$ for a least energy solution u_p . Thus by (3.3), we see there exists $p_0 > 1$ such that

$$\int_{\partial\Omega} u_p^v ds_x \leq (1 + \alpha + \varepsilon)^{v/2} =: M_0$$

for $p > p_0$. Define $\{s_j\}_{j=0,1,2,\dots}$ and $\{M_j\}_{j=0,1,2,\dots}$ such that

$$\begin{cases} p - 1 + 2s_0 = v, \\ p - 1 + 2s_{j+1} = v s_j \quad (j = 0, 1, 2, \dots), \end{cases}$$

and

$$\begin{cases} M_0 = (1 + \alpha + \varepsilon)^{v/2}, \\ M_{j+1} = (C_1 v s_j M_j)^{v/2} \quad (j = 0, 1, 2, \dots). \end{cases}$$

We easily see that $s_0 = \frac{\alpha(p+1)}{2} > 0$, s_j is increasing in j , $s_j \rightarrow +\infty$ as $j \rightarrow \infty$, and actually,

$$s_j = \left(\frac{v}{2}\right)^j (s_0 - x) + x \quad \text{where } x = \frac{p-1}{v-2} > 0.$$

At this moment, we can follow exactly the same argument in [14] to obtain the estimates

$$\|u_p\|_{L^{vs_{j-1}}(\partial\Omega)} \leq M_j^{\frac{1}{vs_{j-1}}} \leq \exp(m(\alpha, p, \varepsilon)),$$

where $m(\alpha, p, \varepsilon)$ is a constant depending on α, p and ε , satisfying

$$\lim_{p \rightarrow \infty} m(\alpha, p, \varepsilon) = \frac{1 + \alpha}{2\alpha} \log(1 + \alpha + \varepsilon).$$

Letting $j \rightarrow \infty, p \rightarrow \infty$ first, we get

$$\limsup_{p \rightarrow \infty} \|u_p\|_{L^\infty(\partial\Omega)} \leq (1 + \alpha + \varepsilon)^{\frac{1+\alpha}{2\alpha}},$$

and then letting $\alpha \rightarrow +0, \varepsilon \rightarrow +0$, we obtain

$$\limsup_{p \rightarrow \infty} \|u_p\|_{L^\infty(\partial\Omega)} \leq \sqrt{e}$$

as desired. \square

By [Theorem 1](#) and Hölder’s inequality, we also obtain

Corollary 8. *There exists $C_1, C_2 > 0$ such that*

$$C_1 \leq p \int_{\partial\Omega} u_p^p ds_x \leq C_2$$

holds.

4. Proof of [Theorem 2](#)

In this section, we prove [Theorem 2](#). First, we recall an L^1 estimate from [\[6\]](#), which is a variant of the one by Brezis and Merle [\[2\]](#).

Lemma 9. *Let u be a solution to*

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = h & \text{on } \partial\Omega \end{cases}$$

with $h \in L^1(\partial\Omega)$, where Ω is a smooth bounded domain in \mathbb{R}^2 . For any $\varepsilon \in (0, \pi)$, there exists a constant $C > 0$ depending only on ε and Ω , independent of u and h , such that

$$\int_{\partial\Omega} \exp\left(\frac{(\pi - \varepsilon)|u(x)|}{\|h\|_{L^1(\partial\Omega)}}\right) ds_x \leq C \tag{4.1}$$

holds true.

Also we need an elliptic L^1 estimate by Brezis and Strauss [\[3\]](#) for weak solutions with the L^1 Neumann data.

Lemma 10. *Let u be a weak solution of*

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

with $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$, where Ω is a smooth bounded domain in $\mathbb{R}^N, N \geq 2$. Then we have $u \in W^{1,q}(\Omega)$ for all $1 \leq q < \frac{N}{N-1}$ and

$$\|u\|_{W^{1,q}(\Omega)} \leq C_q(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)})$$

holds.

For the proof, see [\[3, Lemma 23\]](#).

