J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Continuation and bifurcations of concave central configurations in the four and five body-problems for homogeneous force laws

Alan Almeida Santos [a,*], Marcelo Marchesin [b], Ernesto Pérez-Chavela [c,1], Claudio Vidal [d]

[a] Departamento de Matemática, Campus Prof. Alberto Carvalho, Universidade Federal de Sergipe, Av. Ver. Olímpio Grande, s/n, Itabaiana, SE CEP 49510-200, Brazil
[b] Department of Mathematics, Universidade Federal de Minas Gerais, Av. Antônio Carlos, 6627, Belo Horizonte, MG CEP 31270-901, Brazil
[c] ITAM, Department of Mathematics, Calle Río Hondo No. 1, Col. Progreso Tizapan, México, D.F. 01080, Mexico
[d] Departamento de Matemática, Facultad de Ciencias, Universidad del Bío Bío, Casilla 5-C, Concepción, VIII-Región, Chile

A R T I C L E I N F O

Article history:
Received 18 March 2016
Available online xxxx
Submitted by W. Sarlet

Keywords:
N-body problem
Central configurations
Dziobek configurations
Bifurcation

A B S T R A C T

The central configurations given by an equilateral triangle and a regular tetrahedron with equal masses at the vertices and a body at the barycenter have been widely studied in [9] and [14] due to the phenomena of bifurcation occurring when the central mass has a determined value m^*. We propose a variation of this problem setting the central mass as the critical value m^* and letting a mass at a vertex to be the parameter of bifurcation. In both cases, 2D and 3D, we verify the existence of bifurcation, that is, for a same set of masses we determine two new central configurations. The computation of the bifurcations, as well as their pictures have been performed considering homogeneous force laws with exponent $a < -1$.

1. Introduction

In the classical Newtonian n-body problem the unique explicit solutions which are known until now are the homographic solutions, characterized by the fact that their configurations are invariant up to rotations and scaling, and in which each body describes a Keplerian orbit. These particular solutions are generated by initial configurations called *central configurations* (see [15] for details) and are, certainly the most celebrated of them. More precisely, let be $\mathbb{E}$ a finite dimensional Euclidean vector space; $\mathbf{q}_1, \cdots, \mathbf{q}_n \in \mathbb{E}$ the position vectors; $m_1, \cdots, m_n$ the masses given by n positives numbers; a a negative real number; $M = \sum m_i$ the

* Corresponding author.
E-mail addresses: alan@ufs.br (A.A. Santos), mdm@mat.ufmg.br (M. Marchesin), epc@xanum.uam.mx (E. Pérez-Chavela), clvidal@ubiobio.cl (C. Vidal).

[1] The third author was partially supported by CONACyT grant number 128790.

http://dx.doi.org/10.1016/j.jmaa.2016.09.055
0022-247X/

total mass of the system and $\mathbf{q}_G = \frac{1}{M}\sum m_i\mathbf{q}_i$ the center of mass of the system, we can then give the following definition:

Definition 1. A configuration $\mathbf{q} = (\mathbf{q}_1, \cdots, \mathbf{q}_n) \in \mathbb{E}^n$ is a *central configuration* (cc by short) for the masses $m_1, \cdots, m_n$ if there exists a constant $\lambda \in \mathbb{R}$ such that

$$\lambda(\mathbf{q}_i - \mathbf{q}_G) = \sum_{j \neq i} m_j \|\mathbf{q}_i - \mathbf{q}_j\|^{2a}(\mathbf{q}_i - \mathbf{q}_j), \qquad \forall\ i = 1, \cdots, n. \tag{1}$$

When $a = -3/2$ we have the Newtonian case and when $a = -1$ the vortex case.

Technically, central configurations are zeros of a system of n equations with n vectorial variables and n positive parameters. In some special cases, e.g., when all masses are equal, some quite simple solutions having well-defined positions can be obtained in a trivial way. In effect, we can verify easily that any regular n-gon is a central configuration. If we add any additional mass at the origin, we still get a central configuration. A regular simplex of n points on an affine subspace $n-1$ dimensional is a central configuration no matter the values of the masses at the vertices. Other symmetric configurations like rhombus, kites and pyramids exist for systems with some equal masses [3,4,6,7,11,13,15].

As zeros of systems of equations with many parameters, it is expected that bifurcation phenomena arise. This happens if the Jacobian of the system becomes degenerate for some values of the parameters at a given trivial solution. This type of question has been approached in many works. In [9], the authors proved the existence of bifurcations at the neighborhood of the following planar central configurations: the equilateral triangle with equal masses at the vertices and a fourth mass m^* at the barycenter and the square with equal masses at the vertices and a fifth mass m^{**} at the center. In [8] Meyer studied the continuation of central configurations from the restricted $(3+1)$-body problem with two equal masses $1-\mu$ and a third mass 2μ forming an equilateral triangle to the full 4-body problem. He proved that for small values of the fourth mass, there are central configurations degenerated which undergo bifurcation for specific values of the parameter μ. The same bifurcation analysis was applied in [14] to show the existence of four branches of central configurations which arise from the regular tetrahedron with a critical mass at the barycenter. In [12], by using the S_4-equivariance of equations defining Dziobek's configurations, three new branches of bifurcations were found improving the previous result.

In this paper we investigate bifurcations arising from two concave central configurations in the 4 and 5-body problem. In the former case, we consider the equilateral triangle with masses $m_1 = m_2 = m_4 = 1$ at the vertices and a mass $m_3 = m$ at the barycenter. For any non-negative value of m, this is a central configuration. In [10], Palmore showed that $m^* = \frac{81+64\sqrt{3}}{249}$ is the unique value of m for which, this central configuration is degenerate. Unlike the analysis performed by Schmidt and Meyer [9], which have considered the mass at the barycenter as the bifurcation parameter, we set $m_4 = 1+\varepsilon$ and $m = m^*$, so that at $\varepsilon = 0$ we verify that the system of equations undergo a bifurcation producing symmetric and non symmetric central configurations for ε near zero. The S_2-equivariance of the equations and the Implicit Function Theorem (IFT for shorting) are applied in a singular case as in [5], and they will be the main tools in the proof of existence. For $m \neq m^*$ the equilateral triangle is non-degenerate and so, it can be continued in a neighborhood of $\varepsilon = 0$ as a family of isosceles triangles.

In the five body problem, we consider the regular tetrahedron with masses $m_1 = m_2 = m_3 = m_5 = 1$ at the vertices and a mass $m_4 = m$ at the barycenter. By setting $m_4 = m^{**} = \frac{10368+1701\sqrt{6}}{54952}$ as in [14], we proceed to the bifurcation analysis in a similar way, as in the former case, but taking into account that the equations present an S_3-equivariance. The fact that in this case one has a biggest system entails some extra difficulties, however still in this case, the equations present S_3 instead S_2-equivariance, so the computations can be reduced significantly and allows us to show the existence of several branches of central configurations emanating from the tetrahedron.

Please cite this article in press as: A.A. Santos et al., Continuation and bifurcations of concave central configurations in the four and five body-problems for homogeneous force laws, J. Math. Anal. Appl. (2016), http://dx.doi.org/10.1016/j.jmaa.2016.09.055

A remarkable note about this work is that in both cases, the existence of bifurcations was carried out considering the exponent $a < -1$. That is, our results are valid for any potential of the homogeneous family.

After the introduction where we give the preliminaries of the problems to be studied here, the paper is organized as follows: In Section 2 we define Dziobek's configurations and characterize the central configurations of n bodies in dimension $n-2$ through them. In Section 3 we prove that the *cc* with equilateral triangle shape and a mass other than m^* at the barycenter can be continued to a symmetric family of *cc* and we study the bifurcation phenomena when one of the masses located at one vertex of the equilateral triangle crosses the value 1 and m^* is located at the barycenter. Section 4 is dedicated to the study of concave *cc* formed by a regular tetrahedron with a mass at the barycenter. First we prove that the above *cc* can be continued to a symmetrical family of *cc* and finally we analyze the bifurcation phenomena following the same strategy as in the triangular case.

2. Equations for Dziobek's configurations

Given n points $\mathbf{q} = (q_1, \cdots, q_n)$ in an Euclidean space $\mathbb{E}$, we define the *dimension of the configuration* $\mathbf{q}$ as the number

$$\dim[q_1 - q_i, \cdots, q_n - q_i],$$

which is independent of the chosen point q_i and where the brackets [] mean the vectorial space generated by the list of vectors in there.

If the dimension of $\mathbf{q}$ is exactly $n-2$ then there exists a non-zero n-tuple, $X = (x_1, \cdots, x_n) \in \mathbb{R}^n$ unique up to a factor such that

$$\sum_{i=1}^{n} x_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} x_i q_i = 0. \tag{2}$$

Let $\mathbf{q}$ be a configuration associated to n positive masses, then we can define a Dziobek configuration as following

Definition 2. A Dziobek configuration is a configuration of n particles such that there exists a non-zero $X \in \mathbb{R}^n$ satisfying (2) and such that for some $\lambda \in \mathbb{R}_+$

$$\frac{\lambda}{M} - s_{ij}^{a} = \frac{x_i x_j}{m_i m_j}, \tag{3}$$

where $s_{ij} = \|q_i - q_j\|^2$.

In [1], it is shown that Dziobek's configurations are central configurations of dimension at most $n-2$ and any central configuration of dimension exactly $n-2$ is a Dziobek configuration.

If we define the quantities $t_i = \sum_{j \neq i} s_{ij} x_j$ then equations (2) are equivalent to

$$\sum_{i=1}^{n} x_i = 0 \quad \text{and} \quad t_i = t_j \quad \textit{for all } i, j,$$

so central configurations of dimension $n-2$ associated to n positive masses m_j are given by the following system

$$\sum_{i=1}^{n} x_i = 0,$$
$$t_i = t_j, \tag{4}$$
$$\frac{\lambda}{M} - s_{ij}^a = \frac{x_i x_j}{m_i m_j}.$$

Remark 1. The constant λ required in Definition 2 is the Lagrange multiplier associated to the central configuration. It is not difficult to see that two homothetic Dziobek configurations satisfy the same equations (4) for different values of λ. In the calculations of Dziobek configurations, we set an appropriate value for λ.

Remark 2. If two masses are equal, e.g. $m_1 = m_2$, then for a Dziobek configuration, the equality $x_1 = x_2$ is equivalent to a symmetry with respect to the segment q_1q_2. This is because the function $\varphi(s) = \frac{\lambda}{M} - s^a$ is increasing for $a < -1$ and according to (3), for all $j \neq 1, 2$ we have

$$x_1 = x_2 \quad \Leftrightarrow \quad \frac{x_1 x_j}{m_1 m_j} = \frac{x_2 x_j}{m_2 m_j} \quad \Leftrightarrow \quad \varphi(s_{1j}) = \varphi(s_{2j}),$$

from where follows the identities $s_{1j} = s_{2j}$.

3. The equilateral triangle with a mass at the barycenter

Consider the system (4) with $n = 4$ and masses $m_1 = m_2 = 1$, $m_3 = m$ and $m_4 = 1 + \varepsilon$. In order that the equilateral triangle with the squares of the mutual distances $s_{12}^0 = s_{14}^0 = s_{24}^0 = 3$ and $s_{13}^0 = s_{23}^0 = s_{34}^0 = 1$ be a solution for $\varepsilon = 0$ we must have $\lambda = m + 3^{1+a}$. This central configuration of the 4-body problem will be called simply *concave equilateral triangle* leaving implied the occurrence of a fourth mass at the barycenter. The values for the variables x's corresponding to the concave equilateral triangle are

$$x_1^0 = x_2^0 = x_4^0 = k \quad \text{and} \quad x_3^0 = -3k, \tag{5}$$

where $k = \sqrt{\dfrac{m}{3+m}(1 - 3^a)}$.

