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Symmetry and transformation properties of iterative ordinary differential equation

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ABSTRACT

Symmetries of linear iterative equations and new conditions on the infinitesimals are obtained. Regarding the expressions of the solutions in terms of the parameters of the source equation, an ansatz is made on the original parameters. We have also obtained an expression for the source parameters of the transformed equation under equivalence transformations. We conducted this work with a special emphasis on second-, third- and fourth-order equations, although some of our results are valid for equations of a general order.

1. Introduction

Linear iterative equations are the iterations of a linear first-order equation. They are known as equations that can always be reduced to the canonical form $y^{(n)} = 0$ by point transformations. It is well known that every second-order linear ordinary differential equation can be reduced to the canonical form $y'' = 0$ by an invertible point transformation. However, the corresponding property does not hold for equations of order higher than two and any equation of such an order can be transformed into the canonical form if and only if it is iterative [1]. On the basis of this and a result of S. Lie [3], iterative equations are also the only linear equations that admit a symmetry algebra of maximal dimension. Moreover, the general solution of iterative equations of a general order can be obtained by a simple superposition formula from those of the source equation of the second order.

Linear ordinary differential equations of a general order have been studied in the recent literature and from the symmetry group approach by many authors [1,4,5,7]. It is well known that for the order $n = 2$, the dimension of the symmetry algebra does not exceed 8 and all linear differential equations are locally equivalent to the canonical form $y'' = 0$. For $n \geqslant 3$, Sophus Lie proved that the dimension of the symmetry

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algebra does not exceed $n+4$. One of Lie's main results is that the maximal dimension is reached for equations reducible to the canonical form $y^{(n)}=0$.

In their work, Krause and Michel [1] proved that an equation is reducible to the canonical form if its symmetry algebra has maximal dimension. Then, using the result due to Lie cited above, they showed that for a linear equation of order $n \geqslant 3$ the statements

(a) the equation is reducible to the form $y^{(n)}=0$ by a diffeomorphism of the (x,y)-plane,
(b) the Lie algebra of its symmetry group has maximal dimension,
(c) the equation is iterative,

are equivalent. By definition, iterative equations are the iterations where $\Psi = r\frac{d}{dx}+s$ is a differential operator and r and s are given functions of x referred to as the parameters of the source equation $\Psi y \equiv r(x)y' + s(x)y = 0$.

Let us consider a linear differential equation of a general order n in its standard form

$$y^{(n)} + \sum_{i=0}^{n-1} b_i y^{(i)} = 0, \tag{1}$$

where the b_i are functions of the independent variable x. For $n=3$, Lie [3] and Laguerre [2] showed that the equation is reducible to the form $y^{(n)}=0$, which we shall refer to as the canonical form, if and only if the coefficients in (1) satisfy

$$54b_0 - 18b_1b_2 + 4b_2^3 - 27b_1' + 18b_2b_2' + 9b_2'' = 0. \tag{2}$$

It is well known that one can use the transformation

$$y \mapsto y \exp\left(\frac{1}{n}\int_{x_0}^{x} b_{n-1}(v)\,dv\right) \tag{3}$$

to reduce the general form (1) into the reduced normal form

$$y^{(n)} + \sum_{i=0}^{n-2} a_i y^{(i)} = 0, \tag{4}$$

and in the case of iterative equations, the operator that generates an iterative equation of a general order n in its normal form (4) has been found [7]. We know that up to isomorphism the symmetry algebra of a differential equation does not change under an invertible point transformation, meaning that (1) and (4) have isomorphic symmetry algebras. Therefore, for several considerations we may, without loss of generality, let the iterative equation be of the form (4).

Some properties of iterative equations were obtained and the characterizations of these equations in terms of their coefficients have been considered [5,7]. All the coefficients a_i can naturally be expressed in terms of the parameters r and s of the source equation [7] but surprisingly it is always possible to express the coefficients a_{n-i} for $2 < i \leqslant n$ in terms of the coefficient a_{n-2} and its derivatives [5]. The list of iterative equations in which all the coefficients are given in terms of a_{n-2} and its derivatives for n running between 3 and 8 was obtained in [5]. The first three of them are

$$y^{(3)} + a_1 y^{(1)} + \frac{1}{2}a_1^{(1)} y = 0, \tag{5}$$

$$y^{(4)} + a_2 y^{(2)} + a_2^{(1)} y^{(1)} + \left(\frac{3}{10} a_2^{(2)} + \frac{9}{100} a_2^2\right) y = 0, \tag{6}$$

$$y^{(5)} + a_3 y^{(3)} + \frac{3}{2} a_3' y^{(2)} + \left(\frac{9}{10} a_3'' + \frac{16}{100} a_3^2\right) y' + \left(\frac{1}{5} a_3^{(3)} + \frac{16}{100} a_3 a_3'\right) y = 0. \tag{7}$$

