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Graph completions for impulsive feedback controls

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ABSTRACT

The paper considers impulsive control systems, where the evolution equation depends linearly on the time derivatives of the control function. The basic theory of "graph completions" is here extended to control functions in feedback form. In the case where the control $u = u(t, x)$ is piecewise smooth, with a jump along a hypersurface Σ in the t–x space, results are proved on the existence and uniqueness of solutions, and on their approximation by means of smooth feedbacks. The paper is concluded with a couple of examples, concerning the feedback control of mechanical systems by means of active constraints.

1. Introduction

For a vector-valued function of a scalar variable, the concept of a *graph completion* was introduced in [8]. Its main motivation comes from control theory. As shown in [6,4,14,18], the control of a mechanical system by means of active constraints often leads to a system of ODEs of the form

$$\dot{x} = f_0(t, x, u) + \sum_{k=1}^{m} f_k(t, x, u)\dot{u}_k, \tag{1.1}$$

where $t \mapsto x(t) \in \mathbb{R}^n$ describes the state of the system, while $t \mapsto u(t) \in \mathbb{R}^m$ is the control function. The upper dot denotes a derivative w.r.t. time. A key feature of (1.1) is that the right hand side depends not only on the control values $u = (u_1, \dots, u_m)$, but is also an affine function of the time derivatives $\dot{u}_k$.

To avoid technicalities, we assume that all functions $f_j : \mathbb{R}^{n+m} \mapsto \mathbb{R}^n$, $j = 0, 1, \dots, m$ are smooth and globally bounded. Notice that, by adding the state variables $x_0 = t$, $x_{n+j} = u_j$, $j = 1, \dots, m$, the system (1.2) can be written in the form

$$\dot{x} = f_0(x) + \sum_{k=1}^{m} f_k(x)\dot{u}_k, \tag{1.2}$$

where the vector fields $f_0, \dots, f_m$ depend only on the state variables $x_0, \dots, x_{n+m}$.

Since the right hand side of (1.2) contains the time derivatives $\dot{u}_k$, given an initial data

$$x(0) = \bar{x}, \qquad u(0) = \bar{u}, \tag{1.3}$$

to achieve existence and uniqueness of solutions it is natural to consider control functions $t \mapsto u(t) = (u_1, \dots, u_m)(t)$ which are absolutely continuous. However, as shown in [8], a solution to (1.2)–(1.3) can be uniquely determined also in the case

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where the control function $u(\cdot)$ is a function with bounded variation, provided that we "complete" its graph, bridging the points where a jump occurs.

Definition 1. A *graph completion* of a *BV* function $u : [0,T] \mapsto \mathbb{R}^m$ is a Lipschitz continuous path $\gamma = (\gamma_0;\ \gamma_1, \dots, \gamma_m) : [0,S] \mapsto [0,T] \times \mathbb{R}^m$ such that:

(i) $\gamma(0) = (0, u(0))$, $\gamma(S) = (T, u(T))$,
(ii) $\gamma_0(s_1) \leqslant \gamma_0(s_2)$ whenever $0 \leqslant s_1 < s_2 \leqslant S$,
(iii) for each $t \in [0,T]$ there exists some s such that $\gamma(s) = (t, u(t))$.

Notice that the path γ provides a continuous parameterization of the graph of u in the t–u space. At a time τ where u has a jump, the continuous curve γ must include an arc joining the left and right limits $(\tau, u(\tau-))$, $(\tau, u(\tau+))$. As soon as a graph completion of $u(\cdot)$ is assigned, we can solve the Cauchy problem

$$y(0) = \bar{x}, \qquad \frac{d}{ds} y(s) = f_0\big(y(s)\big) \frac{d}{ds} \gamma_0(s) + \sum_{k=1}^{m} f_k\big(y(s)\big) \frac{d}{ds} \gamma_k(s), \quad s \in [0,S]. \tag{1.4}$$

Notice that (1.4) has a unique Carathéodory solution, $s \mapsto y(s,\gamma)$, because the right hand side of the ODE is measurable w.r.t. s and Lipschitz continuous w.r.t. y. The (possibly multivalued) function

$$t \mapsto x(t;\gamma) = \big\{ y(s,\gamma);\ \gamma_0(s) = t \big\} \tag{1.5}$$

is then called the *generalized solution* of (1.2)–(1.3) determined by the graph completion $\gamma(\cdot)$ of the control function $u(\cdot)$.

Up to now, this theory has been developed in the setting where $u = u(t)$ is a (possibly discontinuous) function of time. Aim of this paper is to study the more general case where the control $u = u(t,x)$ is given in feedback form, depending on the state x as well. More precisely, we consider the equation

$$\dot{x} = f(x,u) + G(x,u)\dot{u}, \quad u = u(t,x). \tag{1.6}$$

Here $f : \mathbb{R}^{n+m} \mapsto \mathbb{R}^n$ and $G : \mathbb{R}^{n+m} \mapsto \mathbb{R}^{n\times m}$ are assumed to be smooth, globally bounded functions.

When the feedback function u is continuously differentiable, by the chain rule (1.6) yields

$$\dot{x} = f(x,u) + G(x,u)u_t + G(x,u)u_x \cdot \dot{x}. \tag{1.7}$$

Hence, assuming that the $n \times n$ matrix $I - G(x,u)u_x$ is invertible, one recovers the ODE in standard form

$$\dot{x} = \big(I - G(x,u)u_x\big)^{-1} \big(f(x,u) + G(x,u)u_t\big). \tag{1.8}$$

Here and throughout the following, for notational convenience we write $u_x \doteq \nabla_x u = (u_{x_1}, \dots, u_{x_n})$ for the gradient of u w.r.t. the space variables. We now rewrite (1.6) in an equivalent form, which allows for the possibility of a trajectory having jumps. Let $U = U(t,x)$ be a bounded, upper semicontinuous multifunction with compact values, so that

$$graph(U) \doteq \big\{ (t,x,u);\ u \in U(t,x) \big\} \subset [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \tag{1.9}$$

is closed.

Definition 2. A *BV* function $t \mapsto X(t)$, $t \in [0,T[$ is a *generalized solution* of the impulsive system

$$\dot{x} = f(x,u) + G(x,u)\dot{u}, \quad u \in U(t,x), \tag{1.10}$$

if there exists $S > 0$ and a Lipschitz continuous map

$$s \mapsto (t,x,u)(s) \in graph(U), \quad s \in [0,S[, \tag{1.11}$$

with $s \mapsto t(s)$ nondecreasing, such that:

(i) For every $\tau \in [0,T[$ there exists $s \in [0,S[$ such that

$$\tau = t(s), \qquad X(\tau) = x(s). \tag{1.12}$$

(ii) For a.e. $s \in [0,S[$ the following system is satisfied:

$$f(x,u)\frac{dt}{ds} + G(x,u)\frac{du}{ds} - \frac{dx}{ds} = 0. \tag{1.13}$$

We say that $t \mapsto X(t)$ is a solution of the initial value problem (1.6), (1.3) if, in addition, when $s = 0$ we have

$$t(0) = 0, \qquad X(0) = x(0) = \bar{x}, \qquad u(0) = \bar{u}. \tag{1.14}$$

In the single-valued case, where $U(t,x) = \{u(t,x)\}$, the function u is continuously differentiable, and the matrix $I - G(x,u)u_x$ is invertible, by a straightforward computation one checks that any generalized solution to (1.6) coincides (for a.e. time t) with a solution of (1.8).

A natural case, studied in Section 2 is where U is piecewise smooth, but multivalued along a hypersurface Σ in the t–x space. Conditions are found, which guarantee the existence and uniqueness of generalized solutions. Under suitable assumptions, in Section 3 we prove that these impulsive solutions can be obtained from a family of Lipschitz continuous approximating feedbacks u^ε, taking the limit as $\varepsilon \to 0$. Finally, Section 4 contains examples coming from mechanical models. The first example describes a skier on a narrow trail, who can modify his speed by changing the height of his barycenter. This problem, well studied in [5], is more naturally recast in the setting of impulsive feedback control. The second example describes a two-dimensional version of this model, showing that a snow disc saucer can be partially controlled by changing the height of the barycenter of the rider.

