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On dually flat (α, β)-metrics ☆

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ABSTRACT

The dual flatness for Riemannian metrics in information geometry has been extended to Finsler metrics. The aim of this paper is to study the dual flatness of the so-called (α, β)-metrics in Finsler geometry. By doing some special deformations, we will show that the dual flatness of an (α, β)-metric always arises from that of some Riemannian metric in dimensional $n \geqslant 3$.

1. Introduction

Dual flatness is a basic notion in information geometry. It was first proposed by S.-I. Amari and H. Nagaoka when they studied the information geometry on Riemannian spaces [2]. Information geometry has been emerged from investigating the geometrical structure of a family of probability distributions, and has been applied successfully to various areas including statistical inference, control system theory and multiterminal information theory [1,2].

In 2007, Z. Shen extended the dual flatness in Finsler geometry [11]. A Finsler metric F on a manifold M is said to be *locally dually flat* if at any point there is a local coordinate system (x^i) in which $F = F(x, y)$ satisfies the following PDEs

$$[F^2]_{x^k y^l} y^k - 2[F^2]_{x^l} = 0.$$

Such a coordinate system is said to be *adapted*.

For a Riemannian metric $\alpha = \sqrt{a_{ij}(x) y^i y^j}$, it is known that α is locally dually flat if and only if in an adapted coordinate system, the fundamental tensor of α is the Hessian of some local smooth function $\psi(x)$ [1,2], i.e.,

$$a_{ij}(x) = \frac{\partial^2 \psi}{\partial x^i \partial x^j}(x).$$

The dual flatness of a Riemannian metric can also be described by its spray [15]: *α is locally dually flat if and only if its spray coefficients could be expressed in an adapted coordinate system* as

$$G^i_\alpha = 2\theta y^i + \alpha^2 \theta^i \tag{1.1}$$

for some 1*-form* $\xi := \xi_i(x) y^i$.

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The first example of non-Riemannian dually flat Finsler metrics is the so-called *Funk metric*

$$F=\frac{\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}{1-|x|^2}+\frac{\langle x,y\rangle}{1-|x|^2}$$

on the unit ball $\mathbb{B}^n(1)$ [6,8], which belongs to a special class of Finsler metrics named *Randers metrics*. Randers metrics are expressed as the sum of a Riemannian metric $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ and a 1-form $\beta=b_i(x)y^i$ with the norm $b:=\|\beta\|_\alpha<1$.

Based on the characterization result for locally dually flat Randers metrics given by X. Cheng et al. [6], the author provides a more direct characterization and proves that the dual flatness of a Randers metric always arises from that of some Riemannian metric [15]: *A Randers metric $F=\alpha+\beta$ is locally dually flat if and only if the Riemannian metric $\bar\alpha=\sqrt{1-b^2}\sqrt{\alpha^2-\beta^2}$ is locally dually flat and the 1-form $\bar\beta=-(1-b^2)\beta$ is dually related with respect to $\bar\alpha$*. In this case, F can be reexpressed as

$$F=\frac{\sqrt{(1-\bar b^2)\bar\alpha^2+\bar\beta^2}}{1-\bar b^2}-\frac{\bar\beta}{1-\bar b^2}. \tag{1.2}$$

Recall that a 1-form β is said to be *dually related* to a locally dually flat Riemannian metric α if in an adapted coordinate system the spray coefficients of α are in the form (1.1) and the covariant derivation of β with respect to α is given by

$$b_{i|j}=2\theta_ib_j+c(x)a_{ij} \tag{1.3}$$

for some scalar function $c(x)$. This concept was first introduced by the author in [15]. In particular, we prove that the Riemannian metrics

$$\bar\alpha=\frac{\sqrt{(1+\mu|x|^2)|y|^2-\mu\langle x,y\rangle^2}}{(1+\mu|x|^2)^{\frac34}} \tag{1.4}$$

are dually flat on the ball $\mathbb{B}^n(r_\mu)$ and the 1-forms

$$\bar\beta=\frac{\lambda\langle x,y\rangle}{(1+\mu|x|^2)^{\frac54}} \tag{1.5}$$

are dually related to $\bar\alpha$ for any constant number μ and λ, where the radius r_μ is given by $r_\mu=\frac{1}{\sqrt{-\mu}}$ if $\mu<0$ and $r_\mu=+\infty$ if $\mu\geqslant 0$.

As a result, we construct many non-trivial dually flat Randers metrics as follows:

$$F(x,y)=\frac{\sqrt[4]{1+(\mu+\lambda^2)|x|^2}\sqrt{(1+\mu|x|^2)|y|^2-\mu\langle x,y\rangle^2}}{1+\mu|x|^2}+\frac{\lambda\langle x,y\rangle}{(1+\mu|x|^2)\sqrt[4]{1+(\mu+\lambda^2)|x|^2}}.$$

It is just the Funk metric when $\mu=-1$ and $\lambda=1$.

(1.2) is just the *navigation expression* for Randers metrics, which play a key role in the research of Randers metrics. For example, D. Bao et al. classified Randers metrics with constant flag curvature [5]: *$F=\alpha+\beta$ is of constant flag curvature if and only if $\bar\alpha$ in (1.2) is of constant sectional curvature and $\bar\beta$ is homothetic to $\bar\alpha$, i.e.*,

$$\frac12(\bar b_{i|j}+\bar b_{j|i})=c\bar a_{ij}$$

for some constant c. Similarly, D. Bao et al. gave a characterization for Einstein metric of Randers type [4]: *$F=\alpha+\beta$ is Einsteinian if and only if $\bar\alpha$ is Einsteinian and $\bar\beta$ is homothetic to $\bar\alpha$*. It seems that most of the properties of Randers metrics become simple and clear if they are described with the navigation form [10].

Except for Randers metrics, there is another important class of Finsler metrics defined also by a Riemannian metric and a 1-form and given in the form

$$F=\alpha\phi\left(\frac{\beta}{\alpha}\right),$$

where $\phi(s)$ is a smooth function. Such kinds of Finsler metrics are called (α,β)-*metrics*. It was proposed by M. Matsumoto in 1972 as a direct generalization of Randers metrics. (α,β)-metrics form a special class of Finsler metrics partly because of its computability [3]. Recently, many encouraging results about (α,β)-metrics, including flag curvature property [9,17], Ricci curvature property [7,13] and projective property [12,16] etc., have been achieved.

