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On linear isometries and ε-isometries between Banach spaces

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ABSTRACT

Let X, Y be two Banach spaces, and $f : X \to Y$ be a standard ε-isometry for some $\varepsilon \geq 0$. Recently, Cheng et al. showed that if $\overline{\text{co}}[f(X) \cup -f(X)] = Y$, then there exists a surjective linear operator $T : Y \to X$ with $\|T\| = 1$ such that the following sharp inequality holds:

$$\|Tf(x) - x\| \leq 2\varepsilon \text{ for all } x \in X.$$

Making use of the above result, we prove the following results: Suppose that $\overline{\text{co}}[f(X) \cup -f(X)] = Y$. Then

(1) if there is a linear isometry $S : X \to Y$ such that $TS = Id_X$, then $T^*S^* : Y^* \to T^*(X^*)$ is a w^*-to-w^* continuous linear projection with $\|T^*S^*\| = 1$,
(2) if there exists a w^*-to-w^* continuous linear projection $P : Y^* \to T^*(X^*)$ with $\|P\| = 1$, then there is an unique linear isometry $S(P) : X \to Y$ such that $TS(P) = Id_X$ and $P = T^*S(P)^*$. Furthermore, if $P_1 \neq P_2$ are two w^*-to-w^* continuous linear projection from Y^* onto $T^*(X^*)$ with $\|P_1\| = \|P_2\| = 1$, then $S(P_1) \neq S(P_2)$.

We apply these results to provide an alternative proof of a recent theorem, which gives an affirmative answer of a question proposed by Vestfrid. We also unify several known theorems concerning the stability of ε-isometries.

1. Introduction

Let X, Y be two Banach spaces, $f : X \to Y$ be a mapping, and $\varepsilon \geq 0$. The mapping $f : X \to Y$ is called an ε-isometry if $|\|f(u) - f(v)\| - \|u - v\|| \leq \varepsilon$ for all $u, v \in X$, and f is said to be standard provided $f(0) = 0$. 0-isometry is simply called isometry.

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Since the seminal theorem of Mazur and Ulam [15] (every surjective isometry between real Banach spaces is affine), which demonstrates the structures of Banach spaces as metric spaces determine their linear structures [1], there is an extensive literature on the study of properties of ε-isometries. Hyers and Ulam [13] first proposed and studied the problem of uniform approximation of a surjective ε-isometry by a surjective linear isometry. After exactly half-a-century of hard work of many mathematicians (see, for instance, Hyers and Ulam [13,14], Bourgin [2,3], Gruber [12] and Gevirtz [10]), the affirmative answer with a sharp estimate had been achieved by Omladič and Šemrl in [16].

Connecting Hyers–Ulam question and Figiel's remarkable theorem [9], Qian [17], Šemrl and Väisälä [18] had investigated the stability problem of non-surjective ε-isometries, i.e., whether there exist bounded linear operator $T : \overline{\text{span}}(f(X)) \to X$ and $\gamma > 0$ such that $Tf - Id_X$ is uniformly bounded by $\gamma\varepsilon$ on X for any standard ε-isometry $f : X \to Y$. The situations in the non-surjective cases are rather different and complicated. [17] and [18] proved some partial affirmative answers of the stability problem, but the general answer of stability problem is negative. Moreover, [5,6] and especially [8] reveal that the possible positive answers are heavily dependent on the complementability of some involved subspaces.

Recently, Cheng et al. [4, Theorem 3.3] showed that if $\overline{\text{co}}[f(X) \cup -f(X)] = Y$, then there exists a surjective linear operator $T : Y \to X$ with $\|T\| = 1$ such that the following sharp inequality holds:

$$\|Tf(x) - x\| \leq 2\varepsilon \text{ for all } x \in X. \tag{1.1}$$

Recall that Figiel [9] proved that if $f : X \to Y$ is a standard isometry, then there is an unique linear operator $F : \overline{\text{span}}(f(X)) \to X$ with $\|F\| = 1$ such that

$$\|Ff(x) - x\| = 0 \text{ for all } x \in X. \tag{1.2}$$

Therefore, (1.1) can be viewed as a sharp extension of Figiel's result (1.2) in the case of ε-isometry with $\varepsilon > 0$. The operator T in (1.1) can also be considered as a generalization of the Figiel operator F defined in (1.2).

For the study of the relationship between isometry and linear isometry, Godefroy and Kalton [11] showed a deep theorem: For any standard isometry $f : X \to Y$, if X is a separable Banach space, then there is a linear isometry $S : X \to \overline{\text{span}}(f(X))$ such that

$$FS = Id_X, \tag{1.3}$$

where F is the Figiel operator defined in (1.2); if X is a non-separable weakly compact generated space, then there exist a Banach space Y and a non-linear isometry $f : X \to Y$, however, X is not linearly isomorphic to any subspace of Y.