It should be noted that if we let

$$y^{(n)} + A_n^2 y^{(n-2)} + \cdots + A_n^{n-1} y^{(1)} + A_n^n y = 0 \tag{8}$$

be the general form of linear iterative equations in normal form for the same source equation, then by a result of [7] we have

$$A_n^2 = \binom{n+1}{3} A_2^2. \tag{9}$$

Using the result from [5] one can generate the list of canonical forms of iterative equations in normal reduced form for any order after a long and sometimes very complicated set of calculations.

Another exceptional property of iterative equation states that if we assume that u and v are the independent solutions of the second-order source equation

$$y'' + p(x)y = 0, \tag{10}$$

where p turns out to be the Wronskian of u and v, then n linearly independent solutions of (4) are given by [1]

$$y_k = u^{n-(k+1)} v^k, \quad 0 \leqslant k \leqslant n-1. \tag{11}$$

Therefore, once we know the general solution of the source equation (10) we can construct the set of solutions to the corresponding nth-order iterative equation. The implication is that there is no need to search for linearly independent solutions for the linear differential equation of order n itself when we know those of its source equation. In other words, finding the general solution of the nth-order equation (4) is equivalent to finding the two linearly independent solutions of the second-order source equation (10).

To rewrite the coefficients of the linear iterative equation in terms of the coefficient A_2^2 of the second-order source equation and its derivatives only, let

$$y^{(n)} + \sum_{i=0}^{n-2} A_n^{n-i} y^{(i)} = 0 \tag{12}$$

be a linear iterative equation in normal form, where A_2^2 are functions of x, and let

$$y'' + A_2^2(x) y = 0 \tag{13}$$

be the corresponding second-order source equation. If we assume that the first-order source equation in standard form is

$$r(x)y' + s(x)y = 0, \tag{14}$$

where $r = r(x)$ and $s = s(x)$ are the parameters of the source equations, it follows [7] that

$$A_2^2(x) = \frac{r'^2 - 2rr''}{4r^2} \tag{15}$$

provided that

$$s = -\frac{1}{2}(n-1)r'. \tag{16}$$

It is well known that it is always possible to express the coefficients of the iterative equation (12) in terms of A_n^2 [5]. On the other hand, it has been proved [7] that the relationship between A_n^2 and A_2^2 is given by

$$A_n^2 = \binom{n+1}{3} A_2^2. \tag{17}$$

It follows from these results that all the coefficients of iterative equations with the same source equation can be written in terms of the coefficient A_2^2 of the second-order source equation. Below is the list of iterative equations of order $n = 3, 4, 5$ involving the coefficient A_2^2, a say, of the second-order source equation and its derivatives only,

$$\begin{aligned}
&y^{(3)} + 4ay^{(1)} + 2a'y = 0,\\
&y^{(4)} + 10ay^{(2)} + 10a'y^{(1)} + \left(3a'' + 9a^2\right)y = 0,\\
&y^{(5)} + 20ay^{(3)} + 30a'y^{(2)} + \left(18a'' + 64a^2\right)y^{(1)} + \left(4a^{(3)} + 64aa'\right)y = 0,\\
&y^{(6)} + 35ay^{(4)} + 70a'y^{(3)} + \left(63a^{(2)} + 259a^2\right)y^{(2)} + \left(28a^{(3)} + 518aa'\right)y'\\
&\quad + \left(5a^{(4)} + 130a'^{\,2} + 155aa^{(2)} + 225a^3\right)y = 0.
\end{aligned}$$

The list can be extended to a general order, although the general formula is not known.