By using the third equation in (4) we can express the six squares of the mutual distances in terms of oriented areas x_i. Besides, from the first equation in (4) we insert $x_3 = -x_1 - x_2 - x_4$ into equations $t_i = t_j$ to get a system of three equations, three variables $X = (x_1, x_2, x_4)$ and one parameter $\varepsilon \in \mathbb{R}$ given by the functions:

$$F_1(X, \varepsilon) = x_2 \left(\frac{\lambda}{M} - x_1 x_2\right)^{1/a} + x_3 \left(\frac{\lambda}{M} - \frac{x_1 x_3}{m}\right)^{1/a} + x_4 \left(\frac{\lambda}{M} - \frac{x_1 x_4}{1+\varepsilon}\right)^{1/a} - t_4,$$
$$F_2(X, \varepsilon) = x_1 \left(\frac{\lambda}{M} - x_1 x_2\right)^{1/a} + x_3 \left(\frac{\lambda}{M} - \frac{x_2 x_3}{m}\right)^{1/a} + x_4 \left(\frac{\lambda}{M} - \frac{x_2 x_4}{1+\varepsilon}\right)^{1/a} - t_4,$$
$$F_3(X, \varepsilon) = \sum_{j=1}^{2} x_j \left(\frac{\lambda}{M} - \frac{x_j x_3}{m}\right)^{1/a} + x_4 \left(\frac{\lambda}{M} - \frac{x_3 x_4}{m(1+\varepsilon)}\right)^{1/a} - t_4,$$

where $m > 0$ and $a < -1$ are fixed and

$$t_4 = x_1 \left(\frac{\lambda}{M} - \frac{x_1 x_4}{1+\varepsilon}\right)^{1/a} + x_2 \left(\frac{\lambda}{M} - \frac{x_2 x_4}{1+\varepsilon}\right)^{1/a} + x_3 \left(\frac{\lambda}{M} - \frac{x_3 x_4}{m(1+\varepsilon)}\right)^{1/a}.$$

Calling $F = (F_1, F_2, F_3) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$, we have $F(X^0, 0) = (0, 0, 0)$ for any $m > 0$. Observe also that F is S_2-equivariant, that is

Please cite this article in press as: A.A. Santos et al., Continuation and bifurcations of concave central configurations in the four and five body-problems for homogeneous force laws, J. Math. Anal. Appl. (2016), http://dx.doi.org/10.1016/j.jmaa.2016.09.055

$$F(\sigma \cdot X, \varepsilon) = \sigma \cdot F(X, \varepsilon) \tag{6}$$

for $\sigma\colon (x_1, x_2, x_4) \mapsto (x_2, x_1, x_4)$ and its first derivative must satisfy

$$D_X F(\sigma \cdot X, \varepsilon) \cdot \sigma = \sigma \cdot D_X F(X, \varepsilon).$$

Take into account that $\sigma \cdot X^0 = X^0$, we have that the derivative $D_X F(X^0, 0)$ is a linear transformation that commutes with σ. Therefore, it is a matrix of the form

$$D_X F(X^0, 0) = \begin{pmatrix} b & c & d \\ c & b & d \\ e & e & f \end{pmatrix}. \tag{7}$$

The calculations of the first partial derivative of F with respect to X evaluated at $(X^0, m, 0)$ give us

$$b = \left.\frac{\partial F_1}{\partial x_1}\right|_{(X^0,0)} = -3 - \frac{k^2}{a}\left(\frac{9}{m} + 3^{1-a}\right) = -d,$$

$$c = \left.\frac{\partial F_1}{\partial x_2}\right|_{(X^0,0)} = 0,$$

$$e = \left.\frac{\partial F_3}{\partial x_1}\right|_{(X^0,0)} = -1 + \frac{k^2}{a}\left(\frac{9}{m} + 3^{1-a}\right) = d - 4,$$

$$f = \left.\frac{\partial F_3}{\partial x_4}\right|_{(X^0,0)} = 2 + 2\frac{k^2}{a}\left(\frac{9}{m} + 3^{1-a}\right) = 2d - 4.$$

With the above values, the determinant below is easily obtained

$$\left|D_X F(X^0, 0)\right| = 4d^2(d - 3). \tag{8}$$

The factor $d - 3$ is negative for any $a < -1$ and $m > 0$, whereas the factor d is zero if, and only if, the mass m is equal to

$$m^* = 3\frac{3^a - a - 1}{3^{-a} + a - 1}. \tag{9}$$

Proposition 1. *For $a < -1$, the value of m^* is in the interval $(0, 1)$.*

Proof. First we observe that $m^* = 3\frac{3^a - a - 1}{3^{-a} + a - 1} > 3\frac{3^a - a - 1}{3^{-a}} > 0$ for all $a < -1$.

Now, on one hand the derivative of m^* with respect to a is

$$\frac{dm^*}{da} = 3\frac{\left(1 - 3^{-a}\right)\left(\log(3)\, a3^a - \log(3)\, 3^a - 3^a + \log(3)\, a + \log(3) + 1\right)}{\left(3^{-a} + a - 1\right)^2}$$

and on the other hand, the derivative of the numerator with respect to a is

$$\log^2(3)\left(3^a(a - 1) + (a + 1)3^{-a}\right),$$

which is negative for $a < -1$. Thus the numerator of $\frac{dm^*}{da}$ is a decreasing function on $(-\infty, -1)$, this implies that it reaches its minimum at $a = -1$. Since $\frac{dm^*}{da}|_{a=-1} = 4(\log(3) - 1)$ we get that $\frac{dm^*}{da} > 0$ on $(-\infty, -1)$.

Finally, it is easy to verify that

$$\lim_{a\to-\infty} m^* = 0 \qquad \text{and} \qquad m^*(-1) = 1,$$

which proves the stated. □

Remark 3. In the Newtonian case with $a = -3/2$, the value of m^* agrees with that found by Meyer–Schmidt in [9] and by Palmore in [10], that is $m^* = \frac{81+64\sqrt{3}}{249}$.

3.1. Continuation

By a straightforward application of the Implicit Function Theorem, we have the

Theorem 1. *For every $a < -1$ and $0 < m \neq m^*$, there exists $\varepsilon_0 \in \mathbb{R}$ such that the concave equilateral triangle $(X^0, 0)$ can be continued to a symmetric family of concave central configurations $X(\varepsilon)$ defined for $|\varepsilon| < \varepsilon_0$ and having isosceles shape given by*

$$\begin{aligned} x_1(\varepsilon) &= k + \alpha\varepsilon + \mathcal{O}(\varepsilon^2), \\ x_2(\varepsilon) &= k + \alpha\varepsilon + \mathcal{O}(\varepsilon^2), \\ x_4(\varepsilon) &= k + \beta\varepsilon + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{10}$$

where α and β are well-defined functions of m and a.

Proof. If $0 < m \neq m^*$, then $|D_X F(X^0, 0)| \neq 0$ and the Implicit Function Theorem can be applied to the equation $F(X, \varepsilon) = (0, 0, 0)$ near the trivial solution $(X^0, 0)$, ensuring the existence of a unique curve $X(\varepsilon) \in \mathbb{R}^3$ defined on an interval $(-\epsilon_0, \epsilon_0)$ and such that

$$\begin{gathered} F(X(\varepsilon), \varepsilon) = (0, 0, 0), \\ X(0) = X^0. \end{gathered}$$

The S_2-equivariance of F implies that $(\sigma \cdot X(\varepsilon), \varepsilon)$ is also solution for $F = 0$. By the local uniqueness of the implicit function we must have

$$\sigma \cdot X(\varepsilon) = X(\varepsilon) \quad \Rightarrow \quad x_1(\varepsilon) = x_2(\varepsilon).$$

By Remark 2, the central configurations given by $F = 0$, which are continuation of X^0 in a neighborhood of $\varepsilon = 0$, are symmetric with respect to the segment q_1q_2, that is

$$s_{13}(\varepsilon) = s_{23}(\varepsilon) \quad \text{and} \quad s_{14}(\varepsilon) = s_{24}(\varepsilon),$$

for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

By multiplying the derivative $\left.\dfrac{\partial F}{\partial \varepsilon}\right|_{(X^0,0)}$ by the matrix $-D_X F(X^0, 0)^{-1}$ we obtain the expression of $X(\varepsilon)$ at the first order in ε

$$\begin{aligned} x_1(\varepsilon) &= k + \alpha\varepsilon + \mathcal{O}(\varepsilon^2), \\ x_2(\varepsilon) &= k + \alpha\varepsilon + \mathcal{O}(\varepsilon^2), \\ x_4(\varepsilon) &= k + \beta\varepsilon + \mathcal{O}(\varepsilon^2). \end{aligned}$$

The expressions for α and β are:

$$\alpha = \frac{k^3 p(m,a)}{6 \cdot 3^a(1-3^a)(m+3^{1+a})(3^{-a}+a-1)(m-m^*)},$$
$$\beta = \frac{k^3 q(m,a)}{6 \cdot 3^a(1-3^a)(m+3^{1+a})(3^{-a}+a-1)(m-m^*)},$$

where

$$p(m,a) = 2a3^a m^2 + \left[3^{1+a}(3a+(a+1)3^a)-3\right] m$$
$$+ 3^{a+2}\left[3^{2a}+(a+1)3^a+3^a+a-3\right] - \frac{2(1-3^a)3^{3+2a}}{m},$$
$$q(m,a) = 2\cdot 3^a m^2(3^{1-a}+a-3) + m\left[3^{a+1}(3^a(a-11)+3(a+2))+15\right]$$
$$+ 3^{a+2}\left[9+a+3^a(a-4)-5\cdot 3^{2a}\right] + \frac{4(1-3^a)3^{3+2a}}{m}.$$

Thus we have proved the theorem. □

Note that $p(m,a)$ is a sum of negative terms for $a < -1$ so that the sign of α is defined by the difference $m - m^*$. On the other hand, it is not possible to give a complete description of the continuation in terms of the parameter ε because the signal of β is undefined due to the third term in the expression of $q(m,a)$.

However, for the particular value $a = -3/2$ which corresponds to the Newtonian case, we can make a complete analysis of the central configurations given by (10). In this case we have

$$x_1(\varepsilon) = k + \alpha\varepsilon + \mathcal{O}(\varepsilon^2),$$
$$x_2(\varepsilon) = k + \alpha\varepsilon + \mathcal{O}(\varepsilon^2),$$
$$x_3(\varepsilon) = -3k + \gamma\varepsilon + \mathcal{O}(\varepsilon^2),$$
$$x_4(\varepsilon) = k + \beta\varepsilon + \mathcal{O}(\varepsilon^2),$$

where

$$\alpha = -\frac{k^3\left[18m^3+(55\sqrt{3}+81)m^2+(241-3\sqrt{3})m+12(3\sqrt{3}-1)\right]}{2(59-21\sqrt{3})m(\sqrt{3}m+1)(m-m^*)},$$
$$\beta = \frac{k^3\left[54(2\sqrt{3}-1)m^3+(245\sqrt{3}+27)m^2+(395-33\sqrt{3})m+24(3\sqrt{3}-1)\right]}{2(59-21\sqrt{3})m(\sqrt{3}m+1)(m-m^*)},$$
$$\gamma = -(2\alpha+\beta) = -\frac{3k^3\left[(36\sqrt{3}-30)m+49\sqrt{3}-27\right]}{2(59-21\sqrt{3})(\sqrt{3}m+1)}.$$

Clearly, we see that the sign of γ is negative for all $m > 0$ while

$$m < m^* \Rightarrow \alpha > 0 \textit{ and } \beta < 0,$$
$$m > m^* \Rightarrow \alpha < 0 \textit{ and } \beta > 0.$$

Considering that

$$s_{ij}(\varepsilon) = \left(\frac{\lambda}{M} - \frac{x_i x_j}{m_i m_j}\right)^{-2/3},$$

and remembering that $M = m + 3 + \varepsilon$, $m_4 = 1 + \varepsilon$ and $\lambda = m + 3^{-1/2}$, we can obtain the derivatives of s_{ij} at $\varepsilon = 0$ in terms of the mass m. After some tedious calculations we obtain

Table 1
Behavior of $s_{ij}(\varepsilon)$ in a small neighborhood of $\varepsilon = 0$.

	$0 < m < m^*$	$m > m^*$
$s_{12}(\varepsilon)$	increasing	decreasing
$s_{13}(\varepsilon)$	decreasing	increasing
$s_{14}(\varepsilon)$	indefinite	increasing
$s_{34}(\varepsilon)$	increasing	decreasing

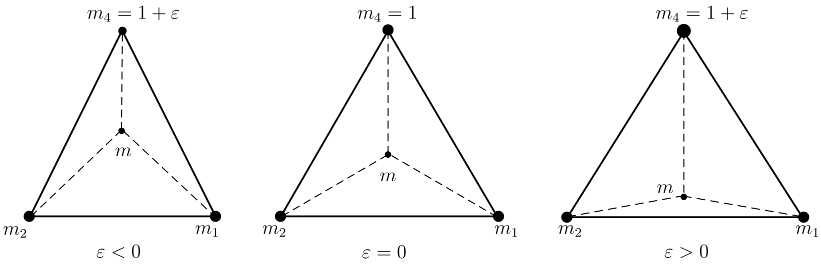

Fig. 1. Continuation for $m < m^*$ and $a = -3/2$.