(1.3) asserts that if X is separable, then the Figiel operator F admits a linearly isometric right inverse. However, let H be non-separable Hilbert space, then Godefroy–Kalton theorem indicates that, even for "the best" non-separable Banach space H, there exist a Banach space Y and an isometry $f : H \to Y$ such that the operator F does not admit linearly isometric right inverse.

In the light of (1.1), (1.2), (1.3) and the above comments, the following question deserves consideration.

Question 1.1. Let X, Y be two Banach spaces, and $f : X \to Y$ be a standard ε-isometry. Suppose that $\overline{\text{co}}[f(X) \cup -f(X)) = Y$, and let $T : Y \to X$ be the operator defined in (1.1).

What are the necessary and sufficient conditions to guarantee the existence of a linear isometry $S : X \to Y$ with

$$TS = Id_X.$$

We present an answer of Question 1.1 in Section 2 (Theorem 2.1). Our main results reveal that the existence of linearly isometric right inverse of the operator T is tightly related with the complementability of the subspace $T^*(X^*)$ in Y^* (Remark 2.2). In Section 3, applying Theorem 2.1, we provide an alternative proof of a recent result firstly presented in [7] (Theorem 3.3), which gives an affirmative answer of a question proposed by Vestfrid in [19]. We also unify several known theorems (such as Omladič–Šemrl Theorem [16]; Šemrl–Väisälä Theorem [18]) concerning the stability of ε-isometries (Theorem 3.4 and Remark 3.5).

All Banach spaces are real, and we use X to denote a Banach space and X^* its dual. $B(X)$ (resp. $S(X)$) represents the closed unit ball (resp. sphere) of X. For a subspace $M \subset X$, $M^\perp$ stands for the annihilator of M, i.e. $M^\perp = \{x^* \in X^*, \langle x^*, x\rangle = 0 \text{ for all } x \in M\}$. If $M \subset X^*$, then ${}^\perp M$, the pre-annihilator of M is defined as ${}^\perp M = \{x \in X, \langle x, x^*\rangle = 0 \text{ for all } x^* \in M\}$. Given a bounded linear operator $T : X \to Y$, $T^* : Y^* \to X^*$ stands for its conjugate operator. For a subset $A \subset X$, $\overline{\text{co}}A$ (resp. $\overline{\text{span}}(A)$) represents the closed convex hull of A (resp. closed subspace linearly generated by A).

2. Linear isometric right inverse

In this section, we mainly show the following results, which provide an answer to Question 1.1.

Theorem 2.1. *Let X, Y be two Banach spaces, $f : X \to Y$ be a standard ε-isometry. Suppose that $\overline{\text{co}}[f(X) \cup -f(X)] = Y$. Let $T : Y \to X$ be the surjective linear operator defined in (1.1), i.e. $\|Tf(x) - x\| \le 2\varepsilon$ for all $x \in X$ and $\|T\| = 1$.*

(1) *If there is a linear isometry $S : X \to Y$ such that*

$$TS = Id_X,$$

then $T^*S^* : Y^* \to T^*(X^*)$ is a w^*-to-w^* continuous linear projection with $\|T^*S^*\| = 1$.

(2) *If there exists a w^*-to-w^* continuous linear projection $P : Y^* \to T^*(X^*)$ with $\|P\| = 1$, then there is an unique linear isometry $S(P) : X \to Y$ such that*

$$TS(P) = Id_X \tag{2.1}$$

and $P = T^*S(P)^*$. Furthermore, if $P_1 \neq P_2$ are two w^*-to-w^* continuous linear projections from Y^* onto $T^*(X^*)$ with $\|P_1\| = \|P_2\| = 1$, then $S(P_1) \neq S(P_2)$.