2. Symmetry generator of third- and nth-order differential equations

The aim of this section is to make a contribution to the results obtained by Krause and Michel [1], i.e. the expression of $\mathbf{v}$ in terms of the solutions u and v of the second-order source equation. Some properties of a linear iterative equation are used in order to generate the vector field that spanned the Lie algebra. For nth-order differential equations, we require the knowledge of the nth extension of $\mathbf{v}$. The following theorem gives the general prolongation formula $pr^{(n)}\mathbf{v}$ of the infinitesimal generator $\mathbf{v}$ [8].

Theorem 1. *Let*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,y)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \Phi_\alpha(x,y)\frac{\partial}{\partial y^\alpha} \tag{18}$$

be a vector field defined on an open subset $M \subset X \times U$. The nth prolongation of $\mathbf{v}$ is the vector field

$$pr^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q}\sum_{J} \Phi_\alpha^J(x,y)\frac{\partial}{\partial y_J^\alpha} \tag{19}$$

defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$, the second summation being over all (unordered) multi-indices $J = (j_1, j_2, \ldots, j_k)$, with $1 \leqslant j_k \leqslant p$, $1 \leqslant k \leqslant n$. The coefficient functions Φ_α^J of $pr^{(n)}\mathbf{v}$ are given by the following formula:

$$\Phi_\alpha^J(x,y) = D_J\left(\Phi_\alpha - \sum_{i=1}^{p}\xi^i y_i^\alpha\right) + \sum_{i=1}^{p}\xi^i y_{J,i}^\alpha, \tag{20}$$

where $y_i^\alpha = \frac{\partial y^\alpha}{\partial x^i}$, and $y_{J,i}^\alpha = \frac{\partial y_J^\alpha}{\partial x^i}$.

The infinitesimal criterion for invariance is given by the following theorem.

Theorem 2. *Suppose*

$$\Delta_\mu\big(x, y^{(n)}\big) = 0, \quad \mu = 1, \ldots, l, \tag{21}$$

is a system of differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M, and

$$pr^{(n)}\mathbf{v}[\Delta_\mu] = 0, \quad \mu = 1, \ldots, l, \quad \textit{whenever } \Delta_\mu\big(x, y^{(n)}\big) = 0, \tag{22}$$

for every infinitesimal generator $\mathbf{v}$ of G, then G is a symmetry group of the system.

For an nth-order ordinary differential equation (ODE) we have up to nth derivatives and so need an nth extension so that we can investigate how the derivatives transform too. Therefore, if we assume that $E = 0$ is an nth-order ODE then the invariance criterion is given by

$$\mathbf{v}^{[n]}E = 0, \quad \text{whenever } E = 0, \tag{23}$$

where $\mathbf{v}^{[n]}$ stands for $pr^{(n)}\mathbf{v}$, that is,

$$\mathbf{v}^{[n]} = \xi\partial_x + \phi\partial_y + \phi^x\partial_{y_x} + \phi^{xx}\partial_{y_{xx}} + \phi^{xxx}\partial_{y_{xxx}} + \cdots + \phi^{[n]}\partial_{y^n}. \tag{24}$$

Here [1],

$$\begin{aligned}
\phi^{[n]} = {} & -\big[(n\xi_x - \phi_y)\big]y^{(n)} - \big[(n+1)\xi_y\big]y'y^{(n)} \\
& - \binom{n+1}{2}\xi_y y'' y^{(n-1)} + n(\phi_{yy} - n\xi_{xy})y'y^{(n-1)} \\
& + n\left(\phi_{xy} - \frac{n-1}{2}\xi_{xx}\right)y^{(n-1)} + \frac{n}{2}\left(\phi_{xxy} - \frac{n-2}{3}\xi xxx\right)y^{(n-2)} + \cdots.
\end{aligned} \tag{25}$$

Theorem 2 leads to a nonlinear partial differential equation in ξ and ϕ. We then equate all the coefficients of all powers of derivatives of y to zero because ξ and ϕ depend only on x and y. The system of determining equations obtained gives the expression of ξ and ϕ. Note that the number of constants found determines the dimension of the Lie group.

