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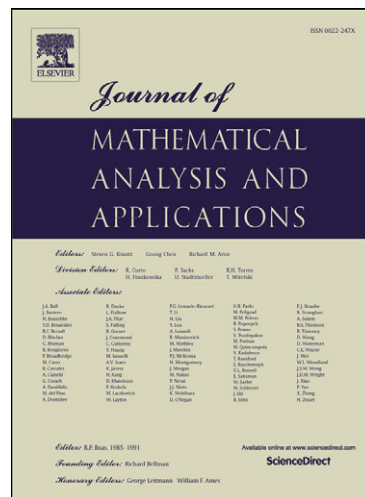
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Walsh and wavelet methods for differential equations on the Cantor group

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Abstract

Ordinary and partial differential equation for unknown functions defined on the Cantor dyadic group are studied. We consider two types of equations: related to the Gibbs derivatives and to the fractional pseudo differential operators. Since the Cantor group is an ultrametric space, pseudo differential operators have spacial properties and are of interest for some applications to models of complex systems, e.g., ultrametric diffusion models in biophysics. We find solutions to the equations in classes of distributions and analyse under what assumptions these solutions are regular functions with some "good" properties. Haar wavelets are used to solve pseudo differential equation. It is very important that the Haar MRA coincides with the Shannon MRA on the Cantor group. To analyse solutions, specific computational method based on the multiresolution structure of the Haar basis was developed.

Keywords Cantor dyadic group; Gibbs derivative; pseudo differential operators; Walsh functions; scaling and wavelet functions; Haar system; ordinary differential equation, partial differential equation.

AMS Subject Classification: 22B99, 42C40,

1 Introduction

Dyadic (Walsh) analysis is actively studied for the last three decades. The foundations of the Walsh theory are presented in monographs [10] and [16]. The theory may be stated in two equivalent forms: the functions under consideration are defined on the real semiline or they are defined on the Cantor group (see [10, §1.2]). We work in the latter form.

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The dyadic wavelet theory is actively studied nowadays. In 1996 the concept of multiresolution analysis (MRA) for the Cantor group was introduced by Lang [14], who also developed a method for the construction MRA-based orthogonal wavelet bases. Later deep investigations of the foundation of the dyadic wavelet theory were added by Farkov and Protasov [7] (see also the recent papers [5, 6] and references therein).

A concept of dyadic derivative was introduced by Gibbs [8] in 1967. Numerous generalizations of this notion in different directions can be found in the literature, the surveys on this topic with extensive bibliographies were given in [19], [18]. A notion of so-called modified Gibbs derivative we use in the present paper was introduced by Golubov in [9].

Dyadic analogues of the classical partial differential equations were considered by Butzer and Wagner [3]. They found a dyadic analogue of d'Alembert's solution to the one-dimensional homogeneous wave equation $\Phi_{x^2}^{[2]} = \Phi_{t^2}^{[2]}$, where $f^{[2]}$ denotes the Gibbs derivative of the second order. Note that both variables x and t are elements of the group G here. In the present paper, we consider PDE's, where the variable x is in G and the variable t , interpreted as time, is real. Equations of such kind in ultrametric spaces may be useful for some applications to models of complex systems, e.g., ultrametric diffusion models in biophysics (see, e.g., [13]), and the Cantor group is just an ultrametric space.

We study ordinary and partial differential equations with respect to both Gibbs and modified Gibbs derivatives (pseudo differential operators). In the case of Gibbs derivative, we consider some linear differential equations and the Cauchy problem for the homogeneous wave equation. The Walsh functions are eigenfunctions for the Gibbs differentiation operator. Due to this property, using Walsh expansions, we find all solutions of the equations in the class of periodic distributions. Also we investigate under what conditions the solution is a regular function with "good" properties (belonging to L_2 or continuity).

Unfortunately, the Gibbs derivative has the following drawback. Consider a simple equation, say $f^{[1]} = g$, where g is 1-periodic. To solve the solution one finds the Fourier-Walsh coefficients of $f^{[1]}$, which allows to restore f . However if g is restricted to the fundamental domain I , then one can repeat this trick using the Fourier-Walsh transform instead of the Fourier-Walsh coefficients, and then restore f . It appears that such a function f is not a compactly supported and it does not coincide on I with the periodic solution. The modified Gibbs differentiability is "more local" property of functions, an analog of the described drawback does not hold for the corresponding differential equations. Namely, if g is a 1-periodic function, f is an unknown function, then the periodic solution of the equation $\mathcal{D}f = g$ coincides on I with the solution of the equation $\mathcal{D}f = g\mathbb{1}_I$, where $\mathbb{1}_I$ is the characteristic function of I . Moreover, the solution of the later equation $\mathcal{D}f = g\mathbb{1}_I$ has the same support as the right-hand side. We consider fractional modified Gibbs derivatives (pseudo differential operators) \mathcal{D}^α , $\alpha \in \mathbb{R}$. Since the Cantor group is an ultrametric space, pseudo differential operators have spacial properties and are of interest for some applications to models of complex systems, e.g., ultrametric diffusion models in biophysics (see, e.g., [15], [2], [13], [12]).

We study ordinary and partial differential equations with respect to the fractional modified Gibbs derivative. In particular, we consider the Cauchy problem for the one-dimensional non-homogeneous heat equation. It appeared that all elements of the Haar basis are eigen-

functions for \mathcal{D}^α . This allows to solve equations using Haar expansions. We find all solutions on a class of distributions and investigate under what conditions the solution is a regular function with "good" properties. The multiresolution structure of the Haar basis is essentially used in our technique. A specific computational method developed in Theorem 1 plays an important role.

The paper is organized as follows. First, we introduce necessary notations and basic facts of the Walsh analysis on the Cantor group. In Subsection 2.2 we discuss the Haar basis, its multiresolution structure, Haar and quasi-Haar expansions. In Section 3 we introduce classes of distributions on the Cantor group and establish some connections between them. In Section 4 we study differential equation with respect to the Gibbs derivative. In Section 5 we study differential equation with respect to the fractional modified Gibbs derivative.

2 Notations and Auxiliary Results

2.1 The Cantor dyadic group

Here and in what follows, we use basic facts on Cantor group and Walsh analysis from [10] and [16].

As usual, by \mathbb{N} , \mathbb{Z} , \mathbb{Z}_+ , \mathbb{R} and \mathbb{C} we denote the sets of positive integers, integers, non-negative integers, real numbers and complex numbers respectively.

Let G be the set of sequences $x = (x_k)_{k \in \mathbb{Z}}$, where $x_k \in \{0, 1\}$ and either there exists $N(x) \in \mathbb{Z}$ such that $x_{N(x)} = 1$ and $x_k = 0$ for $k < N(x)$ or $x_k = 0$ for all $k \in \mathbb{Z}$ (the latter element x we denote by $\mathbf{0}$). We say that x terminates with 1's (0's) if $x_k = 0$ ($x_k = 1$) only for finitely many k . Supply G with coordinate-wise addition, i.e. define the sum of $x \in G$ and $y \in G$ by

$$x \oplus y := (|x_k - y_k|)_{k \in \mathbb{Z}}.$$

Then (G, \oplus) is an abelian group called the **Cantor dyadic group**.

To supply this group with a metric d we introduce a map $\|\cdot\| : G \rightarrow [0, +\infty)$ defined by $\|\mathbf{0}\| := 0$ and $\|x\| := 2^{-N(x)}$ for $x \neq \mathbf{0}$, and let $d(x, y) := \|x \oplus y\|$ for $x, y \in G$. Evidently, d is a metric. Moreover, the strong triangle inequality $d(x, y) \leq \max\{d(x, z), d(y, z)\}$ is satisfied, so d is a non-Archimedean metric. Given $n \in \mathbb{Z}$ and $x \in G$, denote by $I_n(x)$ the ball of radius 2^{-n} with the center at x , i.e.

$$I_n(x) = \{y \in G : d(x, y) < 2^{-n}\}.$$

For convenience we set $I_j := I_j(\mathbf{0})$ and $I := I_0$. Note that I is a subgroup of the group G . Since the metric is non-Archimedean, any two balls in G either are disjoint or one contains the other, and any point of a ball is the center of the ball. The topological group G , with the topology generated by d , is locally compact and totally disconnected.

Let $\lambda(x) := \sum_{j \in \mathbb{Z}} x_j 2^{-j-1}$, then the map $x \mapsto \lambda(x)$ is a one-to-one correspondence taking $G \setminus \mathbb{Q}_0$ onto $[0, \infty)$, where \mathbb{Q}_0 consists of all elements terminating with 1's.

We now define dilation on G by $D : G \rightarrow G$, where $(Dx)_k = x_{k+1}$ for $x \in G$. We let $D^{-1} : G \rightarrow G$ be the inverse mapping $(D^{-1}x)_k = x_{k-1}$. We let D^k be $D \circ \dots \circ D$ (k times) if $k > 0$, and $D^{-1} \circ \dots \circ D^{-1}$ ($-k$ times) if $k < 0$; D^0 is the identity mapping.

We consider functions taking G to \mathbb{C} . Let $\mathbb{1}_E$ denote the characteristic function of a set $E \subset G$. Note that if E is a ball, then $\mathbb{1}_E$ is continuous. For a function $f : G \rightarrow \mathbb{C}$ and a number $h \geq 0$, define $f_{0,h} : G \rightarrow \mathbb{C}$ as follows: for every $x \in G$ we set $f_{0,h}(x) = f(x \oplus \lambda^{-1}(h))$. If additionally $j \in \mathbb{Z}$, then we set

$$f_{j,h}(x) = 2^{j/2} f_{0,h}(D^j x), \quad x \in G.$$

We say that a function f defined on G is 1-periodic if $f(x) = f_{0,1}(x)$ for any $x \in G$. Denote by C^P the space of 1-periodic continuous functions.

Since G is a locally compact group, there exists the Haar measure dx on it (see [11]), which is positive, invariant under the shifts, i.e., $d(x \oplus a) = dx$, and normalized by $\int_G \mathbb{1}_I(x) dx = 1$. Hence the functional spaces $L_q(G)$ and $L_q(E)$, where E is a measurable subset of G , are defined. Denote by L_q^P the space of 1-periodic functions whose restriction to I is in $L_q(I)$.

