



Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa



Application of geometric calculus in numerical analysis and difference sequence spaces

Khierod Boruah, Bipan Hazarika*

Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh 791112, Arunachal Pradesh, India

ARTICLE INFO

Article history:

Received 30 November 2015

Available online xxxx

Submitted by W.L. Wendland

Keywords:

Difference sequence spaces

Dual space

Geometric calculus

Interpolation formula

ABSTRACT

The main purpose of this paper is to introduce the geometric difference sequence space $\ell_\infty^G(\Delta_G)$ and prove that $\ell_\infty^G(\Delta_G)$ is a Banach space with respect to the norm $\|\cdot\|_{\Delta_G}^G$. Also we compute the α -dual, β -dual and γ -dual spaces. Finally we obtain the Geometric Newton–Gregory interpolation formulae.

© 2017 Elsevier Inc. All rights reserved.

1. Introduction and notations

In 1967 Robert Katz and Michael Grossman created the first system of non-Newtonian calculus, which we call the geometric calculus. In 1970 they had created an infinite family of non-Newtonian calculi, each of which differs markedly from the classical calculus of Newton and Leibniz. Among other things, each non-Newtonian calculus possesses four operators: a gradient (i.e. an average rate of change), a derivative, an average, and an integral. For each non-Newtonian calculus there is a characteristic class of functions having a constant derivative.

In view of pioneering work carried out in this area by Grossman and Katz [10] we will call this calculus as multiplicative calculus, although the term of exponential calculus can also be used. The operations of multiplicative calculus will be called as multiplicative derivative and multiplicative integral. We refer to Grossman and Katz [10], Stanley [18], Bashirov et al. [2,3], Grossman [9] for elements of multiplicative calculus and its applications. An extension of multiplicative calculus to functions of complex variables is handled by Bashirov and Rıza [1], Çakmak and Başar [5], Tekin and Başar [19], Türkmen and Başar [20], Uzer [22]. In [13], Kadak and Özlük studied the generalized Runge–Kutta method with respect to non-Newtonian calculus. Kadak and Efe [11] and Kadak et al. [12] studied certain new types of sequence

* Corresponding author.

E-mail addresses: khierodb10@gmail.com (K. Boruah), bh_rgu@yahoo.co.in (B. Hazarika).

spaces over the Non-Newtonian Complex Field. Çakmak and Başar [4] construct the field \mathbb{C}_* of $*$ -complex numbers and the concept of $*$ -metric. Also they give the definitions and the basic important properties of $*$ -boundedness and $*$ -continuity. Later the space $\mathbb{C}_*(\Omega)$ of $*$ -continuous functions which forms a vector space with respect to the non-Newtonian addition and scalar multiplication, and is a Banach space, is introduced. Finally, multiplicative calculus (MC) which is one of the most popular non-Newtonian calculi and created by the famous ‘*exp*’ function, is applied to complex numbers and functions to investigate some advance inner product properties together with the inclusion relationship between $\mathbb{C}_*(\Omega)$ and the set of $\mathbb{C}'_*(\Omega)$ -differentiable functions. The line and double integrals in the sense of non-Newtonian calculus ($*$ -calculus) are given by Çakmak and Başar [6]. Moreover, in the sense of $*$ -calculus, the fundamental theorem of calculus for line integrals and double integrals is stated with some applications. Based on multiplicative calculus matrix transformations in sequence spaces are studied and characterized by Çakmak and Başar [7]. Also, a brief introduction to $*$ -summability based on multiplicative type addition (or just multiplication) is given and the multiplicative dual $*$ -summability methods using $*$ -Stieltjes integral and multiplicative differentiation under the $*$ -integral sign are introduced. The problem of cylindrical wave incidence on a conducting half plane has been considered by Uzer [21]. A modal solution for Green’s function of the problem is transformed into contour integral representations in a complex plane. Some contour deformations and changes of variables are then made for the integrals. The multiplicative calculus is employed in deriving an expression that can be used for obtaining approximate solutions when the observation angles are away from the RSBs of the conducting half plane. The derived expressions are seen to be very simple for implementing in any computational environment.

Geometric calculus is an alternative to the usual calculus of Newton and Leibniz. It provides differentiation and integration tools based on multiplication instead of addition. Every property in Newtonian calculus has an analog in multiplicative calculus. Generally speaking multiplicative calculus is a methodology that allows one to have a different look at problems which can be investigated via calculus. In some cases, for example for growth related problems, the use of multiplicative calculus is advocated instead of a traditional Newtonian one.