2.1. Order 3

Consider the linear iterative equation of order three in reduced normal form

$$y^{(3)} + 4ay' + 2a'y = 0. \tag{26}$$

An application of the infinitesimal criterion of invariance gives

$$\mathbf{v}^{[3]}\big[y^{(3)} + 4ay' + 2a'y\big] = 0, \quad \text{whenever } y^{(3)} + 4ay' + 2a'y = 0, \tag{27}$$

which reduces to

$$\xi\big(4a_x y' + 2a_{xx}y\big) + 2\phi a_x + 4\phi^x a + \phi^{xxx} = 0. \tag{28}$$

Substituting ϕ^x and ϕ^{xxx} in (28) by their expression given by (25), Eq. (28) leads to a single differential equation ($y^{(3)}$ is replaced by $-4ay' - 2a'y$). Equating the coefficients of all powers of derivatives of y to zero yields the system of determining equations given as follows

$$1:\quad 2a_x\phi + 4a\phi_x + \phi_{xxx} + y\big[2\xi a_{xx} - 2a_x(\phi_y - 3\xi_x)\big] = 0, \tag{29}$$

$$y_x:\quad 4\xi a_x + 4a\phi_y - 4a\xi_x - 8\xi_y a_x y + (3\phi_{xxy} - \xi_{xxx}) - 4a(\phi_y - 3\xi_x) = 0, \tag{30}$$

$${y_x}^2:\quad -4a\xi_y + (3\phi_{xyy} - 3\xi_{xxy}) - 16\xi_y a = 0, \tag{31}$$

$${y_x}^5:\quad \phi_{yyy} - 3\xi_{xyy} = 0, \tag{32}$$

$$y_x y_{xx}:\quad 3\phi_{yy} - 7\xi_{xy} = 0, \tag{33}$$

$${y_x}^4:\quad -\xi_{yyy} = 0, \tag{34}$$

$${y_x}^2 y_{xx}:\quad -6\xi_{yy} = 0, \tag{35}$$

$${y_{xx}}^2:\quad -3\xi_y = 0. \tag{36}$$

Therefore, the infinitesimals are given by

$$\xi = f(x), \qquad \phi = \big(f'(x) + c_0\big)y + h(x), \tag{37}$$

where c_0 is an arbitrary constant, with f and g satisfying

$$f^{(3)} + 4af^{(1)} + 2a^{(1)}f = 0, \tag{38}$$

$$h^{(3)} + 4ah^{(1)} + 2a'h = 0 \tag{39}$$

respectively. So, f and h satisfy the original equation.

2.2. Arbitrary order n

Denoting as usual by $v = \xi(x,y)\partial_x + \phi(x,y)\partial_y$ the infinitesimal generator of the iterative equation of order n, [1] showed that its nth prolongation has the form

$$pr^{(n)}\mathbf{v} = \mathbf{v} + \sum_{k=0}^{n} \phi^{[k]}\big(x, y^{(n)}\big)\partial_{y^{(k)}}, \tag{40}$$

where $\phi^{[k]}$ is given by (25).

Using the invariance criterion and separating by the powers of derivatives of y they proved that the most general form of the symmetry generator is given by

$$\mathbf{v} = f(x)\partial_x + \left[\left(\frac{n-1}{2}f'(x) + c\right)y + h(x)\right]\partial_y, \tag{41}$$

where

$$\binom{n+1}{3} f''' + 4A_n^2 f' + 2{A_n^2}' f = 0, \tag{42}$$

$$h^{(n)} + \sum_{i=0}^{n-2} A_n^{n-i} y^{(i)} = 0 \tag{43}$$

and c is an arbitrary constant. To obtain the same condition on h, [3] used the Leibniz's rule of differentiating a product to rewrite (25) fully. They showed that (25) is the same as

$$\phi^{[j]} = \left(\left[\frac{n-1}{2}f' + \alpha\right]y\right)^{(j)} + h^{(j)} - \sum_{i=1}^{j}\binom{n+1}{3}y^{j+1-i}\xi^i, \quad j = 1, \dots, n. \tag{44}$$

Using the above result and the invariance criterion they showed that h satisfies the original equation.