Given $t \in G$, the function

$$\chi_t(x) := (-1)^{\sum_{j \in \mathbb{Z}} t_j x_{-j-1}}, \quad x \in G.$$

is a group character on G . The Pontryagin dual group G^* of G is topologically isomorphic to G , where the isomorphism is given by $t \rightarrow \chi_t$. In the sequel we identify these groups and write G instead of G^* .

The Fourier-Walsh transform of $f \in L_1(G)$ is defined by

$$\widehat{f}(\xi) := \int_G f(x) w(\xi, x) dx,$$

where $w(t, x) := \chi_t(x)$, $t, x \in G$. The Fourier transform is extended to $L_2(G)$ in a standard way, and the Plancherel equality holds

$$\langle f, g \rangle := \int_G f(x) \overline{g(x)} dx = \int_G \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi = \langle \widehat{f}, \widehat{g} \rangle, \quad f, g \in L_2(G).$$

The inversion formula

$$\widehat{\widehat{f}} = f \tag{1}$$

holds true for any $f \in L_2(G)$.

Given $n \in \mathbb{Z}_+$, set $w_n(x) := w(\lambda^{-1}(n), x)$. The functions w_n are called the **Walsh functions**. These functions are continuous on G and 1-periodic, also they form an orthonormal basis for L_2^P . Thus any $f \in L_2^P$ can be decomposed as

$$f \stackrel{L_2}{=} \sum_{k=0}^{\infty} c_k w_k, \quad c_k \in \mathbb{C},$$

(Walsh representation in the sequel). Here c_k is the k -th Walsh-Fourier coefficient of f :

$$c_k = \widehat{f}(k) := \int_I f(x) w_k(x) dx.$$

A function f is called **Walsh polynomial** if only finitely many coefficients in its Walsh representation are non-zero.

The Walsh system is a dyadic analog of the trigonometric system, in particular,

$$\widehat{f_{j,n}}(\xi) = 2^{-j/2} w_n(D^{-j}\xi) \widehat{f}(D^{-j}\xi), \quad n \in \mathbb{Z}_+, j \in \mathbb{Z}. \quad (2)$$

The function

$$f^{[1]}(x) := \lim_{n \rightarrow \infty} \sum_{j=-n}^n 2^{j-1} (f(x) - f_{0,2^{-j-1}}(x))$$

is called the **Gibbs derivative** of f . The inherited properties are

$$\begin{aligned} (f+g)^{[1]} &= f^{[1]} + g^{[1]}, \quad (cf)^{[1]} = cf^{[1]}, \quad \widehat{f^{[1]}} = \lambda \widehat{f} \\ w_n^{[1]} &= nw_n, \end{aligned} \quad (3)$$

Unfortunately the Gibbs derivative does not inherit some natural properties such as the chain rule and the rule $(fg)' = fg' + f'g$. Moreover, Gibbs differentiability is not a local property of a function. Higher order derivatives are defined recursively: $f^{[n+1]} = (f^{[n]})^{[1]}$.

2.2 Haar system

Lang [14] constructed the Haar basis for $L_2(G)$ as one of the MRA-based wavelet systems on G . Not only the basis itself (which also appeared in the literature before [14]), but some details of the construction will be useful for us.

Set $\varphi = \mathbb{1}_I$. This function has two wonderful (from the point of view of wavelet theory) properties: 1) the functions φ_{0k} , $k \in \mathbb{Z}_+$, form an orthonormal system; 2) φ is a solution of the following refinement equation

$$2^{1/2} \varphi(x) = \varphi_{1,0}(x) + \varphi_{1,1}(x), \quad x \in G. \quad (4)$$

It follows that φ generates a multiresolution analysis (MRA) $\{V_j\}_{j \in \mathbb{Z}}$, where

$$V_j = \overline{\text{span}\{\varphi_{j,k} : k \in \mathbb{Z}_+\}}, \quad j \in \mathbb{Z}.$$

The union of all spaces V_j is dense in $L_2(G)$, the intersection of all spaces V_j consists only of zero function, and $V_j \subset V_{j+1}$ for all j .

The wavelet spaces W_j are defined by

$$W_j = V_{j+1} \ominus V_j, \quad m \in \mathbb{Z},$$

which yields the following orthogonal decomposition

$$L_2(G) = \bigoplus_{j \in \mathbb{Z}} W_j. \quad (5)$$

The corresponding wavelet function ψ is given by

$$2^{1/2}\psi(x) := \varphi_{1,0}(x) - \varphi_{1,1}(x), \quad x \in G. \quad (6)$$

The functions $\psi_{j,k}$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}_+$, form an orthonormal basis (Haar basis) for $L_2(G)$, and any function $f \in L_2(G)$ can be decomposed as

$$f \stackrel{L_2}{=} \sum_{j,k} a_{j,k} \psi_{j,k}$$

(Haar representation in the sequel). A function f is called **Haar polynomial** if only finitely many coefficients in its Haar representation are non-zero.

The functions $\psi_{j,k}$, $j \in \mathbb{Z} \setminus \mathbb{Z}_+$, $k \in \mathbb{Z}_+$, form an orthonormal basis for V_0 . Replacing these functions by the functions $\varphi_{0,k}$, $k \in \mathbb{Z}_+$, which also form an orthonormal basis for V_0 , we have another representation for $f \in L_2(G)$:

$$f \stackrel{L_2}{=} \sum_{k \in \mathbb{Z}_+} a_k \varphi_{0,k} + \sum_{j,k \in \mathbb{Z}_+} a_{j,k} \psi_{j,k}$$

(quasi-Haar representation in the sequel).

The MRA we described is an analog of the Haar MRA in the real setting, but it is also an analog of the real Shannon MRA because $\widehat{\varphi} = \varphi$. So, the space V_j consists of all functions $f \in L_2(G)$ whose Fourier-Walsh transform \widehat{f} is supported in I_{-j} , the space W_j consists of the functions whose Fourier-Walsh transform is supported in $I_{-j-1} \setminus I_{-j}$, in particular,

$$\text{supp } \widehat{\varphi_{j,k}} \subset I_{-j}, \quad k \in \mathbb{Z}_+, \quad (7)$$

$$\text{supp } \widehat{\psi_{j,k}} \subset I_{-j-1} \setminus I_{-j}, \quad k \in \mathbb{Z}_+. \quad (8)$$

Moreover, taking into account (2), we have

$$\widehat{\varphi_{j,k}}(\xi) = 2^{-j/2} w_k(D^{-j}\xi) \mathbb{1}_{I_{-j}}(\xi) \quad \widehat{\psi_{j,k}}(\xi) = 2^{-j/2} w_k(D^{-j}\xi) \mathbb{1}_{I_{-j-1} \setminus I_{-j}}(\xi), \quad \xi \in G. \quad (9)$$

3 Distributions and functional classes on the Cantor group

We define distributions on G , which can be considered as an analog of tempered distributions in real analysis. Such a class of distributions was introduced in two different ways in the books [9] and [1]. We follow the latter one.

A function $\phi : G \rightarrow \mathbb{C}$ is said to be locally constant if for every $x \in G$ there exists a ball containing x such that ϕ is constant on this ball. Any locally constant function is continuous on G . A function ϕ is uniformly locally constant with rang n if it is constant on any ball of radius 2^n . We denote by S the class of locally constant compactly supported functions. Evidently, any function in S is uniformly locally constant.

Proposition 1. [10, §6.2] *Let a function $\phi \in L_1(G)$ be continuous. Then ϕ is uniformly locally constant with rang n if and only if $\widehat{\phi}$ is supported in $I_n(0)$; and ϕ is supported in $I_n(0)$ if and only if $\widehat{\phi}$ is uniformly locally constant with rang n .*

Corollary 1. *A continuous function $\phi \in L_1(G)$ is in S if and only if $\widehat{\phi}$ is in S .*

The statement of Corollary follows immediately from Proposition 1. Thus S is invariant with respect to the Walsh-Fourier transform, and it is natural to say that S is an analog of the Schwartz class in the real analysis.

The convergence in S is defined as follows. One says that a sequence $\{\phi_k\}_{k=1}^{\infty} \subset S$ converges to zero in S if

1. there exists a compact set $K \subset G$ such that $\text{supp } \phi_k \subset K$ for all $k \in \mathbb{N}$;
2. there exists $n \in \mathbb{Z}$ such that any ϕ_k is constant on any ball of radius 2^n ;
3. ϕ_k uniformly tends to zero on K as $k \rightarrow \infty$.

Evidently, the space S is complete.

Let S' denote the dual space for S , i.e. S' consists of continuous linear functionals $f : \phi \rightarrow \langle f, \phi \rangle$ on S . The elements of S' are called tempered distributions. As usual, the convergence in S' is defined as the weak convergence, i.e. a sequence $\{f_k\}_{k=1}^{\infty}$ converges to zero in S' , if

$$\langle f_k, \phi \rangle \xrightarrow[k \rightarrow \infty]{} 0 \quad \forall \phi \in S.$$

The completeness of S' can be checked in the standard way.

To give a representation for the tempered distributions we need the following axillary statement.

Proposition 2. *If a function ϕ is in S , then its quasi-Haar representation is finite, i.e. only a finitely many its coefficients $\langle \phi, \psi_{jk} \rangle$ and $\langle \phi, \varphi_{0k} \rangle$ are non-zero. Conversely, any function ϕ with a finite quasi-Haar representation is in S .*

Proof. If $\phi \in S$, then $\phi \in L_2(G)$ and its Fourier-Walsh transform $\widehat{\phi}$ is compactly supported due to Corollary 1. It follows from the Plancherel theorem and (8) that

$$\langle \phi, \psi_{jk} \rangle = \langle \widehat{\phi}, \widehat{\psi}_{jk} \rangle = 0$$

for all large enough j and all $k \in \mathbb{Z}_+$. On the other hand, for every j there exist only finitely many $k \in \mathbb{Z}_+$ such that $\langle \phi, \psi_{jk} \rangle \neq 0$ because $\text{supp } \phi$ is disjoint to $\text{supp } \psi_{jk}$ for large enough k . Similarly, there exist only finitely many $k \in \mathbb{Z}_+$ such that $\langle \phi, \varphi_{0k} \rangle \neq 0$. Hence

$$\phi = \sum_{k \in \mathbb{Z}_+} \langle \phi, \varphi_{0k} \rangle \varphi_{0k} + \sum_{j,k \in \mathbb{Z}_+} \langle \phi, \psi_{jk} \rangle \psi_{jk},$$

where both sums are finite, what proves the first statement. The second statement is trivial because each of the functions φ_{0k}, ψ_{jk} is in S by definition. \square

The Fourier-Walsh transform \widehat{f} of $f \in S'$ is defined by

$$\langle \widehat{f}, \phi \rangle = \langle f, \widehat{\phi} \rangle, \quad \phi \in S.$$

The inversion formula (1) holds true for any $f \in S'$ because the same is valid for any function in S .