In 2011, Q. Xia gave a local characterization of locally dually flat (α,β)-metrics on a manifold with dimension $n\geqslant 3$:

Theorem 1.1. *(See [14].) Let $F=\alpha\phi(\frac{\beta}{\alpha})$ be a Finsler metric on an open subset $U\subseteq\mathbb{R}^n$ with $n\geqslant 3$. Suppose F is not of Riemannian type and $\phi'(0)\neq 0$. Then F is dually flat on U if and only if the following conditions hold*:

$$G^i_\alpha = \big[2\theta + (3k_1 - 2)\tau\beta\big]y^i + \alpha^2\big(\theta^i - \tau b^i\big) + \frac{3}{2}k_3\tau\beta^2 b^i, \tag{1.6}$$

$$r_{00} = 2\theta\beta + \big(3\tau + 2\tau b^2 - 2b_k\theta^k\big)\alpha^2 + \big(3k_2 - 2 - 3k_3 b^2\big)\tau\beta^2, \tag{1.7}$$

$$s_{i0} = \beta\theta_i - \theta b_i, \tag{1.8}$$

$$\tau\big\{s\big(k_2 - k_3 s^2\big)\big(\phi\phi' - s\phi'^2 - s\phi\phi''\big) - \big(\phi'^2 + \phi\phi''\big) + k_1\phi\big(\phi - s\phi'\big)\big\} = 0, \tag{1.9}$$

where θ is a 1-form, τ is a scalar function, and k_1, k_2, k_3 are constants.

The meaning of some notations here can be found in Section 2.

When $\tau = 0$, (1.6) becomes $G^i_\alpha = 2\theta y^i + \alpha^2\theta^i$, which implies α is dually flat. Moreover, (1.7) and (1.8) are equivalent to $b_{i|j} = 2\theta_i b_j - 2b_k\theta^k a_{ij}$, i.e., β is dually related to α with $c(x) + 2b_k\theta^k = 0$. In fact, this is a *trivial* case. Because in this case, $F = \alpha\phi(\frac{\beta}{\alpha})$ will be always dually flat for any suitable function $\phi(s)$ by Theorem 1.1. In this paper, we will focus on the non-trivial case. Thereby, the function $\phi(s)$ must satisfy a 3-parameters equation

$$s\big(k_2 - k_3 s^2\big)\big(\phi\phi' - s\phi'^2 - s\phi\phi''\big) - \big(\phi'^2 + \phi\phi''\big) + k_1\phi\big(\phi - s\phi'\big) = 0. \tag{1.10}$$

It is clear that the geometrical meaning of the original data α and β for the dually flat (α, β)-metrics is very obscure. The main aim of this paper is to provide a luminous description for a non-trivial dually flat (α, β)-metric. By using a special class of metric deformations called *β-deformations*, we prove that *the dual flatness of (α, β)-metrics always arises from that of some Riemannian metric*, just as Randers metrics.

Theorem 1.2. *Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be a Finsler metric on an open subset $U \subseteq \mathbb{R}^n$ with $n \geqslant 3$, where $\phi(s)$ satisfies (1.10). Suppose F is not of Riemannian type and $\phi'(0) \neq 0$. Then F is dually flat if and only if α and β can be expressed as*

$$\alpha = \eta(\bar{b}^2)\sqrt{\bar{\alpha}^2 - \frac{(k_2 - k_3\bar{b}^2)}{1 + k_2\bar{b}^2 - k_3\bar{b}^4}\bar{\beta}^2}, \qquad \beta = -\frac{\eta(\bar{b}^2)}{(1 + k_2\bar{b}^2 - k_3\bar{b}^4)^{\frac{1}{2}}}\bar{\beta},$$

where $\bar{\alpha}$ is a dually flat Riemannian metric on U, $\bar{\beta}$ is dually related to $\bar{\alpha}$, $\bar{b} := \|\bar{\beta}\|_{\bar{\alpha}}$. The deformation factor $\eta(\bar{b}^2)$ is determined by the coefficients k_1, k_2, k_3 and given in the following five cases:

(1) *When $k_3 = 0$, $k_2 = 0$,*

$$\eta\big(\bar{b}^2\big) = \exp\left\{\frac{k_1\bar{b}^2}{4}\right\};$$

(2) *When $k_3 = 0$, $k_2 \neq 0$,*

$$\eta\big(\bar{b}^2\big) = \big\{1 + k_2\bar{b}^2\big\}^{\frac{k_1 - k_2}{4k_2}};$$

(3) *When $k_3 \neq 0$, $\Delta_1 > 0$,*

$$\eta\big(\bar{b}^2\big) = \frac{\big\{\frac{\sqrt{\Delta_1} + k_2}{\sqrt{\Delta_1} - k_2} \cdot \frac{\sqrt{\Delta_1} - k_2 + 2k_3\bar{b}^2}{\sqrt{\Delta_1} + k_2 - 2k_3\bar{b}^2}\big\}^{\frac{2k_1 - k_2}{8\sqrt{\Delta_1}}}}{\sqrt[8]{1 + k_2\bar{b}^2 - k_3\bar{b}^4}};$$

(4) *When $k_3 \neq 0$, $\Delta_1 = 0$,*

$$\eta\big(\bar{b}^2\big) = \frac{\sqrt[4]{2}\exp\{\frac{k_2 - 2k_1}{2k_2}[\frac{1}{2 + k_2\bar{b}^2} - \frac{1}{2}]\}}{\sqrt[4]{2 + k_2\bar{b}^2}};$$

(5) *When $k_3 \neq 0$, $\Delta_1 < 0$,*

$$\eta\big(\bar{b}^2\big) = \frac{\exp\{\frac{2k_1 - k_2}{4\sqrt{-\Delta_1}}(\arctan\frac{k_2 - 2k_3\bar{b}^2}{\sqrt{-\Delta_1}} - \arctan\frac{k_2}{\sqrt{-\Delta_1}})\}}{\sqrt[8]{1 + k_2\bar{b}^2 - k_3\bar{b}^4}},$$

where $\Delta_1 := k_2^2 + 4k_3$.

β-deformations, which play a key role in the proof of Theorem 1.2, are a new method in Riemann–Finsler geometry developed by the author in the research of projectively flat (α, β)-metrics [16]. Roughly speaking, the method of β-deformations is aimed to make clear the latent light. By an analogy, α and β are just like two ropes tangles together, and it is possible to unhitch them using β-deformations. The navigation expression for Randers metrics is a representative example. In fact, it is just a specific kind of β-deformations. In other words, β-deformations can be regarded as a natural generalization of the navigation expression for Randers metrics. See also [13] for more applications.

The argument in this paper is similar to that in [15], but we don't show the original ideas here. One can obtain a more deep analysis in the latter.

In Section 4, we will use a skillful method to solve the basic equation (1.10). As a result, we can construct infinity many non-trivial dually flat (α,β)-metrics combining with (1.4) and (1.5). In particular, the following metrics

$$F=\sqrt{\alpha^2+2\varepsilon\alpha\beta+\kappa\beta^2}$$

are locally dually flat if and only if

$$\alpha=\left(1-\kappa\bar{b}^2\right)^{-1}\sqrt{\left(1-\kappa\bar{b}^2\right)\bar{\alpha}^2+\kappa\bar{\beta}^2},\qquad \beta=-\left(1-\kappa\bar{b}^2\right)^{-1}\bar{\beta},\tag{1.11}$$

where $\bar{\alpha}$ is locally dually flat and $\bar{\beta}$ is dually related to $\bar{\alpha}$.