However, using in

$$\binom{n+1}{3}f''' + 4A_n^2 f' + 2A_n^{2\prime} f = 0 \tag{45}$$

the expression of A_n^2 given in (9), i.e.

$$A_n^2 = \binom{n+1}{3}A_2^2, \tag{46}$$

we get

$$\binom{n+1}{3}f''' + 4\binom{n+1}{3}A_2^2 f' + 2\binom{n+1}{3}A_2^{2\prime} f = 0, \tag{47}$$

which can also be written as

$$f''' + 4A_2^2 f' + 2A_2^{2\prime} f = 0. \tag{48}$$

Hence, f satisfies the third order linear iterative equation as in the cases of order 3 and 4. Therefore, condition on f does not depend on the order of the linear iterative equation. In all, the most general form of the symmetry generator is given by

$$\mathbf{v} = f(x)\partial_x + \left[\left(\frac{n-1}{2}f'(x) + c\right)y + h(x)\right]\partial_y, \tag{49}$$

where the new condition on f is

$$f''' + 4A_2^2 f' + 2A_2^{2\prime} f = 0, \tag{50}$$

and

$$h^{(n)} + \sum_{i=0}^{n-2} A_n^{n-i} y^{(i)} = 0. \tag{51}$$

Eqs. (50) and (51) are linear iterative equations with the same source equation $y'' + A_2^2 y = 0$.

Based on the properties of iterative equations already outlined, finding the solutions of (50) will be reduced to finding the solutions of the second-order source equation. It is well known that if we assume that u and v are solutions of the second-order source equation, then n linearly independent solutions of (12) are given by [1]

$$y_k = u^{n-(k+1)}v^k, \quad 0 \leqslant k \leqslant n-1. \tag{52}$$

Let us verify this known fact for linear iterative equations of order $n = 3, 4, 5, 6$.

- $n=3$. Suppose indeed that u and v are the two linearly independent solutions of

$$y''+ay=0, \tag{53}$$

and let us check that $y_k=u^{2-k}v^k$, $0\leqslant k\leqslant 2$, are linearly independent solutions of the third-order linear iterative equation

$$y^{(3)}+4ay'+2a'y=0. \tag{54}$$

We have

$$\begin{aligned}
y_k^{(3)} &= \left(u^{2-k}\right)^{(3)}v^k+3\left(u^{2-k}\right)^{(2)}v^{k\prime}+3\left(u^{2-k}\right)'v^{k(2)}+u^{2-k}\left(v^k\right)^{(3)}\\
&= \left[(2-k)u^{(3)}u^{1-k}v^k+3(2-k)(1-k)u'u^{(2)}u^{-k}v^k+(2-k)(1-k)(-k)u'^3u^{-k-1}v^k\right]\\
&\quad +3\left[k(2-k)u^{(2)}u^{1-k}v'v^{k-1}+k(2-k)(1-k)u'^2u^{-k}v'v^{k-1}\right]\\
&\quad +3\left[k(2-k)u'u^{1-k}v^{(2)}v^{k-1}+k(2-k)(k-1)u'u^{1-k}v'^2v^{k-2}\right]\\
&\quad +\left[ku^{2-k}v^{(3)}v^{k-1}+3k(k-1)u^{2-k}v'v^{(2)}v^{k-2}+(k-2)(k-1)ku^{2-k}v'^3v^{k-3}\right].
\end{aligned} \tag{55}$$

Using in (55) the substitutions $f''=-af$, $f'''=-(af)'$ for $f=u,v$ gives

$$\begin{aligned}
(y_k)^{(3)} &= -a'(2-k)u^{2-k}v^k-a(2-k)u'u^{1-k}v^k\\
&\quad -3a(2-k)(1-k)u'u^{1-k}v^k-k(1-k)(2-k)u'^3u^{-1-k}v^k\\
&\quad -3ak(2-k)v'u^{2-k}v^{k-1}+3k(2-k)(1-k)u'^2u^{-k}v'v^{k-1}\\
&\quad -3ak(2-k)u'u^{1-k}v^k+3k(k-1)(2-k)u'u^{1-k}v'^2v^{k-2}\\
&\quad -a'ku^{2-k}v^k-akv'u^{2-k}v^{k-1}-3ak(k-1)u^{2-k}v'v^{k-1}\\
&\quad +(k-2)(k-1)(k)u^{2-k}v'^3v^{k-3}.
\end{aligned} \tag{56}$$