The literature on impulsive control systems is presently very extensive. Refs. [3,9,13,15–17] can only provide a partial sample. It is worth mentioning that graph completions have also been used in [11] and in [12] to define "nonconservative products" in connection with hyperbolic systems of PDEs. A first example of set-valued graph completions for functions of several variables can be found in [7]. For the general theory of multifunctions and differential inclusion we refer to [1,2].

2. Impulsive feedback controls

Consider a partition of the space $\mathbb{R} \times \mathbb{R}^n$ consisting of two open sets Ω^-, Ω^+, separated by a smooth hypersurface Σ. The closures of these two sets are thus $\overline{\Omega}^- = \Omega^- \cup \Sigma$, $\overline{\Omega}^+ = \Omega^+ \cup \Sigma$. Let $u^+, u^- : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^m$ be bounded, smooth feedback controls, and consider the piecewise smooth control

$$u(t,x) = \begin{cases} u^-(t,x) & \text{if } (t,x) \in \Omega^-, \\ u^+(t,x) & \text{if } (t,x) \in \Omega^+. \end{cases} \tag{2.1}$$

Definition 3. By a *graph completion* of u we mean a multifunction U of the form

$$U(t,x) = \begin{cases} \{u^-(t,x)\} & \text{if } (t,x) \in \Omega^-, \\ \{u^+(t,x)\} & \text{if } (t,x) \in \Omega^+, \\ \{\gamma(t,x,\theta);\ \theta \in [0,1]\} & \text{if } (t,x) \in \Sigma, \end{cases} \tag{2.2}$$

where $\theta \mapsto \gamma(t,x,\theta)$ is a smooth curve bridging the jump at the point $(t,x) \in \Sigma$. More precisely:

$$\gamma(t,x,0) = u^-(t,x), \qquad \gamma(t,x,1) = u^+(t,x), \tag{2.3}$$

for all $(t,x) \in \Sigma$. We always assume that γ does not have self-intersections: $\gamma(t,x,\theta) \neq \gamma(t,x,\theta')$ whenever $\theta \neq \theta'$.

When U has the special form (2.2), according to Definition 2 solutions to (1.6) are obtained by concatenating three types of trajectories:

(i) Solutions of the implicit ODE

$$\dot{x} = f\big(x,u^-\big) + G\big(x,u^-\big)\big(u_t^- + u_x^- \cdot \dot{x}\big), \quad (t,x) \in \overline{\Omega}^-; \tag{2.4}$$

(ii) Solutions of the implicit ODE

$$\dot{x} = f\big(x,u^+\big) + G\big(x,u^+\big)\big(u_t^+ + u_x^+ \cdot \dot{x}\big), \quad (t,x) \in \overline{\Omega}^+; \tag{2.5}$$

(iii) Trajectories $s \mapsto (t(s), x(s), \theta(s)) \in \mathbb{R} \times \mathbb{R}^n \times [0,1]$ of the system

$$f(x,\gamma)\frac{dt}{ds} + G(x,\gamma)\left[\gamma_t \frac{dt}{ds} + \gamma_x \cdot \frac{dx}{ds} + \gamma_\theta \frac{d\theta}{ds}\right] - \frac{dx}{ds} = 0, \quad (t,x) \in \Sigma, \tag{2.6}$$

with $dt/ds \geqslant 0$.

Remark 1. Since $\gamma = \gamma(t,x,\theta)$ is defined only for $(t,x) \in \Sigma$, the partial derivatives γ_t, γ_x are not defined. Hence the notation used in (2.6) may appear somewhat improper. However, since the vector $(\frac{dt}{ds}, \frac{dx}{ds})$ must be tangent to the hypersurface Σ, the expression $\gamma_t \frac{dt}{ds} + \gamma_x \cdot \frac{dx}{ds}$ is meaningful.

Trajectories of type (2.4) or (2.5) are obtained solving the system of n ODEs (1.8), with $u = u^{\pm}$. On the other hand, if $d\theta/ds > 0$, then we can take $s = \theta$ as the new independent variable along singular trajectories of type (2.6). This leads to the implicit system of $n+1$ ODEs

$$\begin{pmatrix} \mathbf{n}^t & \mathbf{n}^x \\ f + G\gamma_t & G\gamma_x - I \end{pmatrix} \begin{pmatrix} dt/d\theta \\ dx/d\theta \end{pmatrix} = \begin{pmatrix} 0 \\ -G\gamma_\theta \end{pmatrix}. \tag{2.7}$$

Here and throughout the following, $\mathbf{n} = (\mathbf{n}^t, \mathbf{n}^x)(t,x) \in \mathbb{R} \times \mathbb{R}^n$ denotes the unit normal vector to the surface Σ at the point $(t,x) \in \Sigma$, pointing toward the interior of Ω^+.

In order that the implicit ODEs in (2.4)–(2.6) have solutions, it is natural to impose the following conditions.

(A1) The functions f, G in (1.6) are in $\mathcal{C}^1(\mathbb{R}^{n+m})$, while γ and the feedback functions u^-, u^+ are twice continuously differentiable.

(A2) The $n \times n$ matrix $I - G(x, u^+(t,x))u_x^+(t,x)$ is invertible for all $(t,x) \in \overline{\Omega}^+$, while the matrix $I - G(x, u^-(t,x))u_x^-(t,x)$ is invertible for all $(t,x) \in \overline{\Omega}^-$.

(A3) As in Definition 3, the map γ is one-to-one and satisfies (2.3) for all $(t,x) \in \Sigma$.
For each $(t,x,\theta) \in \Sigma \times [0,1]$ the $(1+n) \times (1+n)$ matrix $A(t,x,\theta) = \begin{pmatrix} \mathbf{n}^t & \mathbf{n}^x \\ f+G\gamma_t & G\gamma_x - I \end{pmatrix}$ is invertible. Calling $A^{-1} = \begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix}$ its inverse, one has

$$\frac{dt}{d\theta} = -a^{12} G\gamma_\theta \geqslant 0. \tag{2.8}$$

We mention two cases where the technical condition (A3) is satisfied.

Case 1: As in the case of open-loop controls, the jump occurs at a fixed time τ. In this case the jump surface has the form $\Sigma = \{\tau\} \times \mathbb{R}^n$. This implies $\mathbf{n}^t = 1$, $\mathbf{n}^x = 0$. The system (2.7) thus reduces to

$$(G\gamma_x - I)\frac{dx}{d\theta} = -G\gamma_\theta, \qquad t = \tau. \tag{2.9}$$

This reduced system can be solved provided that the $n \times n$ matrix $B = G\gamma_x - I$ is invertible.