Taking $\kappa=1$ and $\varepsilon=1$, one can see that (1.11) is just the Randers metrics $F=\alpha+\beta$. Taking $\kappa=0$ and $\varepsilon=\frac{1}{2}$, then we can obtain a very simple kind of dually flat (α,β)-metrics given in the form

$$F=\sqrt{\alpha(\alpha+\beta)}.$$

2. Preliminaries

Let M be a smooth n-dimensional manifold. A Finsler metric F on M is a continuous function $F:TM\to[0,+\infty)$ with the following properties:

(i) *Regularity*: F is C^∞ on the entire slit tangent bundle $TM\backslash\{0\}$;
(ii) *Positive homogeneity*: $F(x,\lambda y)=\lambda F(x,y)$ for all $\lambda>0$;
(iii) *Strong convexity*: the fundamental tensor $g_{ij}:=\left[\frac{1}{2}F^2\right]_{y^iy^j}$ is positive definite for all $(x,y)\in TM\backslash\{0\}$.

Here $x=(x^i)$ denotes the coordinates of the point in M and $y=(y^i)$ denotes the coordinates of the vector in T_xM.

For a Finsler metric, the *geodesics* are characterized by the geodesic equation

$$\ddot{c}^i(t)+2G^i\big(c(t),\dot{c}(t)\big)=0,$$

where

$$G^i(x,y):=\frac{1}{4}g^{il}\left\{\left[F^2\right]_{x^ky^l}y^k-\left[F^2\right]_{x^l}\right\}$$

are called the *spray coefficients* of F. Here $(g^{ij}):=(g_{ij})^{-1}$. For a Riemannian metric α, the spray coefficients are given by

$$G^i_\alpha(x,y)=\frac{1}{2}\Gamma^i{}_{jk}(x)y^jy^k$$

in terms of the Christoffel symbols of α.

By definition, an (α,β)-metric is a Finsler metric in the form $F=\alpha\phi\left(\frac{\beta}{\alpha}\right)$, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta=b_i(x)y^i$ is a 1-form and $\phi(s)$ is a positive smooth function on some symmetric open interval $(-b_o,b_o)$.

On the other hand, the so-called β-deformations are a triple of metric deformations in terms of α and β listed below:

$$\tilde{\alpha}=\sqrt{\alpha^2-\kappa\left(b^2\right)\beta^2},\qquad \tilde{\beta}=\beta;$$
$$\hat{\alpha}=e^{\rho(b^2)}\tilde{\alpha},\qquad \hat{\beta}=\tilde{\beta};$$
$$\bar{\alpha}=\hat{\alpha},\qquad \bar{\beta}=\nu\left(b^2\right)\hat{\beta}.$$

Some basic formulas for β-deformations are listed below. Be attention that the notation '$\dot{b}_{i|j}$' always means the covariant derivative of the 1-form '$\dot{\beta}$' with respect to the corresponding Riemannian metric '$\dot{\alpha}$', where the symbol '$\dot{}$' can be nothing, '$\tilde{}$', '$\hat{}$' or '$\bar{}$' in this paper. Moreover, we need the following abbreviations,

$$r_{00}:=r_{ij}y^iy^j,\qquad r_i:=r_{ij}y^j,\qquad r_0:=r_iy^i,\qquad r:=r_ib^i,$$
$$s_{i0}:=s_{ij}y^j,\qquad s^i{}_0:=a^{ij}s_{j0},\qquad s_i:=s_{ij}y^j,\qquad s_0:=s_ib^i,$$

where r_{ij} and s_{ij} are the symmetrization and antisymmetrization of $b_{i|j}$ respectively, i.e.,

$$r_{ij}:=\frac{1}{2}(b_{i|j}+b_{j|i}),\qquad s_{ij}:=\frac{1}{2}(b_{i|j}-b_{j|i}).$$

Roughly speaking, indices are raised and lowered by a_{ij}, vanished by contracted with b^i and changed to be '$_0$' by contracted with y^i. Since $b_{i|j} - b_{j|i} = \frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}$, $s_{ij} = 0$ implies β is closed, and vice versa.

Lemma 2.1. *(See [16].) Let $\tilde{\alpha} = \sqrt{\alpha^2 - \kappa(b^2)\beta^2}$, $\tilde{\beta} = \beta$. Then*

$$\begin{aligned}
\tilde{G}^i_{\tilde{\alpha}} &= G^i_\alpha - \frac{\kappa}{2(1-\kappa b^2)}\left\{2\left(1-\kappa b^2\right)\beta s^i{}_0 + r_{00}b^i + 2\kappa s_0\beta b^i\right\} \\
&\quad + \frac{\kappa'}{2(1-\kappa b^2)}\left\{\left(1-\kappa b^2\right)\beta^2\left(r^i + s^i\right) + \kappa r\beta^2 b^i - 2(r_0+s_0)\beta b^i\right\}, \\
\tilde{b}_{i|j} &= b_{i|j} + \frac{\kappa}{1-\kappa b^2}\left\{b^2 r_{ij} + b_i s_j + b_j s_i\right\} - \frac{\kappa'}{1-\kappa b^2}\left\{r b_i b_j - b^2 b_i(r_j+s_j) - b^2 b_j(r_i+s_i)\right\}.
\end{aligned}$$

Lemma 2.2. *(See [16].) Let $\hat{\alpha} = e^{\rho(b^2)}\tilde{\alpha}$, $\hat{\beta} = \tilde{\beta}$. Then*

$$\begin{aligned}
\hat{G}^i_{\hat{\alpha}} &= \tilde{G}^i_{\tilde{\alpha}} + \rho'\left\{2(r_0+s_0)y^i - \left(\alpha^2 - \kappa\beta^2\right)\left(r^i + s^i + \frac{\kappa}{1-\kappa b^2} r b^i\right)\right\}, \\
\hat{b}_{i|j} &= \tilde{b}_{i|j} - 2\rho'\left\{b_i(r_j+s_j) + b_j(r_i+s_i) - \frac{1}{1-\kappa b^2} r(a_{ij} - \kappa b_i b_j)\right\}.
\end{aligned}$$

Lemma 2.3. *(See [16].) Let $\bar{\alpha} = \hat{\alpha}$, $\bar{\beta} = \nu(b^2)\hat{\beta}$. Then*

$$\begin{aligned}
\bar{G}^i_{\bar{\alpha}} &= \hat{G}^i_{\hat{\alpha}}, \\
\bar{b}_{i|j} &= \nu\hat{b}_{i|j} + 2\nu' b_i(r_j+s_j).
\end{aligned}$$

3. Proof of Theorem 1.2

Suppose that $F = \alpha\phi(\frac{\beta}{\alpha})$ is a non-trivial dually flat (α,β)-metric on U. According to Theorem 1.1, it is easy to obtain the following simple facts:

$$r_{ij} = \theta_i b_j + \theta_j b_i + \left(3\tau + 2\tau b^2 - 2b_k\theta^k\right)a_{ij} + \tau\left(3k_2 - 2 - 3k_3 b^2\right)b_i b_j, \tag{3.1}$$

$$s^i{}_0 = \beta\theta^i - \theta b^i, \tag{3.2}$$

$$s_0 = b_k\theta^k\beta - b^2\theta, \tag{3.3}$$

$$r_i + s_i = 3\tau\left(1 + k_2 b^2 - k_3 b^4\right)b_i, \tag{3.4}$$

$$b_i s_j + b_j s_i = 2b_k\theta^k b_i b_j - b^2(\theta_i b_j + \theta_j b_i), \tag{3.5}$$

$$r = 3\tau\left(1 + k_2 b^2 - k_3 b^4\right)b^2. \tag{3.6}$$

Lemma 3.1. *Take $\kappa(b^2) = -k_2 + k_3 b^2$, then*

$$\tilde{G}^i_{\tilde{\alpha}} = \left[2\theta + \tau\beta(3k_1 - 2)\right]y^i + \tilde{\alpha}^2\theta^i + \frac{\tau(3k_2 - 2 - 3k_3 b^2) - 2(k_2 - k_3 b^2)b_k\theta^k}{2(1 + k_2 b^2 - k_3 b^4)}\tilde{\alpha}^2 b^i.$$