Substituting (56) into (54) and expressing also y_k' and y_k in the resulting equation in terms of u and v gives

$$\begin{aligned}
(y_k)^{(3)}+4a(y_k)'+2a'(y_k) &= k(k-1)(k-2)\left(-u'^3u^{-1-k}v^k+3u'^2u^{-k}v'v^{k-1}\right.\\
&\quad \left.+3u'u^{1-k}v'^2+v^{2-k}+u^{2-k}v'^3v^{k-3}\right)\\
&= k(k-1)(k-2)u^{-1-k}v^{k-3}\left[\left(uv'-u'v\right)^3\right]\\
&= \left(\prod_{j=0}^{2}(k-j)\right)\cdot u^{5-k}v^{k-3}\left[\left(\frac{v}{u}\right)'\right]^3\\
&= 0,\quad \text{for } k=0,1,2.
\end{aligned}$$

Let Ω_n be the linear operator corresponding to the linear iterative equation of order n with source equation $y''+ay=0$. Thus $\Omega_3=\frac{d^3}{dx^3}+4a\frac{d}{dx}+2a'$. Let y_k, for $0\leqslant k\leqslant n-1$ be given as above by $y_k=u^{n-(k+1)}v^k$. Proceeding in the same way for $n=4$, $n=5$ and $n=6$, and for the corresponding values of y_k shows that

- $$\Omega_4(y_k)=\left(\prod_{j=0}^{3}(k-j)\right)\cdot u^{7-k}v^{k-4}\left[\left(\frac{v}{u}\right)'\right]^4,\quad 0\leqslant k\leqslant 3, \tag{57}$$

•

$$\Omega_5(y_k) = \left(\prod_{j=0}^{4}(k-j)\right) \cdot u^{9-k}v^{k-5}\left[\left(\frac{v}{u}\right)'\right]^5, \quad 0 \leqslant k \leqslant 4, \tag{58}$$

and

•

$$\Omega_6(y_k) = \left(\prod_{j=0}^{5}(k-j)\right) \cdot u^{11-k}v^{k-6}\left[\left(\frac{v}{u}\right)'\right]^6, \quad \text{for } 0 \leqslant k \leqslant 5, \tag{59}$$

which are equal to zero for $0 \leqslant k \leqslant 5$. It clearly follows from the expressions of $\Omega_n(y_k)$ obtained for $n = 3, 4, 5, 6$ that the general expression for an arbitrary $n \geqslant 3$ is

$$\Omega_n(y_k) = \left(\prod_{j=0}^{n-1}(k-j)\right) \cdot u^{2n-1-k}v^{k-n}\left[\left(\frac{v}{u}\right)'\right]^n = 0, \tag{60}$$

for $0 \leqslant k \leqslant n-1$, $n \geqslant 3$. A formal proof of the validity of (60) could be done by induction on n.

We deduce from (52) that the solutions of (50) and (51) are given by

$$f(x) = c_1u^2 + c_2uv + c_3v^2, \qquad h(x) = \sum_{k=4}^{n+3} c_k u^{n-1-k}v^k, \tag{61}$$

where u and v are solutions of (13).

Therefore, the general infinitesimal symmetry generator $\mathbf{v} = \xi\partial_x + \phi\partial_y$ of the linear iterative equation of order n is given by

$$\xi(x) = c_1u^2 + c_2uv + c_3v^2, \tag{62}$$

$$\phi(x, y) = \left[\frac{n-1}{2}\left(2c_1u'u + c_2u'v + c_2uv' + 2c_3vv'\right) + c_0\right]y + \sum_{k=4}^{n+3} c_k u^{n-1-k}v^k, \tag{63}$$

where $c_0, \ldots, c_{n+3}$ are arbitrary constants. There are $n+4$ arbitrary constants, meaning that the Lie algebra has maximal dimension. Letting v_k be the generators obtained by setting $c_j = \delta_j^k$ in (62) allows us to find the $n+4$ vector fields [1] (although this result is not an original one of [1])

$$v_0 = y\partial_y, \qquad v_1 = u^2\partial_x + (n-1)uu'y\partial_y, \qquad v_2 = uv\partial_x + \frac{n-1}{2}\left(u'v + uv'\right)y\partial_y,$$
$$v_3 = v^2\partial_x + (n-1)vv'y\partial_y, \qquad v_k = u^{n-1-k}v^k\partial_y, \quad k = 4, \ldots, n+3, \tag{64}$$

that span the Lie algebra. This has been obtained in [1] by a slightly different method. Note indeed that this is simply based on the substitution of (61) into (49), which was clearly obtained in [1].