If now f is in S' , $\phi \in S$, then, by Proposition 2,

$$\langle f, \phi \rangle = \sum_{k \in \mathbb{Z}_+} \langle \varphi_{0k}, \phi \rangle \langle f, \varphi_{0k} \rangle + \sum_{j,k \in \mathbb{Z}_+} \langle \psi_{jk}, \phi \rangle \langle f, \psi_{jk} \rangle,$$

where each of the sums is finite. Hence f can be identified with the formal series

$$\sum_{k \in \mathbb{Z}_+} a_k \varphi_{0k} + \sum_{j,k \in \mathbb{Z}_+} a_{j,k} \psi_{jk}, \quad a_k = \langle f, \varphi_{0k} \rangle, \quad a_{j,k} = \langle f, \psi_{jk} \rangle,$$

which is called **quasi-Haar representation** of f . For convenience we will write $f = \sum_{k \in \mathbb{Z}_+} a_k \varphi_{0k} + \sum_{j,k \in \mathbb{Z}_+} a_{j,k} \psi_{jk}$, and even

$$f(x) = \sum_{k \in \mathbb{Z}_+} a_k \varphi_{0,k}(x) + \sum_{j,k \in \mathbb{Z}_+} a_{j,k} \psi_{j,k}(x), \quad a_k, a_{j,k} \in \mathbb{C}, \quad x \in G. \quad (10)$$

The space S has "sufficiently many" functions to allow us to "tell regularly distributions apart". More precisely, the following analog of the du Bois-Reymond lemma holds.

Proposition 3. *If a function f is locally integrable on G and its quasi-Haar representation is zero, then $f = 0$ almost everywhere on G .*

Proof. Set $n \in \mathbb{Z}_+$, $J_n = I_0(\lambda^{-1}(n))$, $g = f \varphi_{0n}$. Since $\text{supp } g \subset J_n$, $f = g$ on J_n , and each of the functions $\varphi_{0,k}, \psi_{j,k}$, $k, j \in \mathbb{Z}_+$, either is supported or vanishes identically on J_n , the quasi-Haar representation of g is also zero. On the other hand (see [10, Sec. 10.3.1]),

$$g \stackrel{L(J_n)}{=} \sum_{k \in \mathbb{Z}_+} \langle g, \varphi_{0k} \rangle \varphi_{0,k} + \sum_{j,k \in \mathbb{Z}_+} \langle g, \psi_{j,k} \rangle \psi_{j,k}.$$

Hence $f = g = 0$ almost everywhere on J_n , what was to be proved. \square

Denote by \tilde{S} the set of Haar polynomials. Any Haar polynomial is in $L_2(G)$ and compactly supported. Hence its quasi-Haar representation is also finite. It follows from Proposition 2

that $\tilde{S} \subset S$, i.e. \tilde{S} is a subspace of S . Denote by \tilde{S}' the space of continuous linear functionals on \tilde{S} .

Since any $\phi \in \tilde{S}$ is a finite linear combination of the functions ψ_{jk} , $j \in \mathbb{Z}$, $k \in \mathbb{Z}_+$, to define $f \in \tilde{S}'$ on \tilde{S} it suffices to define $\langle f, \psi_{jk} \rangle$ for all $j \in \mathbb{Z}$, $k \in \mathbb{Z}_+$. Hence we can identify f with the formal series

$$\sum_{j,k \in \mathbb{Z}_+} a_{j,k} \psi_{j,k}, \quad a_{j,k} = \langle f, \psi_{j,k} \rangle,$$

which is called **Haar representation** of f . Again we can write for convenience

$$f(x) = \sum_{j,k \in \mathbb{Z}_+} a_{j,k} \psi_{j,k}(x), \quad a_{j,k} \in \mathbb{C}, \quad x \in G. \quad (11)$$

Evidently, if $f \in S'$, then $f|_{\tilde{S}}$ is in \tilde{S}' . But it follows from the next theorem that every element of \tilde{S}' is a restriction of some $f \in S'$ to \tilde{S} , i.e.

$$\tilde{S}' = \{f|_{\tilde{S}}, f \in S'\}.$$

Theorem 1. *If $f \in \tilde{S}'$, then there exists a one-parametric family $\{f_c\}_{c \in \mathbb{C}} \subset S'$ such that $f_c|_{\tilde{S}} = f$ for any $c \in \mathbb{C}$, and $g|_{\tilde{S}} \neq f$ for any $g \in S' \setminus \{f_c : c \in \mathbb{C}\}$. Moreover, if (11) is the Haar representation of f , then the quasi-Haar representation of f_c is given by*

$$f_c(x) = \sum_{k=0}^{\infty} a_k(c) \varphi_{0,k}(x) + \sum_{j,k \in \mathbb{Z}_+} a_{j,k} \psi_{j,k}(x), \quad x \in G,$$

where $a_k = a_k(c) = a_k(0) + c$, $k \in \mathbb{Z}_+$, $c \in \mathbb{C}$, is the general solution of system (12).

Proof. First, we need to find a connection between the functions $\psi_{j,k}$, $j < 0$, and $\varphi_{0,k}$. It follows from (4) and (6) that

$$2^{1/2} \psi_{j-1,k} = \varphi_{j,2k} - \varphi_{j,2k+1}, \quad 2^{1/2} \varphi_{j-1,k} = \varphi_{j,2k} + \varphi_{j,2k+1}.$$

Iterating these recursion formulas, we have

$$2^{j/2} \psi_{-j,k} = \sum_{m=0}^{2^{j-1}-1} \varphi_{0,2^j k+m} - \sum_{m=2^{j-1}}^{2^j-1} \varphi_{0,2^j k+m}, \quad j \in \mathbb{N}, \quad k \in \mathbb{Z}_+.$$

Multiplying this by a function $f_c \in S'$, we obtain

$$2^{j/2} \langle f_c, \psi_{-j,k} \rangle = \sum_{m=0}^{2^{j-1}-1} \langle f_c, \varphi_{0,2^j k+m} \rangle - \sum_{m=2^{j-1}}^{2^j-1} \langle f_c, \varphi_{0,2^j k+m} \rangle, \quad j \in \mathbb{N}, \quad k \in \mathbb{Z}_+.$$

The equality $f_c|_{\tilde{S}} = f$ is equivalent to $\langle f_c, \psi_{j,k} \rangle = \langle f, \psi_{j,k} \rangle = a_{j,k}$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}_+$. So, to find f_c we have to solve the following system with unknowns a_l , $l \in \mathbb{Z}_+$,

$$\sum_{m=0}^{2^{j-1}-1} a_{2^j k+m} - \sum_{m=2^{j-1}}^{2^j-1} a_{2^j k+m} = 2^{j/2} a_{-j,k}, \quad j \in \mathbb{N}, \quad k \in \mathbb{Z}_+, \quad (12)$$

and set $\langle f_c, \varphi_{0,l} \rangle = a_l$, $l \in \mathbb{Z}_+$, $\langle f_c, \psi_{j,k} \rangle = a_{jk}$, $j, k \in \mathbb{Z}_+$.

Choose an arbitrary parameter $c \in \mathbb{C}$ and set $a_0 = c$. Then substituting $(j, k) = (1, 0)$ in (12), we get $2^{1/2}a_{-1,0} = a_0 - a_1$. So, $a_1 = a_0 - 2^{1/2}a_{-1,0}$. On the second step we find the coefficients a_2, a_3 using (12) for $(j, k) = (1, 1)$ and $(j, k) = (2, 0)$

$$\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2^{1/2}a_{-1,1} \\ 2a_{-2,0} - a_0 - a_1 \end{pmatrix}$$

On the J -th step, $J \in \mathbb{N}$, the coefficients a_k , $k = 0, \dots, 2^{J-1} - 1$, are known, and we find the coefficients a_l , $l = 2^{J-1}, \dots, 2^J - 1$, using (12) for (j, k) satisfying

$$2^{J-1} < 2^j(k+1) \leq 2^J, \quad j \in \mathbb{N}, \quad k \in \mathbb{Z}_+. \quad (13)$$

The number of equations, that is the number of all solutions of (13), equals the number of unknown coefficients that is 2^{J-1} .

Consider the $J+1$ -th step, $J \in \mathbb{N}$. It follows from (12) that

$$\begin{aligned} \sum_{m=0}^{2^{j-1}-1} a_{2^j k + m + 2^{J-1}} - \sum_{m=2^{j-1}}^{2^j-1} a_{2^j k + m + 2^{J-1}} &= 2^{j/2} a_{-j, k + 2^{J-1-j}}, \\ \sum_{m=0}^{2^{j-1}-1} a_{2^j k + m + 2^J} - \sum_{m=2^{j-1}}^{2^j-1} a_{2^j k + m + 2^J} &= 2^{j/2} a_{-j, k + 2^{J-j}}, \end{aligned}$$

where $j \in \mathbb{N}$, $k \in \mathbb{Z}_+$, $2^{J-1} < 2^j(k+1) \leq 2^J$, $1 \leq j \leq J-1$. Therefore, on $J+1$ -th step, all but two equations for the unknowns a_{n_1} , $n_1 = 2^J, \dots, 2^J + 2^{J-1} - 1$ and a_{n_2} , $n_2 = 2^J + 2^{J-1}, \dots, 2^{J+1} - 1$ can be written using the equations of the previous J -th step written for the unknowns a_r , $r = 2^{J-1}, \dots, 2^J - 1$, where $n_1 = r + 2^{J-1}$, and $n_2 = r + 2^J$. To do so, it is sufficient to replace the right-hand side $2^{j/2}a_{-j,k}$ with $2^{j/2}a_{-j, k + 2^{J-1-j}}$ and with $2^{j/2}a_{-j, k + 2^{J-j}}$ respectively. The two remaining equations of the $J+1$ -th step correspond to $(j, k) = (J, 1)$ and $(j, k) = (J+1, 0)$ and have the form

$$\begin{aligned} \sum_{m=0}^{2^{J-1}-1} a_{2^J + m} - \sum_{m=2^{J-1}}^{2^J-1} a_{2^J + m} &= 2^{J/2} a_{-J, 1}, \\ - \sum_{m=2^J}^{2^{J+1}-1} a_m &= 2^{(J+1)/2} a_{-J-1, 0} - \sum_{m=0}^{2^J-1} a_m. \end{aligned}$$