The main aim of this paper is to construct the difference sequence space $\ell_\infty^G(\Delta_G)$ over geometric real numbers which forms a Banach space with the norm defined on it and obtain the Geometric Newton–Gregory interpolation formulae which are more useful than Newton–Gregory interpolation formulae in the ordinary sense.

Before we establish new results, we recall the construction of arithmetics generated by different generators and the geometric arithmetic, which is the keyword of the whole article.

2. α -generator and geometric real field

A *generator* is a one-to-one function whose domain is the set of real numbers \mathbb{R} , and range is a set $B \subset \mathbb{R}$. For example, the identity function I and the exponential function *exp* are generators. We consider any generator α with realm i.e. domain, say, A and range B , by α -*arithmetic*, we mean the arithmetic whose operations and ordering relation are defined as follows:

$$\begin{array}{ll} \alpha\text{-addition} & x \dot{+} y = \alpha[\alpha^{-1}(x) + \alpha^{-1}(y)] \\ \alpha\text{-subtraction} & x \dot{-} y = \alpha[\alpha^{-1}(x) - \alpha^{-1}(y)] \\ \alpha\text{-multiplication} & x \dot{\times} y = \alpha[\alpha^{-1}(x) \times \alpha^{-1}(y)] \\ \alpha\text{-division} & x \dot{/} y = \alpha[\alpha^{-1}(x)/\alpha^{-1}(y)] \\ \alpha\text{-order} & x \dot{<} y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y) \end{array}$$

for $x, y \in A$, where A is a domain of the function α .

We should know that all concepts in classical arithmetic have natural counterparts in α -arithmetic. Each generator generates exactly one arithmetic and each arithmetic is generated by exactly one generator. For example, the identity function generates classical arithmetic, and exponential function generates geometric arithmetic. That is, if we choose \exp as an α -generator defined by $\alpha(x) = e^x$ for $x \in \mathbb{R}$ then $\alpha^{-1}(x) = \ln x$ and α -arithmetic turns out to be geometric arithmetic.

α -addition	$x \oplus y = \alpha[\alpha^{-1}(x) + \alpha^{-1}(y)] = e^{(\ln x + \ln y)} = x \cdot y$	geometric addition
α -subtraction	$x \ominus y = \alpha[\alpha^{-1}(x) - \alpha^{-1}(y)] = e^{(\ln x - \ln y)} = \frac{x}{y}, y \neq 0$	geometric subtraction
α -multiplication	$x \odot y = \alpha[\alpha^{-1}(x) \times \alpha^{-1}(y)] = e^{(\ln x \times \ln y)} = x^{\ln y}$	geometric multiplication
α -division	$x \oslash y = \alpha[\alpha^{-1}(x)/\alpha^{-1}(y)] = e^{(\ln x \div \ln y)} = x^{\frac{1}{\ln y}}, y \neq 1$	geometric division.

It is obvious that $\ln(x) < \ln(y)$ if $x < y$ for $x, y \in \mathbb{R}^+$. That is, $x < y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y)$. So, without loss of generality, we use $x < y$ instead of the geometric order $x \dot{<} y$.

Türkmen and Başar [20] defined the sets of geometric integers, geometric real numbers and geometric complex numbers $\mathbb{Z}(G), \mathbb{R}(G)$ and $\mathbb{C}(G)$, respectively, as follows:

$$\begin{aligned}\mathbb{Z}(G) &= \{e^x : x \in \mathbb{Z}\} \\ \mathbb{R}(G) &= \{e^x : x \in \mathbb{R}\} = \mathbb{R}^+ \setminus \{0\} \\ \mathbb{C}(G) &= \{e^z : z \in \mathbb{C}\} = \mathbb{C} \setminus \{0\}.\end{aligned}$$

If we take extended real number line, then $\mathbb{R}(G) = [0, \infty]$.

Remark 2.1. $(\mathbb{R}(G), \oplus, \odot)$ is a field with geometric zero 1 and geometric identity e , since

- (1) $(\mathbb{R}(G), \oplus)$ is a geometric additive Abelian group with geometric zero 1,
- (2) $(\mathbb{R}(G) \setminus \{1\}, \odot)$ is a geometric multiplicative Abelian group with geometric identity e ,
- (3) \odot is distributive over \oplus .

But $(\mathbb{C}(G), \oplus, \odot)$ is not a field, however, geometric binary operation \odot is not associative in $\mathbb{C}(G)$. For, we take $x = e^{1/4}, y = e^4$ and $z = e^{(1+i\pi/2)} = ie$. Then $(x \odot y) \odot z = e \odot z = z = ie$ but $x \odot (y \odot z) = x \odot e^4 = e$.