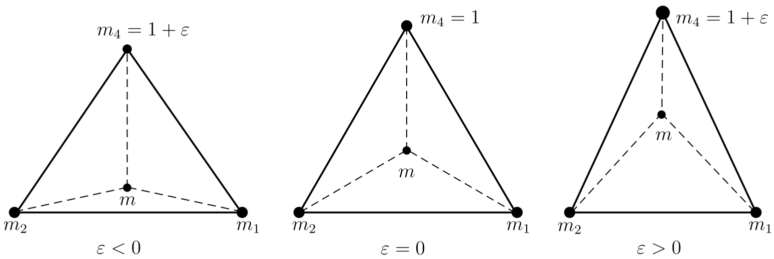

Fig. 2. Continuation for $m > m^*$ and $a = -3/2$.

$$s_{ij}(\varepsilon) = s_{ij}^0 + v_{ij}(m)\varepsilon + \mathcal{O}(\varepsilon^2),$$

where

$$v_{12}(m) = -\frac{2}{9}\frac{(54\sqrt{3}-18)m^2 + (36-\sqrt{3})m + 2 + 3\sqrt{3}}{\lambda(6\sqrt{3}-5)(m-m^*)},$$

$$v_{13}(m) = \frac{2}{27}\frac{(45-8\sqrt{3})m + 6\sqrt{3} - 5}{\lambda(6\sqrt{3}-5)(m-m^*)},$$

$$v_{14}(m) = \frac{2}{9}\frac{(27\sqrt{3}-9)m^2 + (45-8\sqrt{3})m - 2 - 3\sqrt{3}}{\lambda(6\sqrt{3}-5)(m-m^*)},$$

$$v_{34}(m) = -\frac{2}{27}\frac{(36-\sqrt{3})m + 21\sqrt{3} - 4}{\lambda(6\sqrt{3}-5)(m-m^*)}.$$

All coefficients $v_{ij}(m)$ have well-defined sign for each $m > m^*$, whereas for $m < m^*$ only $v_{14}(m)$ has a change of sign on the interval $(0, m^*)$. In fact, to see this just make $m = 0$ and $m = m^*$ in the numerator of the expression. Nevertheless, we can draw the behavior of central configurations given by the equation $F(X, \varepsilon) = 0$ in a small neighborhood of $(X^0, 0)$ for all $m \neq m^*$ (see Table 1, Figs. 1 and 2).

3.2. Bifurcation

Now, we consider the problem

Please cite this article in press as: A.A. Santos et al., Continuation and bifurcations of concave central configurations in the four and five body-problems for homogeneous force laws, J. Math. Anal. Appl. (2016), http://dx.doi.org/10.1016/j.jmaa.2016.09.055

$$F_i(x_1, x_2, x_4, \varepsilon) = 0, \qquad i = 1, 2, 3,$$

where we set $m_3 = m^*$. At the point $(X^0, 0)$, the matrix (7) is given by

$$D_X F(X^0, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & -4 & -4 \end{pmatrix}.$$

We make some transformations, following the Liapunov–Schmidt reduction process. If $L = D_X F(X^0, 0)$ then we have that

$$\ker\{L\} = \{(x_1, x_2, x_4) \in \mathbb{R}^3 \ / \ x_1 + x_2 + x_4 = 0\}, \quad \mathrm{Im}\{L\} = \{(0, 0, x) \ : \ x \in \mathbb{R}\}.$$

By taking, $u_1 = (1, 0, -1)$, $u_2 = (0, 1, -1)$ and $u_3 = (0, 0, -1)$ we proceed the following change of variables

$$(x_1, x_2, x_4) = y_1 \cdot u_1 + y_2 \cdot u_2 + y_3 \cdot u_3, \tag{11}$$

so that

$$x_1 = y_1, \qquad x_2 = y_2, \qquad x_4 = -y_1 - y_2 - y_3 \qquad \text{and} \qquad x_3 = y_3.$$

Let $G(Y, \varepsilon) = F(X(Y), \varepsilon)$ be the new functions defining the Dziobek's configurations on the plane after the change of variables. We get that

$$G_1(y_1, y_2, y_3, \varepsilon) = y_2 \left(\frac{\lambda}{M} - y_1 y_2\right)^{1/a} + y_3 \left(\frac{\lambda}{M} - \frac{y_1 y_3}{m^*}\right)^{1/a}$$
$$- (y_1 + y_2 + y_3) \left(\frac{\lambda}{M} + \frac{y_1(y_1 + y_2 + y_3)}{1 + \varepsilon}\right)^{1/a} - y_1 \left(\frac{\lambda}{M} + \frac{y_1(y_1 + y_2 + y_3)}{1 + \varepsilon}\right)^{1/a}$$
$$- y_2 \left(\frac{\lambda}{M} + \frac{y_2(y_1 + y_2 + y_3)}{1 + \varepsilon}\right)^{1/a} - y_3 \left(\frac{\lambda}{M} + \frac{y_3(y_1 + y_2 + y_3)}{m^*(1 + \varepsilon)}\right)^{1/a},$$
$$G_2(y_1, y_2, y_3, \varepsilon) = y_1 \left(\frac{\lambda}{M} - y_1 y_2\right)^{1/a} + y_3 \left(\frac{\lambda}{M} - \frac{y_2 y_3}{m^*}\right)^{1/a}$$
$$- (y_1 + y_2 + y_3) \left(\frac{\lambda}{M} + \frac{y_2(y_1 + y_2 + y_3)}{1 + \varepsilon}\right)^{1/a} - y_1 \left(\frac{\lambda}{M} + \frac{y_1(y_1 + y_2 + y_3)}{1 + \varepsilon}\right)^{1/a}$$
$$- y_2 \left(\frac{\lambda}{M} + \frac{y_2(y_1 + y_2 + y_3)}{1 + \varepsilon}\right)^{1/a} - y_3 \left(\frac{\lambda}{M} + \frac{y_3(y_1 + y_2 + y_3)}{m^*(1 + \varepsilon)}\right)^{1/a},$$
$$G_3(y_1, y_2, y_3, \varepsilon) = y_1 \left(\frac{\lambda}{M} - \frac{y_1 y_3}{m^*}\right)^{1/a} + y_2 \left(\frac{\lambda}{M} - \frac{y_2 y_3}{m^*}\right)^{1/a}$$
$$- (y_1 + y_2 + y_3) \left(\frac{\lambda}{M} + \frac{y_3(y_1 + y_2 + y_3)}{m^*(1 + \varepsilon)}\right)^{1/a} - y_1 \left(\frac{\lambda}{M} + \frac{y_1(y_1 + y_2 + y_3)}{1 + \varepsilon}\right)^{1/a}$$
$$- y_2 \left(\frac{\lambda}{M} + \frac{y_2(y_1 + y_2 + y_3)}{1 + \varepsilon}\right)^{1/a} - y_3 \left(\frac{\lambda}{M} + \frac{y_3(y_1 + y_2 + y_3)}{m^*(1 + \varepsilon)}\right)^{1/a},$$

where $\lambda = m^* + 3^{1+a}$, $M = 3 + m^* + \varepsilon$ and $k = \sqrt{\frac{m^*}{3 + m^*}(1 - 3^a)}$.

By construction we have

$$D_Y G(Y^0, 0) = D_X F(X^0, 0) \cdot D_Y X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}. \tag{12}$$

Since the change of variables (11) is S_2-equivariant, the S_2-equivariance of F remains in G so that the equation $G_3(Y, \varepsilon) = 0$ is S_2-invariant.

The strategy to calculate the central configurations near the concave equilateral triangle consists of solving the equation $G_3 = 0$ for y_3 in terms of (y_1, y_2, ε), insert it into G_1 and G_2 and solve the remaining bifurcation problem.

In order to study the bifurcation problem

$$G(y_1, y_2, y_3, \varepsilon) = (0, 0, 0), \qquad G(k, k, -3k, 0) = (0, 0, 0), \tag{13}$$

we only need two properties of G to know, the analyticity around the trivial solution $(k, k, -3k, 0)$ and the S_2-equivariance. In view of this, we firstly make a translation $y_i \to y_i + k$ $(i = 1, 2)$ and $y_3 \to y_3 - 3k$ in the expression of G. We have adopted the following notation for its Taylor's series around the trivial root $(0, 0, 0, 0)$,

$$\begin{aligned} G_1(Y, \varepsilon) &= b_4\varepsilon + 2b_{22}y_1y_2 + b_{22}y_2^2 + b_{33}y_3^2 + b_{13}y_1y_3 + b_{23}y_2y_3 \\ &\quad + b_{14}y_1\varepsilon + b_{24}y_2\varepsilon + b_{34}y_3\varepsilon + b_{44}\varepsilon^2 + \mathcal{O}(3), \\ G_2(Y, \varepsilon) &= b_4\varepsilon + 2b_{22}y_1y_2 + b_{22}y_1^2 + b_{33}y_3^2 + b_{13}y_2y_3 + b_{23}y_1y_3 \\ &\quad + b_{14}y_2\varepsilon + b_{24}y_1\varepsilon + b_{34}y_3\varepsilon + b_{44}\varepsilon^2 + \mathcal{O}(3), \\ G_3(Y, \varepsilon) &= 4y_3 + c_4\varepsilon + c_{11}y_1^2 + c_{11}y_2^2 + c_{33}y_3^2 + c_{12}y_1y_2 + c_{13}y_1y_3 \\ &\quad + c_{13}y_2y_3 + c_{14}y_1\varepsilon + c_{14}y_2\varepsilon + c_{34}y_3\varepsilon + c_{44}\varepsilon^2 + \mathcal{O}(3). \end{aligned} \tag{14}$$

The above expressions are due to the S_2-equivariance of G. The derivatives of G_1, G_2 and G_3 at $(0, 0, 0, 0)$ are given by

$$G_1 = G_2 = G_3 = 0,$$

$$\frac{\partial G_1}{\partial y_1} = \frac{\partial G_1}{\partial y_2} = \frac{\partial G_2}{\partial y_1} = \frac{\partial G_2}{\partial y_2} = 0,$$

$$b_4 = \frac{\partial G_1}{\partial \varepsilon} = \frac{\partial G_2}{\partial \varepsilon} = 3\sqrt{\frac{3^a(3^a - a - 1)}{1 - 3^a}} > 0,$$

$$b_{11} = \frac{1}{2}\frac{\partial^2 G_1}{\partial y_1^2} = 0,$$

$$b_{22} = \frac{1}{2}\frac{\partial^2 G_1}{\partial y_2^2} = -\frac{6k3^{1-a}}{2a} + \frac{k^3(1-a)}{2a^2}\left(3^{1-2a} + \frac{27}{m^{*2}}\right) > 0,$$

$$b_{12} = \frac{\partial^2 G_1}{\partial y_2 \partial y_1} = -\frac{6k3^{1-a}}{a} + \frac{k^3(1-a)}{a^2}\left(3^{1-2a} + \frac{27}{m^{*2}}\right) = 2b_{22},$$

$$c_4 = \frac{\partial G_3}{\partial \varepsilon} = \frac{6k}{a}\left[\frac{\lambda(3^{-a} - 1)}{(3 + m^*)^2} - k^2\left(\frac{2}{m^*} + 3^{-a}\right)\right],$$

$$c_{11} = \frac{1}{2}\frac{\partial^2 G_3}{\partial y_1^2} = \frac{k}{2a}\left[\left(\frac{12}{m^*} - 2\cdot 3^{1-a}\right) + \frac{k^2(1-a)}{a}\left(\frac{45}{m^{*2}} - 3^{1-2a}\right)\right],$$

$$c_{12} = \frac{\partial^2 G_3}{\partial y_2 \partial y_1} = \frac{k}{a}\left[\frac{6}{m^*} - 4\cdot 3^{1-a} + \frac{36k^2(1-a)}{am^{*2}}\right].$$

By solving $G_3 = 0$ for y_3 through the IFT, we get a unique analytical function $y_3 = W(y_1, y_2, \varepsilon)$ defined on an invariant neighborhood V of $(0, 0, 0)$.

