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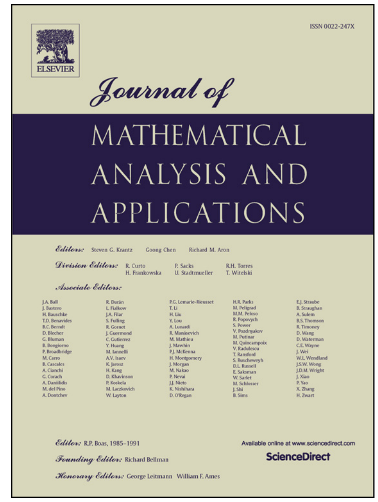

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Rongfei Liu, Dingcheng Wang

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The Ruin Probabilities of a Discrete-Time Risk Model with Dependent Insurance and Financial Risks

Rongfei Liu[a], Dingcheng Wang[b,a,*]

[a] *School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China*

[b] *Center of Financial Engineering, Nanjing Audit University, Nanjing 211815, China*

Abstract

Following the work of Sun and Wei (2014), we investigate the ruin probabilities of a discrete-time insurance risk model with dependent insurance and financial risks. Assume that the one-period net insurance losses and discount factors form a sequence of independent and identically distributed copies of a random pair (X, θ). When the product $X\theta$ is heavy tailed, we establish an asymptotic formula for the finite-time ruin probability without any restriction on the dependence structure of (X, θ) and extend the result to the infinite time ruin probability.

Keywords: asymptotic estimate, ruin probability, dependent insurance and financial risk, heavy tail

1. Introduction

Let x be a positive real number, $\{X_n\}_{n\geq 1}$ be a sequence of real-valued random variables and $\{\theta_n\}_{n\geq 1}$ be a sequence of nonnegative random variables. In this paper, we consider a discrete-time risk model as follows:

$$U_0 = x, U_n = U_{n-1}\theta_n^{-1} - X_n, n \geq 1. \tag{1.1}$$

In the insurance risk context, x is interpreted as the initial wealth of an insurance company and X_n is interpreted as the net insurance loss, i.e., the total claim amount minus the total incoming premium during period n. Assume that the payment of the claims and the collection of the premiums happen at the end of each period. The insurance company can invest surplus into a portfolio consisting of risk-free assets and risky assets at time $n-1$, leading to the stochastic discount factor θ_n from time n to time $n-1$. Thus, U_n is interpreted as the surplus of the insurance company at time n. In the terminology of Norberg (1999), we call $\{X_n\}_{n\geq 1}$ insurance risks and call $\{\theta_n\}_{n\geq 1}$ financial risks. Thus, the sum

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} \theta_j, n \in \mathbb{N},$$

*Corresponding author.

Email address: `wangdc@uestc.edu.cn (D. Wang)` (Rongfei Liu[a], Dingcheng Wang[b,a,])

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represents the stochastic present value of aggregate net losses up to time n.

Now we can define the ruin probability up to time n and the ultimate ruin probability, respectively, by

$$\Psi(x,n) = \mathbb{P}\left\{\min_{1\le m\le n} U_m < 0 \mid U_0 = x\right\} = \mathbb{P}\left\{\max_{1\le m\le n} S_m > x\right\} \tag{1.2}$$

and

$$\Psi(x) = \mathbb{P}\left\{\min_{1\le m<\infty} U_m < 0 \mid U_0 = x\right\} = \mathbb{P}\left\{\max_{1\le m<\infty} S_m > x\right\}. \tag{1.3}$$

Sun and Wei (2014) established an asymptotic formula for (1.2) under the conditions that the distribution of $X\theta$, denoted by H, belongs to the intersection of the dominated variation class ($\mathcal{D}$) and the long-tailed class ($\mathcal{L}$), θ fulfills certain moment condition and

$$\lim_{A\to\infty}\limsup_{x\to\infty}\frac{\mathbb{P}(X\theta > x, \theta > A)}{\mathbb{P}(X\theta > x)} = 0. \tag{1.4}$$

It's not easy to verify (1.4).

In this paper, we obtain an asymptotic estimate for $\Psi(x,n)$ and another for $\Psi(x)$ when H belongs to $\mathcal{D}\cap\mathcal{L}$. Notice that we do not need the specific assumption (1.4), but we need some other certain conditions. Then, the certain conditions also can be simplified when H belongs to the consistent variation class ($\mathcal{C}$). Furthermore, we apply these two results to the special case that H belongs to the regular variation class ($\mathcal{R}$) and get two asymptotic estimates, which are of more transparent forms.

The remaining part of this paper is organized as follows. In Section 2, we introduce some notations and state our main results. In Section 3, we provide some lemmas and prove the main results of the paper.

2. Notations and main results

In this paper, C represents a positive constant without relation to x and may vary from place to place. Hereafter, all limit relations are for $x\to\infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x)\lesssim b(x)$ or $b(x)\gtrsim a(x)$ if $\limsup a(x)/b(x)\le 1$ and write $a(x)\sim b(x)$ if $\lim a(x)/b(x) = 1$. Also, we write $a(x)\asymp b(x)$ if $0<\liminf a(x)/b(x)\le\limsup a(x)/b(x)<\infty$.

In order to facilitate subsequent expression, we denote

$$\kappa_i \triangleq \prod_{j=1}^{i}\theta_j, i\ge 1.$$

Now we recall several classes of heavy-tailed distributions. A distribution F belongs to the dominated variation class (denoted by $\mathcal{D}$) if $\overline{F}(x) = 1 - F(x) > 0$ for all $x\in\mathbb{R}$ and

$$\limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} < \infty \text{ for any } 0<y<1.$$

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A distribution F belongs to the long-tailed class (denoted by $\mathcal{L}$) if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$ and

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1 \text{ for any } y > 0. \tag{2.1}$$

Relation (2.1) holds uniformly for every compact set of y. That is to say, there exists a function $L(x)$, with $0 \le L(x) \le x/2$ and $L(x) \uparrow \infty$, such that (2.1) holds uniformly for $-L(x) \le y \le L(x)$. A distribution F belongs to the consistent variation class (denoted by $\mathcal{C}$) if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$ and

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1, \text{ or equivalently, } \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

A distribution F belongs to the regular variation class (denoted by $\mathcal{R}_{-\alpha}$) if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$ and

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha} \text{ for some } \alpha > 0 \text{ and all } y > 0.$$

It's well known that $\mathcal{R} \subset \mathcal{C} \subset \mathcal{D} \cap \mathcal{L}$.