Now, following [\[13,14\]](#), we define the notion of δ -regular points. Put $u_n = u_{p_n}$ for any subsequence of u_p . Since u_n satisfies

$$\int_{\partial\Omega} \frac{u_n^{p_n}}{\int_{\partial\Omega} u_n^{p_n} ds_x} ds_x = 1,$$

we can select a subsequence $p_n \rightarrow \infty$ (without changing the notation) and a Radon measure $\mu \geq 0$ on $\partial\Omega$ such that

$$f_n := \frac{u_n^{p_n}}{\int_{\partial\Omega} u_n^{p_n} ds_x} \xrightarrow{*} \mu$$

weakly in the sense of Radon measures on $\partial\Omega$, i.e.,

$$\int_{\partial\Omega} f_n \varphi ds_x \rightarrow \int_{\partial\Omega} \varphi d\mu$$

for all $\varphi \in C(\partial\Omega)$. As in [14], we define

$$L_0 = \frac{1}{2\sqrt{e}} \limsup_{p \rightarrow \infty} \left(p \int_{\partial\Omega} u_p^p ds_x \right). \tag{4.2}$$

By Corollary 5 and Hölder’s inequality, we have

$$L_0 \leq \pi \sqrt{e}.$$

For some $\delta > 0$ fixed, we call a point $x_0 \in \partial\Omega$ a δ -regular point if there is a function $\varphi \in C(\partial\Omega)$, $0 \leq \varphi \leq 1$ with $\varphi = 1$ in a neighborhood of x_0 such that

$$\int_{\partial\Omega} \varphi d\mu < \frac{\pi}{L_0 + 2\delta}$$

holds. Define $S = \{x_0 \in \partial\Omega \mid x_0 \text{ is not a } \delta\text{-regular point for any } \delta > 0\}$. Then,

$$\mu(\{x_0\}) \geq \frac{\pi}{L_0 + 2\delta} \tag{4.3}$$

for all $x_0 \in S$ and for any $\delta > 0$.

Here, following the argument in [14], we prove a key lemma in the proof of Theorem 2.

Lemma 11. *Let $x_0 \in \partial\Omega$ be a δ -regular point for some $\delta > 0$. Then $v_n = \frac{u_n}{\int_{\partial\Omega} u_n^{p_n} ds_x}$ is bounded in $L^\infty(B_{R_0}(x_0) \cap \Omega)$ for some $R_0 > 0$.*

Proof. Let $x_0 \in \partial\Omega$ be a δ -regular point. Then by definition, there exists $R > 0$ such that

$$\int_{\partial\Omega \cap B_R(x_0)} f_n ds_x < \frac{\pi}{L_0 + \delta}$$

holds for all n large. Put $a_n = \chi_{B_R(x_0)} f_n$ and $b_n = (1 - \chi_{B_R(x_0)}) f_n$ where $\chi_{B_R(x_0)}$ denotes the characteristic function of $B_R(x_0)$. Split $v_n = v_{1n} + v_{2n}$, where v_{1n}, v_{2n} is a solution to

$$\begin{cases} -\Delta v_{1n} + v_{1n} = 0 & \text{in } \Omega, \\ \frac{\partial v_{1n}}{\partial \nu} = a_n & \text{on } \partial\Omega, \end{cases} \quad \begin{cases} -\Delta v_{2n} + v_{2n} = 0 & \text{in } \Omega, \\ \frac{\partial v_{2n}}{\partial \nu} = b_n & \text{on } \partial\Omega \end{cases}$$

respectively. By the maximum principle, we have $v_{1n}, v_{2n} > 0$. Since $b_n = 0$ on $B_R(x_0)$, elliptic estimates imply that

$$\|v_{2n}\|_{L^\infty(B_{R/2}(x_0) \cap \Omega)} \leq C \|v_{2n}\|_{L^1(B_R(x_0) \cap \Omega)} \leq C,$$

where we used the fact $\|v_{2n}\|_{L^1(\Omega)} = \|\Delta v_{2n}\|_{L^1(\Omega)} = \|b_n\|_{L^1(\partial\Omega)} \leq C$ for the last inequality. Thus we have to consider v_{1n} only.