Lemma 1. *The function $W(y_1, y_2, \varepsilon)$ guaranteed by the IFT is S_2-invariant.*

Proof. Let $\sigma\colon (y_1, y_2) \mapsto (y_2, y_1)$ be the non-trivial permutation of S_2. From S_2-invariance of G_3 we have for all $y = (y_1, y_2) \in V$ and ε small

$$G_3(y, W(\sigma\cdot y, \varepsilon), \varepsilon) = G_3(\sigma\cdot y, W(\sigma\cdot y, \varepsilon), \varepsilon) = 0.$$

That is, $W(\sigma\cdot y, \varepsilon)$ also solves $G_3 = 0$ locally. By uniqueness of the implicit solution, we must have

$$W(\sigma\cdot y, \varepsilon) = W(y, \varepsilon), \qquad \forall\, y \in V \text{ and } \varepsilon \text{ small.} \qquad \square$$

Now we write the power series for W in a small neighborhood of $(0, 0, 0)$ and insert it into the third equation of (14). By comparison of coefficients we get up to order 2

$$\begin{aligned} W(y_1, y_2, \varepsilon) = &-\frac{c_4}{4}\varepsilon - \frac{c_{11}}{4}y_1^2 - \frac{c_{11}}{4}y_2^2 - \frac{c_{12}}{4}y_1 y_2 + \left(\frac{c_4 c_{13}}{16} - \frac{c_{14}}{4}\right) y_1\varepsilon \\ &+ \left(\frac{c_4 c_{13}}{16} - \frac{c_{14}}{4}\right) y_2\varepsilon + \left[\frac{c_{34}c_4}{16} - \frac{c_4^2 c_{33}}{64} - \frac{c_{44}}{4}\right]\varepsilon^2 + \mathcal{O}(3). \end{aligned} \tag{15}$$

With this, we turn back to the bifurcation problem:

$$\begin{aligned} \tilde{G}_1(y_1, y_2, \varepsilon) &= G_1(y_1, y_2, W(y_1, y_2, \varepsilon), \varepsilon) = 0, \\ \tilde{G}_2(y_1, y_2, \varepsilon) &= G_2(y_1, y_2, W(y_1, y_2, \varepsilon), \varepsilon) = 0. \end{aligned} \tag{16}$$

The Taylor's series expansion for these functions are

$$\begin{aligned} \tilde{G}_1(y_1, y_2, \varepsilon) &= b_4\varepsilon + 2b_{22}y_1y_2 + b_{22}y_2^2 + \left(b_{14} - \frac{b_{13}c_4}{4}\right) y_1\varepsilon \\ &+ \left(b_{24} - \frac{b_{23}c_4}{4}\right) y_2\varepsilon + \left(b_{44} - \frac{b_{34}c_4}{4} + \frac{b_{33}c_4^2}{16}\right)\varepsilon^2 + \mathcal{O}(3), \\ \tilde{G}_2(y_1, y_2, \varepsilon) &= b_4\varepsilon + 2b_{22}y_1y_2 + b_{22}y_1^2 + \left(b_{14} - \frac{b_{13}c_4}{4}\right) y_2\varepsilon \\ &+ \left(b_{24} - \frac{b_{23}c_4}{4}\right) y_1\varepsilon + \left(b_{44} - \frac{b_{34}c_4}{4} + \frac{b_{33}c_4^2}{16}\right)\varepsilon^2 + \mathcal{O}(3). \end{aligned}$$

Proposition 2. *The system $\tilde{G}_1 = \tilde{G}_2 = 0$ does not admit a differentiable solution $y(\varepsilon) = (y_1(\varepsilon), y_2(\varepsilon))$ defined around $\varepsilon = 0$*

Proof. If it did, we would have $\tilde{G}(y(\varepsilon), \varepsilon) = 0$ whose differentiation at $(0, 0, 0)$ would furnish

$$D_y\tilde{G}(y(0), 0)\cdot y'(0) + \frac{\partial \tilde{G}}{\partial \varepsilon}(y(0), 0) = 0.$$

However by (12), we have $D_y\tilde{G}(y(0), 0) = 0$ and $\tilde{G}_\varepsilon \neq 0$ so the above equation is impossible. $\square$

Now we solve the equation $\tilde{G}_1(y_1, y_2, \varepsilon) = 0$ for ε by writing it as an analytic function of (y_1, y_2)

$$\varepsilon(y_1, y_2) = -\frac{b_{22}}{b_4} y_2^2 - \frac{2b_{22}}{b_4} y_1 y_2 + \mathcal{O}(3). \tag{17}$$

The last equation to be solved is

$$H(y_1, y_2) = \tilde{G}_2(y_1, y_2, \varepsilon(y_1, y_2)) = 0, \tag{18}$$

but such $H(y_1, y_2)$ is analytic in a small neighborhood of $(0, 0)$ and satisfy

$$H(0, 0) = 0 \quad \text{and} \quad D_y H(0, 0) = (0, 0).$$

Lemma 2. *If $H(x, y)$ is an analytic function defined on a neighborhood of the origin $(0, 0)$, such that $H(x, x) = 0$ then*

$$H(x, y) = (x - y) \cdot h(x, y),$$

where $h(x, y)$ is analytic in a neighborhood of $(0, 0)$.

Proof. By defining $\hat{H}(\xi, \eta) = H(\xi + \eta, \xi - \eta)$ for (ξ, η) near the origin, we have that $\hat{H}$ is analytic in a small neighborhood of $(0, 0)$ and $\hat{H}(\xi, 0) = 0$. Therefore, in the Taylor's series of $\hat{H}(\xi, \eta)$ all coefficients of the powers of ξ^n must be zero so that the series of $\hat{H}$ has a factor η. Thus

$$H(x, y) = (x - y) \cdot h(x, y). \qquad \square$$

The function (18) satisfies the conditions of Lemma 2. In fact, from the S_2-equivariance of $\tilde{G}$ we have the identity

$$\tilde{G}_2(y_2, y_1, \varepsilon(y_1, y_2)) = \tilde{G}_1(y_1, y_2, \varepsilon(y_1, y_2)) = 0,$$

for all (y_1, y_2) near $(0, 0)$. Thus

$$H(t, t) = \tilde{G}_2(t, t, \varepsilon(t, t)) = 0,$$

for all t near 0.

By Lemma 2, we have that

$$H(y_1, y_2) = (y_1 - y_2) \cdot h(y_1, y_2). \tag{19}$$

By replacing the series of $\varepsilon(y_1, y_2)$ into $\tilde{G}_2$ we obtain

$$H(y_1, y_2) = b_{22}(y_1^2 - y_2^2) + \mathcal{O}(3). \tag{20}$$

Comparing with (19) and remembering that the Taylor series is unique, we have that

$$h(y_1, y_2) = b_{22}(y_1 + y_2) + \mathcal{O}(2). \tag{21}$$

Thus, the equation $h(y_1, y_2) = 0$ can be solved for $y_1 = \tau(y_2)$ defined in a neighborhood of $y_2^0 = 0$ such that $\tau(0) = 0$. From the factorization (19), we see that the equation $H(y_1, y_2) = 0$ has two analytic branches at the neighborhood of $y_2^0 = 0$ given by

$$y_1 = y_2 \qquad \text{and} \qquad y_1 = -y_2 + \mathcal{O}(2).$$

In order to write the bifurcation branches, we undo the translation of variables y_i and we introduce a small parameter t by setting $y_2 = k + t$. Writing the expressions of y_1, y_3 and ε in terms of t we have only two families of central configurations given by the system $G(Y, \varepsilon) = 0$ near the concave equilateral triangle with masses $m_1 = m_2 = 1$, $m_3 = m^*$ and $m_4 = 1$, where $Y = (y_1, y_2, y_3)$

$$\text{I:} \begin{cases} y_1 = k + t, \\ y_2 = k + t, \\ \varepsilon = -\dfrac{3b_{22}}{b_4} t^2 + \mathcal{O}(3), \\ y_3 = -3k + \left(\dfrac{3c_4 b_{22}}{4b_4} - \dfrac{c_{11}}{2} - \dfrac{c_{12}}{4} \right) t^2 + \mathcal{O}(3), \end{cases}$$

$$\text{II:} \begin{cases} y_1 = k - t + \mathcal{O}(2), \\ y_2 = k + t, \\ \varepsilon = \dfrac{b_{22}}{b_4} t^2 + \mathcal{O}(3), \\ y_3 = -3k + \left(-\dfrac{c_4 b_{22}}{4b_4} - \dfrac{c_{11}}{2} + \dfrac{c_{12}}{4} \right) t^2 + \mathcal{O}(3). \end{cases}$$

The family I has symmetry with respect to the bisector of the segment q_1q_2 and it exists for $\varepsilon < 0$ whereas the family II is non symmetric and it exists for $\varepsilon > 0$.

However, the concept of bifurcation involves a change in the number of roots of the corresponding equation when the parameter crosses the critical point $\varepsilon = 0$. That is, the bifurcation is verified if, for any $\varepsilon \neq 0$, we have two or more values for $Y = (y_1, y_2, y_3)$ satisfying the equation $G(Y, \varepsilon) = 0$. In order to see this, note that the function $\varepsilon(t)$ has a relative maximum and a relative minimum in family I and family II, respectively. So, e.g., in family II, for every $\varepsilon > 0$ sufficiently small there exist two values $t_1 < 0 < t_2$ such that $\varepsilon(t_1) = \varepsilon = \varepsilon(t_2)$. Consequently, associated to that ε we will have two values for the Y variables. For the family I the argument is analogous.

With all the above we have the following result.

Theorem 2. *For every $a < -1$, the concave equilateral triangle with vertices $m_1 = m_2 = m_4 = 1$ and a mass $m_3 = m^*$ at the barycenter, experiences a bifurcation when the mass m_4 crosses the value 1. More precisely, there is a $\delta > 0$ such that for any $1 - \delta < m_4 < 1$ we have two concave central configurations coming from the bifurcation presenting a symmetry type axes and for any $1 < m_4 < 1 + \delta$ we have two concave central configurations without symmetry.*

In the case of continuation, the coefficients depend on the mass m and the exponent a, while in the study of bifurcation, the coefficients depend only on the exponent. So, we can sketch the behavior of two bifurcation branches. By using that

$$s_{ij} = \left(\frac{\lambda}{m^* + 3 + \varepsilon(t)} - \frac{x_i x_j}{m_i m_j} \right)^{-2/3},$$

we write the power series expansion of t for the two families.

$$s_{ij}^{I,II}(\varepsilon) = s_{ij}^0 + w_{ij}^{I,II} t + \mathcal{O}(t^2).$$

Table 2
Behavior of the two families of bifurcation as a function of parameter t.

	Family I	Family II
$\varepsilon(t)$	negative	positive
$s_{12}(t)$	increasing	indefinite
$s_{13}(t)$	decreasing	increasing
$s_{14}(t)$	decreasing	decreasing
$s_{23}(t)$	decreasing	decreasing
$s_{24}(t)$	decreasing	increasing
$s_{34}(t)$	increasing	indefinite

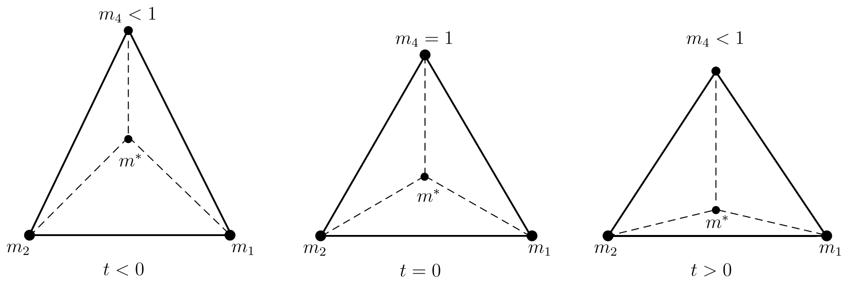

Fig. 3. Family I: with $m_1 = m_2 = 1$, $m_3 = m^*$ and $m_4 = 1+\varepsilon$, for each $\varepsilon < 0$ one has two symmetric concave central configurations.