Denote by M_J the matrix of the system for the J -th step. By observation written in the previous paragraph, we see that the matrix M_{J+1} is organized as follows. We take the first $2^{J-1} - 1$ rows of the matrix M_J and extend each row with 0's. The next $2^{J-1} - 1$ rows are started with 0's and continued with the first $2^{J-1} - 1$ rows of the matrix M_J . The first 2^{J-1} entries of the next row equal 1, the last 2^{J-1} entries of this row equal -1 . Finally, the last

row consists of -1 's. For example,

$$M_3 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \quad M_4 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

To verify that the determinant of the matrix M_J is not equal to zero we prove that $\det M_{J+1} = 2(\det M_J)^2$ for $J > 1$. Indeed, let us consider the matrix M_{J+1} . If we subtract the last row from the next to last one, divide the difference by -2 , and insert the result between the $2^J - 1$ and the 2^J rows, we obtain a matrix that consists of four blocks. The blocks lying on the main diagonal are M_J and one of the remaining blocks consists of 0's. Therefore, the determinant of this matrix is $(\det M_J)^2$. Since $\det M_1 = -1$, $\det M_2 = -2$, we obtain $\det M_J = 2^{2^{J-1}-1}$ for $J > 2$.

Illustrate this for the case $J = 2$:

$$\det M_3 = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 2 & 2 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{vmatrix} = -2(-1) \begin{vmatrix} 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & -1 \end{vmatrix} = 2 \begin{vmatrix} M_2 & 0 \\ \dots & M_2 \end{vmatrix} = 2(\det M_2)^2.$$

Thus, if we fix a parameter $c \in \mathbb{C}$, then system (12) has a unique solution a_k , $k \in \mathbb{Z}_+$, depending on c . Denote by $a_k(0)$ the solution corresponding to $c = 0$ and prove that the general solution is given by $a_0(c) = c$, $a_k(c) = a_k^0 + c$, $k \in \mathbb{N}$, $c \in \mathbb{C}$. Set $b_k := a_k(c) - a_k(0)$. Due to (12), we have

$$B := \sum_{m=0}^{2^{j-1}-1} b_{2^j k+m} - \sum_{m=2^{j-1}}^{2^j-1} b_{2^j k+m} = 0, \quad j \in \mathbb{N}, k \in \mathbb{Z}_+. \quad (14)$$

Let us prove by induction on j that

$$b_{2^j k} = b_{2^j k+1} = \dots = b_{2^j k+2^j-1} \quad (15)$$

for every $k \in \mathbb{Z}_+$. The induction base for $j = 1$ follows from (14) immediately. Let us check the induction step from $j - 1$ to j . Due to the induction hypothesis,

$$B = \sum_{m=0}^{2^{j-1}-1} b_{2^{j-1}(2k)+m} - \sum_{m=2^{j-1}}^{2^j-1} b_{2^{j-1}(2k+1)+m} = 2^{j-1}(b_{2^{j-1}(2k)} - b_{2^{j-1}(2k+1)}).$$

This and (14) yield $b_{2^{j-1}(2k)} = b_{2^{j-1}(2k+1)}$, which together with the induction hypothesis proves (15). It follows that $b_l = b_0 = c$ for every $l \in \mathbb{N}$. Hence the quasi-Haar representation of $f_c - f_0$ is $\sum_{k=0}^{\infty} c \varphi_{0,k}$, what was to be proved. \square

We next introduce a class of periodic distributions. Denote by P the class of 1-periodic locally constant functions. It is clear that any $\phi \in P$ is uniformly locally constant. The convergence in P is defined as follows. One says that a sequence $\{\phi_k\}_{k=1}^\infty \subset P$ converges to zero in P if

1. there exists $n \in \mathbb{Z}$ such that any ϕ_k is constant on any ball of radius 2^n ;
2. ϕ_k uniformly tends to zero on I as $k \rightarrow \infty$.

Evidently, the space P is complete.

Let P' denote the dual space for P , i.e. P' consists of continuous linear functionals $f : \phi \rightarrow \langle f, \phi \rangle$ on P . The convergence in P' is defined as the weak convergence, i.e. a sequence $\{f_k\}_{k=1}^\infty$ converges to zero in P' , if

$$\langle f_k, \phi \rangle \xrightarrow{k \rightarrow \infty} 0 \quad \forall \phi \in P.$$

The completeness of P' can be checked in the standard way.

To give a representation for these distributions we need the following simple statement.

Proposition 4. [10, §1.4, §2.7] *A 1-periodic function ϕ is in P if and only if it is a Walsh polynomial.*

Since, by Proposition 4, any $\phi \in P$ is a finite linear combination of the functions w_k , $k \in \mathbb{Z}_+$, to define $f \in P'$ on P it suffices to define $\langle f, w_k \rangle$ for all $k \in \mathbb{Z}_+$. Hence we can identify f with the formal series

$$\sum_{k=0}^{\infty} a_k w_k, \quad a_k = \langle f, w_k \rangle,$$

which is called its **Walsh representation**. For convenience we write

$$f(x) = \sum_{k=0}^{\infty} c_k w_k(x), \quad c_k \in \mathbb{C}, \quad x \in G. \quad (16)$$

Finally, we introduce an operation of differentiation for periodic distributions. If ϕ is a Walsh polynomial, then its Gibbs derivative $\phi^{[1]}$ is also a Walsh polynomial, and we can well-define the Gibbs derivatives of $f \in P'$ by

$$\langle f^{[k]}, \phi \rangle = \langle f, \phi^{[k]} \rangle, \quad \phi \in P.$$

4 Gibbs differential equations

Consider a linear differential equation of order n

$$\sum_{k=0}^n \alpha_k f^{[k]} = g, \quad n \in \mathbb{N}, \quad \alpha_k \in \mathbb{C}, k = 0, \dots, n. \quad (17)$$

As usual, the characteristic polynomial associated with this equation is $Q_n(y) = \sum_{k=0}^n \alpha_k y^k$.

Theorem 2. Let g be a distribution in P' , $\sum_{m=0}^{\infty} d_m w_m$, be the Walsh representation of g , and let Q_n be the characteristic polynomial associated with (17). Then

1. if Q_n has no integer non-negative roots, then (17) has a unique solution $f = \sum_{m=0}^{\infty} c_m w_m$ in P' , where $c_m = d_m/Q_n(m)$;
2. if $\{m_1, \dots, m_r\}$ is the set of integer non-negative roots of Q_n and $\sum_{l=1}^r |d_{m_l}| = 0$, then (17) has an r -parametric family (enumerated by c_{m_1}, \dots, c_{m_r}) of solutions in P'

$$f(\cdot, c_{m_1}, \dots, c_{m_r}) = \sum_{m \in \mathbb{Z}_+} c_m w_m,$$

where $c_m = d_m/Q_n(m)$ for $m \neq m_l$, $l = 1, \dots, r$, and c_{m_1}, \dots, c_{m_r} are arbitrary complex numbers;

3. if $\{m_1, \dots, m_r\}$ is the set of integer non-negative roots of Q_n and $\sum_{l=1}^r |d_{m_l}| \neq 0$, then (17) has no solutions in P' .

Moreover, all solutions from items 1 and 2 are in L_2^P whenever

$$d_m = O(m^{n-1/2-\varepsilon}), \quad \varepsilon > 0, \quad m \rightarrow \infty; \quad (18)$$

all solutions from items 1 and 2 are in C^P whenever

$$d_m = O(m^{n-1-\varepsilon}), \quad \varepsilon > 0, \quad m \rightarrow \infty. \quad (19)$$

Proof. If $\sum_{m=0}^{\infty} c_m w_m$ is the Walsh representation of $f \in P'$, then, taking into account that, by (3), $w_m^{[k]} = m^k w_m$, we can rewrite (17) in the form

$$\sum_{k=0}^n \alpha_k \sum_{m=0}^{\infty} c_m m^k w_m = \sum_{m=0}^{\infty} d_m w_m,$$

or equivalently

$$\sum_{m=0}^{\infty} c_m Q_n(m) w_m = \sum_{m=0}^{\infty} d_m w_m.$$

Therefore, we get the equations in c_m

$$c_m Q_n(m) = d_m \quad m \in \mathbb{Z}_+.$$

Proof of items 1-3 follows immediately. Also it follows that

$$c_m = d_m/Q_n(m) = O(d_m m^{-n}), \quad m \rightarrow \infty.$$

So, if (18) is fulfilled, then $\sum_{m=0}^{\infty} |c_m|^2 < \infty$, which yields $f \in L_2^P$. If (19) is fulfilled, then $\sum_{m=0}^{\infty} |c_m| < \infty$, which yields the uniform convergence of the series $\sum_{m=0}^{\infty} c_m w_m$, and its sum is continuous. \square

Remark 1. If in (17) the function g is in L_1^P , then $d_m = o(1)$, as $m \rightarrow \infty$. It follows that (18) is fulfilled for any $n \in \mathbb{N}$, and (19) is fulfilled whenever $n > 1$. Hence, in this case all periodic solutions of (17) are in L_2^P , and all periodic solutions are continuous functions whenever $n > 1$.

We now consider the Cauchy problem for the one-dimensional homogeneous wave equation in variables (x, t) , where $x \in G$ and t (time) is real. Let $U = [0, +\infty)$ or $U = [0, T]$. The problem is of the form

$$\begin{cases} \frac{\partial^2 f(x, t)}{\partial t^2} = f_{x^2}^{[2]} f(x, t), \\ f(x, 0) = f^0(x), \quad f'_t(x, 0) = f^1(x), \end{cases} \quad x \in G, \quad t \in U. \quad (20)$$

Theorem 3. Let $f^0, f^1 \in P'$, $\sum_{n=0}^{\infty} p_n w_n$ and $\sum_{n=0}^{\infty} q_n w_n$ be the Walsh representation of f^0 and f^1 respectively. Then

1. the Cauchy problem (20) has a unique solution $f(x, t)$ which is in P' for every $t \in U$;
2. this solution is in C^P whenever

$$p_n = O(e^{-n\theta(n)}), \quad q_n = O(e^{-n\theta(n)}), \quad n \rightarrow \infty, \quad (21)$$

where $\theta(n) \rightarrow \infty$ as $n \rightarrow \infty$.