Claim. *For any $x \in \partial\Omega$, we have*

$$f_n(x) \leq \exp((L_0 + \delta/2)v_n(x)) \tag{4.4}$$

for n large.

Indeed, put

$$\alpha_n = \frac{\|u_n\|_{L^\infty(\partial\Omega)}}{(\int_{\partial\Omega} u_n^{p_n} ds_x)^{1/p_n}}.$$

Then by Lemma 7 and Corollary 8, we have

$$\limsup_{n \rightarrow \infty} \alpha_n \leq \sqrt{e}.$$

Since the function $s \mapsto \frac{\log s}{s}$ is monotone increasing if $0 < s < e$, and $\frac{u_n(x)}{(\int_{\partial\Omega} u_n^{p_n} ds_x)^{1/p_n}} \leq \alpha_n$ for any $x \in \partial\Omega$, we observe that for fixed $\varepsilon > 0$,

$$\frac{\log \frac{u_n(x)}{(\int_{\partial\Omega} u_n^{p_n} ds_x)^{1/p_n}}}{\frac{u_n(x)}{(\int_{\partial\Omega} u_n^{p_n} ds_x)^{1/p_n}}} \leq \frac{\log \alpha_n}{\alpha_n} \leq \frac{1}{2\sqrt{e}} + \varepsilon$$

holds for large n . Thus

$$\begin{aligned} f_n(x) &= \exp\left(p_n \log \frac{u_n(x)}{(\int_{\partial\Omega} u_n^{p_n} ds_x)^{1/p_n}}\right) \leq \exp\left(\frac{p_n u_n(x)}{(\int_{\partial\Omega} u_n^{p_n} ds_x)^{1/p_n}} \left(\frac{1}{2\sqrt{e}} + \varepsilon\right)\right) \\ &= \exp\left(p_n v_n(x) \left(\int_{\partial\Omega} u_n^{p_n} ds_x\right)^{1-1/p_n} \left(\frac{1}{2\sqrt{e}} + \varepsilon\right)\right) \\ &\leq \exp\left(\left(\limsup_{n \rightarrow \infty} p_n \int_{\partial\Omega} u_n^{p_n} ds_x\right) v_n(x) \left(\frac{1}{2\sqrt{e}} + 2\varepsilon\right)\right) \\ &= \exp\left(\left(\frac{1}{2\sqrt{e}} + 2\varepsilon\right) 2\sqrt{e} L_0 v_n(x)\right) = \exp((L_0 + 4\varepsilon\sqrt{e}L_0)v_n(x)). \end{aligned}$$

Thus if we choose $\varepsilon > 0$ so small, we have the claim (4.4).

By this claim and the fact that v_{2n} is uniformly bounded in $B_{R/2}(x_0)$, for sufficiently small $\delta_0 > 0$ so that $(1 + \delta_0) \frac{L_0 + \delta/2}{L_0 + \delta} < 1$, we have

$$\begin{aligned} \int_{B_{R/2}(x_0) \cap \partial\Omega} f_n^{1+\delta_0} ds_x &\leq \int_{B_{R/2}(x_0) \cap \partial\Omega} \exp((1 + \delta_0)(L_0 + \delta/2)v_n(x)) ds_x \\ &\leq C \int_{B_{R/2}(x_0) \cap \partial\Omega} \exp((1 + \delta_0)(L_0 + \delta/2)v_{1n}(x)) ds_x \\ &\leq C \int_{B_{R/2}(x_0) \cap \partial\Omega} \exp\left(\pi(1 + \delta_0) \frac{L_0 + \delta/2}{L_0 + \delta} v_{1n}(x)\right) ds_x \\ &= C \int_{B_{R/2}(x_0) \cap \partial\Omega} \exp(\pi(1 - \varepsilon_0)v_{1n}(x)) ds_x, \end{aligned}$$