Besides that, the upper Matuszewska index $\mathbb{J}_F^+$ and lower Matuszewska index $\mathbb{J}_F^-$ (see Bingham et al.(1987,Ch.2.1)) are used. It's well known that $\mathbb{J}_F^+ < \infty$ if $F \in \mathcal{D}$ and $\mathbb{J}_F^- = \mathbb{J}_F^+ = \alpha$ if $F \in \mathcal{R}_{-\alpha}$.

For making concise statements of our results, we present two assumptions.

Assumption A. There exists some function $L(x) : [0,\infty) \to [0,\infty)$ satisfying that

$$\frac{L(x)}{x^\nu} \to \infty, \frac{L(x)}{x} \to 0,$$

for some $p > \mathbb{J}_H^+$ and $\mathbb{J}_H^+/p < \nu < 1$ such that for each $i \ge 1$,

$$\mathbb{P}\{X_i\kappa_i > x \pm L(x)\} \sim \mathbb{P}\{X_i\kappa_i > x\}. \tag{2.2}$$

Assumption B. There exists some function $L(x) : [0,\infty) \to [0,\infty)$ satisfying that

$$L(x) \to \infty, \frac{L(x)}{x} \to 0,$$

such that

$$\overline{H}(x) \asymp \overline{H}(L(x)). \tag{2.3}$$

Now, we state the main results.

Theorem 2.1. *Let X be a real-valued random variable, θ be a nonnegative random variable and $\{(X_n,\theta_n)\}_{n\ge1}$ be i.i.d. copies of the random pair (X,θ). If the distribution of $X\theta$, denoted by H, belongs to $\mathcal{D}\cap\mathcal{L}$, $\mathbb{E}\theta^p < \infty$ for some $p > \mathbb{J}_H^+$ and there exists $L(x)$ satisfying Assumption A, then it holds that for any integer $n \ge 1$,*

$$\Psi(x,n) \sim \sum_{i=1}^{n} \mathbb{P}\{X_i\kappa_i > x\}. \tag{2.4}$$

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Remark 2.1. In fact, the distributions of $X_i\kappa_i, i \geq 1$ belong to $\mathcal{D} \cap \mathcal{L}$, which will be proved in Section 3.2. So, introducing (2.2) into Theorem 2.1 is reasonable.

Corollary 2.1. *Let X be a real-valued random variable, θ be a nonnegative random variable and $\{(X_n, \theta_n)\}_{n\geq 1}$ be i.i.d. copies of the random pair (X, θ). If the distribution of $X\theta$, denoted by H, belongs to $\mathcal{C}$ and $\mathbb{E}\theta^p < \infty$ for some $p > \mathbb{J}_H^+$, then (2.4) still holds.*

Corollary 2.2. *Let X be a real-valued random variable, θ be a nonnegative random variable and $\{(X_n, \theta_n)\}_{n\geq 1}$ be i.i.d. copies of the random pair (X, θ). If the distribution of $X\theta$, denoted by H, belongs to $\mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$ and $\mathbb{E}\theta^p < \infty$ for some $p > \alpha$, then*

$$\Psi(x, n) \sim \overline{H}(x) \sum_{i=1}^{n} (\mathbb{E}\theta^\alpha)^{i-1} = \overline{H}(x) \frac{1 - (\mathbb{E}\theta^\alpha)^n}{1 - \mathbb{E}\theta^\alpha}. \tag{2.5}$$

Theorem 2.2. *Let X be a real-valued random variable, θ be a nonnegative random variable and $\{(X_n, \theta_n)\}_{n\geq 1}$ be i.i.d. copies of the random pair (X, θ). If the distribution of $X\theta$, denoted by H, belongs to $\mathcal{D} \cap \mathcal{L}$ with $\mathbb{J}_H^- > 0$, $\mathbb{E}\theta^p < 1$ for some $p > \mathbb{J}_H^+$ and there exists $L(x)$ satisfying Assumption B, then it holds that*

$$\Psi(x) \sim \sum_{i=1}^{\infty} \mathbb{P}\{X_i\kappa_i > x\}. \tag{2.6}$$

Corollary 2.3. *Let X be a real-valued random variable, θ be a nonnegative random variable and $\{(X_n, \theta_n)\}_{n\geq 1}$ be i.i.d. copies of the random pair (X, θ). If the distribution of $X\theta$, denoted by H, belongs to $\mathcal{C}$ with $\mathbb{J}_H^- > 0$ and $\mathbb{E}\theta^p < 1$ for some $p > \mathbb{J}_H^+$, then (2.6) still holds.*

Corollary 2.4. *Let X be a real-valued random variable, θ be a nonnegative random variable and $\{(X_n, \theta_n)\}_{n\geq 1}$ be i.i.d. copies of the random pair (X, θ). If the distribution of $X\theta$, denoted by H, belongs to $\mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$ and $\mathbb{E}\theta^p < 1$ for some $p > \alpha$, then*

$$\Psi(x) \sim \overline{H}(x) \sum_{i=1}^{\infty} (\mathbb{E}\theta^\alpha)^{i-1} = \frac{\overline{H}(x)}{1 - \mathbb{E}\theta^\alpha}. \tag{2.7}$$

3. Proofs of the main results

By convention, we have $\sum_\Phi \cdot = 0$, $\prod_\Phi \cdot = 1$ and $X^+ = \max\{X, 0\}$.

3.1. Some lemmas

By Proposition 2.2.1 in Bingham et al.(1987), for a distribution $F \in \mathcal{D}$ and arbitrarily fixed $p > \mathbb{J}_F^+$, there exist positive constants C_p and D_p such that

$$\frac{\overline{F}(y)}{\overline{F}(x)} \leq C_p \left(\frac{x}{y}\right)^p \tag{3.1}$$

holds for all $x \geq y \geq D_p$. Fixing the variable y leads to

$$x^{-p} = o\left(\overline{F}(x)\right) \text{ for any } p > \mathbb{J}_F^+. \tag{3.2}$$

The following fundamental lemmas will be used.