Case 2: The $n \times n$ submatrix $B \doteq G\gamma_x - I$ is invertible for all (t,x,θ). In this case we obtain

$$\frac{dx}{d\theta} = -B^{-1}\left[(f + G\gamma_t)\frac{dt}{d\theta} + G\gamma_\theta\right].$$

Since

$$0 = \mathbf{n}^t \frac{dt}{d\theta} + \left\langle \mathbf{n}^x, \frac{dx}{d\theta} \right\rangle = \left[\mathbf{n}^t - \left\langle \mathbf{n}^x, B^{-1}(f + G\gamma_t)\right\rangle\right]\frac{dt}{d\theta} - \left\langle \mathbf{n}^x, B^{-1} G\gamma_\theta \right\rangle, \tag{2.10}$$

the inequality (2.8) is then satisfied provided that

$$\frac{\langle \mathbf{n}^x, B^{-1} G\gamma_\theta \rangle}{\mathbf{n}^t - \langle \mathbf{n}^x, B^{-1}(f + G\gamma_t)\rangle} \geqslant 0. \tag{2.11}$$

In this connection, we observe that the quantity $\mathbf{n}^t - \langle \mathbf{n}^x, B^{-1}(f + G\gamma_t)\rangle$ in (2.11) is different from zero if and only if the matrix A is invertible. Indeed,

$$A = \begin{pmatrix} \mathbf{n}^t & \mathbf{n}^x \\ f + G\gamma_t & B \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{n}^x \\ 0 & B \end{pmatrix} \begin{pmatrix} \mathbf{n}^t - \langle \mathbf{n}^x, B^{-1}(f + G\gamma_t)\rangle & 0 \\ B^{-1}(f + G\gamma_t) & I \end{pmatrix},$$

and therefore

$$\det A = \det B \cdot \left[\mathbf{n}^t - \left\langle \mathbf{n}^x, B^{-1}(f + G\gamma_t)\right\rangle\right]. \tag{2.12}$$

We can now give the main result on the existence of generalized solutions to the impulsive system (1.6).

Theorem 1. *Assume that the multifunction $U : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^m$ has the form (2.2) and that* (A1)–(A3) *are satisfied. Let the initial data (1.3) be admissible, in the sense that $\bar{u} \in U(0, \bar{x})$.*

(i) *If at every point $(t,x) \in \Sigma$ one has*

$$\mathbf{n}^t + \left\langle \mathbf{n}^x, \left(I - G\left(x, u^+\right)u_x^+\right)^{-1}\left(f + G\left(x, u^+\right)u_t^+\right)\right\rangle \geqslant 0, \tag{2.13}$$

then there exists at least one generalized solution X of (1.6), (1.3) in the sense of Definition 2. This solution is defined on a maximal interval $[0, T[$ such that either $T = \infty$ or else the parameterization in (1.11) satisfies

$$\lim_{s \to S-} \left(t(s) + \left|X(s)\right|\right) = \infty. \tag{2.14}$$

(ii) *If at every point $(t,x)\in\Sigma$ in addition to (2.13) one has*

$$\mathbf{n}^t+\big\langle \mathbf{n}^x, \big(I-G\big(x,u^-\big)u_x^-\big)^{-1}\big(f+G\big(x,u^-\big)u_t^-\big)\big\rangle \geqslant 0, \tag{2.15}$$

then the above solution is unique.

Proof. 1. If $(0,\bar{x})\in\Omega^-$, then by the assumptions (A1)–(A2) the implicit ODE (2.4) has a unique solution $t\mapsto X(t)$ with initial data (1.3). This solution is defined on a maximal interval $[0,\tau[$ such that either $\lim_{t\to\tau-}(t+|X(t)|)=\infty$, or else $\lim_{t\to\tau-}(t,X(t))\in\Sigma$.

2. If $(0,\bar{x})\in\Omega^+$, then by the assumptions (A1)–(A2) the implicit ODE (2.5) has a unique solution $t\mapsto X(t)$ with initial data (1.3). This solution is defined on a maximal interval $[0,\tau[$ such that $\lim_{t\to\tau-}(t+|X(t)|)=\infty$. Notice that, if $\lim_{t\to\tau-}(t,X(t))\in\Sigma$, then by the tangency condition (2.13) the solution $X(\cdot)$ of (2.5) can be prolonged to a strictly larger time interval $[0,\tau+\delta]$, with $(t,X(t))\in\overline{\Omega}^+$ for all t.

3. Consider now the case where $(0,\bar{x})\in\Sigma$. The admissibility condition yields the existence of a unique $\bar{\theta}\in[0,1]$ such that $\gamma(0,\bar{x},\bar{\theta})=\bar{u}$.

By the assumptions (A1) and (A3), the implicit system of ODEs (2.7) has a unique solution defined for θ in a maximal interval $[\bar{\theta},\theta_{max}]$, where either $\theta_{max}=1$ or

$$\lim_{\theta\to\theta_{max}}\big(t(\theta)+\big|X(\theta)\big|\big)=\infty. \tag{2.16}$$

4. Statement (i) of the theorem now follows by putting together the three above steps.

- If $(0,\bar{x})\in\Omega^+$ then by step **2** the solution $t\mapsto X(t)$ is well defined and remains inside the closed, positively invariant set $\Omega^+\cap\Sigma$ for t in a maximal interval $[0,T[$.
- If $(0,\bar{x})\in\Sigma$, by step **3** we obtain a solution $\theta\mapsto(t(\theta),x(\theta),u(\theta))$ of (2.7) on a maximal interval $\theta\in[\bar{\theta},\theta_{max}[$. If (2.16) holds, then this solution already satisfies the conclusion of the theorem. If (2.16) does not hold, then $\theta_{max}=1$. In this case, let $T_2=t(1)$. By step **2** we can then prolong this generalized solution for $t\geqslant T_2$ by solving the Cauchy problem (2.5) with initial data given at $t=T_2$.
- If $(0,\bar{x})\in\Omega^-$, by step **1** we obtain a solution of (2.4) on a maximal interval $[0,T_1[$. If

$$\lim_{t\to T_1-}\big(t+\big|X(t)\big|\big)=\infty, \tag{2.17}$$

  the conclusion of the theorem is satisfied. Otherwise, we must have $\lim_{t\to T_1-}(t,X(t))\in\Sigma$. We can then prolong the solution to a larger time interval by constructing a solution of (2.7) with initial data

$$(t,x,u)(0)=\big(T_1,X(T_1),u^-\big(T_1,X(T_1)\big)\big).$$

  The remainder of the construction is as in the previous case.

5. Finally, we prove that if the additional condition (2.15) holds, then the generalized solution is unique. To fix the ideas, assume

$$(0,\bar{x})\in\Omega^-,\qquad \bar{u}=u^-(0,\bar{x}),$$

the other cases being similar. Let X, $\widetilde{X}$ be two generalized solutions of (1.6), (1.3).

As long as $(t,X(t))\in\Omega^-$, $(t,\widetilde{X}(t))\in\Omega^-$, by (2.4) these two functions provide solutions to the same Cauchy problem

$$\dot{x}=\big(I-G\big(x,u^-\big)u_x^-\big)^{-1}\big(f-G\big(x,u^-\big)u_t^-\big),\qquad x(0)=\bar{x}. \tag{2.18}$$

By assumption, the right hand side of the ODE in (2.18) is Lipschitz continuous. Therefore this solution, defined on a maximal interval $[0,T_1[$, is unique.

If (2.17) holds, we are done. Otherwise $(T_1,X(T_1))=(T_1,\widetilde{X}(T_1))\in\Sigma$. By (2.15) and the regularity of Σ and u^-, there can be no solution of (2.4) that starts at a point $(\bar{t},\bar{x})\in\Sigma$ and enters in the open set Ω^- at a later time. Similarly, by (2.13) and the regularity of Σ and u^+, there can be no solution of (2.4) that starts at a point $(\bar{t},\bar{x})\in\Omega^+$ and reaches Σ at a later time. Therefore, for $t\geqslant T_1$ both X and $\widetilde{X}$ must be a concatenation of a solutions to (2.6), then a solution to (2.5).

We thus consider two solutions $s\mapsto(t(s),x(s),\theta(s))$ and $\sigma\mapsto(\tilde{t}(\sigma),\tilde{x}(\sigma),\tilde{\theta}(\sigma))$ of (2.6), with the same initial data

$$\big(t(0),x(0),\theta(0)\big)=\big(\tilde{t}(0),\tilde{x}(0),\tilde{\theta}(0)\big)=\big(T_1,X(T_1),0\big). \tag{2.19}$$

By the assumption (A3), both of these solutions can be uniquely re-parameterized in terms of the variable θ. In this case, the corresponding functions $\theta\mapsto(t(\theta),x(\theta))$ and $\theta\mapsto(\tilde{t}(\theta),\tilde{x}(\theta))$ provide solutions to the same Cauchy problem (2.7) with initial data

$$t(0)=\tilde{t}(0)=T_1,\qquad x(0)=\tilde{x}(0)=X(T_1).$$

By the assumptions (A1) and (A3), the matrix A has $\mathcal{C}^1$ coefficients and is invertible. Therefore the Cauchy problem for (2.7) has a unique solution.