Proof. Let $T : Y \to X$ be the operator defined in (1.1). (Please note that this is the first time we use the assumption of the theorem, i.e. $\overline{\text{co}}[f(X) \cup -f(X)] = Y$.) This means $T : Y \to X$ with $\|T\| = 1$ such that

$$\|Tf(x) - x\| \le 2\varepsilon \text{ for all } x \in X. \tag{2.2}$$

Since $\|T\| = 1$, $\|T^*\| = 1$. Therefore, on the one hand, for any $x^* \in S(X^*)$, $\|T^*(x^*)\| \le 1$; on the other hand, for any $\delta > 0$, let $x \in B(X)$ so that $\langle x^*, x\rangle > 1 - \delta$. Substituting x by nx in the above (2.2), and dividing both sides by n, we obtain that

$$\|T(\frac{f(nx)}{n}) - x\| \le \frac{2\varepsilon}{n}.$$

Thus,

$$|\langle T^*(x^*), \frac{f(nx)}{n}\rangle - \langle x^*, x\rangle| = |\langle x^*, T(\frac{f(nx)}{n}) - x\rangle| \to 0.$$

This shows that $\|T^*(x^*)\| \geq 1 - \delta$. Since δ is arbitrary, it follows that $\|T^*(x^*)\| \geq 1$. In conclusion, $T^* : X^* \to T^*(X^*) \subset Y^*$ is a w^*-to-w^* continuous linear isometry.

(1): If there exists a linear isometry $S : X \to Y$ such that $TS = Id_X$, then $T^*S^* : Y^* \to T^*(X^*)$ is a w^*-to-w^* continuous linear operator with $\|T^*S^*\| = 1$. Moreover,

$$T^*S^*T^*S^* = T^*(TS)^*S^* = T^*(Id_X)^*S^* = T^*(Id_{X^*})S^* = T^*S^*. \tag{2.3}$$

Therefore, $T^*S^* : Y^* \to T^*(X^*)$ is a w^*-to-w^* continuous linear projection with $\|T^*S^*\| = 1$.

(2): We will complete the proof in several steps.

Step 1: Suppose that there is a w^*-to-w^* continuous projection $P : Y^* \to T^*(X^*)$ with $\|P\| = 1$. We first define the following mapping

$$Q : X \to Y,\ \langle Q(x), y^*\rangle \equiv \langle f(x), P(y^*)\rangle \text{ for all } x \in X, y^* \in Y^*. \tag{2.4}$$

Actually, since $P : Y^* \to T^*(X^*)$ is a w^*-to-w^* continuous projection, $\langle f(x), P(\cdot)\rangle : Y^* \to \mathbb{R}$ is a w^*-continuous linear functional for any $x \in X$. Consequently, $Q(x) : Y^* \to \mathbb{R}$ is also a w^*-continuous linear functional by (2.4). This means that

$$Q(x) \in Y.$$

Due to (2.4), on the one hand,

$$\|Q(x)\| = \sup_{y^*\in B(Y^*)} |\langle y^*, Q(x)\rangle| = \sup_{y^*\in B(Y^*)} |\langle P(y^*), f(x)\rangle| \leq \|P\|\|f(x)\| \leq \|x\| + \varepsilon. \tag{2.5}$$

On the other hand, for any $x \in X$, let $x^* \in S(X^*)$ so that $x^*(x) = \|x\|$, then by (2.4) again,

$$\|Q(x)\| = \sup_{y^*\in B(Y^*)} |\langle y^*, Q(x)\rangle| \geq |\langle Q(x), T^*(x^*)\rangle| = |\langle f(x), T^*(x^*)\rangle| \geq \|x\| - 2\varepsilon. \tag{2.6}$$

Combining (2.5) and (2.6), we deduce that

$$\|x\| - 2\varepsilon \leq \|Q(x)\| \leq \|x\| + \varepsilon \text{ for all } x \in X. \tag{2.7}$$

Further more, for any $u, v \in X$,

$$\begin{aligned}\|Q(u) - Q(v)\| &= \sup_{y^*\in B(Y^*)} |\langle y^*, Q(u) - Q(v)\rangle| = \sup_{y^*\in B(Y^*)} |\langle P(y^*), f(u) - f(v)\rangle| \\ &\leq \|f(u) - f(v)\| \leq \|u - v\| + \varepsilon. \end{aligned} \tag{2.8}$$

Next, we claim that $Q : X \to Y$ satisfies the following inequality

$$\|Q(u + v) - (Q(u) + Q(v))\| \leq 6\varepsilon \text{ for all } u, v \in X. \tag{2.9}$$

Indeed, by Hahn–Banach theorem, we take $\phi \in S(Y^*)$ so that $\langle \phi, Q(u + v) - (Q(u) + Q(v))\rangle = \|Q(u + v) - (Q(u) + Q(v))\|$. Since $T^* : X^* \to T^*(X^*) = P(Y^*)$ is a surjective linear isometry, we further choose $\psi \in X^*$ with $\|\psi\| = \|P(\phi)\| \leq \|P\|\|\phi\| = 1$ such that $T^*(\psi) = P(\phi)$. Due to (2.4) and (2.2),

$$\begin{aligned}&\langle \phi, Q(u + v) - (Q(u) + Q(v))\rangle = \langle f(u + v) - (f(u) + f(v)), P(\phi)\rangle \\ &= \langle f(u + v) - (f(u) + f(v)), T^*(\psi)\rangle = \langle \psi, Tf(u + v) - (Tf(u) + Tf(v))\rangle\end{aligned}$$

$$= \langle \psi, Tf(u+v) - (u+v) - (Tf(u) - u) - (Tf(v) - v) \rangle$$
$$\leq \|\psi\|(\|Tf(u+v) - (u+v)\| + \|Tf(u) - u\| + \|Tf(v) - v\|) \leq 6\varepsilon.$$

Therefore, (2.9) holds.