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Lemma 3.1. *Let X and Y be two independent and nonnegative random variables, where X is distributed by F. If $F \in \mathcal{D}$, then for arbitrarily fixed $\delta > 0$ and $p > \mathbb{J}_F^+$, there exists a positive constant C without relation to δ and Y such that for all large x,*

$$\mathbb{P}(XY > \delta x \mid Y) \leq C\overline{F}(x) \left(\delta^{-p} Y^p \mathbf{1}_{[Y \geq \delta]} + \mathbf{1}_{[Y < \delta]}\right).$$

Proof. *See Lemma 3.2 in Heyde and Wang (2009).*

Lemma 3.2. *Let X and Y be two independent and nonnegative random variables, where X is distributed by F. If $F \in \mathcal{D}$ with $\mathbb{J}_F^- > 0$, then, for any fixed $\delta > 0$ and $0 < p_1 < \mathbb{J}_F^- \leq \mathbb{J}_F^+ < p_2 < \infty$, there exists a positive constant C without relation to δ and Y, such that for all large x,*

$$\mathbb{P}\left(XY > \delta x \mid Y\right) \leq C\overline{F}(x) \left(\delta^{-p_1} Y^{p_1} + \delta^{-p_2} Y^{p_2}\right).$$

Proof. *See Lemma 3 in Guo and Wang (2013).*

Lemma 3.3. *Let X and Y be two independent and nonnegative random variables, where X is distributed by F and Y is nondegenerate at 0. If $F \in \mathcal{D} \cap \mathcal{L}$ and $\mathbb{E}Y^p < \infty$ for some $p > \mathbb{J}_F^+$, then the distribution of XY belongs to $\mathcal{D} \cap \mathcal{L}$ and $\mathbb{P}(XY > x) \asymp \overline{F}(x)$.*

Proof. *As a direct result of Lemma 3.8 and Lemma 3.10 of Tang and Tsitsiashvili (2003) (see also Lemma 4.1.2 in Wang and Tang (2006)).*

Lemma 3.4. *Let X and Y be two independent random variables, X be real-valued and distributed by F and Y be nonnegative and nondegenerate at 0. If $F \in \mathcal{C}$ and $\mathbb{E}Y^p < \infty$ for some $p > \mathbb{J}_F^+$, then the distribution of XY belongs to $\mathcal{C}$ and $\mathbb{P}(XY > x) \asymp \overline{F}(x)$.*

Proof. *See Lemma 2.4 and Lemma 2.5 in Wang et al.(2005).*

Lemma 3.5. *Let X and Y be two independent random variables, X be real-valued and distributed by F and Y be nonnegative and nondegenerate at 0. If $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$ and $\mathbb{E}Y^p < \infty$ for some $p > \alpha$. Then, the distribution of XY belongs to $\mathcal{R}_{-\alpha}$ and $\mathbb{P}(XY > x) \sim \mathbb{E}Y^\alpha \overline{F}(x)$.*

Proof. *The complete proof can be found in Breiman (1965) or Cline and Samorodnitsky (1994).*

The following lemmas will be used in the proof of Theorem 2.1.

Lemma 3.6. *Under the conditions of Theorem 2.1, for any $\epsilon > 0$ and any fixed $n \in \mathbb{N}$,*

(a) there exists x^* such that

$$\sum_{i=1}^{n} \sum_{1 \leq l \leq n, l \neq i} \mathbb{P}\left\{X_l^+ \kappa_l > \frac{x}{n}, X_i^+ \kappa_i > \frac{L(x)}{n-1}\right\} \leq C\epsilon \overline{H}(x) \tag{3.3}$$

holds for all $x > x^*$, where $L(x)$ is as in Assumption A;

(b) there exists x^* such that for all $x > x^*$,

$$\sum_{i=1}^{n} \sum_{1 \leq l \leq n, l \neq i} \mathbb{P}\left\{X_l \kappa_l > x, X_i \kappa_i > x\right\} \leq C\epsilon \overline{H}(x). \tag{3.4}$$

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Proof. *(a)By the fact $\mathbb{P}\{X^+\theta > x\} = \mathbb{P}\{X\theta > x\}$ for all $x > 0$, we know that the distribution of $X^+\theta$, denoted by H^+, belongs to $\mathcal{D}\cap\mathcal{L}$ and*

$$\overline{H^+}(x) = \overline{H}(x), x > 0. \tag{3.5}$$

If $i > l$, by Lemma 3.1, Chebyshev's inequality and (3.2), we can derive

$$\begin{aligned}
&\mathbb{P}\left\{X_l^+\kappa_l > \frac{x}{n}, X_i^+\kappa_i > \frac{L(x)}{n-1}\right\}\\
&\le \mathbb{P}\left\{X_l^+\kappa_l > \frac{x}{n}, X_i^+\kappa_i > \frac{L(x)}{n-1}, \theta_l \le x^\nu\right\} + \mathbb{P}\left\{\theta_l > x^\nu\right\}\\
&\le \mathbb{P}\left\{X_l^+\theta_l\left(\prod_{j=1}^{l-1}\theta_j\right) > \frac{x}{n}, X_i^+\left(\prod_{1\le j\le i, j\ne l}\theta_j\right) > \frac{L(x)}{(n-1)x^\nu}\right\} + \frac{\mathbb{E}\theta^p}{x^{p\nu}}\\
&\le C\overline{H}(x)\mathbb{E}\left[n^p\left(\prod_{j=1}^{l-1}\theta_j^p\right)1_{\left\{X_i^+(\prod_{j=1,j\ne l}^i\theta_j)>\frac{L(x)}{(n-1)x^\nu}\right\}} + 1_{\left\{X_i^+(\prod_{j=1,j\ne l}^i\theta_j)>\frac{L(x)}{(n-1)x^\nu}\right\}}\right]\\
&\quad + C\epsilon\overline{H}(x).
\end{aligned}$$

Because $\{\theta_n\}_{n\ge1}$ are independent and $\mathbb{E}\theta^p < \infty$,we can obtain

$$\mathbb{E}\left(\prod_{j=1}^{l-1}\theta_j^p\right) = (\mathbb{E}\theta^p)^{l-1} < \infty. \tag{3.6}$$