Proof. 1. Fix $t \in U$. Suppose $f(\cdot, t) = \sum_{n=0}^{\infty} c_n(t) w_n$ is the Walsh representation of $f(\cdot, t)$. Then, taking into account that, by (3), $w_n^{[k]} = n^k w_n$, we can rewrite (20) in the form

$$\begin{cases} \sum_{n=0}^{\infty} \ddot{c}_n(t) w_n = \sum_{n=0}^{\infty} n^2 c_n(t) w_n, \\ \sum_{n=0}^{\infty} c_n(0) w_n = \sum_{n=0}^{\infty} p_n w_n, \quad \sum_{n=0}^{\infty} \dot{c}_n(0) w_n = \sum_{n=0}^{\infty} q_n w_n \end{cases}$$

where \dot{c}_n and \ddot{c}_n denotes the ordinary first and second derivatives of c_n with respect to t . Therefore, for every $n \in \mathbb{N}$ we obtain the Cauchy problem for a linear ordinary differential equation of the second order

$$\begin{cases} \ddot{c}_n(t) = n^2 c_n(t), \\ c_n(0) = p_n, \quad \dot{c}_n(0) = q_n. \end{cases}$$

The solution to the problem is

$$c_n(t) = p_n \cosh(nt) + \frac{q_n}{n} \sinh(nt).$$

Thus the coefficients in the Walsh representation of f are found.

2. If (21) is fulfilled, then $\sum_{n=0}^{\infty} |c_n| < \infty$, which yields $f \in C^P$. □

5 Modified Gibbs derivatives

In the previous section we described a method for finding 1-periodic solutions to linear differential equations. Let us analyse this method for a simple equation $f^{[1]} = g$, where g is 1-periodic, say $g = w_k$. We find the Fourier-Walsh coefficients of $f^{[1]}$, that are $\widehat{f^{[1]}}(n) = \delta_{n,k}$, then, using $\widehat{f^{[1]}}(n) = n\widehat{f}(n)$, find the Fourier-Walsh coefficients of f , which allows to restore f , that is $f = w_k/k$.

Let us try to repeat our trick for a non-periodic function g , using the Fourier-Walsh transform instead of the Fourier-Walsh coefficients, i.e. find \widehat{f} and then restore f . Suppose $g = w_k\varphi$, where, as above, $\varphi = \mathbb{1}_I$, and try to find a solution f to the equation $f^{[1]} = g$. It would be desirable to obtain a solution supported on I . Since $\widehat{f^{[1]}} = \lambda\widehat{f}$, and $\widehat{w_k\varphi} = \varphi_{0,k}$, we have $\widehat{f} = (1/\lambda)\varphi_{0,k}$. Therefore,

$$f(x) = \int_G (\lambda(\xi))^{-1} \varphi_{0,k}(\xi) w(x, \xi) d\xi.$$

The solution is not supported on I because

$$f(x) = w_k(x) \int_I (\lambda(\xi \oplus k))^{-1} w_{[\lambda(x)]}(\xi) d\xi,$$

and $f(x) = w_k(x) \log((k+1/2)^2 k^{-1} (k+1)^{-1})$ for $x \in I_{-1} \setminus I$. It is not difficult to check that f is even not a compactly supported function. Moreover, $f(x) = w_k(x) \log((k+1)/k)$ for $x \in I$, which yields that f does not coincide on I with the periodic solution of the equation $f^{[1]} = w_k$.

Actually, it is easy to see that the restriction of the periodic solution on I (the solution of the equation $f^{[1]} = w_k$) can not coincide with a solution to the equation restricted on I , that is $f^{[1]} = w_k \mathbb{1}_I$. Indeed, let us take a solution of the first equation $f_0 = w_k/k$ and substitute the restricted function $f_1 = w_k/k \cdot \mathbb{1}_I$ into the second equation. We switch to the Fourier domain, then we get $\widehat{f_1} = \varphi_{0,k}/k$, while the equation takes the form $\lambda\widehat{f} = \varphi_{0,k}$. Since $\varphi_{0,k}/k \neq \varphi_{0,k}/\lambda$, it follows that $f_1^{[1]} \neq w_k \mathbb{1}_I$.

However, if we modify a little bit the definition of the derivative, then the situation cardinally changes. Let us use an operator \mathcal{D} defined (on an appropriate class of functions) by

$$\mathcal{D}f(x) = \int_G \|\xi\| \widehat{f}(\xi) w(\xi, x) d\xi$$

instead of the Gibbs derivative. Then the solution of the equation $\mathcal{D}f = w_k\varphi$ is compactly supported. It takes the form

$$f(x) = \int_G \|\xi\|^{-1} \varphi_{0,k}(\xi) w(x, \xi) d\xi = \|\lambda^{-1}(k)\|^{-1} \int_G \varphi_{0,k}(\xi) w(x, \xi) d\xi = \|\lambda^{-1}(k)\|^{-1} w_k(x) \varphi(x).$$

In a similar way one can introduce the modified derivative \mathcal{D} for the 1-periodic function. Indeed, suppose f is a 1-periodic function defined on an appropriate class, $\sum_{k \in \mathbb{Z}_+} \widehat{f}(k) w_k$ is

its Walsh series; then

$$\mathcal{D}f = \sum_{k \in \mathbb{Z}_+} \|\lambda^{-1}(k)\| \widehat{f}(k) w_k$$

It is clear that $\mathcal{D}w_k = \|\lambda^{-1}(k)\| w_k$, i.e. the Walsh functions are eigenfunctions of \mathcal{D} too. In contrast to the Gibbs derivative, if g is a 1-periodic function, then the periodic solution f of the equation $\mathcal{D}f = g$ coincides on I with the solution of the equation $\mathcal{D}f = g \mathbb{1}_I$. Indeed, if $\sum_{k \in \mathbb{Z}_+} \widehat{g}(k) w_k$ is the Walsh series of g , then the equation $\mathcal{D}f = g$ can be rewritten as

$$\sum_{k \in \mathbb{Z}_+} \|\lambda^{-1}(k)\| \widehat{f}(k) w_k = \sum_{k \in \mathbb{Z}_+} \widehat{g}(k) w_k,$$

that is $\widehat{f}(k) = \|\lambda^{-1}(k)\|^{-1} \widehat{g}(k)$, which yields $f = \sum_{k \in \mathbb{Z}_+} \|\lambda^{-1}(k)\|^{-1} \widehat{g}(k) w_k$. On the other hand, the equation $\mathcal{D}f = g \mathbb{1}_I$ can be rewritten as $\|\xi\| \widehat{f} = \widehat{g \mathbb{1}_I}$, that is $\|\xi\| \widehat{f} = \sum_{k \in \mathbb{Z}_+} \widehat{g}(k) \varphi_{0,k}$. Hence

$$\begin{aligned} f(x) &= \int_G \sum_{k \in \mathbb{Z}_+} \|\xi\|^{-1} \widehat{g}(k) \varphi_{0,k}(\xi) w(\xi, x) d\xi = \sum_{k \in \mathbb{Z}_+} \widehat{g}(k) \int_G \|\xi\|^{-1} \varphi_{0,k}(\xi) w(\xi, x) d\xi \\ &= \sum_{k \in \mathbb{Z}_+} \widehat{g}(k) \int_I \|\xi \oplus \lambda^{-1}(k)\|^{-1} w(\xi \oplus \lambda^{-1}(k), x) d\xi = \sum_{k \in \mathbb{Z}_+} \widehat{g}(k) w_k(x) \int_I \|\xi \oplus \lambda^{-1}(k)\|^{-1} w(\xi, x) d\xi \end{aligned}$$

Taking into account that $\|\xi \oplus \lambda^{-1}(k)\| = \|\lambda^{-1}(k)\|$ for $\xi \in I$, we finally get

$$f(x) = \sum_{k \in \mathbb{Z}_+} \widehat{g}(k) w_k(x) \|\lambda^{-1}(k)\|^{-1} \int_I w(\xi, x) d\xi = \left(\sum_{k \in \mathbb{Z}_+} \|\lambda^{-1}(k)\|^{-1} \widehat{g}(k) w_k(x) \right) \mathbb{1}_I.$$

Since this is just an illustration of the general idea we suppose that all operations with sums, integrals and so on are justified.

We now introduce fractional modified Gibbs derivatives \mathcal{D}^α defined on \tilde{S}' . This operator was introduced on $L_1(G)$ in [9]. Such kind of operators are often called pseudo-differential.

First we need the following axillary statement.