where $1 - \varepsilon_0 = (1 + \delta_0) \frac{L_0 + \delta/2}{L_0 + \delta}$. Thus by Lemma 9, we have

$$\int_{B_{R/2}(x_0) \cap \partial\Omega} f_n^{1+\delta_0} ds_x \leq C$$

for some $C > 0$ independent of n . This fact and elliptic estimates imply that

$$\limsup_{n \rightarrow \infty} \|v_n\|_{L^\infty(\Omega \cap B_{R/4}(x_0))} \leq C,$$

which proves lemma. \square

Now, we estimate the cardinality of the set S . By Theorem 1, we have

$$v_n(x_n) = \frac{\|u_n\|_{L^\infty(\partial\Omega)}}{\int_{\partial\Omega} u_n^{p_n} ds_x} \geq \frac{C_1}{\int_{\partial\Omega} u_n^{p_n} ds_x} \rightarrow \infty$$

for a sequence $x_n \in \partial\Omega$ such that $u_n(x_n) = \|u_n\|_{L^\infty(\partial\Omega)}$. Thus by Lemma 11, we see $x_0 = \lim_{n \rightarrow \infty} x_n \in S$ and $\#S \geq 1$. On the other hand, by (4.3) we have

$$1 = \lim_{n \rightarrow \infty} \|f_n\|_{L^1(\partial\Omega)} \geq \mu(\partial\Omega) \geq \frac{\pi}{L_0 + 2\delta} \#S,$$

which leads to

$$1 \leq \#S \leq \frac{L_0 + 2\delta}{\pi} \leq \sqrt{e} + \frac{2\delta}{\pi} \simeq 1.64\dots + \frac{2\delta}{\pi}.$$

Thus we have $\#S = 1$ if $\delta > 0$ is chosen small.

Let $S = \{x_0\}$ for some point $x_0 \in \partial\Omega$. By Lemma 11, we can conclude easily that $f_n \xrightarrow{*} \delta_{x_0}$ in the sense of Radon measures on $\partial\Omega$:

$$\int_{\partial\Omega} f_n \varphi ds_x \rightarrow \varphi(x_0), \quad \text{as } n \rightarrow \infty$$

for any $\varphi \in C(\partial\Omega)$, since v_n is locally uniformly bounded on $\partial\Omega \setminus \{x_0\}$ and $f_n \rightarrow 0$ uniformly on any compact sets of $\partial\Omega \setminus \{x_0\}$.

Now, by the L^1 estimate in Lemma 10, we have v_n is uniformly bounded in $W^{1,q}(\Omega)$ for any $1 \leq q < 2$. Thus, by choosing a subsequence, we have a function \bar{G} such that $v_n \rightharpoonup \bar{G}$ weakly in $W^{1,q}(\Omega)$ for any $1 \leq q < 2$, $v_n \rightarrow \bar{G}$ strongly in $L^t(\Omega)$ and $L^t(\partial\Omega)$ respectively for any $1 \leq t < \infty$. The last convergence follows by the compact embedding $W^{1,q}(\Omega) \hookrightarrow L^t(\Omega)$ for any $1 \leq t < \frac{q}{2-q}$. Thus by taking the limit in the equation

$$\int_{\Omega} (-\Delta\varphi + \varphi)v_n dx = \int_{\partial\Omega} f_n \varphi ds_x - \int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu} v_n ds_x$$

for any $\varphi \in C^1(\bar{\Omega})$, we obtain

$$\int_{\Omega} (-\Delta\varphi + \varphi)\bar{G} dx + \int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu} \bar{G} ds_x = \varphi(x_0),$$

which implies \bar{G} is the solution of (1.4) with $y = x_0$.