Step 2: It follows from (2.9) that $\{\frac{Q(2^n x)}{2^n}\}_{n=1}^{\infty}$ is a Cauchy sequence for any $x \in X$. Therefore, we shall define the desired mapping $S(P) : X \to Y$ as follows:

$$S(P) : X \to Y : S(P)(x) \equiv \lim_n \frac{Q(2^n x)}{2^n} \text{ for all } x \in X. \tag{2.10}$$

In fact, by (2.9) and (2.10), we obtain

$$\|S(P)(u+v) - (S(P)(u) + S(P)(v))\| = \lim_n \| \frac{Q(2^n(u+v)) - (Q(2^n u) + Q(2^n v))}{2^n} \| = 0, \tag{2.11}$$

which entails $S(P)$ is additive.

(2.8) and (2.10) show that

$$\|S(P)(u) - S(P)(v)\| = \lim_n \| \frac{Q(2^n u) - Q(2^n v)}{2^n} \|$$
$$\leq \lim_n (\frac{\|2^n u - 2^n v\|}{2^n} + \frac{\varepsilon}{2^n}) = \|u - v\|. \tag{2.12}$$

Hence, $S(P)$ is 1-Lipschitz. (2.11) and (2.12) together imply $S(P)$ is a bounded linear operator. According to (2.7), we have that

$$\|x\| = \lim_n \frac{\|2^n x\| - 2\varepsilon}{2^n} \leq \|S(P)(x)\| = \lim_n \| \frac{Q(2^n x)}{2^n} \| \leq \lim_n \frac{\|2^n x\| + \varepsilon}{2^n} = \|x\|. \tag{2.13}$$

Therefore, $S(P) : X \to Y$ is a linear isometry. Consequently, in view of (2.4) and (2.2), we have

$$\|TS(P)(x) - x\| = \lim_n \|T(\frac{Q(2^n x)}{2^n}) - x\| = \lim_n \sup_{x^* \in B(X^*)} |\langle x^*, T(\frac{Q(2^n x)}{2^n}) - x \rangle|$$
$$= \lim_n \sup_{x^* \in B(X^*)} |\langle T^* x^*, \frac{Q(2^n x)}{2^n} \rangle - \langle x^*, x \rangle|$$
$$= \lim_n \sup_{x^* \in B(X^*)} |\langle P(T^* x^*), \frac{f(2^n x)}{2^n} \rangle - \langle x^*, x \rangle|$$
$$= \lim_n \sup_{x^* \in B(X^*)} |\langle x^*, \frac{Tf(2^n x)}{2^n} \rangle - \langle x^*, x \rangle|$$
$$= \lim_n \sup_{x^* \in B(X^*)} |\langle x^*, \frac{Tf(2^n x) - 2^n x}{2^n} \rangle| \leq \lim_n \frac{2\varepsilon}{2^n} = 0.$$

This means $TS(P) = Id_X$, i.e. (2.1) is shown.

Step 3: We will prove $P = T^* S(P)^*$. Note that $T^*(X^*) = \cup_{n=1}^{\infty} nT^*(B(X^*))$. Thus $T^*(X^*)$ is w^*-closed by Krein–Šmulian Theorem. This entails $T^*(X^*) = ({}^{\perp}T^*(X^*))^{\perp} = (Y/{}^{\perp}T^*(X^*))^*$.