Lemma 1. *Let $\phi \in S$. For ϕ to be in \tilde{S} it is necessary and sufficient that*

$$\text{supp } \widehat{\phi} \cap I_n = \emptyset \quad \text{for some } n \in \mathbb{Z}. \quad (22)$$

Proof. The necessity follows from (8). Let (22) be satisfied for $\phi \in S$. It follows from the Plancherel equality and (8) that

$$\langle \phi, \psi_{jk} \rangle = \langle \widehat{\phi}, \widehat{\psi}_{jk} \rangle = 0, \quad k \in \mathbb{Z}_+,$$

whenever $-j$ is large enough. On the other hand, it was shown in the proof of Proposition 2 that $\langle \phi, \psi_{jk} \rangle = 0$ for all $k \in \mathbb{Z}_+$, whenever j is large enough, and for every j there exist only finitely many $k \in \mathbb{Z}_+$ such that $\langle \phi, \psi_{jk} \rangle \neq 0$. Therefore, ϕ is a Haar polynomial, which proves the sufficiency. \square

Let $\alpha \in \mathbb{R}$, $\mathbb{D}^\alpha(x) := \|x\|^\alpha$ for $x \in G$, $x \neq \mathbf{0}$, and $\mathbb{D}^\alpha(\mathbf{0}) := 1$. Define the fractional modified Gibbs derivative \mathcal{D}^α on \tilde{S} by

$$\widehat{\mathcal{D}^\alpha \phi} = \mathbb{D}^\alpha \widehat{\phi}, \quad \phi \in \tilde{S}.$$

Due to Lemma 1, \mathcal{D}^α is well defined and $\mathcal{D}^\alpha : \tilde{S} \rightarrow \tilde{S}$. Moreover, $\mathcal{D}^{-\alpha}$ is the inverse operator to \mathcal{D}^α , which yields that \mathcal{D}^α is a one-to-one map taking \tilde{S} onto \tilde{S} . This allows to extend fractional modified Gibbs derivatives to \tilde{S}' . For $f \in \tilde{S}'$, we can well-define $\mathcal{D}^\alpha f \in \tilde{S}'$ by

$$\langle \mathcal{D}^\alpha f, \phi \rangle = \langle f, \mathcal{D}^\alpha \phi \rangle, \quad \phi \in \tilde{S}.$$

Remark 2. Consider the equation

$$\mathcal{D}^0 f = g, \quad g \in \tilde{S}'. \quad (23)$$

Since \mathcal{D}^0 is the identical operator on \tilde{S}' , evidently, there is a unique solution $f = g$ in \tilde{S}' . Any locally integrable function, in particular, a continuous one, is in \tilde{S}' . Assume that g is a continuous function. In this case it is natural to say that the solution f is also a continuous function. However, by Theorem 1, there exist infinitely many different continuous functions f satisfying (23). All these functions have the same Haar representation, but they have different quasi-Haar representations. On the other hand, if $g \in L_2(G)$, then only one function $f \in L_2(G)$ satisfies (23).

Proposition 5. Suppose $g, \widehat{g}, \mathbb{D}^\alpha \widehat{g}$ are locally integrable on G , $j \in \mathbb{Z}$. Then the assertion $\text{supp } \widehat{g} \subset I_{-j-1} \setminus I_{-j}$ is necessary and sufficient for g to be an eigenfunction of \mathcal{D}^α corresponding to the eigenvalue $2^{j\alpha}$.

Proof. Since the function g is locally integrable, it is also in \tilde{S}' . Assume that $\text{supp } \widehat{g} \subset I_{-j-1} \setminus I_{-j}$. Using the Plancherel equality, for any $\phi \in \tilde{S}$ we obtain

$$\langle \mathcal{D}^\alpha g, \phi \rangle = \langle g, \mathcal{D}^\alpha \phi \rangle = \langle \widehat{g}, \widehat{\mathcal{D}^\alpha \phi} \rangle = \int_{I_{-j-1} \setminus I_{-j}} \widehat{g}(\xi) \|\xi\|^\alpha \widehat{\phi}(\xi) d\xi = 2^{j\alpha} \int_G \widehat{g}(\xi) \widehat{\phi}(\xi) d\xi = 2^{j\alpha} \langle g, \phi \rangle,$$

which proves the necessity.

Let now g be an eigenfunction of \mathcal{D}^α corresponding to the eigenvalue $2^{j_0\alpha}$. Again by the Plancherel equality, we have

$$0 = \langle \mathcal{D}^\alpha g, \phi \rangle - 2^{j_0\alpha} \langle g, \phi \rangle = \int_G \widehat{g}(\xi) (\|\xi\|^\alpha - 2^{j_0\alpha}) \widehat{\phi}(\xi) d\xi$$

for every $\phi \in \tilde{S}$. Therefore, for $\phi = \psi_{j,k}$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}_+$, we have

$$\begin{aligned} 0 &= \int_G \widehat{g}(\xi) (\|\xi\|^\alpha - 2^{j_0\alpha}) \widehat{\psi}_{j,k}(\xi) d\xi = 2^{-j/2} \int_G \widehat{g}(\xi) (\|\xi\|^\alpha - 2^{j_0\alpha}) w_k(D^{-j}\xi) \mathbb{1}_{I_{-j-1} \setminus I_{-j}}(\xi) d\xi \\ &= 2^{-j/2} (2^{j\alpha} - 2^{j_0\alpha}) \int_{I_{-j-1} \setminus I_{-j}} \widehat{g}(\xi) w_k(D^{-j}\xi) d\xi = 2^{j/2} (2^{j\alpha} - 2^{j_0\alpha}) \int_I \widehat{g}(2^j(\eta \oplus 1)) w_k(\eta) d\eta. \end{aligned}$$

Since \widehat{g} is locally integrable on G , the function $h(\eta) = \widehat{g}(2^j(\eta \oplus 1))$ is integrable on I for every fixed j , and if $j \neq j_0$, then all Fourier-Walsh coefficients of h equal zero. It follows that $h = 0$ almost everywhere on I for $j \neq j_0$ (see [10, Sec. 10.2.1]). Hence $\widehat{g} = 0$ almost everywhere on $I_{-j-1} \setminus I_{-j}$ whenever $j \neq j_0$, which proves the sufficiency. \square

Remark 3. If $g \in L_2(G)$, then the necessity of the conclusion of Proposition 5 holds for any $\alpha \in \mathbb{R}$, and the sufficiency holds for $\alpha > 0$.

Corollary 2. Any Haar function $\psi_{j,k}$ is an eigenfunction of \mathcal{D}^α corresponding to the eigenvalue $2^{j\alpha}$.

Proof The statement follows from Proposition 5 and (9). \square

5.1 Modified Gibbs differential equations

Consider the equation

$$\mathcal{D}^\alpha f + \beta f = g, \quad \alpha, \beta \in \mathbb{R} \quad (24)$$

with respect to the unknown function f .

Theorem 4. Let g be a distribution in \tilde{S}' , $\beta \neq -2^{j\alpha}$ for all $j \in \mathbb{Z}$. Then

1. equation (24) has a unique solution in \tilde{S}' ;
2. if $g \in L_2(G)$, and either $\beta \neq 0$ or $\beta = 0$, $(\langle g, \psi_{j,k} \rangle 2^{-j\alpha})_{j \in \mathbb{Z}} \in l_2$, then the solution is in $L_2(G)$;
3. if g is continuous on G , and either $\beta \neq 0$ or $\beta = 0$, $\alpha > -1/2$, then all continuous solutions of (24) are given by

$$f_c = \sum_{k=0}^{\infty} a_k \varphi_{0,k} + \sum_{j,k \in \mathbb{Z}_+} a_{j,k} \psi_{j,k} + c, \quad (25)$$

where $a_{j,k} = (2^{j\alpha} + \beta)^{-1} \langle g, \psi_{j,k} \rangle$, c is an arbitrary complex number, and a_k , $k \in \mathbb{Z}_+$, is a solution of system (12). All the functions f_c have the same Haar representation.

Proof. 1. Let $f = \sum_{j,k} a_{j,k} \psi_{j,k}$, be the Haar representations of $f \in \tilde{S}'$. It follows from Corollary 2 that equation (24) can be rewritten as

$$\sum_{j,k} (2^{j\alpha} + \beta) a_{j,k} \psi_{j,k} = \sum_{j,k} \langle g, \psi_{j,k} \rangle \psi_{j,k},$$

or equivalently

$$(2^{j\alpha} + \beta) a_{j,k} = \langle g, \psi_{j,k} \rangle, \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}_+. \quad (26)$$

It follows that

$$f = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}_+} \frac{\langle g, \psi_{j,k} \rangle}{2^{j\alpha} + \beta} \psi_{j,k}.$$

This proves item 1. It follows from the above formula, that if $\beta = 0$, and $(\langle g, \psi_{j,k} \rangle 2^{-j\alpha})_{j \in \mathbb{Z}} \in l_2$, then $f \in L_2(G)$. If $\beta \neq 0$, then $|2^{j\alpha} + \beta| \geq \delta$ for some $\delta > 0$ and all $j \in \mathbb{Z}$, which yields $f \in L_2(G)$. Item 2 is proved.

Let now the assumptions of item 3 be fulfilled. According to Theorem 1, the functions f_c given by (25), and only these functions in S' , have the same Haar representation as f . Let us prove that f_0 is continuous.

Fix $x_0 \in G$ and the compact set $I_0(x_0)$. By the definition of the Haar function, for every $x \in I_0(x_0)$ there is a unique $k = k(x) = k(x_0) \in \mathbb{Z}_+$ such that $x \in \text{supp } \varphi_{0,k(x)}$, and for every $j \in \mathbb{Z}_+$ there is a unique $k = k(j, x) \in \mathbb{Z}_+$ such that $x \in \text{supp } \psi_{j,k(j,x)}$. So, using the quasi-Haar representation for f_0 , we have

$$f_0(x) = a_{k(x)} \varphi_{0,k(x)}(x) + \sum_{j \in \mathbb{Z}_+} \frac{\langle g, \psi_{j,k(j,x)} \rangle}{2^{j\alpha} + \beta} \psi_{j,k(j,x)}(x).$$

Since g is continuous, g is bounded on the compact set $I_0(x_0)$, that is $|g(x)| \leq M$. Therefore,

$$|\langle g, \psi_{j,k(j,x)} \rangle| = \left| \int_G g \psi_{j,k(j,x)} \right| \leq M \int_G |\psi_{j,k(j,x)}| = M 2^{-j/2}.$$

Thus

$$\frac{|\langle g, \psi_{j,k(j,x)} \rangle|}{2^{j\alpha} + \beta} \leq M \frac{2^{-j/2}}{2^{j\alpha} + \beta} = \begin{cases} O(2^{j(-1/2-\alpha)}), & \text{if } \alpha > 0 \text{ or } \beta = 0, \\ O(2^{-j/2}), & \text{if } \alpha \leq 0, \beta \neq 0, \end{cases} \quad \text{as } j \rightarrow \infty.$$

Since the latter estimate is uniform on $I_0(x_0)$, the series

$$\sum_{j \in \mathbb{Z}_+} \frac{\langle g, \psi_{j,k(j,x)} \rangle}{2^{j\alpha} + \beta} \psi_{j,k(j,x)}(x)$$

uniformly converges on $I_0(x_0)$, which yields that f_0 is continuous on $I_0(x_0)$. It remains to note that x_0 is an arbitrary element of G , thus f_0 is continuous on G . \square

Remark 4. It is clear from the proof of Theorem 4, that the assumption of continuity of g in item 3 can be replaced by the boundedness of g on any compact set.