Let us define geometric positive real numbers and geometric negative real numbers as follows:

$$\begin{aligned}\mathbb{R}^+(G) &= \{x \in \mathbb{R}(G) : x > 1\}; \\ \mathbb{R}^-(G) &= \{x \in \mathbb{R}(G) : x < 1\}.\end{aligned}$$

2.1. Some useful relations between geometric operations and ordinary arithmetic operations

For all $x, y \in \mathbb{R}(G)$

- $x \oplus y = xy$.
- $x \ominus y = x/y$.
- $x \odot y = x^{\ln y} = y^{\ln x}$.
- $\left(\frac{a}{b}\right) \odot y = \left(\frac{a}{b}\right)^{\ln y} = \frac{a^{\ln y}}{b^{\ln y}} = \frac{a \odot y}{b \odot y}$.
- $\frac{a}{b} \odot \frac{c}{d} = \left(\frac{a}{b}\right)^{\ln(\frac{c}{d})} = \left(\frac{b}{a}\right)^{-\ln(\frac{c}{d})} = \left(\frac{b}{a}\right)^{\ln(\frac{d}{c})} = \frac{b}{a} \odot \frac{d}{c}$.
- $x \oslash y$ or $\frac{x}{y} \oslash = x^{\frac{1}{\ln y}}, y \neq 1$.
- $x^{2G} = x \odot x = x^{\ln x}$.

- $x^{p_G} = x^{\ln^{p-1} x}$.
- $\sqrt{x}^G = e^{(\ln x)^{\frac{1}{2}}}$.
- $x^{-1_G} = e^{\frac{1}{\log x}}$.
- $x \odot e = x$ and $x \oplus 1 = x$.
- $e^n \odot x = x \oplus x \oplus \dots$ (upto n number of x) $= x^n$.
-

$$|x|^G = \begin{cases} x, & \text{if } x > 1; \\ 1, & \text{if } x = 1; \\ \frac{1}{x}, & \text{if } 0 < x < 1. \end{cases}$$

- Thus $|x|^G \geq 1$.
- $\sqrt{x^{2_G}}^G = |x|^G$.
 - $|e^y|^G = e^{|y|}$.
 - $|x \odot y|^G = |x|^G \odot |y|^G$.
 - $|x \oplus y|^G \leq |x|^G \oplus |y|^G$.
 - $|x \otimes y|^G = |x|^G \otimes |y|^G$.
 - $|x \ominus y|^G \geq |x|^G \ominus |y|^G$.
 - $0_G \ominus 1_G \odot (x \ominus y) = y \ominus x$, i.e. in short $\ominus (x \ominus y) = y \ominus x$.

Further $e^{-x} = \ominus e^x$ holds for all $x \in \mathbb{Z}^+$. Thus the set of all geometric integers turns out to the following:

$$\mathbb{Z}(G) = \{\dots, e^{-3}, e^{-2}, e^{-1}, e^0, e^1, e^2, e^3, \dots\} = \{\dots, \ominus e^3, \ominus e^2, \ominus e, 1, e, e^2, e^3, \dots\}.$$

A sequence space X is called a FK -space if it is complete linear metric space with continuous coordinates $p_n : X \rightarrow \mathbb{R} (n \in \mathbb{N})$, where \mathbb{R} denotes the real field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$. A BK -space is a normed FK -space, that is a BK -space is a Banach space with continuous coordinates.

Let ℓ_∞, c and c_0 be the linear spaces of real bounded, convergent and null sequences, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|.$$

Then

$$\omega(G) = \{(x_k) : x_k \in \mathbb{R}(G) \text{ for all } k \in \mathbb{N}\}$$

is a vector space over $\mathbb{R}(G)$ with respect to the algebraic operations addition \oplus and scalar multiplication \odot

$$\oplus : \omega(G) \times \omega(G) \rightarrow \omega(G)$$

$$(x, y) \rightarrow x \oplus y = (x_k) \oplus (y_k) = (x_k y_k)$$

$$\odot : \mathbb{R}(G) \times \omega(G) \rightarrow \omega(G)$$

$$(\alpha, y) \rightarrow \alpha \odot y = \alpha \odot (y_k) = (\alpha^{\ln y_k}),$$

where $x = (x_k), y = (y_k) \in \omega(G)$ and $\alpha \in \mathbb{R}$. Then

$$\ell_\infty(G) = \{x = (x_k) \in \omega(G) : \sup_{k \in \mathbb{N}} |x_k|^G < \infty\}$$

$$c(G) = \{x = (x_k) \in \omega(G) : \lim_{k \rightarrow \infty} |x_k \ominus l|^G = 1\}$$