Proof. By (1.6), (3.1)–(3.6) and Lemma 2.1, we have

$$\begin{aligned}
\tilde{G}^i_{\tilde{\alpha}} &= \left[2\theta + (3k_1 - 2)\tau\beta\right]y^i + \alpha^2\left(\theta^i - \tau b^i\right) + \frac{3}{2}k_3\tau\beta^2 b^i \\
&\quad - \frac{\kappa}{2(1-\kappa b^2)}\left\{2\left(1-\kappa b^2\right)\beta\left(\beta\theta^i - \theta b^i\right) + 2\theta\beta b^i + \left(3\tau + 2\tau b^2 - b_k\theta^k\right)\alpha^2 b^i\right. \\
&\quad \left. + \tau\left(3k_2 - 2 - 3k_3 b^2\right)\beta^2 b^i + 2\kappa\left(b_k\theta^k\beta - b^2\theta\right)\beta b^i\right\} \\
&\quad + \frac{\kappa'}{2(1-\kappa b^2)}\left\{3\tau\left(1-\kappa b^2\right)\left(1 + k_2 b^2 - k_3 b^4\right)\beta^2 b^i\right. \\
&\quad \left. + 3\tau\kappa\left(1 + k_2 b^2 - k_3 b^4\right)b^2\beta^2 b^i - 6\tau\left(1 + k_2 b^2 - k_3 b^4\right)\beta^2 b^i\right\} \\
&= \left[2\theta + (3k_1 - 2)\tau\beta\right]y^i + \tilde{\alpha}^2\theta^i - \frac{1}{2(1-\kappa b^2)}\left\{\left(3\tau\kappa + 2\tau - 2\kappa b_k\theta^k\right)\alpha^2\right. \\
&\quad \left. + \left[2\kappa^2 b_k\theta^k - 3\tau k_3\left(1-\kappa b^2\right) + \tau\kappa\left(3k_2 - 2 - 3k_3 b^2\right) + 3\tau\kappa'\left(1 - k_2 b^2 + k_3 b^4\right)\right]\beta^2\right\}b^i.
\end{aligned}$$

When $\kappa = -k_2 + k_3 b^2$, it is easy to verify that

$$\kappa^2 + k_2\kappa - k_3 = -\kappa'\left(1 + k_2 b^2 - k_3 b^4\right),$$

and hence $\tilde{G}^i_{\tilde{\alpha}}$ is given in the following form,

$$\tilde{G}^i_{\tilde{\alpha}} = \left[2\theta + \tau\beta(3k_1 - 2)\right]y^i + \tilde{\alpha}^2\theta^i - \frac{3\tau\kappa + 2\tau - 2\kappa b_k\theta^k}{2(1-\kappa b^2)}\tilde{\alpha}^2 b^i. \qquad \square \tag{3.7}$$

Lemma 3.2. *Take* $\rho(b^2) = -\frac{1}{4}\int \frac{k_1 - k_2 + k_3 b^2}{1 + k_2 b^2 - k_3 b^4}\,\mathrm{d}b^2$, *then*

$$\hat{G}^i_{\hat{\alpha}} = 2\hat{\theta}y^i + \hat{\alpha}^2\hat{\theta}^i,$$

where $\hat{\theta} = \theta - \frac{1}{4}\tau[4 - 3(k_1 + k_2 - k_3 b^2)]\beta$. *In particular,* $\hat{\alpha}$ *is dually flat on* U.

Proof. By (3.4), (3.6), (3.7) and Lemma 2.2 we have

$$\begin{aligned}
\hat{G}^i_{\hat{\alpha}} &= \tilde{G}^i_{\tilde{\alpha}} + \rho'\left\{6\tau\left(1 + k_2 b^2 - k_3 b^4\right)\beta y^i - \tilde{\alpha}^2\left(3\tau\left(1 + k_2 b^2 - k_3 b^4\right)b^i + \frac{\kappa}{1-\kappa b^2}\cdot 3\tau\left(1 + k_2 b^2 - k_3 b^4\right)b^2 b^i\right)\right\} \\
&= \left\{2\theta + \tau\left[3k_1 - 2 + 6\rho'\left(1 + k_2 b^2 - k_3 b^4\right)\right]\beta\right\}y^i + \tilde{\alpha}^2\theta^i \\
&\quad - \frac{1}{2(1-\kappa b^2)}\left\{3\tau\kappa + 2\tau + 6\tau\rho'\left(1 + k_2 b^2 - k_3 b^4\right) - 2\kappa b_k\theta^k\right\}\tilde{\alpha}^2 b^i.
\end{aligned}$$

Let

$$\hat{\theta} := \theta + \frac{1}{2}\tau\left[3k_1 - 2 + 6\rho'\left(1 + k_2 b^2 - k_3 b^4\right)\right]\beta.$$

It is easy to verify that the inverse of $(\hat{a}_{ij})$ is given by

$$\hat{a}^{ij} = e^{-2\rho}\left(a^{ij} + \frac{\kappa}{1-\kappa b^2}b^i b^j\right), \tag{3.8}$$

so $\hat{\theta}^i := \hat{a}^{ij}\hat{\theta}_j$ are given by

$$\hat{\theta}^i = e^{-2\rho}\left\{\theta^i + \frac{1}{2(1-\kappa b^2)}\left[2\kappa b_k\theta^k + \tau(3k_1 - 2) + 6\tau\rho'\left(1 + k_2 b^2 - k_3 b^4\right)\right]b^i\right\}.$$

Hence $\hat{G}^i_{\hat{\alpha}}$ can be reexpressed as

$$\hat{G}^i_{\hat{\alpha}} = 2\hat{\theta}y^i + \hat{\alpha}^2\hat{\theta}^i - \frac{3\tau e^{-2\rho}}{2(1-\kappa b^2)}\left\{k_1 + \kappa + 4\rho'\left(1 + k_2 b^2 - k_3 b^4\right)\right\}\hat{\alpha}^2 b^i.$$

Obviously, the deformation factor given in the lemma satisfies

$$\rho' = -\frac{k_1 + \kappa}{4(1 + k_2 b^2 - k_3 b^4)}, \tag{3.9}$$

thus $\hat{G}^i_{\hat{\alpha}} = 2\hat{\theta}y^i + \hat{\alpha}^2\hat{\theta}^i$. $\square$

Lemma 3.3. *Take* $\nu(b^2) = -\sqrt{1 + k_2 b^2 - k_3 b^4}\,e^{\rho(b^2)}$, *then*

$$\bar{G}^i_{\bar{\alpha}} = 2\bar{\theta}y^i + \bar{\alpha}^2\bar{\theta}^i,$$
$$\bar{b}_{i|j} = 2\bar{\theta}_i\bar{b}_j + \bar{c}(x)\bar{a}_{ij},$$

where $\bar{c}(x)$ *is a scalar function. In particular,* $\bar{\beta}$ *is dually related to* $\bar{\alpha}$.