Then, we can take x_1^* such that for all $x > x_1^*$,

$$\mathbb{E}\left[n^p\left(\prod_{j=1}^{l-1}\theta_j^p\right)1_{\left\{X_i^+(\prod_{j=1,j\ne l}^i\theta_j)>\frac{L(x)}{(n-1)x^\nu}\right\}}\right] < \epsilon$$

and

$$\mathbb{P}\left\{X_i^+(\prod_{j=1,j\ne l}^{i}\theta_j) > \frac{L(x)}{(n-1)x^\nu}\right\} < \epsilon.$$

If $i < l$, we can derive

$$\begin{aligned}
&\mathbb{P}\left\{X_l^+\kappa_l > \frac{x}{n}, X_i^+\kappa_i > \frac{L(x)}{n-1}\right\}\\
&= \mathbb{P}\left\{X_l^+\theta_l\left(\prod_{j=1}^{l-1}\theta_j\right) > \frac{x}{n}, X_i^+\kappa_i > \frac{L(x)}{n-1}\right\}\\
&\le C\overline{H}(x)\mathbb{E}\left[n^p\left(\prod_{j=1}^{l-1}\theta_j^p\right)1_{\left\{X_i^+\kappa_i>\frac{L(x)}{n-1}\right\}} + 1_{\left\{X_i^+\kappa_i>\frac{L(x)}{n-1}\right\}}\right].
\end{aligned}$$

Similarly, we can take x_2^* such that for all $x > x_2^*$,

$$\mathbb{E}\left[n^p\left(\prod_{j=1}^{l-1}\theta_j^p\right)1_{\left\{X_i^+\kappa_i>\frac{L(x)}{n-1}\right\}}\right] < \epsilon$$

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and

$$\mathbb{P}\left\{X_i^+\kappa_i > \frac{L(x)}{n-1}\right\} < \epsilon.$$

Hence, taking $x^* = \max\{x_1^*, x_2^*\}$, *it holds that for all* $x > x^*$ *and* $1 \le l \ne i \le n$,

$$\mathbb{P}\left\{X_l^+\kappa_l > \frac{x}{n}, X_i^+\kappa_i > \frac{L(x)}{n-1}\right\} < C\epsilon\overline{H}(x).$$

Thus, we know that (3.3) *holds for all* $x > x^*$.

(b)If we notice

$$\begin{aligned}\mathbb{P}\{X_l\kappa_l > x, X_i\kappa_i > x\} &\le \mathbb{P}\left\{X_l\kappa_l > \frac{x}{n}, X_i\kappa_i > \frac{L(x)}{n-1}\right\}\\ &= \mathbb{P}\left\{X_l^+\kappa_l > \frac{x}{n}, X_i^+\kappa_i > \frac{L(x)}{n-1}\right\},\end{aligned}$$

we can get (3.4) *easily.*

Lemma 3.7. *Under the conditions of Theorem 2.1, for any* $\epsilon > 0$ *and any fixed* $n \in \mathbb{N}$, *there exists* x^* *such that*

$$\sum_{k=1}^{n}\sum_{i=1,i\ne k}^{n}\mathbb{P}\left\{|X_i|\kappa_i \ge \frac{L(x)}{n-1}, X_k\kappa_k > x\right\} \le C\epsilon\overline{H}(x) \tag{3.7}$$

holds for all $x > x^*$, *where* $L(x)$ *is as in Assumption A.*

Proof. *If* $i > k$, *by* (3.5), *Lemma 3.1, Chebyshev's inequality and* (3.2), *we can derive*

$$\begin{aligned}&\mathbb{P}\left\{|X_i|\kappa_i \ge \frac{L(x)}{n-1}, X_k\kappa_k > x\right\}\\ &\le \mathbb{P}\left\{X_k^+\kappa_k > x, |X_i|\kappa_i \ge \frac{L(x)}{n-1}, \theta_k \le x^\nu\right\} + \mathbb{P}\{\theta_k > x^\nu\}\\ &\le \mathbb{P}\left\{X_k^+\theta_k\left(\prod_{j=1}^{k-1}\theta_j\right) > x, |X_i|\left(\prod_{1\le j\le i, j\ne k}\theta_j\right) \ge \frac{L(x)}{(n-1)x^\nu}\right\} + \frac{\mathbb{E}\theta^p}{x^{p\nu}}\\ &\le C\overline{H}(x)\mathbb{E}\left[\left(\prod_{j=1}^{k-1}\theta_j^p\right)1_{\left\{|X_i|\left(\prod_{1\le j\le i,j\ne k}\theta_j\right)\ge\frac{L(x)}{(n-1)x^\nu}\right\}} + 1_{\left\{|X_i|\left(\prod_{1\le j\le i,j\ne k}\theta_j\right)\ge\frac{L(x)}{(n-1)x^\nu}\right\}}\right]\\ &\quad + C\epsilon\overline{H}(x).\end{aligned}$$

By (3.6), *we can take* x_1^* *such that for all* $x > x_1^*$,

$$\mathbb{E}\left[\left(\prod_{j=1}^{k-1}\theta_j^p\right)1_{\left\{|X_i|(\prod_{j=1,j\ne k}^{i}\theta_j)\ge\frac{L(x)}{(n-1)x^\nu}\right\}}\right] < \epsilon$$

and

$$\mathbb{P}\left\{|X_i|(\prod_{j=1,j\ne k}^{i}\theta_j) \ge \frac{L(x)}{(n-1)x^\nu}\right\} < \epsilon.$$

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If $i < k$, we can derive

$$
\begin{aligned}
&\mathbb{P}\left\{|X_i|\kappa_i \geq \frac{L(x)}{n-1}, X_k\kappa_k > x\right\} \\
&= \mathbb{P}\left\{X_k^+\theta_k\left(\prod_{j=1}^{k-1}\theta_j\right) > x, |X_i|\kappa_i \geq \frac{L(x)}{n-1}\right\} \\
&\leq C\overline{H}(x)\mathbb{E}\left[\left(\prod_{j=1}^{k-1}\theta_j^p\right)1_{\left\{|X_i|\kappa_i \geq \frac{L(x)}{n-1}\right\}} + 1_{\left\{|X_i|\kappa_i \geq \frac{L(x)}{n-1}\right\}}\right].
\end{aligned}
$$