If this solution is defined for θ in a maximal interval $[0, \theta_{max}[$ with $\theta_{max} \leqslant 1$ and

$$\lim_{\theta \to \theta_{max}-} \big(t(\theta) + \big|X(\theta)\big|\big) = \infty,$$

then we are done. Otherwise, consider the terminal value of this solution for $\theta = 1$, namely $(T_2, X(T_2)) = (T_2, \widetilde{X}(T_2)) = (t(1), x(1))$. Then the two solutions X, $\widetilde{X}$ can be extended beyond the time $t = T_2$ only as solutions of the implicit ODE (2.5), with the same initial data at $t = T_2$. By (A2), (2.5) is equivalent to the ODE

$$\dot{x} = \big(I - G\big(x, u^+\big)u_x^+\big)^{-1}\big(f - G\big(x, u^+\big)u_t^+\big). \tag{2.20}$$

By (A1)–(A2), the right hand side of (2.20) is locally $\mathcal{C}^1$, hence this Cauchy problem has a unique solution. □

3. Lipschitz approximations of impulsive feedbacks

Let the multifunction U have the form (2.2). In the same setting of Theorem 1, we shall construct a family of Lipschitz continuous feedback controls $u^\varepsilon(t, x)$ such that the classical solutions of the implicit ODE

$$\dot{x} = f\big(x, u^\varepsilon\big) + G\big(x, u^\varepsilon\big)u_t^\varepsilon + G\big(x, u^\varepsilon\big)u_x^\varepsilon \dot{x} \tag{3.1}$$

are well defined and converge as $\varepsilon \to 0$ to the generalized solutions of the impulsive equation (1.10), in a suitable sense.

The signed distance from the point (t, x) to the hypersurface Σ will be denoted by

$$d_\Sigma(t, x) \doteq \begin{cases} d((t, x), \Sigma) & \text{if } (t, x) \in \Omega^+ \cup \Sigma, \\ -d((t, x), \Sigma) & \text{if } (t, x) \in \Omega^-. \end{cases}$$

We assume that this function d_Σ is of class $\mathcal{C}^{2,1}$ on the set $\{(t, x);\ 0 \leqslant d_\Sigma(t, x) < \bar{\varepsilon}\}$, for some $\bar{\varepsilon} > 0$. For a given $\varepsilon \in]0, \bar{\varepsilon}]$, consider the region

$$\Sigma^\varepsilon \doteq \big\{(t, x);\ 0 \leqslant d_\Sigma(t, x) < \varepsilon\big\}.$$

Notice that every point $(t, x) \in \Sigma^\varepsilon$ can be uniquely represented as

$$(t, x) = (t_0, x_0) + \theta\varepsilon\mathbf{n}(t_0, x_0) \quad \text{with } (t_0, x_0) \in \Sigma,\ \theta \in [0, 1]. \tag{3.2}$$

As before, $\mathbf{n}(t_0, x_0) \in \mathbb{R}^{1+n}$ denotes the unit normal vector to Σ at the point (t_0, x_0), pointing toward the interior of Ω^+. We define the perpendicular projection $\pi : \Sigma^\varepsilon \mapsto \Sigma$ by setting

$$\pi(t, x) \doteq (t_0, x_0) \tag{3.3}$$

whenever (3.2) holds. Moreover, we define the map $\tilde{\gamma} : \Sigma^\varepsilon \times [0, 1] \mapsto \mathbb{R}^m$ as

$$\tilde{\gamma}(t, x, \theta) \doteq \gamma\big(\pi(t, x), \theta\big) + u^+(t, x) - u^+\big(\pi(t, x)\big). \tag{3.4}$$

Notice that $\tilde{\gamma}$ does not depend on ε. We can now define an approximating feedback control u^ε by setting

$$u^\varepsilon(t, x) \doteq \begin{cases} u^-(t, x) & \text{if } (t, x) \in \Omega^-, \\ \tilde{\gamma}(t, x, \varepsilon^{-1}d_\Sigma(t, x)) & \text{if } (t, x) \in \Sigma^\varepsilon, \\ u^+(t, x) & \text{if } (t, x) \in \Omega^+ \setminus \Sigma^\varepsilon. \end{cases} \tag{3.5}$$

For every given $\varepsilon > 0$, by (3.4) the feedback function u^ε is Lipschitz continuous.

Next, consider a set of admissible initial data (1.3) for (1.10). The admissibility condition means that

$$\begin{cases} (0, \bar{x}) \in \Omega^- & \Longrightarrow \quad \bar{u} = u^-(0, \bar{x}), \\ (0, \bar{x}) \in \Omega^+ & \Longrightarrow \quad \bar{u} = u^+(0, \bar{x}), \\ (0, \bar{x}) \in \Sigma & \Longrightarrow \quad \bar{u} = \gamma(0, \bar{x}, \bar{\theta}) \quad \text{for some } \bar{\theta} \in [0, 1]. \end{cases} \tag{3.6}$$

In connection with this initial data, we consider a corresponding set of initial data for (3.1):

$$\begin{cases} x(0) = \bar{x} & \text{if } (0, \bar{x}) \notin \Sigma, \\ x(\bar{t}_\varepsilon) = \bar{x}_\varepsilon & \text{if } (0, \bar{x}) \in \Sigma \end{cases} \tag{3.7}$$

where

$$\big(\bar{t}^\varepsilon, \bar{x}^\varepsilon\big) = (0, \bar{x}) + \bar{\theta}\varepsilon\mathbf{n}(0, \bar{x}). \tag{3.8}$$

The next result shows that the Cauchy problem (3.1), (3.7) has a well defined solution, converging to the generalized solution of (1.10), (1.3) in the sense of the graph, as $\varepsilon \to 0$.

Theorem 2. *Assume that the multifunction $U:[0,T]\times\mathbb{R}^n\to\mathbb{R}^m$ has the form (2.2) and that (A1)–(A3) hold. In addition, assume that*

$$\mathbf{n}^t-\bigl\langle\mathbf{n}^x,\bigl(Gu_x^\pm-I\bigr)^{-1}\bigl(f+Gu_t^\pm\bigr)\bigr\rangle\geqslant 0\quad\text{for all }(t,x)\in\Sigma,\tag{3.9}$$

and that the $n\times n$ matrix $\widetilde{B}\doteq G\tilde{\gamma}_x-I$ is invertible for all $(t,x,\theta)\in\Sigma\times[0,1]$.

Let $t\mapsto X(t)$ be a generalized solution to (1.10), (1.3), defined for $t\in[0,T]$. As in Definition 2, let this solution be parameterized by $s\mapsto(t(s),x(s))$, $s\in[0,S]$.

Then, for every $\varepsilon>0$ sufficiently small, the Cauchy problem (3.1), (3.7) has a well defined solution $t\mapsto X^\varepsilon(t)$, defined on some interval $[\bar{t}_\varepsilon,T_\varepsilon]$.