Finally, we prove the statement (3) of Theorem 2. We borrow the idea of [6] and derive Pohozaev-type identities in balls around the peak point. We may assume $x_0 = 0$ without loss of generality. As in [6], we use a conformal diffeomorphism $\Psi : H \cap B_{R_0} \rightarrow \Omega \cap B_r$ which flattens the boundary $\partial\Omega$, where $H = \{(y_1, y_2) \mid y_2 > 0\}$ denotes the upper half space and $R_0 > 0$ is a radius sufficiently small. We may choose Ψ is at least C^3 , up to $\partial H \cap B_{R_0}$, $\Psi(0) = 0$ and $D\Psi(0) = Id$. Set $\tilde{u}_n(y) = u_n(\Psi(y))$ for $y = (y_1, y_2) \in H \cap B_{R_0}$. Then by the conformality of Ψ , \tilde{u}_n satisfies

$$\begin{cases} -\Delta\tilde{u}_n + b(y)\tilde{u}_n = 0 & \text{in } H \cap B_{R_0}, \\ \frac{\partial\tilde{u}_n}{\partial\tilde{\nu}} = h(y)\tilde{u}_n^{p_n} & \text{on } \partial H \cap B_{R_0}, \end{cases} \tag{4.5}$$

where $\tilde{\nu}$ is the unit outer normal vector to $\partial(H \cap B_{R_0})$, b and h are defined

$$b(y) = |\det D\Psi(y)|, \quad h(y) = |D\Psi(y)e|$$

with $e = (0, -1)$. Note that $\tilde{\nu}(y) = \nu(\Psi(y))$ for $y \in \partial H \cap B_{R_0}$. Note also that, by using a clever idea of [6], we can modify Ψ to prescribe the number

$$\alpha = \left. \frac{(\frac{\partial h}{\partial y_1})}{h(y)^2} \right|_{y=0} = \left(\frac{\partial h}{\partial y_1} \right)(0).$$

Let $D \subset \mathbb{R}^N$ be a bounded domain and recall the Pohozaev identity for the equation $-\Delta u = f(y, u)$, $y \in D$:

$$\begin{aligned} N \int_D F(y, u) dy - \left(\frac{N-2}{2} \right) \int_D |\nabla u|^2 dy + \int_D (y - y_0, \nabla_y F(y, u)) dy \\ = \int_{\partial D} (y - y_0, \nu) F(y, u) ds_y + \int_{\partial D} (y - y_0, \nabla u) \left(\frac{\partial u}{\partial \nu} \right) ds_y - \frac{1}{2} \int_{\partial D} (y - y_0, \nu) |\nabla u|^2 ds_y \end{aligned}$$

for any $y_0 \in \mathbb{R}^N$, where u is a smooth solution. Applying this to (4.5) for $N = 2$, $D = H \cap B_R$ for $0 < R < R_0$, $f(y, \tilde{u}_n) = -b(y)\tilde{u}_n$ and $F(y, \tilde{u}_n) = -\frac{b(y)}{2}\tilde{u}_n^2$, we obtain

$$\begin{aligned} & \int_{H \cap B_R} b(y)\tilde{u}_n^2(y) dy + \int_{H \cap B_R} (y - y_0, \nabla b(y)) \frac{1}{2}\tilde{u}_n^2(y) dy \\ &= \int_{\partial(H \cap B_R)} (y - y_0, \tilde{\nu}) \frac{1}{2}b(y)\tilde{u}_n^2(y) ds_y - \int_{\partial(H \cap B_R)} (y - y_0, \nabla \tilde{u}_n(y)) \left(\frac{\partial \tilde{u}_n}{\partial \tilde{\nu}} \right) ds_y + \frac{1}{2} \int_{\partial(H \cap B_R)} (y - y_0, \tilde{\nu}) |\nabla \tilde{u}_n|^2 ds_y, \end{aligned}$$

where and from now on, $\tilde{\nu}$ will be used again to denote the unit normal to $\partial(H \cap B_R)$. Differentiating with respect to y_0 , we have, in turn,

$$\int_{\partial(H \cap B_R)} \nabla \tilde{u}_n(y) \left(\frac{\partial \tilde{u}_n}{\partial \tilde{\nu}} \right) ds_y = \frac{1}{2} \int_{\partial(H \cap B_R)} (|\nabla \tilde{u}_n|^2 + b(y)\tilde{u}_n^2) \tilde{\nu} ds_y - \frac{1}{2} \int_{H \cap B_R} \nabla b(y)\tilde{u}_n^2(y) dy.$$