According to 2.9, we show that for any $x \in X$

$$\begin{aligned}\|S(P)(x) - Q(x)\| &= \lim_n \|\frac{Q(2^n x)}{2^n} - Q(x)\| = \lim_n \frac{\|Q(2^n x) - 2^n Q(x)\|}{2^n} \\ &\le \lim_n \frac{1}{2^n} \sum_{i=0}^{n-1} \|2^i Q(2^{n-i}x) - 2^{i+1} Q(2^{n-i-1}x)\| \\ &\le \lim_n \frac{1}{2^n} \sum_{i=0}^{n-1} 2^i 6\varepsilon \le 6\varepsilon. \end{aligned} \tag{2.14}$$

(2.14), (2.2) and $T^*(X^*) = (Y/^{\perp}T^*(X^*))^*$ together imply that for any $y^* \in Y^*$, $n \in \mathbb{N}$, $\lambda_i \in \mathbb{R}$ with $\sum_{i=1}^n |\lambda_i| = 1$, and $\{x_i\}_{i=1}^n \subset X$

$$\begin{aligned}&\langle T^* S(P)^*(y^*), \sum_{i=1}^n \lambda_i f(x_i) +^{\perp} T^*(X^*)\rangle = \langle T^* S(P)^*(y^*), \sum_{i=1}^n \lambda_i f(x_i)\rangle \\ &= \langle S(P)^*(y^*), T(\sum_{i=1}^n \lambda_i f(x_i))\rangle \\ &= \langle S(P)^*(y^*), T(\sum_{i=1}^n \lambda_i f(x_i)) - \sum_{i=1}^n \lambda_i x_i + \sum_{i=1}^n \lambda_i x_i\rangle \\ &\le \langle S(P)^*(y^*), \sum_{i=1}^n \lambda_i x_i\rangle + \sum_{i=1}^n |\lambda_i| 2\varepsilon \|y^*\| = \langle y^*, \sum_{i=1}^n \lambda_i S(P)(x_i)\rangle + 2\varepsilon\|y^*\| \\ &= \langle y^*, \sum_{i=1}^n \lambda_i S(P)(x_i) - \sum_{i=1}^n \lambda_i Q(x_i) + \sum_{i=1}^n \lambda_i Q(x_i)\rangle + 2\varepsilon\|y^*\| \\ &\le \langle y^*, \sum_{i=1}^n \lambda_i Q(x_i)\rangle + 6\varepsilon\|y^*\| + 2\varepsilon\|y^*\| = \langle P(y^*), \sum_{i=1}^n \lambda_i f(x_i)\rangle + 8\varepsilon\|y^*\| \\ &= \langle P(y^*), \sum_{i=1}^n \lambda_i f(x_i) +^{\perp} T^*(X^*)\rangle + 8\varepsilon\|y^*\|. \end{aligned} \tag{2.15}$$

Next, we will use the assumption of the theorem again, i.e.

$$\overline{\mathrm{co}}(f(X) \cup -f(X)) = Y. \tag{2.16}$$

(Please note that this is the second time and the last time that we use the assumption of the theorem, i.e. $\overline{\mathrm{co}}(f(X) \cup -f(X)) = Y$.)

(2.15) and (2.16) ensure that

$$T^* S(P)^* = P. \tag{2.17}$$

Step 4: Finally, suppose that there exist two linear isometry $S(P)_1 : X \to Y$ and $S(P)_2 : X \to Y$ with $T^* S(P)_1^* = P = T^* S(P)_2^*$. If $S(P)_1 \ne S(P)_2$, then there is $0 \ne x \in X$ such that $S(P)_1(x) \ne S(P)_2(x)$. According to separation theorem, taking $y^* \in Y^*$ with $y^*(S(P)_1(x)) \ne y^*(S(P)_2(x))$, we observe that $S(P)_1^*(y^*)(x) \ne S(P)_2^*(y^*)(x)$. Therefore,

$$S(P)_1^*(y^*) \ne S(P)_2^*(y^*). \tag{2.18}$$

(2.18) and the fact that $T^* : X^* \to T^*(X^*)$ is a linear isometry imply that $T^*S(P)_1^*(y^*) \neq T^*S(P)_2^*(y^*)$, which leads to a contradiction with the fact $T^*S(P)_1^* = P = T^*S(P)_2^*$. Suppose that $P_1 \neq P_2$ are two w^*-to-w^* continuous linear projection from Y^* onto $T^*(X^*)$ with $\|P_1\| = \|P_2\| = 1$. By (2.17), $T^*S(P_1)^* = P_1 \neq P_2 = T^*S(P_2)^*$. Consequently, $S(P_1)^* \neq S(P_2)^*$ since T^* is a linear isometry. This shows $S(P_1) \neq S(P_2)$. The proof is completed. □

Remark 2.2. (i) Suppose that $T : Y \to X$ is the linear operator defined in (1.1). Let Φ be the set of all linearly isometric right inverse of T, Ψ be the set of all w^*-to-w^* continuous linear projection $P : Y^* \to T^*(X^*)$ with $\|P\| = 1$. Theorem 2.1 shows that $\lambda : \Phi \to \Psi, \lambda(S(P)) = T^*S(P)^*$ is a surjective one-to-one mapping. This reveals the closed relationship between the existence of linearly isometric right inverse of T and the w^*-complementability of the subspace $T^*(X^*)$ in Y^*.