Proof. Under the deformations used above, combining with (3.1), (3.4), (3.5) and Lemma 2.2 we can see that

$$\begin{aligned}
\tilde{r}_{ij} &= \frac{1}{1-\kappa b^2}\left\{r_{ij} + 2\kappa b_k\theta^k b_i b_j - \kappa b^2(\theta_i b_j + \theta_j b_i) + 3\tau\kappa'\left(1 + k_2 b^2 - k_3 b^4\right)b^2 b_i b_j\right\} \\
&= \theta_i b_j + \theta_j b_i + \frac{1}{1-\kappa b^2}\left\{\left(3\tau + 2\tau b^2 - 2b_k\theta^k\right)a_{ij} + \left[\tau\left(3k_2 - 2 - 3k_3 b^2\right) + 2\kappa b_k\theta^k\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + 3\tau\kappa'\left(1+k_2b^2-k_3b^4\right)b^2\big]b_ib_j\big\}\\
&= \theta_ib_j+\theta_jb_i+\frac{1}{1-\kappa b^2}\left(3\tau+2\tau b^2-2b_k\theta^k\right)\tilde{a}_{ij}+\tau(3\kappa+3k_2-2)b_ib_j,\\
\tilde{s}_{ij} &= s_{ij}=\theta_ib_j-\theta_jb_i.
\end{aligned}
$$

Similarly, by (3.4), (3.9) and Lemma 2.2 we get

$$
\begin{aligned}
\hat{r}_{ij} &= \tilde{r}_{ij}+\frac{k_1+\kappa}{2(1+k_2b^2-k_3b^4)}\left\{6\tau\left(1+k_2b^2-k_3b^4\right)b_ib_j-\frac{1}{1-\kappa b^2}\cdot 3\tau\left(1+k_2b^2-k_3b^4\right)b^2\tilde{a}_{ij}\right\}\\
&= \theta_ib_j+\theta_jb_i+\frac{e^{-2\rho}}{2(1-\kappa b^2)}\left\{6\tau+(4-3k_1)\tau b^2-3\tau\kappa b^2-4b_k\theta^k\right\}\hat{a}_{ij}+\tau(6\kappa+3k_1+3k_2-2)b_ib_j,\\
\hat{s}_{ij} &= s_{ij}=\theta_ib_j-\theta_jb_i.
\end{aligned}
$$

If we use $\hat{\theta}$ instead of θ to express $\hat{r}_{ij}$ and $\hat{s}_{ij}$, then

$$
\begin{aligned}
\hat{r}_{ij} &= \hat{\theta}_i\hat{b}_j+\hat{\theta}_j\hat{b}_i+\frac{e^{-2\rho}}{2(1-\kappa b^2)}\left\{6\tau+\tau b^2-3\tau\kappa b^2-4b_k\theta^k\right\}\hat{a}_{ij}+\frac{3}{2}\tau(5\kappa+k_1+2k_2)\hat{b}_i\hat{b}_j,\\
\hat{s}_{ij} &= \hat{\theta}_i\hat{b}_j-\hat{\theta}_j\hat{b}_i,
\end{aligned}
$$

where $\hat{b}_i=b_i$ according to the definition of β-deformations.

Finally, by (3.4) and Lemma 2.3 we have

$$
\begin{aligned}
\bar{r}_{ij} &= \nu\hat{r}_{ij}+6\tau\nu'\left(1+k_2b^2-k_3b^4\right)b_ib_j\\
&= \bar{\theta}_i\bar{b}_j+\bar{\theta}_j\bar{b}_i+\frac{e^{-2\rho}\nu}{2(1-\kappa b^2)}\left\{6\tau+\tau b^2-3\tau\kappa b^2-4b_k\theta^k\right\}\bar{a}_{ij}\\
&\quad+\frac{3}{2}\tau\left\{(5\kappa+k_1+2k_2)\nu+4\left(1+k_2b^2-k_3b^4\right)\nu'\right\}\hat{b}_i\hat{b}_j,\\
\bar{s}_{ij} &= \nu s_{ij}=\nu(\hat{\theta}_i\hat{b}_j-\hat{\theta}_j\hat{b}_i)=\bar{\theta}_i\bar{b}_j-\bar{\theta}_j\bar{b}_i,
\end{aligned}
$$

where $\bar{\theta}:=\hat{\theta}$. It is easy to verify that the deformation factor in the lemma satisfies

$$
(5\kappa+k_1+2k_2)\nu+4\left(1+k_2b^2-k_3b^4\right)\nu'=0, \tag{3.10}
$$

So

$$
\bar{r}_{ij}=\bar{\theta}_i\bar{b}_j+\bar{\theta}_j\bar{b}_i+\bar{c}(x)\bar{a}_{ij}
$$

where $\bar{c}(x)$ is a scalar function and can be reexpressed as

$$
\bar{c}(x)=-2\bar{b}_k\bar{\theta}^k+\frac{3\tau e^{-2\rho}\nu}{2(1-\kappa b^2)}\left\{2\left(1-\kappa b^2\right)+(k_1-1)b^2\right\}. \tag{3.11}
$$

Combining with $\bar{s}_{ij}$, we have $\bar{b}_{i|j}=2\bar{\theta}_i\bar{b}_j+\bar{c}(x)\bar{a}_{ij}$. $\square$

From the equality (3.11) we can see that $\bar{c}(x)\neq-2\bar{b}_k\bar{\theta}^k$ unless $\tau=0$. In other words, when $\tau\neq 0$, $\bar{\beta}$ is non-trivial (see the statements below Theorem 1.1 for the reason).

Proof of Theorem 1.2. Due to the above lemmas, we have shown that if $F=\alpha\phi(\frac{\beta}{\alpha})$ is a non-trivial dually flat Finsler metric with dimension $n\geqslant 3$, then the output Riemannian metric $\bar{\alpha}$ is dually flat and the output 1-form $\bar{\beta}$ is dually related to $\bar{\alpha}$.