As $\varepsilon\to0$, one has $\bar{t}_\varepsilon\to0$, $T_\varepsilon\to T$ and the solution X^ε converges to X in the sense of graphs. Indeed, there exist continuous increasing functions $s\mapsto t^\varepsilon(s)$ from $[0,S]$ to $[\bar{t}_\varepsilon,T_\varepsilon]$ such that

$$\lim_{\varepsilon\to0}\sup_{s\in[0,S]}\bigl\{\bigl|t^\varepsilon(s)-t(s)\bigr|+\bigl|X^\varepsilon\bigl(t^\varepsilon(s)\bigr)-X\bigl(t(s)\bigr)\bigr|\bigr\}=0.\tag{3.10}$$

Proof. Before working out details, we explain the main idea. The solution X^ε of (3.1), (3.7) is given by a continuous concatenation of trajectories of the implicit ODEs

$$\dot{x}=f\bigl(x,u^-\bigr)+G\bigl(x,u^-\bigr)\bigl(u_t^-+u_x^-\dot{x}\bigr),\quad (t,x)\in\Omega^-,\tag{3.11}$$

$$\dot{x}=f(x,\tilde{\gamma})+G(x,\tilde{\gamma})\left[\tilde{\gamma}_t+\frac{1}{\varepsilon}\tilde{\gamma}_\theta\mathbf{n}^t+\left(\tilde{\gamma}_x+\frac{1}{\varepsilon}\tilde{\gamma}_\theta\otimes\mathbf{n}^x\right)\dot{x}\right],\quad (t,x)\in\Sigma^\varepsilon,\tag{3.12}$$

$$\dot{x}=f\bigl(x,u^+\bigr)+G\bigl(x,u^+\bigr)\bigl(u_t^++u_x^+\dot{x}\bigr),\quad (t,x)\in\Omega^+\setminus\Sigma^\varepsilon.\tag{3.13}$$

Here the function $\mathbf{n}=(\mathbf{n}^t,\mathbf{n}^x)$ is defined also for $(t,x)\in\Sigma^\varepsilon$ by setting $\mathbf{n}(t,x)\doteq\mathbf{n}(\pi(t,x))$, where π is the perpendicular projection introduced at (3.2)–(3.3).

Clearly, (3.11) and (3.13) are the same as (2.4)–(2.5). The main goal of the proof is to show that any trajectory $\theta\mapsto(t(\theta),x(\theta))$ of (2.7) can be uniformly approximated by trajectories of (3.12), parameterized by

$$\theta\mapsto\bigl(t^\varepsilon(\theta),x^\varepsilon(\theta)\bigr)\in\Sigma^\varepsilon,\tag{3.14}$$

where the parameter $\theta\in$ is chosen so that

$$\theta\doteq\frac{1}{\varepsilon}d_\Sigma\bigl(t,x^\varepsilon(t)\bigr).\tag{3.15}$$

More precisely, let $t\mapsto x^\varepsilon(t)$ be a solution to (3.12). We will show that:

(i) For $\varepsilon>0$ sufficiently small, the map $t\mapsto\theta(t)$ defined by (3.15) is a strictly increasing function of time. Therefore, the parameterization of (3.14) is well defined.

(ii) In terms of the variable θ, the solution $(t,x(t))$ of (3.12) satisfies the implicit system of ODEs

$$\begin{pmatrix}\mathbf{n}^t & \mathbf{n}^x\\ f+G\tilde{\gamma}_t & G\tilde{\gamma}_x-I\end{pmatrix}\begin{pmatrix}dt/d\theta\\ dx/d\theta\end{pmatrix}=\begin{pmatrix}\varepsilon\\ -G\tilde{\gamma}_\theta\end{pmatrix}.\tag{3.16}$$

As $\varepsilon\to0$, this system approaches (2.7). Since the Cauchy problem for (2.7) is well posed, this will yield the convergence of solutions to (3.16) to the solutions of (2.7), uniformly for θ in compact sets.

1. From (3.15) it follows

$$\frac{d\theta}{dt}=\frac{1}{\varepsilon}\bigl(\mathbf{n}^t+\langle\mathbf{n}^x,\dot{x}\rangle\bigr),$$

which is equivalent to the first equation in (3.16). Moreover, using (3.12) and then the first equation in (3.16) we obtain

$$\frac{dx}{d\theta}=\left\{f(x,\tilde{\gamma})+G(x,\tilde{\gamma})\left(\tilde{\gamma}_t+\frac{1}{\varepsilon}\tilde{\gamma}_\theta\mathbf{n}^t\right)\right\}\frac{dt}{d\theta}+G(x,\tilde{\gamma})\left(\tilde{\gamma}_x+\frac{1}{\varepsilon}\tilde{\gamma}_\theta\otimes\mathbf{n}^x\right)\frac{dx}{d\theta},$$

$$\bigl[f(x,\tilde{\gamma})+G(x,\tilde{\gamma})\tilde{\gamma}_t\bigr]\frac{dt}{d\theta}+\bigl[G(x,\tilde{\gamma})\tilde{\gamma}_x-I\bigr]\frac{dx}{d\theta}=-G\tilde{\gamma}_\theta\cdot\frac{1}{\varepsilon}\left[\mathbf{n}^t\frac{dt}{d\theta}+\left\langle\mathbf{n}^x,\frac{dx}{d\theta}\right\rangle\right]=-G\tilde{\gamma}_\theta,$$

proving that (3.16) holds.

2. The following computations will show that, for $\varepsilon>0$ sufficiently small, the solution of (3.16) satisfies the inequality $dt/d\theta>0$. We remark that, for solutions of (3.12), the corresponding inequality (2.8) is contained in the assumption (A3).

As a first step, we prove that the assumption (2.8) implies

$$\bigl\langle\mathbf{n}^x,\widetilde{B}^{-1}G\tilde{\gamma}_\theta\bigr\rangle\geqslant0,\tag{3.17}$$

where $\tilde{\gamma}$ and $\widetilde{B} = G\tilde{\gamma}_x - I$ and are evaluated at $(t, x, \theta) \in \Sigma \times [0, 1]$. Observe that, by choosing $\bar{\varepsilon} > 0$ sufficiently small, we can assume that the matrix $\widetilde{B}$ is invertible on $\Sigma^{\bar{\varepsilon}}$.

To prove (3.17), observing that the determinant of the matrix $A = \begin{pmatrix} \mathbf{n}^t & \mathbf{n}^x \\ f+G\gamma_t & G\gamma_x - I \end{pmatrix}$ does not depend on the extension of the map γ over a neighborhood of Σ, as in (2.12) we compute

$$\det A = \det \widetilde{B} \cdot \big[\mathbf{n}^t - \big\langle \mathbf{n}^x, \widetilde{B}^{-1}(f + G\tilde{\gamma}_t)\big\rangle\big].$$

Since A is invertible, the continuous function

$$\eta(t, x, \theta) \doteq \frac{\det A}{\det \widetilde{B}} = \mathbf{n}^t - \big\langle \mathbf{n}^x, \widetilde{B}^{-1}(f + G\tilde{\gamma}_t)\big\rangle$$

is different from zero for every $(t, x, \theta) \in \Sigma \times [0, 1]$. Moreover, for each $(t, x) \in \Sigma$ one has

$$\tilde{\gamma}_t(t, x, 1) = u_t^+(t, x), \qquad \tilde{\gamma}_x(t, x, 1) = u_x^+(t, x).$$

Hence (3.9) yields $\eta(t, x, 1) > 0$. The continuity of η implies

$$\mathbf{n}^t - \big\langle \mathbf{n}^x, \widetilde{B}^{-1}(f + G\tilde{\gamma}_t)\big\rangle > 0 \quad \text{for all } (t, x, \theta) \in \Sigma \times [0, 1]. \tag{3.18}$$

From (2.7) one obtains

$$\mathbf{n}^t \cdot \frac{dt}{d\theta} + \langle \mathbf{n}^x, \dot{x}\rangle = \mathbf{n}^t \cdot \frac{dt}{d\theta} - \left\langle \mathbf{n}^x, \widetilde{B}^{-1}(f + G\tilde{\gamma}_t)\frac{dt}{d\theta} + \widetilde{B}^{-1}G\tilde{\gamma}_\theta \right\rangle = 0.$$

By the assumption (2.8), the derivative $dt/d\theta$ is non-negative. Hence

$$\frac{dt}{d\theta} = \frac{\langle \mathbf{n}^x, \widetilde{B}^{-1}G\tilde{\gamma}_\theta\rangle}{\mathbf{n}^t - \langle \mathbf{n}^x, \widetilde{B}^{-1}(f + G\tilde{\gamma}_t)\rangle} \geqslant 0. \tag{3.19}$$

The inequality (3.17) thus follows from (3.18) and (3.19).