Similarly, we can take x_2^* such that for all $x > x_2^*$,

$$\mathbb{E}\left[\left(\prod_{j=1}^{k-1}\theta_j^p\right)1_{\left\{|X_i|\kappa_i \geq \frac{L(x)}{n-1}\right\}}\right] < \epsilon$$

and

$$\mathbb{P}\left\{|X_i|\kappa_i \geq \frac{L(x)}{n-1}\right\} < \epsilon.$$

Hence, taking $x^* = \max\{x_1^*, x_2^*\}$, it holds that for all $x > x^*$ and $1 \leq k \neq i \leq n$,

$$\mathbb{P}\left\{|X_i|\kappa_i \geq \frac{L(x)}{n-1}, X_k\kappa_k > x\right\} < C\epsilon\overline{H}(x).$$

Thus, we know that (3.7) holds for all $x > x^*$.

The following lemmas will be used in the proof of Theorem 2.2.

Lemma 3.8. *Under the conditions of Theorem 2.2, for any $\epsilon > 0$,*
(a) there exist x^* and k^* such that the relation

$$\mathbb{P}\left\{\sum_{i=k+1}^{\infty} X_i^+\kappa_i > L(x)\right\} \leq C\epsilon\overline{H}(x) \tag{3.8}$$

holds for any fixed $k \geq k^*$ and all $x > x^*$, where $L(x)$ is as in Assumption B;
(b) there exist x^* and k^* such that the relation

$$\sum_{i=k+1}^{\infty} \mathbb{P}\left\{X_i\kappa_i > x\right\} \leq C\epsilon\overline{H}(x) \tag{3.9}$$

holds for any fixed $k \geq k^*$ and all $x > x^*$.

Proof. *(a)By Lyapounov's inequality and $\mathbb{E}\theta^p < 1$, we can get that for any p_1, p_2 satisfying $0 < p_1 < \mathbb{J}_H^- \leq \mathbb{J}_H^+ < p_2 < p$,*

$$\mathbb{E}\theta^{p_l} \leq (\mathbb{E}\theta^p)^{\frac{p_l}{p}} < 1, l = 1, 2.$$

Then, we can derive

$$\sum_{i=1}^{\infty} i^{(1+\eta)p_l}\left(\mathbb{E}\theta^{p_l}\right)^{i-1} < \infty, l = 1, 2. \tag{3.10}$$

We can take k_1 such that $\sum_{i=k+1}^{\infty} 1/i^{1+\eta} < 1$ holds for any $\eta > 0$ and $k \geq k_1$. Thus, we can get that for any $\eta > 0$ and fixed $k \geq k_1$,

$$\mathbb{P}\left\{\sum_{i=k+1}^{\infty} X_i^+\kappa_i > L(x)\right\} \leq \mathbb{P}\left\{\sum_{i=k+1}^{\infty} X_i^+\kappa_i > \sum_{i=k+1}^{\infty} \frac{1}{i^{1+\eta}} L(x)\right\}$$
$$\leq \sum_{i=k+1}^{\infty} \mathbb{P}\left\{X_i^+\kappa_i > \frac{L(x)}{i^{1+\eta}}\right\}. \tag{3.11}$$

By Lemma 3.2, there exist x_1 such that for all $x > x_1$,

$$\mathbb{P}\left\{X_i^+\kappa_i > \frac{L(x)}{i^{1+\eta}}\right\} \leq C\overline{H^+}(L(x))\left\{i^{(1+\eta)p_1}\mathbb{E}\left(\prod_{j=1}^{i-1}\theta_j^{p_1}\right) + i^{(1+\eta)p_2}\mathbb{E}\left(\prod_{j=1}^{i-1}\theta_j^{p_2}\right)\right\}$$
$$= C\overline{H}(L(x))\left\{i^{(1+\eta)p_1}\left(\mathbb{E}\theta^{p_1}\right)^{i-1} + i^{(1+\eta)p_2}\left(\mathbb{E}\theta^{p_2}\right)^{i-1}\right\}, \tag{3.12}$$

where C has no relation to i.

By (3.10)-(3.12) and (2.3), there exist $x^* \geq x_1$ and $k^* \geq k_1$ such that for any fixed $k \geq k^*$ and all $x > x^*$,

$$\mathbb{P}\left\{\sum_{i=k+1}^{\infty} X_i^+\kappa_i > L(x)\right\} \leq C\overline{H}(L(x))\sum_{i=k+1}^{\infty}\left\{i^{(1+\eta)p_1}\left(\mathbb{E}\theta^{p_1}\right)^{i-1} + i^{(1+\eta)p_2}\left(\mathbb{E}\theta^{p_2}\right)^{i-1}\right\}$$
$$< C\epsilon\overline{H}(x).$$

(b)By the method used in (3.12), there exists x^* such that for all $x > x^*$ and any p_1, p_2 satisfying $0 < p_1 < \mathbb{J}_H^- \leq \mathbb{J}_H^+ < p_2 < p$,

$$\mathbb{P}\left\{X_i\kappa_i > x\right\} = \mathbb{P}\left\{X_i^+\kappa_i > x\right\}$$
$$\leq C\overline{H^+}(x)\left\{\mathbb{E}\left(\prod_{j=1}^{i-1}\theta_j^{p_1}\right) + \mathbb{E}\left(\prod_{j=1}^{i-1}\theta_j^{p_2}\right)\right\}$$
$$= C\overline{H}(x)\left\{\left(\mathbb{E}\theta^{p_1}\right)^{i-1} + \left(\mathbb{E}\theta^{p_2}\right)^{i-1}\right\}, \tag{3.13}$$

where C has no relation to i. Then, by (3.10), there exists k^* such that for any fixed $k \geq k^*$ and all $x > x^*$,

$$\sum_{i=k+1}^{\infty}\mathbb{P}\left\{X_i\kappa_i > x\right\} \leq C\overline{H}(x)\sum_{i=k+1}^{\infty}\left\{\left(\mathbb{E}\theta^{p_1}\right)^{i-1} + \left(\mathbb{E}\theta^{p_2}\right)^{i-1}\right\}$$
$$\leq C\overline{H}(x)\sum_{i=k+1}^{\infty}\left\{i^{(1+\eta)p_1}\left(\mathbb{E}\theta^{p_1}\right)^{i-1} + i^{(1+\eta)p_2}\left(\mathbb{E}\theta^{p_2}\right)^{i-1}\right\}$$
$$\leq C\epsilon\overline{H}(x).$$