Since $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2) = (0, -1)$ on $\partial H \cap B_R$, the first component of the above vector equation reads

$$\begin{aligned} & \int_{\partial H \cap B_R} (\tilde{u}_n)_{y_1} h(y) \tilde{u}_n^{p_n}(y) ds_y + \int_{H \cap \partial B_R} (\tilde{u}_n)_{y_1}(y) \left(\frac{\partial \tilde{u}_n}{\partial \tilde{\nu}} \right) ds_y \\ &= \frac{1}{2} \int_{H \cap \partial B_R} (|\nabla \tilde{u}_n|^2 + b(y)\tilde{u}_n^2) \tilde{\nu}_1 ds_y - \frac{1}{2} \int_{H \cap B_R} b_{y_1}(y) \tilde{u}_n^2(y) dy, \end{aligned} \tag{4.6}$$

where $(\)_{y_1}$ denotes the derivative with respect to y_1 . Let $\gamma_n = \int_{\partial \Omega} u_n^{p_n} ds_x$. From the fact that $\tilde{f}_n(y) = \frac{\tilde{u}_n^{p_n}}{\gamma_n} \xrightarrow{*} \delta_0$ in the sense of Radon measures on $\partial H \cap B_R$, Corollary 8 and $\|\tilde{u}_n\|_{L^\infty(\partial H \cap B_R)} = O(1)$ uniformly in n , we see

$$\tilde{g}_n(y) = \frac{1}{\gamma_n^2} \frac{\tilde{u}_n^{p_n+1}(y)}{p_n+1} = \frac{1}{(p_n+1)\gamma_n} \tilde{f}_n(y) \tilde{u}_n(y)$$

satisfies that $\text{supp}(\tilde{g}_n) \rightarrow \{0\}$ and $\int_{\partial H \cap B_R} \tilde{g}_n ds_y = O(1)$ as $n \rightarrow \infty$. Thus, by choosing a subsequence, we have the convergence

$$\tilde{g}_n(y) = \frac{1}{\gamma_n^2} \frac{\tilde{u}_n^{p_n+1}(y)}{p_n+1} \xrightarrow{*} C_0 \delta_0$$

in the sense of Radon measures on $\partial H \cap B_R$, where $C_0 = \lim_{n \rightarrow \infty} \int_{\partial H \cap B_R} \tilde{g}_n ds_y$ (up to a subsequence). By using this fact, we have

$$\frac{1}{\gamma_n^2} \int_{\partial H \cap B_R} (\tilde{u}_n)_{y_1} h(y) \tilde{u}_n^{p_n}(y) ds_y = \left[\frac{h(y)}{\gamma_n^2} \frac{\tilde{u}_n^{p_n+1}}{p_n+1} \right]_{y_1=-R}^{y_1=R} - \int_{\partial H \cap B_R} h_{y_1}(y) \frac{\tilde{u}_n^{p_n+1}(y)}{(p_n+1)\gamma_n^2} ds_y \rightarrow 0 - C_0 h_{y_1}(0) = -C_0 \alpha$$

as $n \rightarrow \infty$. Thus after dividing (4.6) by γ_n^2 and then letting $n \rightarrow \infty$, we obtain

$$-C_0 \alpha + \int_{H \cap \partial B_R} \tilde{G}_{y_1}(y) \left(\frac{\partial \tilde{G}}{\partial \tilde{\nu}} \right) ds_y = \frac{1}{2} \int_{H \cap \partial B_R} (|\nabla \tilde{G}|^2 + b(y)\tilde{G}^2) \tilde{\nu}_1 ds_y - \frac{1}{2} \int_{H \cap B_R} b_{y_1}(y) \tilde{G}^2(y) dy, \tag{4.7}$$