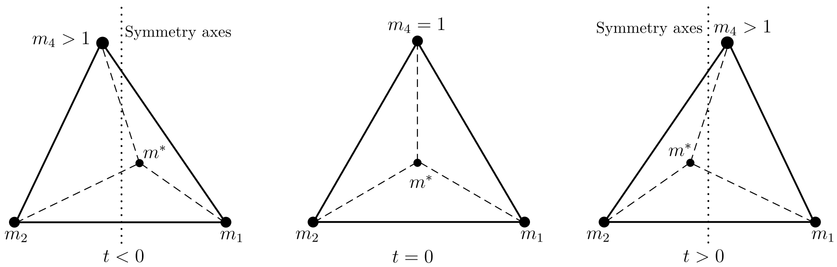

Fig. 4. Family II: with $m_1 = m_2 = 1$, $m_3 = m^*$ and $m_4 = 1 + \varepsilon$, for each $\varepsilon > 0$ one has two non symmetric concave central configurations.

$$\text{I:}\begin{cases} w^I_{12} = -\frac{2\cdot 3^{1-a}\cdot k}{a} > 0, \\ w^I_{13} = w^I_{23} = \frac{3\cdot k}{m^*\cdot a} < 0, \\ w^I_{14} = w^I_{24} = \frac{k\cdot 3^{1-a}}{a} < 0, \\ w^I_{34} = -\frac{6\cdot k}{m^*\cdot a} > 0, \end{cases} \qquad \text{II:}\begin{cases} w^{II}_{12} = 0, \\ w^{II}_{13} = -\frac{3\cdot k}{m^*\cdot a} > 0, \\ w^{II}_{23} = \frac{3\cdot k}{m^*\cdot a} < 0, \\ w^{II}_{14} = \frac{k\cdot 3^{1-a}}{a} < 0, \\ w^{II}_{24} = -\frac{k\cdot 3^{1-a}}{a} > 0, \\ w^{II}_{34} = 0. \end{cases}$$

In Table 2 we show the behavior of both families in terms of the sign of ε and the mutual distances. In Fig. 3 we show the two symmetric concave families of central configurations coming from Family 1. Fig. 4 shows two non symmetric families of concave central configurations.

In [2], the authors found numerically a non-symmetric concave central configuration in the four body problem. We observe that, since in their example $m_1 = m_2 = m_3 = 1$ and $m_4 = 0.992$, it does not belong to our Family II of non-symmetric concave central configurations.

4. The regular tetrahedron with a mass at the barycenter

Now, consider the system (4) with $n = 5$ and masses $m_1 = m_2 = m_3 = 1$, $m_4 = m$ and $m_5 = 1 + \varepsilon$. In order that the regular tetrahedron with the squares of the mutual distances $s^0_{12} = s^0_{13} = s^0_{15} = s^0_{23} = s^0_{25} = s^0_{35} = 8/3$ and $s^0_{14} = s^0_{24} = s^0_{34} = s^0_{45} = 1$ be a solution for $\varepsilon = 0$ we must have $\lambda = m + 4 \cdot \rho^a$ in which $\rho = 8/3$. This central configuration of the 5-body problem will be called simply *concave regular tetrahedron* leaving implied the occurrence of a fifth mass at the barycenter. The values for variables x's corresponding to the concave regular tetrahedron are

$$x^0_1 = x^0_2 = x^0_3 = x^0_5 = k \quad \text{and} \quad x^0_4 = -4k, \tag{22}$$

where $k = \sqrt{\dfrac{m}{4+m}(1-\rho^a)}$.

In this case, we insert $x_4 = -x_1 - x_2 - x_3 - x_5$ into the equations $t_i = t_j$ in (4) to get a system of four equations, four variables $X = (x_1, x_2, x_3, x_5)$ and one parameter $\varepsilon \in \mathbb{R}$ given by the functions

$$F_1(X,\varepsilon) = \sum_{\substack{j=1\\ j\neq 1}}^{4} x_j \left(\frac{\lambda}{M} - \frac{x_1x_j}{m_j}\right)^{1/a} + x_5\left(\frac{\lambda}{M} - \frac{x_1x_5}{1+\varepsilon}\right)^{1/a} - t_5,$$

$$F_2(X,\varepsilon) = \sum_{\substack{j=1\\ j\neq 2}}^{4} x_j \left(\frac{\lambda}{M} - \frac{x_jx_2}{m_j}\right)^{1/a} + x_5\left(\frac{\lambda}{M} - \frac{x_2x_5}{1+\varepsilon}\right)^{1/a} - t_5,$$

$$F_3(X,\varepsilon) = \sum_{\substack{j=1\\ j\neq 3}}^{4} x_j \left(\frac{\lambda}{M} - \frac{x_jx_3}{m_j}\right)^{1/a} + x_5\left(\frac{\lambda}{M} - \frac{x_3x_5}{1+\varepsilon}\right)^{1/a} - t_5,$$

$$F_4(X,\varepsilon) = \sum_{j=1}^{3} x_j \left(\frac{\lambda}{M} - \frac{x_jx_4}{m}\right)^{1/a} + x_5\left(\frac{\lambda}{M} - \frac{x_4x_5}{m(1+\varepsilon)}\right)^{1/a} - t_5,$$

where $m > 0$ and $a < -1$ are fixed and

$$t_5 = \sum_{j=1}^{3} x_j \left(\frac{\lambda}{M} - \frac{x_jx_5}{1+\varepsilon}\right)^{1/a} + x_4\left(\frac{\lambda}{M} - \frac{x_4x_5}{m(1+\varepsilon)}\right)^{1/a}.$$

Calling $F = (F_1, F_2, F_3, F_4) : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$, we have $F(X^0, 0) = (0, 0, 0, 0)$ for any $m > 0$. Observe also that F is S_3-equivariant, that is

$$F(\sigma \cdot X, \varepsilon) = \sigma \cdot F(X, \varepsilon) \tag{23}$$

for any permutation in $\Sigma_4 = \{\sigma \in S_4 \;:\; \sigma(4) = 4\}$ and so, its first derivative must satisfy

$$D_X F(\sigma \cdot X, \varepsilon) \cdot \sigma = \sigma \cdot D_X F(X, \varepsilon).$$

Taking into account that $\sigma \cdot X^0 = X^0$, we have that the derivative $D_X F(X^0, 0)$ is a linear transformation that commutes with every $\sigma \in \Sigma_4$. Therefore, it can be represented as a matrix of the form

$$D_X F(X^0, 0) = \begin{pmatrix} b & c & c & d \\ c & b & c & d \\ c & c & b & d \\ e & e & e & f \end{pmatrix} \tag{24}$$

The computation of the first partial derivatives of F with respect to X evaluated at $(X^0, m, 0)$ gives us

$$\begin{aligned}
b &= \left.\frac{\partial F_1}{\partial x_1}\right|_{(X^0,0)} = -\rho - \frac{k^2}{a}\left(\frac{16}{m} + 2\rho^{1-a}\right) = -d,\\
c &= \left.\frac{\partial F_1}{\partial x_2}\right|_{(X^0,0)} = 0,\\
e &= \left.\frac{\partial F_4}{\partial x_1}\right|_{(X^0,0)} = 2 - \rho + \frac{k^2}{a}\left(\frac{12}{m} + \rho^{1-a}\right),\\
f &= \left.\frac{\partial F_4}{\partial x_5}\right|_{(X^0,0)} = 2 + \frac{k^2}{a}\left(\frac{28}{m} + 3\rho^{1-a}\right) = e - b.
\end{aligned}$$

With the above values, the determinant is easily obtained

$$\left|D_X F(X^0, 0)\right| = b^3(4e - b). \tag{25}$$

The factor $b - 4e$ is positive for any $a < -1$ and $m > 0$, whereas the factor b is zero if, and only if, the mass m is equal to

$$m^{**} = 2\frac{3\rho^a - 2a - 3}{2\rho^{-a} + a - 2}. \tag{26}$$

Proposition 3. *For all $a < -1$, the value of m^{**} is positive.*

Proof. By labeling $n(a) = 3\rho^a - 2a - 3$ and $d(a) = 2\rho^{-a} + a - 2$ we have for any $a < -1$

$$\begin{aligned}
n'(a) &= 3\rho^a \ln(\rho) - 2 < 3\frac{3}{8}\ln(e) - 2 < 0,\\
d'(a) &= 1 - 2\rho^{-a}\ln(\rho) < 1 - 2\rho\ln(\rho) < 0,
\end{aligned}$$

so that both functions are decreasing on $(-\infty, -1)$ implying that they reach their minima at $a = -1$. Since $n(-1) = 3\rho^{-1} - 1 > 0$ and $d(-1) = 2\rho - 3 > 0$, we get that $m^{**} > 0$ on $(-\infty, -1)$. □

Remark 4. In the Newtonian case with $a = -3/2$, the value of m^{**} agrees with that found by Meyer and Schmidt in [14], that is $m^{**} = \frac{10368+1701\sqrt{6}}{54952}$. Moreover, it is worth noting that in terms of m^{**}, the expression for b becomes

$$b = -\frac{(2\rho^{-a} + a - 2)\cdot(m - m^{**})\cdot\rho}{a\cdot(4 + m)},$$

which shows that $b > 0$ if and only if $m > m^{**}$.

4.1. Continuation

Analogously to Theorem 1, we have the

Theorem 3. *For every $a < -1$ and $0 < m \neq m^{**}$, there exists $\varepsilon_0 \in \mathbb{R}$ such that the concave regular tetrahedron $(X^0, 0)$ can be continued to a symmetric family of concave central configurations $X(\varepsilon)$ defined for $|\varepsilon| < \varepsilon_0$ and with symmetry axis type given by*

$$\begin{aligned} x_1(\varepsilon) &= k + \alpha\varepsilon + \mathcal{O}(\varepsilon^2), \\ x_2(\varepsilon) &= k + \alpha\varepsilon + \mathcal{O}(\varepsilon^2), \\ x_3(\varepsilon) &= k + \alpha\varepsilon + \mathcal{O}(\varepsilon^2), \\ x_4(\varepsilon) &= -4k + \gamma\varepsilon + \mathcal{O}(\varepsilon^2), \\ x_5(\varepsilon) &= k + \beta\varepsilon + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{27}$$

where $\gamma = -(3\alpha + \beta)$ and α and β are well-defined functions of m and a.

Proof. If $0 < m \neq m^{**}$, then $|D_X F(X^0, 0)| \neq 0$ and the IFT can be applied to the equation $F(X, \varepsilon) = (0, 0, 0)$ near the trivial solution $(X^0, 0)$, ensuring the existence of a unique curve $X(\varepsilon) \in \mathbb{R}^4$ defined on an interval $(-\varepsilon_0, \varepsilon_0)$ and such that

$$F(X(\varepsilon), \varepsilon) = (0, 0, 0),$$
$$X(0) = X^0.$$

The S_3-equivariance of F implies that $(\sigma \cdot X(\varepsilon), \varepsilon)$ is also a solution for $F = 0$. By the local uniqueness of the implicit function we must have

$$\forall\ \sigma \in \Sigma_4, \quad \sigma \cdot X(\varepsilon) = X(\varepsilon) \quad \Rightarrow \quad x_1(\varepsilon) = x_2(\varepsilon) = x_3(\varepsilon).$$

By Remark 2, the central configurations given by $F = 0$, which are continuation of X^0 in a neighborhood of $\varepsilon = 0$, present a symmetry axis type, that is

$$s_{1i}(\varepsilon) = s_{2i}(\varepsilon) = s_{3i}(\varepsilon), \quad i \in \{4, 5\}$$

for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

By multiplying the derivative $\left.\dfrac{\partial F}{\partial \varepsilon}\right|_{(X^0,0)}$ by the matrix $-D_X F(X^0, 0)^{-1}$ we obtain the expression for $\alpha = \left.\dfrac{dx_1}{d\varepsilon}\right|_{(X^0,0)}$ and $\beta = \left.\dfrac{dx_5}{d\varepsilon}\right|_{(X^0,0)}$:

$$\alpha = \frac{32k(1-\rho^a)p(m,a)}{(b-4e)ba^2(4+m)^3\rho^a},$$
$$\beta = \frac{32k(1-\rho^a)q(m,a)}{(b-4e)ba^2(4+m)^3\rho^a},$$

where

$$\begin{aligned} p(m,a) &= \frac{a}{3}m^3 + \frac{2}{3}\left[a(3+\rho^a) - \frac{2(1-\rho^{2a})}{\rho^a}\right]m^2 \\ &\quad + \frac{4}{3}\left[2a(1+\rho^a) - (1-\rho^a)(11+3\rho^a)\right]m - 32(1-\rho^a)\rho^a, \\ q(m,a) &= \frac{1}{3}\left[a + 8\frac{1-\rho^a}{\rho^a}\right]m^3 + \frac{2}{3}\left[\frac{2(7+13\rho^a)(1-\rho^a)}{\rho^a} + a(3+\rho^a)\right]m^2 \\ &\quad + 4\left[(15+7\rho^a)(1-\rho^a) + \frac{2a(1+\rho^a)}{3}\right]m + 96(1-\rho^a)\rho^a. \end{aligned}$$