Conversely, by (3.8) we can see that the norm of $\bar{b}$ is related to b as

$$
\bar{b}^2=\nu b_i\nu b_je^{-2\rho}\left(a^{ij}+\frac{\kappa}{1-\kappa b^2}b^ib^j\right)=b^2,
$$

which implies that the β-deformations given above are reversible. More specifically, we have

$$
\beta=\nu^{-1}\left(\bar{b}^2\right)\bar{\beta}=-\frac{e^{-\rho(\bar{b}^2)}}{\sqrt{1+k_2\bar{b}^2-k_3\bar{b}^4}}\bar{\beta}
$$

and

$$
\alpha=\sqrt{e^{-2\rho(\bar{b}^2)}\bar{\alpha}^2+\kappa\left(\bar{b}^2\right)\beta^2}=e^{-\rho(\bar{b}^2)}\sqrt{\bar{\alpha}^2-\frac{(k_2-k_3\bar{b}^2)}{1+k_2\bar{b}^2-k_3\bar{b}^4}\bar{\beta}^2}.
$$

Denote $\eta(\bar{b}^2) := e^{-\rho(\bar{b}^2)}$. By (3.9), η can be chosen as

$$\eta(\bar{b}^2) = \exp\left\{\frac{1}{4}\int_0^{\bar{b}^2} \frac{k_1 - k_2 + k_3 t}{1 + k_2 t - k_3 t^2}\,dt\right\}.$$

Combining with the discussions in the proofs of Lemma 3.1, Lemma 3.2 and Lemma 3.3, it is not hard to see that if $\bar{\alpha}$ is dually flat and $\bar{\beta}$ is dually related to $\bar{\alpha}$, then the output data α and β of the reverse β-deformations satisfy (1.6)–(1.8) and hence $F = \alpha\phi(\frac{\beta}{\alpha})$ is dually flat. □

4. Symmetry and solutions of Eq. (1.10)

In this section, we will solve the basic equation (1.10) in a nonconventional way. Firstly, let us introduce two special transformations for the function ϕ:

$$g_u\big(\phi(s)\big) := \sqrt{1+us^2}\,\phi\left(\frac{s}{\sqrt{1+us^2}}\right), \qquad h_v\big(\phi(s)\big) := \phi(vs),$$

where u and v are constants with $v \neq 0$. The motivation of the above transformations can be found in [16], here we just need to know that such transformations satisfy

$$g_{u_1} \circ g_{u_2} = g_{u_1+u_2}, \qquad h_{v_1} \circ h_{v_2} = h_{v_1 v_2}, \qquad h_v \circ g_u = g_{v^2 u} \circ h_v,$$

and hence generate a transformation group G under the above generation relationship, which is isomorphism to $(\mathbb{R} \times \mathbb{R}\backslash\{0\}, \cdot)$ under the map $\pi(g_u \circ h_v) = (u, v)$. For the later, the operation is given by $(u_1, v_1) \cdot (u_2, v_2) = (u_1 + v_1^2 u_2, v_1 v_2)$. In particular,

$$g_u^{-1} = g_{-u}, \qquad h_v^{-1} = h_{v^{-1}}.$$

The importance of the transformation group G for our question is that the solution space of the 3-parameters equation (1.10) is invariant under the action of G as below. The computations are elementary and hence omitted here.

Lemma 4.1. *If $\phi(s)$ satisfies (1.10), then the function $\psi(s) := g_u(\phi)$ satisfies the same kind of equation*

$$s\big(k_2' - k_3' s^2\big)\big(\psi\psi' - s\psi'^2 - s\psi\psi''\big) - \big(\psi'^2 + \psi\psi''\big) + k_1'\psi\big(\psi - s\psi'\big) = 0,$$

where

$$k_1' = k_1 + u, \qquad k_2' = k_2 + 2u, \qquad k_3' = k_3 - k_2 u - u^2.$$

Moreover, $\phi(0) = \psi(0)$ and $\phi'(0) = \psi'(0)$.

Lemma 4.2. *If $\phi(s)$ satisfies (1.10), then the function $\varphi(s) := h_v(\phi)$ satisfies the same kind of equation*

$$s\big(k_2'' - k_3'' s^2\big)\big(\varphi\varphi' - s\varphi'^2 - s\varphi\varphi''\big) - \big(\varphi'^2 + \varphi\varphi''\big) + k_1''\varphi\big(\varphi - s\varphi'\big) = 0,$$

where

$$k_1'' = v^2 k_1, \qquad k_2'' = v^2 k_2, \qquad k_3'' = v^4 k_3.$$

Moreover, $\phi(0) = \varphi(0)$ and $\phi'(0) = v\varphi'(0)$.

Furthermore, there are some invariants. Denote

$$\Delta_1 = k_2^2 + 4k_3, \qquad \Delta_2 = k_2 - 2k_1, \qquad \Delta_3 = k_1^2 - k_1 k_2 - k_3.$$

Then we have

Lemma 4.3. $\mathrm{Sgn}(\Delta_i)$ $(i = 1, 2, 3)$ *are all invariants under the action of G.*

Proof. We only need to show that $\mathrm{Sgn}(\Delta_i)$ are invariant for $g_u(\phi)$ and $h_v(\phi)$. It is obvious, because by Lemma 4.1 and Lemma 4.2 we have $\Delta_1' = \Delta_1$, $\Delta_2' = \Delta_2$, $\Delta_3' = \Delta_3$ and $\Delta_1'' = v^4\Delta_1$, $\Delta_2'' = v^2\Delta_2$, $\Delta_3'' = v^4\Delta_3$. □

Furthermore, Δ_i satisfy $\Delta_2^2 - 4\Delta_3 = \Delta_1$. They will play a basic role for the further research.

Next, we will solve Eq. (1.10) with the initial conditions

$$\phi(0) = 1, \qquad \phi'(0) = \varepsilon$$

combining with the transformation group G. Note that for (α,β)-metrics $F=\alpha\phi(\frac{\beta}{\alpha})$, the function $\phi(s)$ must be positive near $s=0$ and hence we can always assume $\phi(0)=1$ after necessary scaling. On the other hand, $\varepsilon\neq 0$ by the assumption of Theorem 1.1.

Let $\psi(s)=g_{-k_1}(\phi)$. According to Lemma 4.1, the function $\psi(s)$ will satisfy the following equation

$$s\{k_2-2k_1-(k_3+k_1k_2-k_1^2)s^2\}(\psi\psi'-s\psi'^2-s\psi\psi'')-\psi'^2+\psi\psi''=0 \tag{4.1}$$

with the initial conditions

$$\psi(0)=1, \qquad \psi'(0)=\varepsilon.$$

Let $u(s)=\psi^2(s)$. It is easy to see that (4.1) becomes

$$\{1+\Delta_2s^2+\Delta_3s^4\}u''=s\{\Delta_2+\Delta_3s^2\}u' \tag{4.2}$$

with the initial conditions

$$u(0)=1, \qquad u'(0)=2\epsilon.$$

Hence, $u'(s)$ is given by

$$u'(s)=\exp\left\{\frac{1}{2}\int\frac{\Delta_2+\Delta_3s^2}{1+\Delta_2s^2+\Delta_3s^4}\,\mathrm{d}s^2\right\}:=2\varepsilon f(s),$$

where $f(s)$ satisfying $f(0)=1$ can be expressed as elementary functions. So we have

Lemma 4.4. *The solutions of Eq.* (4.2) *with the initial conditions $u(0)=1$, $u'(0)=2\epsilon$ are given by*

$$u(s)=1+2\epsilon\int_0^s f(\sigma)\,\mathrm{d}\sigma,$$

where $f(s)$ satisfying $f(0)=1$ are given in the following:

1. *when* $\Delta_3=0$, $\Delta_1=0$,
$$f(s)=1;$$
2. *when* $\Delta_3=0$, $\Delta_1\neq 0$,
$$f(s)=\sqrt{1+\Delta_2s^2};$$
3. *when* $\Delta_3\neq 0$, $\Delta_1>0$,
$$f(s)=\sqrt[4]{1+\Delta_2s^2+\Delta_3s^4}\left\{\frac{2+(\Delta_2+\sqrt{\Delta_1})s^2}{2+(\Delta_2-\sqrt{\Delta_1})s^2}\right\}^{\frac{\Delta_2}{4\sqrt{\Delta_1}}};$$
4. *when* $\Delta_3\neq 0$, $\Delta_1=0$,
$$f(s)=\sqrt{1+\frac{\Delta_2}{2}s^2}\exp\left\{\frac{1}{2+\Delta_2s^2}-\frac{1}{2}\right\};$$
5. *when* $\Delta_3\neq 0$, $\Delta_1<0$,
$$f(s)=\sqrt[4]{1+\Delta_2s^2+\Delta_3s^4}\exp\left\{\frac{\Delta_2}{2\sqrt{-\Delta_1}}\left[\arctan\frac{\Delta_2+2\Delta_3s^2}{\sqrt{-\Delta_1}}-\arctan\frac{\Delta_2}{\sqrt{-\Delta_1}}\right]\right\}.$$

Theorem 4.5. *The solutions of Eq.* (1.10) *with the initial conditions $\phi(0)=1$, $\phi'(0)=\epsilon$ are given by*

$$\phi(s)=\sqrt{(1+k_1s^2)\left\{1+2\epsilon\int_0^s(1+k_1\sigma^2)^{-\frac{3}{2}}f\left(\frac{\sigma}{\sqrt{1+k_1\sigma^2}}\right)\mathrm{d}\sigma\right\}}.$$

Proof. By assumption,

$$\psi(s)=\sqrt{u}=\sqrt{1+2\varepsilon\int_0^s f(\sigma)\,\mathrm{d}\sigma},$$

so

$$\phi(s) = g_{k_1}(\psi) = \sqrt{1+k_1s^2}\,\psi\left(\frac{s}{\sqrt{1+k_1s^2}}\right) = \sqrt{\left(1+k_1s^2\right)\left(1+2\varepsilon\int\limits_0^{\frac{s}{\sqrt{1+k_1s^2}}} f(\sigma)\,\mathrm{d}\sigma\right)},$$

which can also be expressed as the form given in the theorem. □

Most of the solutions of (1.10) are non-elementary. Some elementary solutions are listed below (except for the last two items). Notice that there is no sum of formula when the sum index $n = 1$, and we rule $m!! = 1$ when $m \leqslant 0$.

- When $k_1 = 0$, $k_2 = 0$, $k_3 = 0$,
$$\phi(s) = \sqrt{1+2\epsilon s};$$
- When $k_1 = 0$, $k_2 < 0$, $k_3 = 0$,
$$\phi(s) = \sqrt{1+\epsilon\left(s\sqrt{1+k_2s^2} + \frac{1}{\sqrt{-k_2}}\arcsin\sqrt{-k_2}s\right)};$$
- When $k_1 = 0$, $k_2 > 0$, $k_3 = 0$,
$$\phi(s) = \sqrt{1+\epsilon\left(s\sqrt{1+k_2s^2} + \frac{1}{\sqrt{k_2}}\operatorname{arcsinh}\sqrt{k_2}s\right)};$$
- When $k_3 = 0$, $k_1 + k_2 = 0$,
$$\phi(s) = \sqrt{1+2\epsilon s + k_1s^2};$$
- When $k_1 \neq 0$, $k_2 = \frac{1}{2n}k_1$ $(n = 1, 2, 3, \ldots)$, $k_3 = 0$,
$$\phi(s) = \sqrt{1+k_1s^2+\epsilon s\sqrt{1+k_2s^2}\left[\frac{(2n)!!}{(2n-1)!!} - \sum_{k=1}^{n-1}\frac{2(n-k)(2n-2)!!(2k-3)!!}{(2n-1)!!(2k)!!}\left(1+k_2s^2\right)^{-k}\right]};$$
- When $k_1 > 0$, $k_2 = \frac{1}{2n+1}k_1$ $(n = 1, 2, 3, \ldots)$, $k_3 = 0$,
$$\phi(s) = \left\{\left(1+k_1s^2\right)\left[1+\frac{(2n-1)!!}{(2n)!!}\frac{\epsilon}{\sqrt{k_2}}\arctan\sqrt{k_2}s\right] + \epsilon s\left[\frac{(2n+1)!!}{(2n)!!} - \sum_{k=1}^{n-1}\frac{2(n-k)(2n-1)!!(2k-2)!!}{(2n)!!(2k+1)!!}\left(1+k_2s^2\right)^{-k}\right]\right\}^{\frac{1}{2}};$$
- When $k_1 < 0$, $k_2 = \frac{1}{2n+1}k_1$ $(n = 1, 2, 3, \ldots)$, $k_3 = 0$,
$$\phi(s) = \left\{\left(1+k_1s^2\right)\left[1+\frac{(2n-1)!!}{(2n)!!}\frac{\epsilon}{\sqrt{-k_2}}\operatorname{arctanh}\sqrt{-k_2}s\right] + \epsilon s\left[\frac{(2n+1)!!}{(2n)!!} - \sum_{k=1}^{n-1}\frac{2(n-k)(2n-1)!!(2k-2)!!}{(2n)!!(2k+1)!!}\left(1+k_2s^2\right)^{-k}\right]\right\}^{\frac{1}{2}};$$
- When $k_1 \neq 0$, $k_2 = -\frac{1}{2n+1}k_1$ $(n = 1, 2, 3, \ldots)$, $k_3 = 0$,
$$\phi(s) = \sqrt{1+k_1s^2+\epsilon s\left[\frac{(2n+2)!!}{(2n+1)!!} - \sum_{k=1}^{n}\frac{2(n-k+1)(2n)!!(2k-3)!!}{(2n+1)!!(2k)!!}\left(1+k_2s^2\right)^{k}\right]};$$
- When $k_1 > 0$, $k_2 = -\frac{1}{2n}k_1$ $(n = 1, 2, 3, \ldots)$, $k_3 = 0$,

$$\phi(s)=\Bigg\{\left(1+k_1s^2\right)\left[1+\frac{(2n-1)!!}{(2n)!!}\frac{\epsilon}{\sqrt{-k_2}}\arcsin\sqrt{-k_2}s\right]$$
$$+\epsilon s\sqrt{1+k_2s^2}\left[\frac{(2n+1)!!}{(2n)!!}-\sum_{k=1}^{n-1}\frac{2(n-k)(2n-1)!!(2k-2)!!}{(2n)!!(2k+1)!!}\left(1+k_2s^2\right)^k\right]\Bigg\}^{\frac{1}{2}};$$