To avoid confusion, we shall denote by $\theta \mapsto (t^\varepsilon(\theta), x^\varepsilon(\theta))$ a solution to (3.16). By assumption, $\widetilde{B} = G\tilde{\gamma} - I$ is invertible on $\Sigma^\varepsilon \times [0, 1]$, for $\varepsilon > 0$ sufficiently small. Using (3.19) we thus obtain

$$\begin{aligned} &\frac{dx^\varepsilon}{d\theta} = \widetilde{B}^{-1}\left[-G\tilde{\gamma}_\theta - (f + G\tilde{\gamma}_t)\frac{dt^\varepsilon}{d\theta}\right], \\ &\frac{dt^\varepsilon}{d\theta} = \frac{\varepsilon + \langle \mathbf{n}^x, \widetilde{B}^{-1}G\tilde{\gamma}_\theta\rangle}{\mathbf{n}^t - \langle \mathbf{n}^x, \widetilde{B}^{-1}(f + G\tilde{\gamma}_t)\rangle} > 0. \end{aligned} \tag{3.20}$$

3. We now conclude the proof, showing the convergence (3.10).

(i) If $(\bar{t}, \bar{x}) \in \Omega^+$ the result is trivial. Indeed, the solutions $X(\cdot)$ and X^ε coincide for all $t \geqslant 0$, provided that $\varepsilon > 0$ is small enough.

(ii) If $(\bar{t}, \bar{x}) \in \Omega^-$, then the solution of (1.10), (2.2) is obviously the same the solution of (3.1), (3.5), up to the first time τ where $(\tau, x(\tau)) \in \Sigma$.

(iii) To prove the theorem, it thus suffices to examine the case of an initial data $(0, \bar{x}) \in \Sigma$, and $\bar{u} = \gamma(0, \bar{x}, \bar{\theta})$ for some $\bar{\theta} \in [0, 1]$.

Since $\mathbf{n}(0, \bar{x})$ in (3.8) is a unit vector, it is clear that $|\bar{t}^\varepsilon| = |\bar{\theta}\varepsilon\mathbf{n}^t| \leqslant \varepsilon$, hence $\bar{t}^\varepsilon \to 0$ as $\varepsilon \to 0$. By the previous analysis, for each $\varepsilon > 0$ sufficiently small the implicit ODEs (3.16) determine a unique solution $\theta \mapsto (t^\varepsilon(\theta), x^\varepsilon(\theta))$. As $\varepsilon \to 0$, this solution converges to the solution $\theta \mapsto (t(\theta), x(\theta))$ of (2.7), uniformly for $\theta \in [\bar{\theta}, 1]$. In particular, when $\theta = 1$ we have the convergence

$$t^\varepsilon(1) \to t(1), \qquad x^\varepsilon(1) \to x(1). \tag{3.21}$$

Let $q \doteq T - t(1)$ be the length of the time interval where the reference trajectory X lies in Ω^+, and set $T_\varepsilon = t^\varepsilon(1) + q$.

For $t \in [t^\varepsilon(1), T_\varepsilon]$, each trajectory X^ε can be continued as a solution to the Cauchy problem (3.13) with initial data $X(t^\varepsilon(1)) = x^\varepsilon(1)$. Since this implicit ODE is well posed, as $\varepsilon \to 0$ this solution converges to the solution X of (3.13) with data $X(t(1)) = x(1)$. $\square$

4. Examples

In this last section we consider two examples of mechanical systems controlled by active constraints given in feedback form. A comprehensive introduction to the theory of control of mechanical systems can be found in [10].

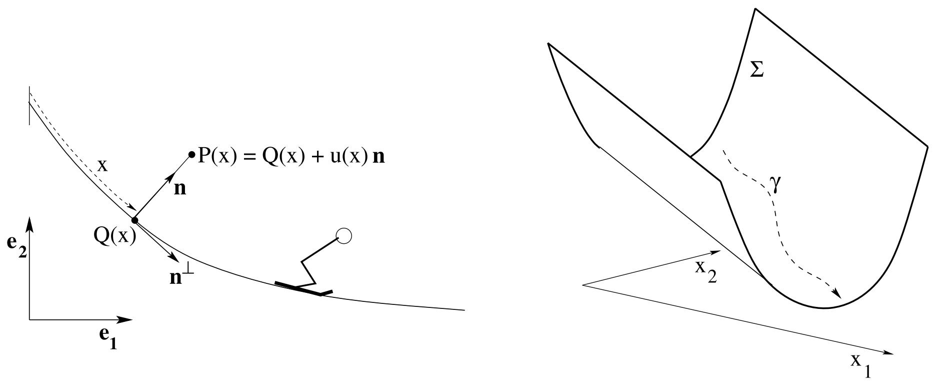

Fig. 1. Left: in Example 1, $P(x)$ denotes the barycenter of the skier, on a vertical plane. Right: in Example 2, given a curve γ on the surface Σ, we seek a solution of the system (4.16) such that the corresponding point $Q(x) = (x_1, x_1, h(x_1, x_2))$ moves along γ.

Example 1 *(Skier on a narrow trail).* Following [5], consider a skier sliding downhill on a narrow trail. As shown in Fig. 1 left, let x be the arc-length coordinate of a point $Q(x)$ along the trail, and let $\mathbf{n}(x)$, $\mathbf{t}(x) = \mathbf{n}^\perp(x)$ denote the normal and the tangent unit vector to the trail, at $Q(x)$.

Assume that the skier, by standing up or bending down, can change the position of his barycenter w.r.t. the trail. Calling P the position of the barycenter, we assume that this can be assigned as a function of x, in the form

$$P(x) = Q(x) + u(x)\mathbf{n}(x), \tag{4.1}$$

where $x \mapsto u(x)$ is regarded as a feedback control, depending only on the position of the skier. Differentiating w.r.t. time one obtains

$$\dot{P} = \left[\mathbf{n}^\perp(x) + \frac{\partial u}{\partial x}\mathbf{n}(x) + u(x)\frac{\partial \mathbf{n}}{\partial x}\right]\dot{x}. \tag{4.2}$$

Approximating the skier by a point mass concentrated at the barycenter, we obtain the equation of motion

$$\ddot{P}(t) = \lambda(t)\mathbf{n}\big(x(t)\big) - g\mathbf{e}_2, \tag{4.3}$$

where the first term on the right hand side accounts for the constraint reactions and the second term represents gravity. To derive a scalar ODE for the function $x = x(t)$, consider the curvature of the trail at the point $Q(x)$:

$$\kappa(x) = \left\langle \frac{\partial \mathbf{n}^\perp}{\partial x}, \mathbf{n} \right\rangle = -\left\langle \frac{\partial \mathbf{n}}{\partial x}, \mathbf{n}^\perp \right\rangle.$$

From (4.2) it follows

$$\frac{d}{dt}\langle \dot{P}, \mathbf{n}^\perp\rangle = \frac{d}{dt}\big[(1-\kappa u)\dot{x}\big] = (1-\kappa u)\ddot{x} - \kappa u_x \dot{x}^2 - \kappa_x u \dot{x}^2. \tag{4.4}$$

On the other hand, (4.3) yields

$$\frac{d}{dt}\langle \dot{P}, \mathbf{n}^\perp\rangle = -\langle g\mathbf{e}_2, \mathbf{n}^\perp\rangle + \langle u_x \mathbf{n}\dot{x}, \mathbf{n}_x^\perp \dot{x}\rangle = -\langle g\mathbf{e}_2, \mathbf{n}^\perp\rangle + \kappa u_x \dot{x}^2. \tag{4.5}$$