3.2. Proof of Theorem 2.1

After rewriting the expression (1.2), we have that for $n \geq 1$,

$$\Psi(x, n) = \mathbb{P}\left\{\max_{1\leq m\leq n}\sum_{i=1}^{m} X_i\kappa_i > x\right\}. \tag{3.14}$$

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By (3.6), Lemma 3.3 and (3.5), the distributions of $X_i\kappa_i = X_i\theta_i(\prod_{j=1}^{i-1}\theta_j)$ and $X_i^+\kappa_i = X_i^+\theta_i(\prod_{j=1}^{i-1}\theta_j)$ belongs to $\mathcal{D}\cap\mathcal{L}$ and

$$\mathbb{P}\{X_i\kappa_i > x\} = \mathbb{P}\left\{X_i^+\kappa_i > x\right\} \asymp \overline{H}(x) = \overline{H^+}(x). \tag{3.15}$$

Then, by (2.2), it holds that for all $i = 1, 2, ..., n$,

$$\mathbb{P}\{X_i\kappa_i > x \pm L(x)\} = \mathbb{P}\left\{X_i^+\kappa_i > x \pm L(x)\right\} \sim \mathbb{P}\left\{X_i^+\kappa_i > x\right\} = \mathbb{P}\{X_i\kappa_i > x\}. \tag{3.16}$$

Firstly, we deal with the upper bound. For any $\epsilon > 0$, we get

$$\begin{aligned}
\Psi(x,n) &\le \mathbb{P}\left\{\sum_{i=1}^n X_i^+\kappa_i > x\right\}\\
&\le \mathbb{P}\left\{\bigcup_{i=1}^n \{X_i^+\kappa_i > x - L(x)\}\right\} + \mathbb{P}\left\{\sum_{i=1}^n X_i^+\kappa_i > x, \bigcap_{i=1}^n \{X_i^+\kappa_i \le x - L(x)\}\right\}\\
&:= P_1 + P_2.
\end{aligned} \tag{3.17}$$

Thus, by (3.16), there exists x_1^{up} such that for all $x > x_1^{up}$ and $1 \le i \le n$,

$$\mathbb{P}\left\{X_i^+\kappa_i > x - L(x)\right\} \le (1+\epsilon)\mathbb{P}\{X_i\kappa_i > x\}.$$

Then, we can get

$$P_1 \le \sum_{i=1}^n \mathbb{P}\left\{X_i^+\kappa_i > x - L(x)\right\} \le (1+\epsilon)\sum_{i=1}^n \mathbb{P}\{X_i\kappa_i > x\} \tag{3.18}$$

holds for all $x > x_1^{up}$. By Lemma 3.6(a), there exists $x_2^{up} > x_1^{up}$ such that for all $x > x_2^{up}$,

$$\begin{aligned}
P_2 &= \mathbb{P}\left\{\sum_{i=1}^n X_i^+\kappa_i > x, \bigcap_{i=1}^n \{X_i^+\kappa_i \le x - L(x)\}, \bigcup_{l=1}^n \left\{X_l^+\kappa_l > \frac{x}{n}\right\}\right\}\\
&\le \sum_{l=1}^n \mathbb{P}\left\{X_l^+\kappa_l > \frac{x}{n}, \sum_{1\le i\le n, i\ne l} X_i^+\kappa_i > L(x)\right\}\\
&\le \sum_{l=1}^n \sum_{1\le i\le n, i\ne l} \mathbb{P}\left\{X_l^+\kappa_l > \frac{x}{n}, X_i^+\kappa_i > \frac{L(x)}{n-1}\right\}\\
&\le C\epsilon\overline{H}(x).
\end{aligned} \tag{3.19}$$

Combining (3.17), (3.18), (3.19) and using (3.15), we can get that for all $x > x_2^{up}$,

$$\Psi(x,n) \le (1+C\epsilon)\sum_{i=1}^n \mathbb{P}\{X_i\kappa_i > x\}.$$

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Secondly, we deal with the lower bound. For any $\epsilon > 0$, we have

$$
\begin{aligned}
\Psi(x,n) &\geq \mathbb{P}\left\{\sum_{i=1}^{n} X_i\kappa_i > x\right\} \\
&\geq \mathbb{P}\left\{\sum_{i=1}^{n} X_i\kappa_i > x, \bigcup_{k=1}^{n}\{X_k\kappa_k > x + L(x)\}\right\} \\
&\geq \sum_{k=1}^{n}\mathbb{P}\left\{\sum_{i=1}^{n} X_i\kappa_i > x, X_k\kappa_k > x + L(x)\right\} - \sum_{k=1}^{n}\sum_{1\leq l\leq n, l\neq k}\mathbb{P}\{X_k\kappa_k > x, X_l\kappa_l > x\} \\
&= \sum_{k=1}^{n}\mathbb{P}\{X_k\kappa_k > x + L(x)\} - \sum_{k=1}^{n}\mathbb{P}\left\{\sum_{i=1}^{n} X_i\kappa_i \leq x, X_k\kappa_k > x + L(x)\right\} \\
&\quad - \sum_{k=1}^{n}\sum_{1\leq l\leq n, l\neq k}\mathbb{P}\{X_k\kappa_k > x, X_l\kappa_l > x\} \\
&:= L_1 - L_2 - L_3.
\end{aligned} \tag{3.20}
$$

By (3.16), there exists x_1^{low} such that for all $x > x_1^{low}$ and $1 \leq k \leq n$,

$$\mathbb{P}\{X_k\kappa_k > x + L(x)\} \geq (1-\epsilon)\mathbb{P}\{X_k\kappa_k > x\}.$$