Theorem 5. Let g be a distribution in \tilde{S}' , $\beta = -2^{j_0\alpha}$, $j_0 \in \mathbb{Z}_+$. If $\langle g, \psi_{j_0,k} \rangle \neq 0$ for at least one $k \in \mathbb{Z}_+$, then (24) has no solutions in \tilde{S}' . If $\langle g, \psi_{j_0,k} \rangle = 0$ for all $k \in \mathbb{Z}_+$, then

1. equation (24) has a family of solutions in \tilde{S}'

$$f_{\{c_k\}} = \sum_{k \in \mathbb{Z}_+, j \in \mathbb{Z}, j \neq j_0} \frac{\langle g, \psi_{j,k} \rangle}{2^{j\alpha} + \beta} \psi_{j,k} + \sum_{k=0}^{\infty} c_k \psi_{j_0,k},$$

where c_k , $k \in \mathbb{Z}_+$, are arbitrary complex numbers.

2. if g is continuous on G , then all continuous solutions of (24) are given by

$$f_{c,\{c_k\}} = \sum_{k=0}^{\infty} a_k \varphi_{0,k} + \sum_{k,j \in \mathbb{Z}_+, j \neq j_0} a_{j,k} \psi_{j,k} + \sum_{k=0}^{\infty} c_k \psi_{j_0,k} + c,$$

where $a_{j,k} = (2^{j\alpha} + \beta)^{-1} \langle g, \psi_{j,k} \rangle$, $c, c_k, k \in \mathbb{Z}_+$, are arbitrary complex numbers, and $a_k, k \in \mathbb{Z}_+$, is a solution of system (12).

Proof It suffices to repeat the proof of Theorem 4, and take into account that equation (26) with $j = j_0$ has no solution if there exists $k_0 \in \mathbb{Z}_+$ such that $\langle g, \psi_{j_0,k_0} \rangle \neq 0$, and $\alpha_{j_0,k} = c_k$ is a solution for any $c_k \in \mathbb{C}$ if $\langle g, \psi_{j_0,k} \rangle = 0$ for all $k \in \mathbb{Z}_+$. \square

We now consider the Cauchy problem for the one-dimensional non-homogeneous heat equation in variables (x, t) , where $x \in G$ and t (time) is real. Let $U = [0, +\infty)$ or $U = [0, T]$. The problem is of the form

$$\begin{cases} \frac{\partial f(x,t)}{\partial t} = \mathcal{D}_x^\alpha f(x, t) + g(x, t), & x \in G, \quad t \in U. \\ f(x, 0) = f^0(x), \end{cases} \quad (27)$$

Theorem 6. Let $f^0 \in \tilde{S}'$, $g_t := g(\cdot, t) \in \tilde{S}'$ for each $t \in U$, and $g(x, \cdot)$ is continuous on U . Then

1. the Cauchy problem (27) has a unique solution $f(x, t)$ which is in \tilde{S}' for every $t \in U$;
2. this solution is in $L_2(G)$ for each $t \in U$ whenever $f^0 \in L_2(G)$,

$$\widehat{f^0}(\xi) = O(e^{-\|\xi\|^{\alpha\theta(\log_2\|\xi\|)}), \quad \|\xi\| \rightarrow \infty, \quad (28)$$

for every $t \in U$ the distribution g_t is in $L_2(G)$ and

$$\widehat{g_t}(\xi) = O(e^{-\|\xi\|^{\alpha\theta(\log_2\|\xi\|)}), \quad \|\xi\| \rightarrow \infty, \quad (29)$$

where $\theta(\nu) \rightarrow \infty$ as $\nu \rightarrow \infty$;

3. if either $\alpha > 0$ and conditions (28), (29) are fulfilled, or $\alpha < 0$ and for every $t \in U$

$$\widehat{f^0}(\xi) = O(\|\xi\|^{-(\varepsilon+1/2)}), \quad \widehat{g_t}(\xi) = O(\|\xi\|^{-(\varepsilon+1/2)}), \quad \|\xi\| \rightarrow \infty, \quad (30)$$

where $\varepsilon > 0$, then the solution is continuous on G for every $t \in U$, all continuous solutions are given by

$$f_c(x, t) = a_{k(x)}(t) \varphi_{0,k(x)}(x) + \sum_{j \in \mathbb{Z}_+} a_{j,k(j,x)}(t) \psi_{j,k(j,x)}(x) + c,$$

where $k(x)$ is a number $k \in \mathbb{N}$ such that $x \in \text{supp } \varphi_{0,k(x)}$, $k(j, x)$ is a number $k \in \mathbb{N}$ such that $x \in \text{supp } \psi_{j,k(j,x)}$, $a_{j,k}(t) = \int_0^t e^{2^{j\alpha}(t-\tau)} \langle g_\tau, \psi_{j,k} \rangle d\tau + \langle f^0, \psi_{j,k} \rangle e^{2^{j\alpha}t}$, c is an arbitrary complex number, and $a_k(t), k \in \mathbb{Z}_+$, is a solution of system (12) with $a_{j,k} = a_{j,k}(t)$.

Proof. 1. Suppose $f^0 = \sum_{j,k} b_{j,k} \psi_{j,k}$, $g_t = \sum_{j,k} d_{j,k}(t) \psi_{j,k}$ and $f(\cdot, t) = \sum_{j,k} a_{j,k}(t) \psi_{j,k}$ are the Haar representations of f^0 , g_t , and $f(\cdot, t)$ respectively. Using Corollary 2, we can rewrite (27) in the form

$$\begin{cases} \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}_+} (\dot{a}_{j,k}(t) - 2^{j\alpha} a_{j,k}(t) - d_{j,k}(t)) \psi_{j,k}(x) = 0, \\ \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}_+} a_{j,k}(0) \psi_{j,k}(x) = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}_+} b_{j,k} \psi_{j,k}(x), \end{cases}$$

where $\dot{a}_{j,k}$ denotes the ordinary derivative of $a_{j,k}$ with respect to t . Therefore, for every $j \in \mathbb{Z}$ and $k \in \mathbb{Z}_+$ we obtain the Cauchy problem for a linear ordinary differential equation of the first order

$$\begin{cases} \dot{a}_{j,k}(t) - 2^{j\alpha} a_{j,k}(t) - d_{j,k}(t) = 0, \\ a_{j,k}(0) = b_{j,k}, \end{cases}$$

The solution to the problem is

$$a_{j,k}(t) = \int_0^t e^{2^{j\alpha}(t-\tau)} d_{j,k}(\tau) d\tau + b_{j,k} e^{2^{j\alpha}t}.$$

Thus the coefficients in the Haar representation of f are found.

2. Let $\alpha > 0$. The case $\alpha < 0$ is quite analogous to the first one. Suppose $h \in L_2(G)$, then

$$\sum_{k \in \mathbb{Z}_+} |\langle h, \psi_{j,k} \rangle|^2 = \int_{I_{-j-1} \setminus I_{-j}} |\widehat{h}|^2 \quad (31)$$

Indeed, let h_j be the orthogonal projection of h on W_j . Then, taking into account that the functions $\psi_{j,k}$ form a basis for W_j , using the Parseval and Plancherel equalities, and (8), we obtain

$$\sum_{k \in \mathbb{Z}_+} |\langle h, \psi_{j,k} \rangle|^2 = \sum_{k \in \mathbb{Z}_+} |\langle h_j, \psi_{j,k} \rangle|^2 = \|h_j\|^2 = \|\widehat{h}_j\|^2 = \int_{I_{-j-1} \setminus I_{-j}} |\widehat{h}_j|^2 = \int_{I_{-j-1} \setminus I_{-j}} |\widehat{h}|^2.$$

Fix $t \in U$. Suppose that all conditions of item 2 are fulfilled. Recall that $\{\psi_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{Z}_+}$ is an orthonormal basis in $L_2(G)$ and using the Haar representation of f from item 1, we have

$$\|f(\cdot, t)\|_{L_2(G)}^2 = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}_+} |a_{j,k}|^2 \leq \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}_+} e^{2^{j\alpha+1}t} |b_{j,k}|^2 + \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}_+} e^{2^{j\alpha+1}t} \left| \int_0^t e^{-2^{j\alpha}\tau} d_{j,k} d\tau \right|^2$$

Applying the Cauchy-Bunyakovskii inequality and calculating the integral $\int_0^t e^{2^{j\alpha+1}\tau} d\tau$, we get

$$\|f(\cdot, t)\|_{L_2(G)}^2 \leq \sum_{j \in \mathbb{Z}} e^{2^{j\alpha+1}t} \sum_{k \in \mathbb{Z}_+} |\langle f^0, \psi_{j,k} \rangle|^2 + \sum_{j \in \mathbb{Z}} e^{2^{j\alpha+1}t} \frac{1 - e^{-2^{j\alpha+1}t}}{2^{j\alpha+1}} \int_0^t \sum_{k \in \mathbb{Z}_+} |(g_\tau, \psi_{j,k})|^2 d\tau$$

Thus, by (31) with $h = f^0$ and $h = g_\tau$,

$$\|f(\cdot, t)\|_{L_2(G)}^2 \leq \sum_{j \in \mathbb{Z}} e^{2^{j\alpha+1}t} \int_{I_{-j-1} \setminus I_{-j}} |\widehat{f^0}(\xi)|^2 d\xi + \sum_{j \in \mathbb{Z}} \frac{e^{2^{j\alpha+1}t} - 1}{2^{j\alpha+1}} \int_0^t \int_{I_{-j-1} \setminus I_{-j}} |\widehat{g_\tau}(\xi)|^2 d\xi d\tau.$$

Consider 2 cases. Case 1: $j \geq 0$. By (28),

$$\sum_{j \in \mathbb{Z}_+} e^{2^{j\alpha+1}t} \int_{I_{-j-1} \setminus I_{-j}} |\widehat{f^0}(\xi)|^2 d\xi \leq C \sum_{j \in \mathbb{Z}_+} 2^j e^{2^{j\alpha+1}(t-\theta(j))} < \infty.$$

By (29),

$$\sum_{j \in \mathbb{Z}_+} \frac{e^{2^{j\alpha+1}t} - 1}{2^{j\alpha+1}} \int_0^t \int_{I_{-j-1} \setminus I_{-j}} |\widehat{g_\tau}(\xi)|^2 d\xi d\tau \leq Ct \sum_{j \in \mathbb{Z}_+} \frac{(e^{2^{j\alpha+1}t} - 1)e^{-2^{j\alpha+1}\theta(j)}}{2^{j(\alpha-1)+1}}.$$

It is clear that the latter sum is finite.