(ii) As noted in the proof of Theorem 2.1, we assume that $\overline{\text{co}}[f(X) \cup -f(X)] = Y$ only for two purposes. Namely, (1) to entail the existence of a linear operator $T : Y \to Y$ with $\|T\| = 1$ and $\|Tf(x) - x\| \le 2\varepsilon$ for all $x \in X$ by equation (1.1), (2) to make sure $P = T^*S(P)^*$ (in Step 3). Therefore, if we replace the assumption that $\overline{\text{co}}[f(X) \cup -f(X)] = Y$ by the existence of a linear operator $T : Y \to Y$ with $\|T\| = 1$ and $\|Tf(x) - x\| \le 2\varepsilon$ for all $x \in X$, then the conclusion of (1) of Theorem 2.1 and equation (2.1) are still true. We do not know whether we still have $P = T^*S(P)^*$.

However, if $\varepsilon = 0$, i.e., $f : X \to Y$ is a standard isometry, then equation (1.2) ensures that there exists linear operator $F : Y \to Y$ with $Ff(x) = x$ for all $x \in X$, even though we drop the assumption that $\overline{\text{co}}[f(X) \cup -f(X)] = Y$. We close this section by giving the following corollary. We omit its proof since it is similar to the proof of Theorem 2.1.

Corollary 2.3. *Let X, Y be two Banach spaces, $f : X \to Y$ be a standard isometry. Let $F : \overline{\text{span}}(f(X)) \to X$ with $\|F\| = 1$ and $Ff = Id_X$ be the operator defined in (1.2).*

(1) *If there is a linear isometry $S : X \to \overline{\text{span}}(f(X))$ such that*
$$FS = Id_X,$$
then $F^*S^* : [\overline{\text{span}}(f(X))]^* \to F^*(X^*)$ is a w^*-to-w^* continuous linear projection with $\|F^*S^*\| = 1$.

(2) *If there is a w^*-to-w^* continuous linear projection*
$$P : [\overline{\text{span}}(f(X))]^* \to F^*(X^*) \text{ with } \|p\| = 1,$$
then there exists an unique linear isometry $S(P) : X \to Y$ such that
$$FS(P) = Id_X \text{ and } P = F^*S(P)^*.$$
Furthermore, if $P_1 \neq P_2$ are two w^*-to-w^* continuous linear projections from Y^* onto $F^*(X^*)$ with $\|P_1\| = \|P_2\| = 1$, then $S(P_1) \neq S(P_2)$.

3. Some applications

In this section we will present some applications of our main results Theorem 2.1. We first provide an alternative proof of the following Theorem 3.3, which is firstly presented in [7]. Then we apply Theorem 3.3 to unify several known results concerning the stability of ε-isometry.

We first recall some preliminaries, (see [19]). For any $y \in S(Y)$ and $\emptyset \neq A \subset Y$, let

$$\varrho(y, A) \equiv \liminf_{|t|\to\infty} \operatorname{dist}(ty, A)/|t|$$
$$\tau(A) \equiv \sup_{y\in S_Y} \varrho(y, A) \equiv \sup_{y\in S_Y} \liminf_{|t|\to\infty} \operatorname{dist}(ty, A)/|t|.$$

By making use of some results from [6], Vestfrid [19] proved the following interesting result: For any standard ε-isometry $f : X \to Y$, if $\tau(f(X)) < 1/2$, then there is a surjective linear isometry $S : X \to Y$ such that

$$\|f(x) - S(x)\| \le 2\varepsilon \text{ for all } x \in X.$$

Vestfrid further asked "whether the condition $\varrho(y, f(X)) < 1/2$ for every $y \in S_Y$ is enough to guarantee the existence of an approximating isometry."

Before describing our main results, we also need the following lemma.

Lemma 3.1. *Let X, Y be Banach spaces, and $f : X \to Y$ be a standard ε-isometry for some $\varepsilon \ge 0$. Then $\tau(\overline{\text{co}}[f(X) \cup -f(X)]) < 1$ if and only if $\overline{\text{co}}[f(X) \cup -f(X)] = Y$.*

Proof. Sufficiency: It is trivial.