Then, we can get

$$L_1 \geq (1-\epsilon)\sum_{k=1}^{n}\mathbb{P}\{X_k\kappa_k > x\} \tag{3.21}$$

holds for all $x > x_1^{low}$. By Lemma 3.7, there exists $x_2^{low} > x_1^{low}$ such that for all $x > x_2^{low}$,

$$
\begin{aligned}
L_2 &\leq \sum_{k=1}^{n}\mathbb{P}\left\{\sum_{i=1, i\neq k}^{n} X_i\kappa_i \leq -L(x), X_k\kappa_k > x + L(x)\right\} \\
&\leq \sum_{k=1}^{n}\sum_{i=1, i\neq k}^{n}\mathbb{P}\left\{X_i\kappa_i \leq \frac{-L(x)}{n-1}, X_k\kappa_k > x\right\} \\
&\leq \sum_{k=1}^{n}\sum_{i=1, i\neq k}^{n}\mathbb{P}\left\{|X_i|\kappa_i \geq \frac{L(x)}{n-1}, X_k\kappa_k > x\right\} \\
&\leq C\epsilon\overline{H}(x).
\end{aligned} \tag{3.22}
$$

By Lemma 3.6(b), there exists $x_3^{low} > x_2^{low}$ such that for all $x > x_3^{low}$,

$$L_3 \leq C\epsilon\overline{H}(x). \tag{3.23}$$

Combining (3.20)-(3.23) and using (3.15), we can obtain that for all $x > x_3^{low}$,

$$\Psi(x,n) \geq (1-C\epsilon)\sum_{k=1}^{n}\mathbb{P}\{X_k\kappa_k > x\}.$$

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3.3. Proof of Corollary 2.1

The proof is parallel to that of Theorem 2.1 with some modifications. By (3.6), Lemma 3.4 and (3.5), the distributions of $X_i\kappa_i$ and $X_i^+\kappa_i$ belongs to $\mathcal{C}$ and

$$\mathbb{P}\left\{X_i\kappa_i > x\right\} = \mathbb{P}\left\{X_i^+\kappa_i > x\right\} \asymp \overline{H}(x) = \overline{H^+}(x). \tag{3.24}$$

For any μ, ν satisfying $\mathbb{J}_H^+/p < \nu < \mu < 1$,

$$\Psi(x,n) \leq \mathbb{P}\left\{\bigcup_{i=1}^{n}\left\{X_i^+\kappa_i > x - x^\mu\right\}\right\} + \mathbb{P}\left\{\sum_{i=1}^{n} X_i^+\kappa_i > x, \bigcap_{i=1}^{n}\left\{X_i^+\kappa_i \leq x - x^\mu\right\}\right\}.$$

Because the distribution of $X_i^+\kappa_i$ belongs to $\mathcal{C}$, it holds that for $1 \leq i \leq n$,

$$\mathbb{P}\left\{X_i^+\kappa_i > x - x^\mu\right\} \lesssim \mathbb{P}\left\{X_i\kappa_i > x\right\},$$

where we notice $x - x^\mu = x(1 - x^{\mu-1})$ and $x^{\mu-1} \to 0$. Clearly Lemma 3.6 still holds if we replace $L(x)$ with x^μ. Hence, following the proof of Theorem 2.1, we can derive

$$\Psi(x,n) \lesssim \sum_{i=1}^{n} \mathbb{P}\left\{X_i\kappa_i > x\right\}.$$

For any μ, ν satisfying $\mathbb{J}_H^+/p < \nu < \mu < 1$, we have

$$\begin{aligned}\Psi(x,n) \geq &\sum_{k=1}^{n} \mathbb{P}\left\{X_k\kappa_k > x + x^\mu\right\} - \sum_{k=1}^{n} \mathbb{P}\left\{\sum_{i=1}^{n} X_i\kappa_i \leq x, X_k\kappa_k > x + x^\mu\right\}\\ &- \sum_{k=1}^{n} \sum_{1\leq l\leq n, l\neq k} \mathbb{P}\left\{X_k\kappa_k > x, X_l\kappa_l > x\right\}.\end{aligned}$$

Because the distribution of $X_i^+\kappa_i$ belongs to $\mathcal{C}$, it holds that for $1 \leq k \leq n$,

$$\mathbb{P}\left\{X_k\kappa_k > x + x^\mu\right\} \gtrsim \mathbb{P}\left\{X_k\kappa_k > x\right\},$$

where we notice $x + x^\mu = x(1 + x^{\mu-1})$ and $x^{\mu-1} \to 0$. Clearly Lemma 3.7 still holds if we replace $L(x)$ with x^μ. Hence, following the proof of Theorem 2.1, we can derive

$$\Psi(x,n) \gtrsim \sum_{i=1}^{n} \mathbb{P}\left\{X_i\kappa_i > x\right\}.$$

3.4. Proof of Corollary 2.2

By $\mathcal{R}_{-\alpha} \subset \mathcal{C}$ and Corollary 2.1, we can get (2.4). By $H \in \mathcal{R}_{-\alpha}$, (3.6) and Lemma 3.5, we can get that the distribution of $X_i\kappa_i$ belongs to $\mathcal{R}_{-\alpha}$ and

$$\mathbb{P}\left\{X_i\kappa_i > x\right\} \sim \left(\mathbb{E}\theta^\alpha\right)^{i-1} \overline{H}(x) \tag{3.25}$$

holds for $1 \leq i \leq n$. Substituting (3.25) into (2.4), we can get (2.5).

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3.5. Proof of Theorem 2.2

After rewriting the expression (1.3), we have

$$\Psi(x) = \mathbb{P}\left\{\max_{1\le m<\infty}\sum_{i=1}^{m} X_i\kappa_i > x\right\}. \tag{3.26}$$

Firstly, we deal with the upper bound. For any $\epsilon > 0$ and fixed $k \ge 1$,

$$\begin{aligned}\Psi(x) &\le \mathbb{P}\left\{\sum_{i=1}^{\infty} X_i^+\kappa_i > x\right\}\\ &\le \mathbb{P}\left\{\sum_{i=1}^{k} X_i^+\kappa_i > x - L(x)\right\} + \mathbb{P}\left\{\sum_{i=k+1}^{\infty} X_i^+\kappa_i > L(x)\right\}\\ &:= P_1' + P_2'.\end{aligned} \tag{3.27}$$