Case 2: $j < 0$. We now have

$$\begin{aligned} & \sum_{j \in \mathbb{Z} \setminus \mathbb{Z}_+} e^{2^{j\alpha+1}t} \int_{I_{-j-1} \setminus I_{-j}} |\widehat{f^0}(\xi)|^2 d\xi + \sum_{j \in \mathbb{Z} \setminus \mathbb{Z}_+} \frac{e^{2^{j\alpha+1}t} - 1}{2^{j\alpha+1}} \int_0^t \int_{I_{-j-1} \setminus I_{-j}} |\widehat{g_\tau}(\xi)|^2 d\xi d\tau \\ & \leq C_1(t) \sum_{j \in \mathbb{Z} \setminus \mathbb{Z}_+} \int_{I_{-j-1} \setminus I_{-j}} |\widehat{f^0}(\xi)|^2 d\xi + C_2(t) \sum_{j \in \mathbb{Z} \setminus \mathbb{Z}_+} \int_0^t \int_{I_{-j-1} \setminus I_{-j}} |\widehat{g_\tau}(\xi)|^2 d\xi d\tau \\ & \leq C_1(t) \int_I |\widehat{f^0}(\xi)|^2 d\xi + C_2(t) \int_0^t \int_I |\widehat{g_\tau}(\xi)|^2 d\xi d\tau \leq C_1(t) \|\widehat{f^0}\|^2 + C_2(t) \int_0^t \|\widehat{g_\tau}\|^2 d\tau \\ & = C_1(t) \|f^0\|^2 + C_2(t) \int_0^t \|g_\tau\|^2 d\tau. \end{aligned}$$

It remains to note that, because of the continuity of $g(x, \cdot)$,

$$\int_0^t \|g_\tau\|^2 d\tau = \int_G \int_0^t |g(x, \tau)|^2 d\tau dx = t \int_G |g(x, \tau_0)|^2 dx, \quad 0 \leq \tau_0 \leq t,$$

If $\alpha < 0$, then cases 1 and 2 change places.

3. Fix $t \in U$, $x_0 \in G$ and the compact set $I_0(x_0)$. By the definition of the Haar function, for every $x \in I_0(x_0)$ there is a unique $k = k(x) = k(x_0) \in \mathbb{Z}_+$ such that $x \in \text{supp } \varphi_{0,k(x_0)}$, and for every $j \in \mathbb{Z}_+$ there is a unique $k = k(j, x) \in \mathbb{Z}_+$ such that $x \in \text{supp } \psi_{j,k(j,x)}$. According to Theorem 1, there exists a family of functions $f_c(\cdot, t) \in S'$ which and only which have the same

Haar representation as f . Using the quasi-Haar representation for $f_c(\cdot, t)$ (see Theorem 1), we have

$$f_c(x, t) = a_{k(x)}(t)\varphi_{0,k(x)}(x) + \sum_{j \in \mathbb{Z}_+} a_{j,k(j,x)}(t)\psi_{j,k(j,x)}(x) + c,$$

Let us prove that for every $t \in U$ the series $\sum_{j \in \mathbb{Z}_+} |a_{j,k(j,x)}(t)|$ converges uniformly with respect to $x \in I_0(x_0)$. Using the Plancherel equality and (9), we get

$$|b_{j,k}| = |\langle f^0, \psi_{j,k} \rangle| = 2^{-j/2} \left| \int_{I_{-j-1} \setminus I_{-j}} \widehat{f^0}(\xi) \mathbf{w}_k(D^j \xi) d\xi \right| \leq 2^{-j/2} \int_{I_{-j-1} \setminus I_{-j}} |\widehat{f^0}|$$

and analogously

$$|d_{j,k}(t)| = |\langle g_t, \psi_{j,k} \rangle| \leq 2^{-j/2} \int_{I_{-j-1} \setminus I_{-j}} |\widehat{g_t}|.$$

Therefore, if $\alpha > 0$, then by (28) and (29), we have

$$\begin{aligned} \sum_{j \in \mathbb{Z}_+} |a_{j,k(j,x)}(t)| &\leq \sum_{j \in \mathbb{Z}_+} e^{2^{j\alpha}t} |b_{j,k(j,x)}| + \sum_{j \in \mathbb{Z}_+} e^{2^{j\alpha}t} \left| \int_0^t e^{-2^{j\alpha}\tau} d_{j,k(j,x)}(\tau) d\tau \right| \\ &\leq C \sum_{j \in \mathbb{Z}_+} 2^{j/2} e^{2^{j\alpha}(t-\theta(j))} + C \sum_{j \in \mathbb{Z}_+} \frac{e^{2^{j\alpha}t} - 1}{2^{j(\alpha-1/2)}} e^{-2^{j\alpha}\theta(j)}. \end{aligned}$$

It is clear that both the sums are finite.

If $\alpha < 0$, then by (30) and elementary properties of the exponential function, we have

$$\begin{aligned} \sum_{j \in \mathbb{Z}_+} |a_{j,k(j,x)}(t)| &\leq \sum_{j \in \mathbb{Z}_+} e^{2^{j\alpha}t} |b_{j,k(j,x)}| + \sum_{j \in \mathbb{Z}_+} e^{2^{j\alpha}t} \left| \int_0^t e^{-2^{j\alpha}\tau} d_{j,k(j,x)}(\tau) d\tau \right| \\ &\leq C \sum_{j \in \mathbb{Z}_+} e^{2^{j\alpha}t} 2^{-j\epsilon} + C \sum_{j \in \mathbb{Z}_+} \frac{e^{2^{j\alpha}t} - 1}{2^{j\alpha}} 2^{-j\epsilon} \leq C e^t \sum_{j \in \mathbb{Z}_+} 2^{-j\epsilon} + C \sum_{j \in \mathbb{Z}_+} \left(t + \frac{e^{t^2}}{2} 2^{j\alpha} \right) 2^{-j\epsilon}. \end{aligned}$$

It is clear that both the sums are finite.

Therefore $\sum_{j \in \mathbb{Z}_+} |a_{j,k(j,x)}(t)|$ uniformly converges on $I_0(x_0)$. Thus $f_c(\cdot, t)$ is continuous at x_0 . It remains to note that x_0 is an arbitrary element of G , thus $f_c(\cdot, t)$ is continuous on G . \square

Finally note that a statement analogous to item 1 of Theorem 6 was proved by Shelkovich in [17] for the functions defined on the field of p -adic numbers. The idea to use the class of distributions \tilde{S}' and the Haar representation of its elements was borrowed from [17]. Several other PDEs were also considered in this paper. We restrict ourselves to the study of the heat equation, because actually a repetition of all arguments are needed to solve other equations, for which this technique works.

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References

- [1] G.N. Agaev, N.Y. Vilenkin, G.M. Dzharfali, A.I. Rubinshtein, Multiplicative systems of functions and harmonic analysis on zero-dimensional groups, [in Russian], Baku, ELM, 1981.
- [2] A. Blumen, J. Klafter, and G. Zumofen, "Relaxation behavior in ultrametric spaces," J. Phys. A 19, L77 (1986).
- [3] P.L. Butzer and H.J. Wagner, A calculus for Walsh functions defined on \mathbb{R}_+ , Proc. Sympos. Applic. Walsh Functions, Washington, D.C., 75-81, (1973).
- [4] P.L. Butzer and H.J. Wagner, Walsh series and the concept of a derivative, Appl. Anal. 3 (1973), 29-46.
- [5] Yu. A. Farkov, Examples of frames on the Cantor dyadic group, J. Math. Sc. (2012), 187(1): 22-34.
- [6] Yu. A. Farkov, Wavelet expansions on the Cantor group, Math. Notes (2014), 96(6): 996-1007.
- [7] Yu. A. Farkov, V. Yu. Protasov, Dyadic wavelets and refinable functions on a half-line, Sbornik: Mathematics (2006), 197(10): 1529-1558.
- [8] Gibbs J.E. Walsh Spectrometry, a Form of Spectral Analysis Well Suited to Binary Digital Computation. Teddington: Nat. Phys. Lab., UK, 1967.
- [9] B.I. Golubov, Elements of dyadic analysis, [in Russian], Moscow, LKI, 2007.
- [10] B.I. Golubov, A.V. Efimov, and V.A. Skvortsov, Walsh series and transforms, English transl.: Kluwer, Dordrecht, 1991.
- [11] E. Hewitt, K. A. Ross, Abstract Harmonic Analysis. Springer-Verlag, New York, 1963, 1979.
- [12] A. N. Kochubei, Pseudo-Differential Equations and Stochastics over Non-Archimedean Fields, Marcel Dekker, New York, 2001.
- [13] Kozyrev, S. V., Osipov, V. Al., Avetisov, V. A., Nondegenerate ultrametric diffusion, J. Math. Phys. 46 (2005), no. 6.
- [14] W.C.Lang, Orthogonal wavelets on the Cantor dyadic group, SIAM J. Math. Anal., 27 (1996), 305-312.

- [15] A. T. Ogielski and D. L. Stein, "Dynamics on ultrametric spaces," *Phys. Rev. Lett.* 55, 1634 (1985).
- [16] F. Schipp, W. R. Wade, P. Simon, *Walsh series. An introduction to dyadic harmonic analysis*, Akademiai Kiado, Budapest, 1990.
- [17] Shelkovich, V. M. *p*-adic evolution pseudodifferential equations and *p*-adic wavelets. (Russian) *Izv. Ross. Akad. Nauk Ser. Mat.* 75 (2011), no. 6, 163–194; translation in *Izv. Math.* 75 (2011), no. 6, 1249-1278
- [18] R.S.Stankovic, J. Astola, *Gibbs Derivative — the First Forty Years*, T.I.C.S.P.#39, Tampere, 2008.
- [19] Stankovic R. S., Gibbs J.E. *Bibliography of Gibbs derivatives, Theory and Applications of Gibbs Derivatives*. Proc. First Intern. Workshop on Gibbs Derivatives (Kuopari-Dubrovnik, September 26-28, 1989). Beograd: Math. Institute, 1989, XIV-XXIV