- When $k_1<0$, $k_2=-\frac{1}{2n}k_1$ $(n=1,2,3,\dots)$, $k_3=0$,

$$\phi(s)=\Bigg\{\left(1+k_1s^2\right)\left[1+\frac{(2n-1)!!}{(2n)!!}\frac{\epsilon}{\sqrt{k_2}}\operatorname{arcsinh}\sqrt{k_2}s\right]$$
$$+\epsilon s\sqrt{1+k_2s^2}\left[\frac{(2n+1)!!}{(2n)!!}-\sum_{k=1}^{n-1}\frac{2(n-k)(2n-1)!!(2k-2)!!}{(2n)!!(2k+1)!!}\left(1+k_2s^2\right)^k\right]\Bigg\}^{\frac{1}{2}};$$

- When $k_1=0$, $k_2=0$, $k_3\neq 0$,

$$\phi(s)=\sqrt{1+2\epsilon\int_0^s\sqrt[4]{1-k_3\sigma^4}\,\mathrm{d}\sigma};$$

- When $k_1\neq 0$, $k_2=0$, $k_3=0$,

$$\phi(s)=\sqrt{\left(1+k_1s^2\right)\left[1+2\epsilon\int_0^s\frac{e^{\frac{k_1}{2}\sigma^2}}{(1+k_1\sigma^2)^2}\,\mathrm{d}\sigma\right]}.$$

5. Some explicit examples

We can construct some typical examples below.

Example 5.1. Take $k_1=k_2=k_3=0$ and $\varepsilon=\frac{1}{2}$, then $\phi(s)=\sqrt{1+s}$ satisfies (1.10). By Theorem 1.2, the Finsler metric

$$F=\sqrt{\alpha(\alpha+\beta)}$$

is locally dually flat if and only if α is locally dually flat and β is dually related to α. In particular, the following metrics

$$F=\sqrt{\frac{\sqrt{(1+\mu|x|^2)|y|^2-\mu\langle x,y\rangle^2}}{(1+\mu|x|^2)^{\frac{3}{4}}}\left(\frac{\sqrt{(1+\mu|x|^2)|y|^2-\mu\langle x,y\rangle^2}}{(1+\mu|x|^2)^{\frac{3}{4}}}+\frac{\lambda\langle x,y\rangle}{(1+\mu|x|^2)^{\frac{5}{4}}}\right)}$$

are dually flat.

Example 5.2. Take $k_1=-k_2=\kappa$, $k_3=0$, then $\phi(s)=\sqrt{1+2\varepsilon s+\kappa s^2}$ satisfies (1.10). By Theorem 1.2, the Finsler metric

$$F=\sqrt{\alpha^2+2\varepsilon\alpha\beta+\kappa\beta^2}$$

is locally dually flat if and only if

$$\alpha=\left(1-\kappa\bar{b}^2\right)^{-1}\sqrt{\left(1-\kappa\bar{b}^2\right)\bar{\alpha}^2+\kappa\bar{\beta}^2},\qquad \beta=-\left(1-\kappa\bar{b}^2\right)^{-1}\bar{\beta},$$

where $\bar{\alpha}$ is locally dually flat and $\bar{\beta}$ is dually related to $\bar{\alpha}$.

Example 5.3. Take $k_1=k_3=0$, $k_2=-1$ and $\varepsilon=1$, then $\phi(s)=\sqrt{1+s\sqrt{1-s^2}+\arcsin s}$ satisfies (1.10). By Theorem 1.2, the Finsler metric

$$F=\sqrt{\alpha^2+\sqrt{\alpha^2-\beta^2}\beta+\alpha^2\arcsin\frac{\beta}{\alpha}}$$

is locally dually flat if and only if

$$\alpha=\left(1-\bar{b}^2\right)^{-\frac{3}{4}}\sqrt{\left(1-\bar{b}^2\right)\bar{\alpha}^2+\bar{\beta}^2},\qquad \beta=-\left(1-\bar{b}^2\right)^{-\frac{3}{4}}\bar{\beta},$$

where $\bar{\alpha}$ is locally dually flat and $\bar{\beta}$ is dually related to $\bar{\alpha}$.

Example 5.4. Take $k_1 = k_3 = 0$, $k_2 = 1$ and $\varepsilon = 1$, then $\phi(s) = \sqrt{1 + s\sqrt{1+s^2} + \operatorname{arcsinh} s}$ satisfies (1.10). By Theorem 1.2, the Finsler metric

$$F = \sqrt{\alpha^2 + \sqrt{\alpha^2 + \beta^2}\beta + \alpha^2 \operatorname{arcsinh}\frac{\beta}{\alpha}}$$

is locally dually flat if and only if

$$\alpha = (1+\bar{b}^2)^{-\frac{3}{4}}\sqrt{(1+\bar{b}^2)\bar{\alpha}^2 - \bar{\beta}^2}, \qquad \beta = -(1+\bar{b}^2)^{-\frac{3}{4}}\bar{\beta},$$

where $\bar{\alpha}$ is locally dually flat and $\bar{\beta}$ is dually related to $\bar{\alpha}$.

Example 5.5. Take $k_1 = k_2 = 0$, $k_3 = \pm 1$ and $\varepsilon = \frac{1}{2}$, then $\phi(s) = \sqrt{1 + \int_0^s \sqrt[4]{1 \pm \sigma^4}\, d\sigma}$ satisfies (1.10). By Theorem 1.2, the Finsler metric

$$F = \sqrt{1 + \int_0^{\frac{\beta}{\alpha}} \sqrt[4]{1 \pm \sigma^4}\, d\sigma}$$

is locally dually flat if and only if

$$\alpha = (1 \mp \bar{b}^4)^{-\frac{5}{8}}\sqrt{(1 \mp \bar{b}^4)\bar{\alpha}^2 \pm \bar{b}^2\bar{\beta}^2}, \qquad \beta = -(1 \mp \bar{b}^4)^{-\frac{5}{8}}\bar{\beta},$$

where $\bar{\alpha}$ is locally dually flat and $\bar{\beta}$ is dually related to $\bar{\alpha}$.

Example 5.6. Take $k_2 = k_3 = 0$, $k_1 = \pm 1$ and $\varepsilon = \frac{1}{2}$, then $\phi(s) = \sqrt{(1 \pm s^2)(1 + \int_0^s \frac{e^{\pm\frac{\sigma^2}{2}}}{(1\pm\sigma^2)^2}\, d\sigma)}$ satisfies (1.10). By Theorem 1.2, the Finsler metric

$$F = \sqrt{(\alpha^2 \pm \beta^2)\left(1 + \int_0^{\frac{\beta}{\alpha}} \frac{e^{\pm\frac{\sigma^2}{2}}}{(1 \pm \sigma^2)^2}\, d\sigma\right)}$$

is locally dually flat if and only if

$$\alpha = e^{\pm\frac{\bar{b}^2}{4}}\bar{\alpha}, \qquad \beta = -e^{\pm\frac{\bar{b}^2}{4}}\bar{\beta},$$

where $\bar{\alpha}$ is locally dually flat and $\bar{\beta}$ is dually related to $\bar{\alpha}$.

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