Thus we have proved the theorem. □

As in Theorem 1, it's not possible to give a complete description for the behavior of the solution $X(\varepsilon)$ independently of the exponent a, so in order to show how the solution $X(\varepsilon)$ behaves, we analyze the particular value $a = -3/2$ which corresponds to the Newtonian case, in this case we can make a complete analysis of the central configurations given by (27). We compute

$$\alpha = -\frac{k(1-\rho^a)\tilde{p}(m)}{2^2 \cdot 3^3 \cdot (b-4e)ba^2(4+m)^3\rho^a},$$
$$\beta = \frac{k \cdot (1-\rho^a)\tilde{q}(m)}{2^2 \cdot 3^3 \cdot (b-4e)ba^2(4+m)^3\rho^a},$$
$$\gamma = -(3\alpha+\beta) = -\frac{4k(1-\rho^a)m(m-m^{**})(16m+32+3\sqrt{6})(64\sqrt{6}-63)}{3^3 \cdot (b-4e)ba^2(4+m)^3\rho^a},$$

where

$$\tilde{p}(m) = 1728m^3 + (10368 + 8084\sqrt{6})m^2 + (63783 - 2160\sqrt{6})m + 10368\sqrt{6} - 5832,$$
$$\tilde{q}(m) = (16384\sqrt{6} - 10944)m^3 + (51404\sqrt{6} + 17280)m^2 + (188433 - 11664\sqrt{6})m + 31104\sqrt{6} - 17496.$$

Since $b > 0$ if, and only if $m > m^{**}$, we see that the sign of γ is negative for all $m > 0$ while

$$m < m^{**} \Rightarrow \alpha > 0 \text{ and } \beta < 0,$$
$$m > m^{**} \Rightarrow \alpha < 0 \text{ and } \beta > 0.$$

Considering that

$$s_{ij}(\varepsilon) = \left(\frac{\lambda}{M} - \frac{x_i(\varepsilon)x_j(\varepsilon)}{m_i m_j}\right)^{1/a},$$

and remembering that $M = m + 4 + \varepsilon$, $m_5 = 1 + \varepsilon$ and $\lambda = m + 4\rho^{-3/2}$, we can obtain the derivatives of s_{ij} at $\varepsilon = 0$ in terms of the mass m. After some tedious calculations we get

$$s_{ij}(\varepsilon) = s_{ij}^0 + v_{ij}(m)\varepsilon + \mathcal{O}(\varepsilon^2),$$

where

$$v_{12}(m) = -\frac{64(16\sqrt{6}-9)m^2 + 12(3\sqrt{6}+64)m + 243}{3 \cdot (64\sqrt{6}-63) \cdot (8m+3\sqrt{6}) \cdot (m-m^{**})},$$
$$v_{14}(m) = -\frac{8(56\sqrt{6}-45)m + 27(16-3\sqrt{6})}{4 \cdot (21\sqrt{6}-128) \cdot (8m+3\sqrt{6}) \cdot (m-m^{**})},$$
$$v_{15}(m) = \frac{64(16\sqrt{6}-9)m^2 + 12(320-39\sqrt{6})m - 243}{3 \cdot (64\sqrt{6}-63) \cdot (8m+3\sqrt{6}) \cdot (m-m^{**})},$$
$$v_{45}(m) = \frac{8(40\sqrt{6}-9)m + 81(16-\sqrt{6})}{4 \cdot (21\sqrt{6}-128) \cdot (8m+3\sqrt{6}) \cdot (m-m^{**})}.$$

All coefficients $v_{ij}(m)$ have well-defined sign for each $m > m^{**}$, whereas for $m < m^{**}$ only $v_{15}(m)$ has a change of sign on the interval $(0, m^{**})$. In fact, to see this, just make $m = 0$ and $m = m^{**}$ in the numerator

Table 3
Behavior of $s_{ij}(\varepsilon)$ at the neighborhood of $\varepsilon = 0$.

	$0 < m < m^{**}$	$m > m^{**}$
$s_{12}(\varepsilon), s_{13}(\varepsilon), s_{23}(\varepsilon)$	increasing	decreasing
$s_{14}(\varepsilon), s_{24}(\varepsilon), s_{34}(\varepsilon)$	decreasing	increasing
$s_{15}(\varepsilon), s_{25}(\varepsilon), s_{35}(\varepsilon)$	indefinite	increasing
$s_{45}(\varepsilon)$	increasing	decreasing

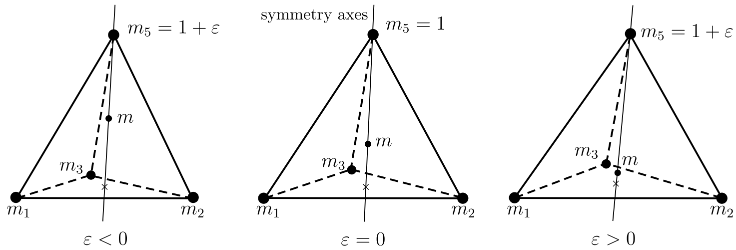

Fig. 5. Continuation for $m < m^{**}$ and $a = -3/2$.

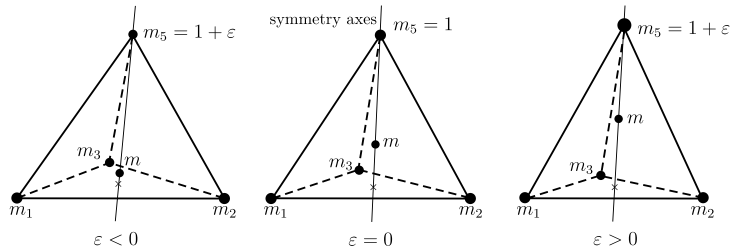

Fig. 6. Continuation for $m > m^{**}$ and $a = -3/2$.

of the expression. Nevertheless, we can draw the behavior of central configurations given by the equation $F(X, \varepsilon) = 0$ at the neighborhood of $(X^0, 0)$ for all $m \neq m^{**}$ (see Table 3, Figs. 5 and 6).

4.2. Bifurcation

Now, we consider the problem

$$F_i(x_1, x_2, x_3, x_5, \varepsilon) = 0, \qquad i = 1, 2, 3, 4,$$

where we set $m_4 = m^{**}$. At the point $(X^0, 0)$, the matrix (24) is

$$D_X F(X^0, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f & f & f & f \end{pmatrix}$$

where $f = -\frac{4(1-\rho^a)(1-2a)}{a(4-3\rho^a)}$. As in the planar triangular case we make some transformations, following the Liapunov–Schmidt reduction process. Let L be the matrix $D_X F(X^0, 0)$. We have that

$$\ker\{L\} = \{(a_1, a_2, a_3, a_4)/\sum a_i = 0\}, \quad \mathrm{Im}\{L\} = \{(0, 0, 0, x) \ : \ x \in \mathbb{R}\}.$$

By taking,

$$u_1 = (1,0,0,-1),\ u_2 = (0,1,0,-1),\ u_3 = (0,0,1,-1) \text{ and } u_4 = (0,0,0,-1)$$

we proceed the change of variables

$$(x_1, x_2, x_3, x_5) = y_1 \cdot u_1 + y_2 \cdot u_2 + y_3 \cdot u_3 + y_4 \cdot u_4, \tag{28}$$

so that

$$x_1 = y_1, \qquad x_2 = y_2, \qquad x_3 = y_3 \qquad \text{and} \qquad x_5 = -y_1 - y_2 - y_3 - y_4.$$

Let $G(Y, \varepsilon) = F(X(Y), \varepsilon)$ be the new functions defining the Dziobek configurations on the space after the change of variables. We have that

$$\begin{aligned}
G_1(Y,\varepsilon) &= y_2\left(\frac{\lambda}{M} - y_1y_2\right)^{1/a} + y_3\left(\frac{\lambda}{M} - y_1y_3\right)^{1/a} + y_4\left(\frac{\lambda}{M} - \frac{y_1y_4}{m^{**}}\right)^{1/a} \\
&- y_1\left(\frac{\lambda}{M} + \frac{y_1(y_1+y_2+y_3+y_4)}{\varepsilon+1}\right)^{1/a} - y_2\left(\frac{\lambda}{M} + \frac{y_2(y_1+y_2+y_3+y_4)}{\varepsilon+1}\right)^{1/a} \\
&- y_3\left(\frac{\lambda}{M} + \frac{y_3(y_1+y_2+y_3+y_4)}{\varepsilon+1}\right)^{1/a} - y_4\left(\frac{\lambda}{M} + \frac{y_4(y_1+y_2+y_3+y_4)}{m^{**}(\varepsilon+1)}\right)^{1/a} \\
&- (y_1+y_2+y_3+y_4)\left(\frac{\lambda}{M} + \frac{y_1(y_1+y_2+y_3+y_4)}{\varepsilon+1}\right)^{1/a}, \\
G_2(Y,\varepsilon) &= y_1\left(\frac{\lambda}{M} - y_1y_2\right)^{1/a} + y_3\left(\frac{\lambda}{M} - y_2y_3\right)^{1/a} + y_4\left(\frac{\lambda}{M} - \frac{y_2y_4}{m^{**}}\right)^{1/a} \\
&- y_1\left(\frac{\lambda}{M} + \frac{y_1(y_1+y_2+y_3+y_4)}{\varepsilon+1}\right)^{1/a} - y_2\left(\frac{\lambda}{M} + \frac{y_2(y_1+y_2+y_3+y_4)}{\varepsilon+1}\right)^{1/a} \\
&- y_3\left(\frac{\lambda}{M} + \frac{y_3(y_1+y_2+y_3+y_4)}{\varepsilon+1}\right)^{1/a} - y_4\left(\frac{\lambda}{M} + \frac{y_4(y_1+y_2+y_3+y_4)}{m^{**}(\varepsilon+1)}\right)^{1/a} \\
&- (y_1+y_2+y_3+y_4)\left(\frac{\lambda}{M} + \frac{y_2(y_1+y_2+y_3+y_4)}{\varepsilon+1}\right)^{1/a}, \\
G_3(Y,\varepsilon) &= y_1\left(\frac{\lambda}{M} - y_1y_3\right)^{1/a} + y_2\left(\frac{\lambda}{M} - y_2y_3\right)^{1/a} + y_4\left(\frac{\lambda}{M} - \frac{y_3y_4}{m^{**}}\right)^{1/a} \\
&- y_1\left(\frac{\lambda}{M} + \frac{y_1(y_1+y_2+y_3+y_4)}{\varepsilon+1}\right)^{1/a} - y_2\left(\frac{\lambda}{M} + \frac{y_2(y_1+y_2+y_3+y_4)}{\varepsilon+1}\right)^{1/a} \\
&- y_3\left(\frac{\lambda}{M} + \frac{y_3(y_1+y_2+y_3+y_4)}{\varepsilon+1}\right)^{1/a} - y_4\left(\frac{\lambda}{M} + \frac{y_4(y_1+y_2+y_3+y_4)}{m^{**}(\varepsilon+1)}\right)^{1/a} \\
&- (y_1+y_2+y_3+y_4)\left(\frac{\lambda}{M} + \frac{y_3(y_1+y_2+y_3+y_4)}{\varepsilon+1}\right)^{1/a}, \\
G_4(Y,\varepsilon) &= y_1\left(\frac{\lambda}{M} - \frac{y_1y_4}{m^{**}}\right)^{1/a} + y_2\left(\frac{\lambda}{M} - \frac{y_2y_4}{m^{**}}\right)^{1/a} + y_3\left(\frac{\lambda}{M} - \frac{y_3y_4}{m^{**}}\right)^{1/a} \\
&- y_1\left(\frac{\lambda}{M} + \frac{y_1(y_1+y_2+y_3+y_4)}{\varepsilon+1}\right)^{1/a} - y_2\left(\frac{\lambda}{M} + \frac{y_2(y_1+y_2+y_3+y_4)}{\varepsilon+1}\right)^{1/a}
\end{aligned}$$

$$- y_3 \left(\frac{\lambda}{M} + \frac{y_3 (y_1 + y_2 + y_3 + y_4)}{\varepsilon + 1} \right)^{1/a} - y_4 \left(\frac{\lambda}{M} + \frac{y_4 (y_1 + y_2 + y_3 + y_4)}{m^{**}(\varepsilon + 1)} \right)^{1/a}$$

$$- (y_1 + y_2 + y_3 + y_4) \left(\frac{\lambda}{M} + \frac{y_4 (y_1 + y_2 + y_3 + y_4)}{m^{**}(\varepsilon + 1)} \right)^{1/a},$$

where $\lambda = m^{**} + 4\rho^a$, $M = 4 + m^{**} + \varepsilon$ and $k = \sqrt{\frac{m^{**}}{4 + m^{**}}(1 - \rho^a)}$.