Necessity: Suppose, to the contrary, that $C \equiv \overline{\text{co}}(f(X) \cup -f(X)) \neq Y$. Let $1 - \tau(C) = 2\delta > 0$, then by separation theorem, there exist $\psi \in S_{Y^*}$ and $\alpha \in \mathbb{R}$ such that

$$\langle \psi, z\rangle \le \alpha, \text{ for all } z \in C.$$

Let $y \in S_Y$ so that $\langle \psi, y\rangle > 1 - \delta$. Then for all $t \in \mathbb{R}$, since C is symmetric,

$$\operatorname{dist}(ty, C) = \operatorname{dist}(|t|y, C) \ge \inf\langle \psi, |t|y - C\rangle > (1 - \delta)|t| - \sup\langle \psi, C\rangle \ge (1 - \delta)|t| - \alpha.$$

Thus,

$$\liminf_{|t|\to\infty} \operatorname{dist}(ty, C)/|t| \ge 1 - \delta = \tau(C) + \delta,$$

which leads to a contradiction. $\square$

Remark 3.2. It follows from Lemma 3.1 that

$$\begin{aligned} \tau(\overline{\text{co}}[f(X) \cup -f(X)]) < 1 &\Leftrightarrow \overline{\text{co}}[f(X) \cup -f(X)] = Y \\ &\Leftrightarrow \tau(\overline{\text{co}}[f(X) \cup -f(X)]) = 0 \\ &\Leftrightarrow \varrho(y, \overline{\text{co}}[f(X) \cup -f(X)]) = 0 \text{ for all } y \in S(Y). \end{aligned} \tag{3.1}$$

By (3.1), we can substitute the assumption $\overline{\text{co}}[f(X) \cup -f(X)] = Y$ by any one of the equivalent conditions mentioned as above, and the conclusions of Theorem 2.1 and equation (1.1) are still true.

Theorem 3.3. *Let X, Y be Banach spaces, and $f : X \to Y$ be a standard ε-isometry for some $\varepsilon \ge 0$. Then there is an unique linear surjective isometry $S : X \to Y$ such that*

$$\|f(x) - S(x)\| \le 2\varepsilon \text{ for all } x \in X, \tag{3.2}$$

if and only if

$$\varrho(y, f(X)) \equiv \liminf_{|t|\to\infty} \operatorname{dist}(ty, f(X))/|t| < 1/2 \textit{ for all } y \in S_Y. \tag{3.3}$$

Proof. Necessity: The proof of necessity have been given in [7]. We show the detailed proof for the sake of convenience.

If there is a linear surjective isometry $S : X \to Y$ such that (3.2) holds, then for any $y \in Y$ and $t \in \mathbb{R}$, there exist $\{x_t\} \in X$ such that $ty = S(x_t)$ and

$$\|f(x_t) - S(x_t)\| \le 2\varepsilon,$$

which entails $\operatorname{dist}(ty, f(X)) \le 2\varepsilon$. Therefore,

$$\varrho(y, f(X)) \equiv \liminf_{|t|\to\infty} \operatorname{dist}(ty, f(X))/|t| \le \liminf_{|t|\to\infty} 2\varepsilon/|t| = 0 < 1/2.$$

(3.3) is shown.

Sufficiency: Suppose that $\varrho(y, f(X)) \equiv \liminf_{|t|\to\infty} \operatorname{dist}(ty, f(X))/|t| < 1/2$ for all $y \in S(Y)$. Clearly,

$$\begin{aligned} \tau(\overline{\text{co}}[f(X) \cup -f(X)]) &= \sup_{y\in S_Y} \varrho(y, \overline{\text{co}}[f(X) \cup -f(X)]) \\ &\le \sup_{y\in S_Y} \varrho(y, f(X)) \le 1/2 < 1. \end{aligned}$$

According to Lemma 3.1, $\overline{\text{co}}[f(X) \cup -f(X)] = Y$. Hence, (1.1) entails that there is a surjective linear operator $T : Y \to X$ with $\|T\| = 1$ such that

$$\|Tf(x) - x\| \le 2\varepsilon \text{ for all } x \in X. \tag{3.4}$$

We then assert that $Y^* = T^*(X^*)$. Otherwise, $T^*(X^*)$ is a proper w^*-closed subspace. By separation theorem, there exist $\psi \in S_{Y^*} \setminus (T^*(X^*))$ and $y \in S_Y$ such that

$$\langle \psi, y\rangle > 0, \text{ and } \langle \phi, y\rangle = 0 \text{ for all } \phi \in T^*(X^*). \tag{3.5}$$