Together, (4.4)–(4.5) imply

$$\big(1 - \kappa(x)u\big)\ddot{x} = 2\kappa(x)u_x\dot{x}^2 + u\kappa_x\dot{x}^2 - \big\langle g\mathbf{e}_2, \mathbf{n}^\perp(x)\big\rangle. \tag{4.6}$$

If we assume that the feedback control is always smaller than the radius of curvature, so that $u(x) < \kappa^{-1}(x)$ for all x, then Eq. (4.6) can be written as a first order system of the same type as (1.1):

$$\begin{cases} \dot{x} = v, \\ \dot{v} = \dfrac{1}{1-\kappa(x)u}\big[2\kappa(x)v\dot{u} + u\kappa_x(x)v^2 + f(x)\big], \end{cases} \tag{4.7}$$

where $f(x) \doteq -\langle g\mathbf{e}_2, \mathbf{n}^\perp(x)\rangle$. Under the natural assumption $v > 0$, meaning that the skier always moves forward, one can eliminate time from the equation and write one single ODE for $v = v(x)$, namely

$$\frac{dv}{dx} = \frac{1}{1-\kappa(x)u}\left[2v\kappa u_x + u\kappa_x v + \frac{f(x)}{v}\right]. \tag{4.8}$$

Regarding x as the new time variable, the map $x \mapsto u(x)$ becomes an open-loop control. In this special case, when u is a control of bounded variation, the graph completion of u is a special case of the theory developed in [8].

Example 2 *(Snow disc).* We now consider a two-dimensional extension of the previous model. Let $x = (x_1, x_2)$ be the two-dimensional horizontal coordinate, and let

$$\Sigma \doteq \{Q(x) = (x_1, x_2, z);\ z = h(x_1, x_2)\}$$

describe the surface of a ski slope. Here $h(x)$ is the height at the point x. Let $\mathbf{n}(x)$ be a vector perpendicular to the surface, pointing upward. Assume that a feedback control $u = u(x)$ can be implemented, so that the barycenter of the snow boarder is located at

$$P(x) = Q(x) + u(x)\mathbf{n}(x). \tag{4.9}$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard orthonormal basis in $\mathbb{R}^3$, with $\mathbf{e}_3$ pointing in the upward vertical direction. Differentiating (4.9) w.r.t. time one obtains

$$\dot{P}(x) = \dot{x}_1\mathbf{e}_1 + \dot{x}_2\mathbf{e}_2 + \dot{h}\mathbf{e}_3 + \dot{u}\mathbf{n} + u\dot{\mathbf{n}}. \tag{4.10}$$

As before, the equations of motion can be written as

$$\ddot{P} = \lambda(t)\mathbf{n}(x) - g\mathbf{e}_3, \tag{4.11}$$

where the first term accounts for the force produced by the active constraint, while the second is due to gravity. Taking the inner product of (4.11) with the two vectors

$$\mathbf{t}_1 = \mathbf{e}_1 + h_{x_1}\mathbf{e}_3, \qquad \mathbf{t}_2 = \mathbf{e}_2 + h_{x_2}\mathbf{e}_3,$$

both tangent to the surface Σ and hence orthogonal to $\mathbf{n}$, we obtain a system of equations for the components x_1, x_2, $v_1 = \dot{x}_1$, $v_2 = \dot{x}_2$.

In the following, we consider the case where

$$h(x_1, x_2) = -x_1 + \frac{x_2^2}{2}, \tag{4.12}$$

shown in Fig. 1, right. Since

$$\mathbf{n}(x) = \mathbf{e}_1 - x_2\mathbf{e}_2 + \mathbf{e}_3, \qquad \dot{\mathbf{n}} = -\dot{x}_2\mathbf{e}_2,$$

Eq. (4.10) takes the form

$$\dot{P}(x) = (\dot{x}_1 + \dot{u})\mathbf{e}_1 + (\dot{x}_2 - x_2\dot{u} - \dot{x}_2 u)\mathbf{e}_2 + (-\dot{x}_1 + x_2\dot{x}_2 + \dot{u})\mathbf{e}_3. \tag{4.13}$$

Taking the inner product of (4.13) with the vector $\mathbf{t}_1 = \mathbf{e}_1 - \mathbf{e}_3$ and differentiating w.r.t. time, we obtain

$$\frac{d}{dt}\langle \dot{P}(x), \mathbf{t}_1\rangle = 2\ddot{x}_1 - \dot{x}_2^2 - x_2\ddot{x}_2 = g. \tag{4.14}$$

Here the last equality follows from (4.11). Performing a similar computation with the tangent vector $\mathbf{t}_2 = \mathbf{e}_2 + x_2\mathbf{e}_3$, one obtains

$$\begin{aligned}\frac{d}{dt}\langle \dot{P}(x), \mathbf{t}_2\rangle &= \ddot{x}_2 - \ddot{x}_2 u - \dot{x}_2\dot{u} - \ddot{x}_1 x_2 - \dot{x}_1\dot{x}_2 + 2x_2\dot{x}_2^2 + x_2^2\ddot{x}_2 \\ &= -gx_2 + (-\dot{x}_1 + x_2\dot{x}_2 + \dot{u})\dot{x}_2.\end{aligned} \tag{4.15}$$

Eqs. (4.14) and (4.15) yield a system of ODEs for the velocities $(v_1, v_2) = (\dot{x}_1, \dot{x}_2)$, namely

$$\begin{pmatrix}\dot{v}_1\\ \dot{v}_2\end{pmatrix} = \left(1 + \frac{x_2^2}{2} - u\right)^{-1}\left[\frac{v_2^2 + g}{2}\begin{pmatrix}1-u\\ -x_2\end{pmatrix} + v_2\begin{pmatrix}x_2\\ 2\end{pmatrix}\dot{u}\right]. \tag{4.16}$$

We are primarily interested in a controllability issue. Let a smooth curve

$$\gamma \doteq \{(x_1, x_2, h(x_1, x_2));\ x_2 = \varphi(x_1)\} \subset \Sigma \tag{4.17}$$

be given. We seek an impulsive feedback control of the type

$$u(x) = \begin{cases} u^-(x) & \text{if } x_2 < \varphi(x_1), \\ u^+(x) & \text{if } x_2 > \varphi(x_1), \end{cases} \tag{4.18}$$

keeping the trajectory of the impulsive system (4.16) on the curve γ. More precisely,

$$Q(t) = (x_1, x_2, h(x_1, x_2))(t) \in \gamma \quad \text{for all } t \in [0, T].$$

Consider first the case where the initial data lies on γ, with velocity tangent to the curve:

$$x_2(0) = \varphi\big(x_1(0)\big), \qquad v_2(0) = \varphi'\big(x_1(0)\big)v_1(0). \tag{4.19}$$

In this case, the point $Q(t)$ remains on γ provided that

$$\frac{v_2}{v_1} = \varphi'(x_1), \qquad \frac{d}{dt}\left(\frac{v_2}{v_1}\right) = \frac{\dot v_2 v_1 - v_2\dot v_1}{v_1^2} = \varphi''(x_1)v_1. \tag{4.20}$$

Introducing the quantity $\beta \doteq 1 + \frac{[\varphi(x_1)]^2}{2} - u$ and using (4.16), (4.19), we obtain

$$\begin{cases} \dot x_1 = v_1, \\ 2\beta\dot v_1 = \big(\big[\varphi'(x_1)\big]^2 v_1^2 + g\big)(1-u) + 2v_1\varphi(x_1)\varphi'(x_1)\dot u, \\ 2\beta\varphi'(x_1)\dot v_1 + 2\beta\varphi''(x_1)v_1^2 = -\big(\big[\varphi'(x_1)\big]^2 v_1^2 + g\big)\varphi(x_1) + 4v_1\varphi'(x_1)\dot u. \end{cases}$$