For P_1', by the proof of the upper bound of Theorem 2.1, we can get $x_{1'}^{up} > 0$ such that for any fixed $k \ge 1$,

$$P_1' \le (1 + C\epsilon)\sum_{i=1}^{k}\mathbb{P}\left\{X_i\kappa_i > x - L(x)\right\}$$

holds for all $x > x_{1'}^{up}$. Because the distribution of $X_i\kappa_i$ belongs to $\mathcal{L}$ (see (3.15)), there exists $x_{2'}^{up} > x_{1'}^{up}$ such that for all $x > x_{2'}^{up}$,

$$\mathbb{P}\left\{X_i\kappa_i > x - L(x)\right\} \le (1+\epsilon)\mathbb{P}\left\{X_i\kappa_i > x\right\}$$

holds for $1 \le i \le k$. Thus, we can obtain that for all $x > x_{2'}^{up}$,

$$\begin{aligned}P_1' &\le (1 + C\epsilon)(1+\epsilon)\sum_{i=1}^{k}\mathbb{P}\left\{X_i\kappa_i > x\right\}\\ &\le (1 + C\epsilon)\sum_{i=1}^{k}\mathbb{P}\left\{X_i\kappa_i > x\right\}\\ &\le (1 + C\epsilon)\sum_{i=1}^{\infty}\mathbb{P}\left\{X_i\kappa_i > x\right\}.\end{aligned} \tag{3.28}$$

For P_2', by Lemma 3.8(a), there exist $x_{3'}^{up} > x_{2'}^{up}$ and $k^{up} \ge 1$ such that for $k = k^{up}$ and all $x > x_{3'}^{up}$,

$$P_2' \le C\epsilon\overline{H}(x). \tag{3.29}$$

Combining (3.27), (3.28) and (3.29), we can obtain that for all $x > x_{3'}^{up}$,

$$\begin{aligned}\Psi(x) &\le (1 + C\epsilon)\sum_{i=1}^{\infty}\mathbb{P}\left\{X_i\kappa_i > x\right\} + C\epsilon\overline{H}(x)\\ &\le (1 + C\epsilon)\sum_{i=1}^{\infty}\mathbb{P}\left\{X_i\kappa_i > x\right\},\end{aligned}$$

where we used (3.15) in the last step.

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Secondly, we deal with the lower bound. By the proof of the lower bound of Theorem 2.1, we can get that for any $\epsilon > 0$ and fixed $k \geq 1$,

$$\begin{aligned}\Psi(x) &\geq \mathbb{P}\left\{\sum_{i=1}^{k} X_i\kappa_i > x\right\} \\ &\geq (1-C\epsilon)\sum_{i=1}^{k}\mathbb{P}\left\{X_i\kappa_i > x\right\} \\ &= (1-C\epsilon)\left(\sum_{i=1}^{\infty} - \sum_{i=k+1}^{\infty}\right)\mathbb{P}\left\{X_i\kappa_i > x\right\}. \qquad (3.30)\end{aligned}$$

In the last equality of (3.30), by Lemma 3.8(b), there exist $k^{low} > 1$ and $x_{3'}^{low} > x_{2'}^{low}$ such that for $k = k^{low}$ and all $x > x_{3'}^{low}$,

$$\sum_{i=k+1}^{\infty}\mathbb{P}\left\{X_i\kappa_i > x\right\} \leq C\epsilon\overline{H}(x).$$

Then, by (3.15), we can get that for all $x > x_{3'}^{low}$,

$$\begin{aligned}\Psi(x) &\geq (1-C\epsilon)\left(\sum_{i=1}^{\infty}\mathbb{P}\left\{X_i\kappa_i > x\right\} - C\epsilon\overline{H}(x)\right) \\ &\geq (1-C\epsilon)\sum_{i=1}^{\infty}\mathbb{P}\left\{X_i\kappa_i > x\right\}.\end{aligned}$$

3.6. Proof of Corollary 2.3

The proof is parallel to that of Theorem 2.2 with some modifications. The distributions of $X_i\kappa_i$ and $X_i^+\kappa_i$ belongs to $\mathcal{C}$ and we have (3.24).

For any fixed $0 < \delta < 1/2$ and$k \geq 1$,

$$\Psi(x) \leq \mathbb{P}\left\{\sum_{i=1}^{k} X_i^+\kappa_i > (1-\delta)x\right\} + \mathbb{P}\left\{\sum_{i=k+1}^{\infty} X_i^+\kappa_i > \delta x\right\}.$$

Because the distribution of $X_i\kappa_i$ belongs to $\mathcal{C}$, there exists δ^* such that for any fixed $0 < \delta \leq \delta^*$ and all $1 \leq i \leq k$,

$$\mathbb{P}\left\{X_i\kappa_i > (1-\delta)x\right\} \lesssim \mathbb{P}\left\{X_i\kappa_i > x\right\}.$$

Lemma 3.8(a) still holds if we replace $L(x)$ with δx and notice (3.1). That is to say, in the proof of Lemma 3.8(a), there exists k^* such that for any fixed $k \geq k^*$,

$$\begin{aligned}\mathbb{P}\left\{\sum_{i=k+1}^{\infty} X_i^+\kappa_i > \delta x\right\} &\leq C\overline{H}(\delta x)\sum_{i=k+1}^{\infty}\left\{i^{(1+\eta)p_1}\left(\mathbb{E}\theta^{p_1}\right)^{i-1} + i^{(1+\eta)p_2}\left(\mathbb{E}\theta^{p_2}\right)^{i-1}\right\} \\ &< C\epsilon\overline{H}(x).\end{aligned}$$

Hence, following the proof of Theorem 2.2, we can obtain

$$\Psi(x) \lesssim \sum_{i=1}^{\infty}\mathbb{P}\left\{X_i\kappa_i > x\right\}.$$

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Lemma 3.8(b) still holds without $L(x)$. Hence, following the proof of Theorem 2.2 and using (3.24) instead of (3.15), we can get

$$\Psi(x) \gtrsim \sum_{i=1}^{\infty} \mathbb{P}\left\{X_i \kappa_i > x\right\}.$$

3.7. Proof of Corollary 2.4

By $\mathcal{R}_{-\alpha} \subset \mathcal{C}$ and Corollary 2.3, we can get (2.6). Substituting (3.25) into (2.6), we can get (2.7).

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