Since $\varrho(y, f(X)) \equiv \liminf_{|t|\to\infty} \operatorname{dist}(ty, f(X))/|t| < 1/2$, there exist $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ with $|t_n| \to \infty$ and a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that

$$\lim_n \frac{\|t_n y - f(x_n)\|}{|t_n|} \equiv \beta < 1/2. \tag{3.6}$$

For any x_n, let $x_n^* \in S(X^*)$ with $\langle x_n^*, x_n\rangle = \|x_n\|$, and let $\phi_n = T^* x_n^*$. By (3.4), (3.5) and (3.6),

$$\begin{aligned} 2\varepsilon &\ge \|Tf(x_n) - x_n\| \ge \langle x_n^*, x_n\rangle - \langle \phi_n, f(x_n)\rangle \\ &= \langle x_n^*, x_n\rangle - \langle \phi_n, f(x_n) - t_n y\rangle = \|x_n\| - \langle \phi_n, f(x_n) - t_n y\rangle \\ &\ge \|f(x_n)\| - \varepsilon - \|f(x_n) - t_n y\| \ge \|t_n y\| - 2\|f(x_n) - t_n y\| - \varepsilon \\ &= |t_n|(1 - 2\frac{\|f(x_n) - t_n y\|}{|t_n|}) - \varepsilon \xrightarrow{n\to\infty} \infty, \end{aligned} \tag{3.7}$$

which leads to a contradiction. Thus, $T^*(X^*) = Y^*$. Let $P = Id_{Y^*}$ be the identity of Y^*. Obviously, $P : Y^* \to T^*(X^*) = Y^*$ is a w^*-to-w^* continuous linear projection with $\|P\| = 1$. Due to (2) of Theorem 2.1, there exist an unique linear isometry $S : X \to Y$ such that

$$TS = Id_X \text{ and } Id_{Y^*} = P = T^*S^* = (ST)^*. \tag{3.8}$$

Consequently, (3.8) shows

$$T = S^{-1}. \tag{3.9}$$

(3.4) and (3.9) together show $S : X \to Y$ is a surjective linear isometry and

$$\|S(x) - f(x)\| \leq 2\varepsilon \text{ for all } x \in X. \tag{3.10}$$

Clearly, the surjective linear isometry S satisfying (3.10) is unique. □

Theorem 3.3 is not only an improvement of Vestfrid [19, Theorem 2 (iii)], but also an affirmative answer to Vestfrid's question.

Recall that [4] a subset N in a metric space (Ω, ρ) is called a sublinear growth net in metric ρ provided for any fixed $\omega_0 \in \Omega$,

$$\lim_{\rho(\omega,\omega_0)\to\infty} \frac{\rho(\omega, N)}{\rho(\omega, \omega_0)} = 0.$$

(See [18].) A subset $A \subset X$ is said to be δ-surjective for some $\delta \geq 0$ if for any $x \in X$ there is $a(x) \in A$ such that $\|a(x) - x\| \leq \delta$. Clearly, if $f(X)$ contains a sublinear growth net of Y or $f(X)$ is δ-surjective, then $\varrho(y, f(X)) = \liminf_{|t|\to\infty} \operatorname{dist}(ty, f(X))/|t| = 0$ for all $y \in S_Y$. Due to Theorem 3.3, we derive the following results.

Theorem 3.4. *Let X, Y be Banach spaces, and $f : X \to Y$ be a standard ε-isometry for some $\varepsilon \geq 0$. Then the following statements are equivalent.*

(1) *There is an unique linear surjective isometry $S : X \to Y$ such that $\|f(x) - S(x)\| \leq 2\varepsilon$ for all $x \in X$,*
(2) $\varrho(y, f(X)) \equiv \liminf_{|t|\to\infty} \operatorname{dist}(ty, f(X))/|t| = 0$ *for all $y \in S_Y$,*
(3) $\varrho(y, f(X)) \equiv \liminf_{|t|\to\infty} \operatorname{dist}(ty, f(X))/|t| < 1/2$ *for all $y \in S_Y$,*
(4) *$f(X)$ contains a sublinear growth net of Y,*
(5) *$f(X)$ is δ-surjective for some $\delta \geq 0$,*
(6) $\tau(f(X)) < 1/2$,
(7) $\tau(f(X)) = 0$.

Proof. It is trivial that (1)⇒(2), (1)⇒(3), (1)⇒(4), (1)⇒(5), (2)⇒(7)⇒(6) ⇒(3), (5)⇒(2)⇒(3) and (4)⇒(2)⇒(3). Theorem 3.2 implies that (3)⇒(1). Therefore, all of them are equivalent. □

Remark 3.5. Theorem 3.4 unifies several known results such as Omladič–Šemrl Theorem [16] ((5)⇒(1) when $\delta = 0$); Šemrl–Väisälä Theorem [18] ((5)⇒(1) for all $\delta \geq 0$); Vestfrid Theorem [19] ((6)⇒(1)); and Cheng–Cheng–Tu–Zhang Theorem [4] ((4)⇒(1)).

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