3. Parameters of the transformed equation under equivalence transformations

The group of equivalence transformations G of a family $\mathcal{A}$ of differential equations of a specified form and labeled by a set of arbitrary functions is the largest group of invertible point transformations that map each element of $\mathcal{A}$ to another element of $\mathcal{A}$. On the other hand, we know that two equations are said to be equivalent if they can be mapped to each other by an invertible point transformation. In this section, we shall be interested in finding the parameters of the source equation for the transformed equation under an equivalence transformation of a given iterative equation.

3.1. Equivalence transformations

Let us consider the linear equation

$$y^{(n)} + A_n^1(x)y^{(n-1)} + A_n^2(x)y^{(n-2)} + \cdots + A_n^n(x)y = 0, \tag{65}$$

where x is the independent variable and y the dependent variable. If we suppose that

$$x = \alpha(z, w), \qquad y = \beta(z, w) \tag{66}$$

is an equivalence transformation mapping (65) to an equivalent equation then the latter must have the same form as (65). The substitution of x and y in terms of new variables z and w in (65) must yields an equation of the form

$$w^{(n)} + B_n^1(z)w^{(n-1)} + B_n^2(z)w^{(n-2)} + \cdots + B_n^n(z)w = 0, \tag{67}$$

where z is the independent variable and w is the dependent variable. However, it is well known [5,6] that the group of equivalence transformations of the general linear equation in standard form (65) is given by transformations of the form

$$x = f(z), \qquad y = g(z)w, \tag{68}$$

where f and g are arbitrary functions. Moreover, we note that by assuming Eq. (65) to be in its normal form, (68) reduces to

$$x = f(z), \qquad y = \lambda\big[f'(z)\big]^{\frac{n-1}{2}} w, \tag{69}$$

where λ is an arbitrary constant while f is an arbitrary function.

Also note that a symmetry group transforms the differential equation into the same equation. So, the transformed equation of (65) will then be of the form

$$w^{(n)} + A_n^1(z)w^{(n-1)} + A_n^2(z)w^{(n-2)} + \cdots + A_n^n(z)w = 0 \tag{70}$$

under a symmetry group. We can say that a symmetry transformation is a special case of an equivalence transformation because it preserves not only the form but also the equation itself, as it leaves the equation locally unchanged.

As we already mentioned, the symmetry algebra of two given equivalent equations are isomorphic, and thus if one of them has maximal dimension, the other one will also be of maximal dimension, but having maximal dimension is equivalent to being iterative. Therefore under an equivalence transformation an iterative equation remains iterative. In the next section, we shall be interested in finding the parameters of the source equation for the transformed equation under an equivalence transformation of a given iterative equation.

3.2. Parameters of the transformed equations

Consider the linear iterative equation in the standard form

$$\Psi^n y \equiv K_n^0 y^{(n)} + K_n^1 y^{(n-1)} + K_n^2 y^{(n-2)} + \cdots + K_n^{n-1} y' + K_n^n y = 0 \tag{71}$$

and let

$$y^{(n)} + A_n^2 y^{(n-2)} + \cdots + A_n^j y^{(n-j)} + \cdots + A_n^n y = 0 \tag{72}$$

be the normal reduced form of (71). Suppose that Eq. (71), which may be written again as

$$\Delta_n(y) \equiv \Psi^n y = 0 \tag{73}$$

has the first-order source equation

$$r(x)y' + s(x)y \equiv \Psi(y). \tag{74}$$

Let

$$\Omega_n(w) \equiv \Phi^n w = 0 \tag{75}$$

be an equivalent equation with source equation

$$R(z)w' + S(z)w = \Phi(w) \tag{76}$$

obtained from $\Delta_n(y) = 0$ by the transformations (68). We may assume that

$$\Phi^n(w) \equiv Z_n^0 w^{(n)} + Z_n^1 w^{(n-1)} + Z_n^2 w^{(n-2)} + \cdots + Z_n^{n-1} w' + Z_n^n w = 0 \tag{77}$$

and let

$$w^{(n)} + B_n^2 w^{(n-2)} + \cdots + B_n^{n-1} w' + B_n^n w = 0 \tag{78}$$

be its normal reduced form. We want to find out the parameters R and S of the first-order source equation of the transformed equation $R(z)w' + S(z)w = \Phi(w)$ in terms of the parameters r, s defined in (74). To do so we may assume that the equation is in its reduced form, which also assumes the equality $S = -(n-1)R'/2$.