By construction we have

$$D_Y G(Y^0, 0) = D_X F(X^0, 0) \cdot D_Y X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -f \end{pmatrix}. \tag{29}$$

Since the change of variables (11) is S_3-invariant, the S_3-equivariance of F remains in G so that the equation $G_4(Y, \varepsilon) = 0$ is S_3-invariant.

As before, we will solve the equation $G_4 = 0$ for y_4 in terms of $(y_1, y_2, y_3, \varepsilon)$, inserting it into G_1, G_2 and G_3 and finally solve the remaining bifurcation problem in order to describe the bifurcations from the concave regular tetrahedron.

In order to study the bifurcation problem

$$G(y_1, y_2, y_3, y_4, \varepsilon) = (0, 0, 0, 0), \qquad G(k, k, k, -4k, 0) = (0, 0, 0, 0), \tag{30}$$

we only need to know two properties of G, the analyticity around the trivial solution $(k, k, k, -4k, 0)$ and the S_3-equivariance. In view of this, we first make a translation $y_i \to y_i + k$ $(i = 1, 2, 3)$ and $y_4 \to y_4 - 4k$ in the expression of G. Observe that G_1 is invariant by $\sigma = (23)$ so that its analytical expression is given by:

$$G_1(Y, \varepsilon) = b_5 \varepsilon + b_{11} y_1^2 + b_{22}(y_2^2 + y_3^2) + b_{44} y_4^2 + b_{55} \varepsilon^2 + b_{12} y_1 (y_2 + y_3)$$
$$+ b_{23} y_2 y_3 + b_{14} y_1 y_4 + b_{24} (y_2 + y_3) y_4 + b_{15} y_1 \varepsilon + b_{25} (y_2 + y_3) \varepsilon + b_{45} y_4 \varepsilon + \mathcal{O}(3),$$

and due to S_3-equivariance of G we must have

$$\begin{aligned} G_2(y_1, y_2, y_3, y_4, \varepsilon) &= G_1(y_2, y_1, y_3, y_4, \varepsilon), \\ G_3(y_1, y_2, y_3, y_4, \varepsilon) &= G_1(y_3, y_2, y_1, y_4, \varepsilon), \\ G_4(Y, \varepsilon) &= -f y_4 + c_5 \varepsilon + c_{11}(y_1^2 + y_2^2 + y_3^2) + c_{44} y_4^2 + c_{12}(y_1 y_2 + y_1 y_3 + y_2 y_3) \\ &\quad + c_{14}(y_1 + y_2 + y_3) y_4 + c_{15}(y_1 + y_2 + y_3)\varepsilon + c_{45} y_4 \varepsilon + c_{55} \varepsilon^2 + \mathcal{O}(3). \end{aligned} \tag{31}$$

The derivatives of G at $(0, 0, 0, 0, 0)$ are given by

$$\begin{aligned} & G_1 = G_2 = G_3 = G_4 = 0, \\ & b_1 = \frac{\partial G_1}{\partial y_1} = \frac{\partial G_1}{\partial y_2} = \frac{\partial G_2}{\partial y_1} = \frac{\partial G_2}{\partial y_2} = 0, \\ & b_5 = \frac{\partial G_1}{\partial \varepsilon} = \frac{\partial G_2}{\partial \varepsilon} = \frac{-2k^3 (m^{**} \rho^{1-a} + 8)}{a m^{**}} > 0, \\ & b_{11} = \frac{1}{2} \frac{\partial^2 G_1}{\partial y_1^2} = 0, \\ & b_{12} = \frac{\partial^2 G_1}{\partial y_2 \partial y_1} = -\frac{6k\rho^{1-a}}{a} + \frac{64k^3(1-a)}{a^2 m^{*2}} > 0, \end{aligned}$$

$$b_{22} = \frac{1}{2}\frac{\partial^2 G_1}{\partial y_2^2} = \frac{b_{12}}{2} > 0,$$

$$b_{23} = \frac{\partial^2 G_1}{\partial y_2 \partial y_3} = b_{12} > 0.$$

By solving G_4 for y_4 through the IFT, we get a unique analytical function $y_4 = W(y_1, y_2, y_3, \varepsilon)$ defined on an invariant neighborhood V of $(0,0,0,0)$ such that $W(0,0,0,0) = 0$.

Lemma 3. *The function $W(y_1, y_2, y_3, \varepsilon)$ guaranteed by the IFT is S_3-invariant.*

Proof. Let $\sigma \in S_3$. From the S_3-invariance of G_4, for all $y = (y_1, y_2, y_3) \in V$ and ε small, we have that

$$G_4(y, W(\sigma \cdot y, \varepsilon), \varepsilon) = G_4(\sigma \cdot y, W(\sigma \cdot y, \varepsilon), \varepsilon) = 0.$$

That is, $W(\sigma \cdot y, \varepsilon)$ also solves $G_4 = 0$ locally. By the uniqueness of the implicit solution, we must have

$$W(\sigma \cdot y, \varepsilon) = W(y, \varepsilon), \qquad \forall\, y \in V \text{ and } \varepsilon \text{ small.} \qquad \square$$

Now we write the power series for W in a small neighborhood of $(0,0,0,0)$ and insert it into the third equation of (31). By comparison of coefficients we get up to order 2.

$$W(y_1, y_2, y_3, \varepsilon) = \frac{1}{f}(c_5\varepsilon + c_{11}(y_1^2 + y_2^2 + y_3^2) + c_{12}(y_1y_2 + y_1y_3 + y_2y_3)$$
$$+ \varepsilon(c_{14}c_5 + c_{15})(y_1 + y_2 + y_3) + (c_{44}c_5^2 + c_{45}c_5 + c_{55})\varepsilon^2 + \mathcal{O}(3).$$

With this we go back to the bifurcation problem

$$\begin{aligned} \tilde{G}_1(y_1, y_2, y_3, \varepsilon) &= G_1(y_1, y_2, y_3, W(y_1, y_2, y_3, \varepsilon), \varepsilon) = 0, \\ \tilde{G}_2(y_1, y_2, y_3, \varepsilon) &= G_2(y_1, y_2, y_3, W(y_1, y_2, y_3, \varepsilon), \varepsilon) = 0, \\ \tilde{G}_3(y_1, y_2, y_3, \varepsilon) &= G_3(y_1, y_2, y_3, W(y_1, y_2, y_3, \varepsilon), \varepsilon) = 0. \end{aligned} \tag{32}$$

The Taylor's series expansion for these functions are

$$\tilde{G}_1(y_1, y_2, y_3, \varepsilon) = b_5\varepsilon + b_{22}(y_2 + y_3)(2y_1 + y_2 + y_3) + \left(b_{15} + \frac{b_{14}c_5}{f}\right) y_1\varepsilon$$
$$+ \left(b_{25} + \frac{b_{24}c_5}{f}\right)(y_2 + y_3)\varepsilon + \left(b_{55} + \frac{b_{45}c_5}{f} + \frac{b_{44}c_5^2}{f^2}\right)\varepsilon^2 + \mathcal{O}(3),$$
$$\tilde{G}_2(y_1, y_2, y_3, \varepsilon) = \tilde{G}_1(y_2, y_1, y_3, \varepsilon),$$
$$\tilde{G}_3(y_1, y_2, y_3, \varepsilon) = \tilde{G}_1(y_3, y_2, y_1, \varepsilon).$$

Since $D_y\tilde{G}(y(0), 0) = 0$ and $\tilde{G}_\varepsilon \neq 0$, the system $\tilde{G}_1 = \tilde{G}_2 = \tilde{G}_3 = 0$ does not admit a differentiable solution $y(\varepsilon) = (y_1(\varepsilon), y_2(\varepsilon), y_3(\varepsilon))$ defined around $\varepsilon = 0$. So, we solve the equation $\tilde{G}_3(y_1, y_2, y_3, \varepsilon) = 0$ for ε by writing it as an analytic function of (y_1, y_2, y_3) given by

$$\varepsilon(y_1, y_2, y_3) = -\frac{b_{22}}{b_5}(y_1 + y_2)(2y_3 + y_2 + y_1) + \mathcal{O}(3). \tag{33}$$

Finally we define the last two equations to be solved:

$$H_1(y_1,y_2,y_3)=\tilde{G}_1(y_1,y_2,y_3,\varepsilon(y_1,y_2,y_3))=0,$$
$$H_2(y_1,y_2,y_3)=\tilde{G}_2(y_1,y_2,y_3,\varepsilon(y_1,y_2,y_3))=0.$$

Note that the functions H_i, $i=1,2$ are analytic in a small neighborhood of $(0,0,0)$ and satisfy

$$H_i(0,0,0)=0 \quad \text{and} \quad D_yH_i(0,0,0)=(0,0,0) \quad \textit{for} \;\; i=1,2.$$

We also observe that $H_i(\tau,y_2,\tau)=0$ for $i=1,2$. In fact, by definition

$$H_1(\tau,y_2,\tau)=\tilde{G}_1(\tau,y_2,\tau,\varepsilon(\tau,y_2,\tau))$$

and since $\tilde{G}_1(y_3,y_2,y_1,\varepsilon(y_1,y_2,y_3))=\tilde{G}_3(y_1,y_2,y_3,\varepsilon(y_1,y_2,y_3))=0$, we get

$$H_1(\tau,y_2,\tau)=0.$$

Clearly the same argument works for H_2 so that we can apply Lemma 2 to both H_1 and H_2 to show that they can be written as

$$H_1(y_1,y_2,y_3)=(y_1-y_3)\cdot h_1(y_1,y_2,y_3), \tag{34}$$
$$H_2(y_1,y_2,y_3)=(y_2-y_3)\cdot h_2(y_1,y_2,y_3),$$

where

$$h_1(y_1,y_2,y_3)=-b_{22}(y_1+y_3)+\mathcal{O}(2), \tag{35}$$
$$h_2(y_1,y_2,y_3)=-b_{22}(y_2+y_3)+\mathcal{O}(2).$$

Thus, the equations $h_1(y_1,y_2,y_3)=0$ and $h_2(y_1,y_2,y_3)=0$ can be solved for $y_1=\tau_1(y_3)$ and $y_2=\tau_2(y_3)$ defined on a neighborhood of $y_3^0=0$ such that $\tau_i(0)=0$. From the factorization (34), we see that the equations $H_1(y_1,y_2,y_3)=0$ and $H_2(y_1,y_2,y_3)=0$ have four analytic branches at the neighborhood of $y_3^0=0$ given by

$$\begin{aligned}
&\text{(I)} \;\; y_1=y_3 \quad \text{and} \quad y_2=y_3,\\
&\text{(II)} \;\; y_1=-y_3+\mathcal{O}(2) \quad \text{and} \quad y_2=-y_3+\mathcal{O}(3),\\
&\text{(III)} \;\; y_1=y_3 \quad \text{and} \quad y_2=-y_3+\mathcal{O}(2),\\
&\text{(IV)} \;\; y_1=-y_3+\mathcal{O}(2) \quad \text{and} \quad y_2=y_3.
\end{aligned}$$