This can be written as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2\beta & -2v_1\varphi(x_1)\varphi'(x_1) \\ 0 & 2\beta\varphi'(x_1) & -4v_1\varphi'(x_1) \end{pmatrix} \begin{pmatrix} \dot x_1 \\ \dot v_1 \\ \dot u \end{pmatrix} = \begin{pmatrix} v_1 \\ ([\varphi'(x_1)]^2 v_1^2 + g)(1-u) \\ -([\varphi'(x_1)]^2 v_1^2 + g)\varphi(x_1) - 2\beta\varphi''(x_1)v_1^2 \end{pmatrix}. \tag{4.21}$$

The system (4.21) has a unique solution with initial value $(x_1(0), v_1(0), \bar u)$ whenever the matrix on the left hand side is invertible, i.e.

$$\beta v_1\varphi'(x_1)\big[\varphi(x_1)\varphi'(x_1) - 2\big] \neq 0. \tag{4.22}$$

Next, consider a feedback impulsive control of the form (4.18), and its graph completion

$$U(x) = \begin{cases} \{u^-(x)\} & \text{if } x_2 < \varphi(x_1), \\ \{u^+(x)\} & \text{if } x_2 > \varphi(x_1), \\ \{su^+(x) + (1-s)u^-(x);\ s \in [0,1]\} & \text{if } x_2 = \varphi(x_1). \end{cases}$$

Let $t \mapsto (x_1, v_1, u)(t)$ be a solution to (4.21). Then the map

$$t \mapsto (x, v, u)(t) = (x_1, x_2, v_1, v_2, u)(t) = \big(x_1, \varphi(x_1), v_1, \varphi'(x_1)v_1, u\big)(t)$$

is a solution to the impulsive feedback equation (4.16) with $u \in U(x)$, as long as

$$u(t) \in \big\{su^+\big(x(t)\big) + (1-s)u^-\big(x(t)\big);\ s \in [0,1]\big\}.$$

Next, we study what happens if a trajectory starts on the curve γ with a non-tangential velocity. To fix the ideas, assume

$$u(0) = \bar u, \qquad \begin{cases} x_1(0) = \bar x_1 > 0, \\ x_2(0) = \bar x_2 = \varphi(\bar x_1), \end{cases} \qquad \begin{cases} v_1(0) = \bar v_1 > 0, \\ v_2(0) = \bar v_2 > \varphi'(\bar x_1)\bar v_1. \end{cases} \tag{4.23}$$

In this case, the initial motion has impulsive character. The size of the initial jump in the velocity v and in the control u, at time $t = 0$, can be computed as follows. Take $\theta = u$ as independent variable. Keeping in mind that $t \equiv 0$, from (4.16) we obtain

$$\frac{d}{d\theta}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \left(1 + \frac{\bar x_2^2}{2} - \theta\right)^{-1} \begin{pmatrix} \bar x_2 \\ 2 \end{pmatrix} v_2, \qquad \begin{pmatrix} v_1(\bar u) \\ v_2(\bar u) \end{pmatrix} = \begin{pmatrix} \bar v_1 \\ \bar v_2 \end{pmatrix}. \tag{4.24}$$

Notice that the solution of (4.24) is well defined, provided that

$$1 + \frac{\bar x_2^2}{2} - \theta > 0 \quad \text{for all } \theta \in I \doteq \big\{su^+(\bar x) + (1-s)u^-(\bar x);\ s \in \,]0,1[\big\}.$$

Two cases must be considered.

Case 1: For every $\theta \in I$ one has $v_2(\theta) > \varphi'(\bar x_1)v_1(\theta)$.

In this case, the solution of the original problem leaves the curve γ immediately at time $t = 0$. If u^+ satisfies (2.13), the corresponding trajectory can be computed by solving the Cauchy problem

$$\begin{pmatrix} \dot v_1 \\ \dot v_2 \end{pmatrix} = \left(1 + \frac{x_2^2}{2} - u^+(x)\right)^{-1} \left[\frac{v_2^2 + g}{2}\begin{pmatrix} 1 - u^+(x) \\ -x_2 \end{pmatrix} + v_2\begin{pmatrix} x_2 \\ 2 \end{pmatrix}\dot u^+(x)\right] \tag{4.25}$$

with initial data at $t = 0+$ computed by

$$\begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix} = \begin{pmatrix} v_1(u^+(\bar{x})) \\ v_2(u^+(\bar{x})) \end{pmatrix}, \qquad u(0) = u^+(\bar{x}). \tag{4.26}$$

Here $v_i(u^+(\bar{x}))$, $i = 1, 2$, are the components of the solution of (4.24) for $\theta = u^+(\bar{x})$.

Case 2: There exists $\bar{\theta} \in I$ such that $v_2(\bar{\theta}) = \varphi'(\bar{x}_1)v_1(\bar{\theta})$.

In this case the system can move along the curve γ for a positive interval of time. Its motion is described by Eqs. (4.16), with initial data at $t = 0+$ computed by

$$\begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix} = \begin{pmatrix} v_1(\bar{\theta}) \\ v_2(\bar{\theta}) \end{pmatrix}, \qquad u(0) = \bar{\theta}. \tag{4.27}$$

To understand when these cases occur, assume for example that $\bar{u} = u^-(\bar{x}) < u^+(\bar{x})$ and $\bar{v}_2 > 0$. Clearly, the solution of (4.24) then satisfies $v_2(\theta) > 0$ for all $\theta \in I$. Defining

$$\beta(\theta) \doteq 1 + \frac{\bar{x}_2^2}{2} - \theta, \qquad \alpha \doteq 2 - \bar{x}_2\varphi'(\bar{x}_1),$$

we find that Case 1 occurs either if $\alpha \geqslant 0$, or else if $\alpha < 0$ and $\bar{v}_1$, $\bar{v}_2$, u^-, u^+ satisfy

$$\left(\frac{\beta(u^-(\bar{x}))}{\beta(u^+(\bar{x}))}\right)^2 < 1 - \frac{2}{\alpha}\left(1 - \varphi'(\bar{x}_1)\frac{\bar{v}_1}{\bar{v}_2}\right). \tag{4.28}$$

Indeed, if $\alpha \geqslant 0$, using (4.24) we find

$$\frac{d}{d\theta}\big[v_2(\theta) - \varphi'(\bar{x}_2)v_1(\theta)\big] \geqslant 0.$$

Hence $v_2(\theta) - \varphi'(\bar{x}_2)v_1(\theta) > 0$ for all $\theta \in I$. As a consequence, the velocity remains non-tangential to the curve γ and the unique solution of the feedback impulsive system immediately leaves such curve.

Next, assume that $\alpha < 0$. Then by (4.24) it follows

$$v_2(\theta) - \varphi'(\bar{x}_1)v_1(\theta) = \bar{v}_2 - \varphi'(\bar{x}_1)\bar{v}_1 + \alpha \int_{u^-}^{\theta} \beta^{-1}(\sigma)v_2(\sigma)\,d\sigma = \bar{v}_2 - \varphi'(\bar{x}_1)\bar{v}_1 + \frac{\alpha}{2}\big[v_2(\theta) - \bar{v}_2\big].$$

Therefore, the inequality $v_2(\theta) > \varphi'(\bar{x}_1)v_1(\theta)$ remains valid as long as

$$\frac{2}{-\alpha}\big(\bar{v}_2 - \varphi'(\bar{x}_1)\bar{v}_1\big) > v_2(\theta) - \bar{v}_2 = \left(\frac{\beta(\bar{u})}{\beta(\theta)}\right)^2 \bar{v}_2 - \bar{v}_2.$$

Since $\bar{u} = u^-(\bar{x})$, the above inequality is equivalent to (4.28).

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