For simplicity, but without loss of generality, we may assume that the equations are in their reduced normal form (72) and (78). As already mentioned, suppose that the parameter of the source equation generating (72) is $r = r(x)$. Given that equivalent equations have isomorphic symmetry algebras, Eq. (78) is also iterative and we wish to find the corresponding parameter $R = R(z)$ of its source equation. We also need to recall that by considering the equation to be in its normal form, the point transformations (68) reduces to (69), i.e.

$$x = f(z), \qquad y = \lambda\big[f'(z)\big]^{\frac{n-1}{2}} w. \tag{79}$$

A direct calculation (we let $g = [f'(z)]^{\frac{n-1}{2}} w$ and $B_n^i = \mathbb{Z}_n^i|_{\mathbb{Z}_n^1=0}$ in the above calculation) shows that in terms of the parameters λ and f of the equivalence transformation and the coefficients A_n^i of the original equation, we have

$$B_2^2 = \frac{1}{f'^2}\left[A_2^2 f'^4 - \frac{3}{4}f''^2 + \frac{1}{2}f' f^{(3)}\right], \quad \text{for } n = 2, \tag{80}$$

$$B_3^2 = \frac{1}{f'^2}\left[A_3^2 f'^4 - 3f''^2 + 2f' f^{(3)}\right], \quad \text{for } n = 3, \tag{81}$$

$$B_4^2 = \frac{1}{2f'^2}\left[2A_4^2 f'^4 - 15f''^2 + 10f' f^{(3)}\right], \quad \text{for } n = 4. \tag{82}$$

On the other hand we know that by assuming r and R to be the parameters of the source equations for (72) and (78) respectively, we have for $n \geqslant 2$

$$A_n^2(x) = \binom{n+1}{3} A(r), \qquad B_n^2(z) = \binom{n+1}{3} A(R), \tag{83}$$

where

$$A\bigl(r(x)\bigr) = \frac{r'^2 - 2rr''}{4r^2}. \tag{84}$$

Consequently substituting the above expressions for A_n^2 and B_n^2 in terms of r and R respectively in (81) would yield the determining equation for R when $n = 3$. Namely, we have

$$\frac{R'^2 - 2RR''}{R^2} = \frac{1}{f'^2}\left[\frac{r'(f)^2 - 2r(f)r''(f)}{r(f)^2} f'^4 - 3f''^2 + 2f' f^{(3)}\right], \tag{85}$$

where $f = f(z)$. Similarly, for $n = 4$, the determining equation for R takes the form

$$\frac{10}{4}\frac{R'^2 - 2RR''}{R^2} = \frac{1}{2f'^2}\left[\frac{10}{4}\cdot 2\frac{r'(f)^2 - 2r(f)r''(f)}{r(f)^2} f'^4 - 15f''^2 + 10f' f^{(3)}\right] \tag{86}$$

which is equivalent to

$$\frac{R'^2 - 2RR''}{R^2} = \frac{1}{f'^2}\left[\frac{r'(f)^2 - 2r(f)r''(f)}{r(f)^2} f'^4 - 3f''^2 + 2f' f^{(3)}\right]. \tag{87}$$

As it should be expected, Eqs. (85) and (87) are the same and correspond to that derived from (80), which is due to the fact that in reality the expression for R does not depend on the order of the equation. In order word, we only need to know this expression for the second-order source equation.

Note that Eq. (85) has the form

$$\frac{R'^2 - 2RR''}{R^2} = H(z), \tag{88}$$

where H is a given function. Therefore, if we let r or f be arbitrary functions, we may not be able to solve (85) for R, because the solution of the differential equation (88) is not available for B_2^2 an arbitrary function.

4. Conclusion

In this paper, we have reviewed the results obtained by Krause and Michel [1], i.e. the expression of the symmetry generator of the linear iterative equation in terms of the solutions of the second-order source equation. We have obtained their results by a slightly different method which consists of substituting (61) into (49). We made use of the expression (9) to reduce the condition on the infinitesimal ξ. We have proved that the condition on the infinitesimal function $\xi = f(x)$ does not depend on the order of the linear iterative equation. Some results concerning the parameters of the transformed equation under equivalence transformation were obtained for the linear iterative equation of order n.

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