In order to write the bifurcation branches, we first introduce a small parameter t by setting $y_3=t$ and after we undo the translation of variables $y_i\mapsto y_i-k$. Writing the expressions of y_1, y_2 and ε in terms of t we have four families of central configurations given by the system $G(Y,\varepsilon)=0$ near the regular tetrahedron with masses $m_1=m_2=m_3=1$, $m_4=m^{**}$ and $m_5=1+\varepsilon$, where

$$\text{I:}\begin{cases} y_1=k+t,\\ y_2=k+t,\\ y_3=k+t,\\ \varepsilon=-\frac{8b_{22}}{b_5}t^2+\mathcal{O}(3),\\ y_4=\frac{-8c_5b_{22}+3b_5(c_{11}+c_{12})}{fb_5}t^2+\mathcal{O}(3),\end{cases} \qquad \text{II:}\begin{cases} y_1=k-t+\mathcal{O}(2),\\ y_2=k-t+\mathcal{O}(2),\\ y_3=k+t,\\ \varepsilon=\mathcal{O}(3),\\ y_4=\frac{3c_{11}-c_{12}}{f}t^2+\mathcal{O}(3),\end{cases}$$

$$\text{III:}\begin{cases} y_1 = k + t, \\ y_2 = k - t + \mathcal{O}(2), \\ y_3 = k + t, \\ \varepsilon = \mathcal{O}(3), \\ y_4 = \frac{3c_{11}-c_{12}}{f}t^2 + \mathcal{O}(3), \end{cases} \qquad \text{IV:}\begin{cases} y_1 = k - t + \mathcal{O}(2), \\ y_2 = k + t, \\ y_3 = k + t, \\ \varepsilon = \mathcal{O}(3), \\ y_4 = \frac{3c_{11}-c_{12}}{f}t^2 + \mathcal{O}(3). \end{cases}$$

The family I has the segment q_4q_5 as an axis of symmetry and it is defined for $\varepsilon < 0$. Again, we observe a bifurcation for the function $\varepsilon(t)$, since it has a local maximum at $t = 0$ we get that for each ε sufficiently small, there are two values $t_1 < 0 < t_2$ such that $\varepsilon(t_1) = \varepsilon(t_2)$ and $Y(t_1) \neq Y(t_2)$, that is, we have two central configurations for each $\varepsilon < 0$. On the other hand the families II, III, IV are essentially the same up to a reordering of the bodies at the vertices. According to the Remark 2, each of them has a plane of symmetry, e.g., the family II is symmetric with respect to the plane perpendicular to the segment q_1q_2 passing by its middle point.

With all the above we have the following result:

Theorem 4. *For every $a < -1$, the regular tetrahedron with vertices $m_1 = m_2 = m_3 = 1$, $m_5 = 1 + \varepsilon$ and a mass $m_4 = m^{**}$ at the barycenter, presents a bifurcation when the mass m_5 crosses the value 1. More precisely, there is a $\delta > 0$ such that for any $1 - \delta < m_5 < 1$ we have two central configurations coming from the bifurcation presenting a symmetry axis type and for any $m_5 \in (1 - \delta, 1 + \delta)$ one has three families of central configurations with symmetry plane type.*

We can draw the behavior of four bifurcations branches. Using that

$$s_{ij} = \left(\frac{\lambda}{m^{**} + 4 + \varepsilon(t)} - \frac{x_i x_j}{m_i m_j}\right)^{-2/3},$$

we write the power series expansion of t for the two families.

$$s^{\nu}_{ij}(\varepsilon) = s^0_{ij} + v^{\nu}_{ij}t + \mathcal{O}(t^2),$$

where $\nu = I, II, III, IV$.

$$\text{I:}\begin{cases} v^I_{12} = v^I_{13} = v^I_{23} = -\frac{2k\rho^{1-a}}{a} > 0, \\ v^I_{15} = v^I_{25} = v^I_{35} = \frac{2k\rho^{1-a}}{a} < 0, \\ v^I_{14} = v^I_{24} = v^I_{34} = \frac{4k}{am^{**}} < 0, \\ v^I_{45} = -\frac{12k}{am^{**}} > 0, \end{cases} \qquad \text{II:}\begin{cases} v^{II}_{13} = v^{II}_{23} = 0, \\ v^{II}_{12} = -\frac{2k\rho^{1-a}}{a} > 0, \\ v^{II}_{15} = v^{II}_{25} = 0, \\ v^{II}_{35} = \frac{2k\rho^{1-a}}{a} < 0, \\ v^{II}_{14} = v^{II}_{24} = \frac{4k}{am^{**}} < 0, \\ v^{II}_{34} = v^{II}_{45} = -\frac{4k}{am^{**}} > 0, \end{cases}$$

$$\text{III:}\begin{cases} v^{III}_{12} = v^{III}_{23} = 0, \\ v^{III}_{13} = -\frac{2k\rho^{1-a}}{a} > 0, \\ v^{III}_{15} = v^{III}_{35} = 0, \\ v^{III}_{25} = \frac{2k\rho^{1-a}}{a} < 0, \\ v^{III}_{14} = v^{III}_{34} = \frac{4k}{am^{**}} < 0, \\ v^{III}_{24} = v^{III}_{45} = -\frac{4k}{am^{**}} > 0, \end{cases} \qquad \text{IV:}\begin{cases} v^{IV}_{12} = v^{IV}_{13} = 0, \\ v^{IV}_{23} = -\frac{2k\rho^{1-a}}{a} > 0, \\ v^{IV}_{25} = v^{IV}_{35} = 0, \\ v^{IV}_{15} = \frac{2k\rho^{1-a}}{a} < 0, \\ v^{IV}_{24} = v^{IV}_{34} = \frac{4k}{am^{**}} < 0, \\ v^{IV}_{14} = v^{IV}_{45} = -\frac{4k}{am^{**}} > 0. \end{cases}$$

Table 4
Behavior of the two families of bifurcation as a function of parameter t in the tetrahedron case.

	Family I	Family II	Family III	Family IV
$\varepsilon(t)$	negative	indefinite	indefinite	indefinite
$s_{12}(t)$	increasing	increasing	indefinite	indefinite
$s_{13}(t)$	increasing	indefinite	increasing	indefinite
$s_{14}(t)$	decreasing	decreasing	decreasing	increasing
$s_{15}(t)$	decreasing	indefinite	indefinite	decreasing
$s_{23}(t)$	increasing	indefinite	indefinite	increasing
$s_{24}(t)$	decreasing	decreasing	increasing	decreasing
$s_{25}(t)$	decreasing	indefinite	decreasing	indefinite
$s_{34}(t)$	decreasing	increasing	decreasing	decreasing
$s_{35}(t)$	decreasing	decreasing	indefinite	indefinite
$s_{45}(t)$	increasing	increasing	increasing	increasing

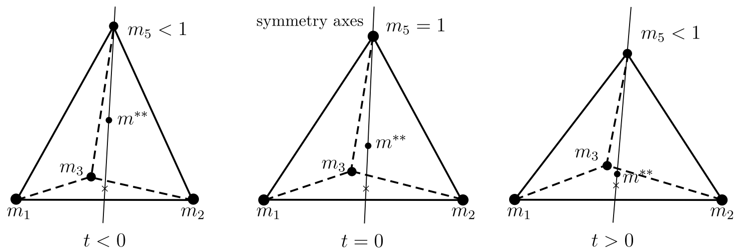

Fig. 7. Family I: with $m_1 = m_2 = m_3 = 1$, $m_4 = m^{**}$ and $m_5 = 1 + \varepsilon$, with symmetry type axis.

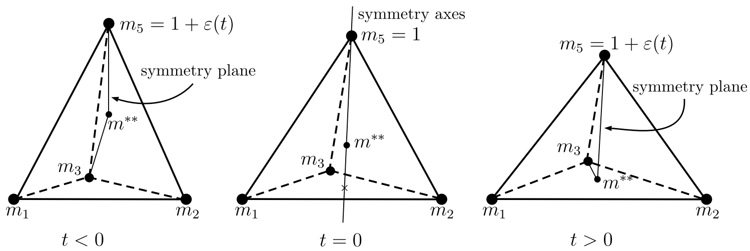

Fig. 8. Family II: with $m_1 = m_2 = m_3 = 1$, $m_4 = m^{**}$ and $m_5 \approx 1$, symmetric with respect the plane orthogonal to the segment q_1q_2.

In Table 4 we show the behavior of both families in terms of the sign of ε and the mutual distances. In Fig. 7 we show the two symmetric concave branches of central configurations coming from Family 1. In Fig. 8 we show the branches of concave central configurations coming from family II with its symmetry planes. For the families III and IV the shapes are the same up to a reordering of the bodies at the base of tetrahedron.

The central configurations coming from the bifurcation when the parameter is the mass at one of the vertices present the same symmetries occurring in the bifurcations found by Santos in [12] where the parameter of bifurcation was the mass at the barycenter.

5. Conclusions

We succeeded carry out the explicit calculations of central configurations near the equilateral triangle and the regular tetrahedron, both with a specific mass at their barycenter. In the analysis, the Lemma 2 allowed us write the complete factorization of the equations, thus we can draw the diagram of bifurcation. In the four body problem, in the neighborhood of the equilateral triangle with $m_1 = m_2 = 1$ at the vertices, $m_3 = m^*$ at the barycenter and m_4 near 1, we have two symmetric central configurations for each $m_4 < 1$ and two non-symmetric central configurations for each $m_4 > 1$. In the five body problem, in the neighborhood of the

Please cite this article in press as: A.A. Santos et al., Continuation and bifurcations of concave central configurations in the four and five body-problems for homogeneous force laws, J. Math. Anal. Appl. (2016), http://dx.doi.org/10.1016/j.jmaa.2016.09.055

regular tetrahedron with $m_1 = m_2 = m_3 = 1$ at the vertices, $m_4 = m^{**}$ at the barycenter and m_5 near 1, we have two central configurations with symmetry type axis for each $m_5 < 1$ and two central configurations with symmetry type plane for each $m_5 \approx 1$.

Acknowledgments

This project started when the authors agreed in the summer school of the math department at the University Federal of Minas Gerais, Brazil. We would like to thank FAPEMIG for partially supporting the development of this paper through the "PROJETO UNIVERSAL 20076". We also want to thank the Project 14 Math–AmSud 02 of International Cooperation.

References

[1] A. Albouy, On a paper of Moeckel on central configurations, Regul. Chaotic Dyn. 8 (2) (2003) 133–138.
[2] J. Bernat, J. Llibre, E. Perez-Chavela, On a the planar central configurations of the 4-body problem with three equal masses, Dyn. Contin. Discrete Impuls. Syst. 16 (2009) 1–3.
[3] N. Fayçal, On the classification of pyramidal central configurations, Proc. Amer. Math. Soc. 124 (1996) 249–258.
[4] M. Hampton, Stacked central configurations: new examples in the planar five-body problem, Nonlinearity 18 (2005) 2299–2304.
[5] S. Krantz, H. Parks, The Implicit Function Theorem, Birkhäuser, 2002.
[6] M. Kulesza, M. Marchesin, C. Vidal, Restricted rhomboidal five-body problem, J. Phys. A: Math. Theor. 44 (2011) 485204–485236.
[7] W. Longley, Some particular solutions in the problem of n bodies, Bull. Amer. Math. Soc. 13 (1906) 324–335.
[8] K. Meyer, Bifurcation of a central configuration, Celestial Mech. Dynam. Astronom. 40 (1987) 273–282.
[9] K. Meyer, D. Schmidt, Bifurcations of relative equilibria in the 4- and 5-body problem, Ergodic Theory Dynam. Systems 8 (1988) 215–225.
[10] J. Palmore, Relative equilibria of the n-body problem, Ph.D. thesis, 1973.
[11] E. Perez-Chavela, M. Santoprete, Convex four-body central configurations with some equal masses, Arch. Ration. Mech. Anal. 185 (2007) 481–494.
[12] A. Santos, Dziobek's configurations in restricted problems and bifurcations, Celestial Mech. Dynam. Astronom. 90 (2004) 213–238.
[13] A. Santos, C. Vidal, Symmetry of the restricted $4 + 1$ body problem with equal masses, Regul. Chaotic Dyn. 12 (2007) 27–38.
[14] D. Schmidt, Central configurations in $\mathbb{R}^2$ and $\mathbb{R}^3$, Contemp. Math. 81 (1988) 59–76.
[15] A. Wintner, The Analytical Foundations of Celestial Mechanics, Princeton Univ. Press, 1941.