where $\tilde{G}(y) = G(\Psi(y), 0)$ is a limit function of $\tilde{v}_n(y) = v_n(\Psi(y)) = \frac{\tilde{u}_n(y)}{\gamma_n}$. At this point, we have the same formula as Eq. (117) in [6], thus we obtain the result. Indeed, decompose $G(x, 0) = s(x) + w(x)$ where

$$s(x) = \frac{1}{\pi} \log|x|^{-1}, \quad w(x) = H(x, 0),$$

and put $\tilde{s}(y) = s(\Psi(y))$, $\tilde{w}(y) = H(\Psi(y), 0)$ so that $\tilde{G} = \tilde{s} + \tilde{w}$. Then after some computation using the fact that \tilde{w} satisfies

$$-\Delta \tilde{w} + b(y)\tilde{w} = -b(y)\tilde{s}(y) \quad \text{in } H \cap B_R,$$

we have from (4.7) that

$$\begin{aligned}
& -C_0\alpha + \int_{H\cap\partial B_R} (\tilde{s}_{\tilde{v}}\tilde{s}_{y_1} + \tilde{s}_{\tilde{v}}\tilde{w}_{y_1} + \tilde{s}_{y_1}\tilde{w}_{\tilde{v}}) ds_y \\
& = \int_{H\cap\partial B_R} \left(\frac{1}{2}|\nabla\tilde{s}|^2 + \nabla\tilde{s} \cdot \nabla\tilde{w}\right)\tilde{v}_1 ds_y + \int_{H\cap\partial B_R} \left(\frac{1}{2}\tilde{s}^2 + \tilde{s}\tilde{w}\right)b(y)\tilde{v}_1 ds_y \\
& \quad - \int_{\partial H\cap B_R} b_{y_1}(y)\left(\frac{1}{2}\tilde{s}^2 + \tilde{s}\tilde{w}\right) ds_y + \int_{\partial H\cap B_R} \tilde{w}_{\tilde{v}}\tilde{w}_{y_1} ds_y - \int_{H\cap B_R} b(y)\tilde{s}(y)\tilde{w}_{y_1} dy.
\end{aligned} \tag{4.8}$$

By Lemma 9.3 in [6], we know estimates

$$\begin{aligned}
\lim_{R\rightarrow 0} \int_{H\cap\partial B_R} \tilde{s}_{\tilde{v}}\tilde{s}_{y_1} ds_x &= \frac{3\alpha}{4\pi}, & \lim_{R\rightarrow 0} \int_{H\cap\partial B_R} \tilde{s}_{\tilde{v}}\tilde{w}_{y_1} ds_x &= -\tilde{w}_{y_1}(0), \\
\lim_{R\rightarrow 0} \frac{1}{2} \int_{H\cap\partial B_R} |\nabla\tilde{s}|^2\tilde{v}_1 ds_x &= \frac{\alpha}{4\pi}, & \lim_{R\rightarrow 0} \int_{H\cap\partial B_R} \nabla\tilde{s} \cdot \nabla\tilde{w}\tilde{v}_1 ds_x &= -\frac{1}{2}\tilde{w}_{y_1}(0)
\end{aligned}$$

and other terms in (4.8) go to 0 as $R \rightarrow 0$. Thus we take the limit in (4.8) as $R \rightarrow 0$ to obtain the relation

$$-C_0\alpha + \frac{3\alpha}{4\pi} - \tilde{w}_{y_1}(0) = \frac{\alpha}{4\pi} - \frac{1}{2}\tilde{w}_{y_1}(0),$$

which leads to

$$\alpha\left(\frac{1}{2\pi} - C_0\right) = \frac{1}{2}\tilde{w}_{y_1}(0).$$

Since $\alpha \in \mathbb{R}$ can be chosen arbitrary, we conclude that $C_0 = \frac{1}{2\pi}$ and $\tilde{w}_{y_1}(0) = 0$. This last equation means the desired conclusion of Theorem 2(3). \square

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