$c_0(G) = \{x = (x_k) \in \omega(G) : {}_G\lim_{k \rightarrow \infty} x_k = 1\}$, where ${}_G\lim$ is the geometric limit

$\ell_p(G) = \{x = (x_k) \in \omega(G) : {}_G\sum_{k=0}^{\infty} (|x_k|^G)^{p_G} < \infty\}$, where ${}_G\sum$ is the geometric sum,

are classical sequence spaces over the field $\mathbb{R}(G)$. Also it is shown that $\ell_\infty(G)$, $c(G)$ and $c_0(G)$ are Banach spaces with the norm

$$\|x\|_\infty^G = \sup_k |x_k|^G, x = (x_1, x_2, x_3, \dots) \in \lambda(G), \lambda \in \{\ell_\infty, c, c_0\}.$$

For the convenience, in this paper we denote $\ell_\infty(G)$, $c(G)$, $c_0(G)$, respectively as ℓ_∞^G , c^G , c_0^G .

3. Main results: geometric sequence space

In 1981, Kizmaz [14] introduced the notion of difference sequence spaces using forward difference operator Δ and studied the classical difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$. In this section we define the following new geometric sequence space

$$\ell_\infty^G(\Delta_G) = \{x = (x_k) \in \omega(G) : \Delta_G x \in \ell_\infty^G\}, \text{ where } \Delta_G x = x_k \ominus x_{k+1}.$$

Theorem 3.1. *The space $\ell_\infty^G(\Delta_G)$ is a normed linear space with respect to the norm*

$$\|x\|_{\Delta_G}^G = |x_1|^G \oplus \|\Delta_G x\|_\infty^G.$$

Proof. For $x = (x_k), y = (y_k) \in \ell_\infty^G(\Delta_G)$,

N1.

$$\begin{aligned} \|x\|_{\Delta_G}^G &= |x_1|^G \oplus \|\Delta_G x\|_\infty^G \\ &= |x_1|^G \cdot \sup_k |x_k \ominus x_{k+1}|^G \\ &\geq 1, \quad \text{since } |x_1|^G \geq 1 \text{ and } |x_k \ominus x_{k+1}|^G \geq 1. \end{aligned}$$

N2.

$$\begin{aligned} \|x\|_{\Delta_G}^G = 1 &\Leftrightarrow |x_1|^G \oplus \|\Delta_G x\|_\infty^G = 1 \\ &\Leftrightarrow |x_1|^G \cdot \sup_k |x_k \ominus x_{k+1}|^G = 1, \quad \forall k \\ &\Leftrightarrow |x_1|^G = 1 \text{ and } |x_k \ominus x_{k+1}|^G = 1 \\ &\Leftrightarrow x_1 = 1 \text{ and } x_k \ominus x_{k+1} = 1, \quad \forall k \\ &\Leftrightarrow x_1 = 1 \text{ and } x_k/x_{k+1} = 1, \quad \forall k \\ &\Leftrightarrow x_1 = 1 \text{ and } x_k = x_{k+1}, \quad \forall k \\ &\Leftrightarrow x_k = 1, \quad \forall k \\ &\Leftrightarrow x = (1, 1, 1, 1, \dots) = 1_G. \end{aligned}$$

N3.

$$\begin{aligned}
\|x \oplus y\|_{\Delta_G}^G &= |x_1 \oplus y_1|^G \oplus \|\Delta_G(x_k \oplus y_k)\|_{\infty}^G \\
&= |x_1 \oplus y_1|^G \oplus \|\Delta_G(x_k y_k)\|_{\infty}^G \\
&= |x_1 \oplus y_1|^G \oplus \sup_k |x_k y_k \ominus x_{k+1} y_{k+1}|^G \\
&= |x_1 \oplus y_1|^G \oplus \sup_k \left| \frac{x_k y_k}{x_{k+1} y_{k+1}} \right|^G \\
&= |x_1 \oplus y_1|^G \oplus \sup_k \left| \frac{x_k}{x_{k+1}} \cdot \frac{y_k}{y_{k+1}} \right|^G \\
&= |x_1 \oplus y_1|^G \oplus \sup_k \left| \frac{x_k}{x_{k+1}} \oplus \frac{y_k}{y_{k+1}} \right|^G \\
&\leq |x_1 \oplus y_1|^G \oplus \sup_k \left\{ \left| \frac{x_k}{x_{k+1}} \right|^G \oplus \left| \frac{y_k}{y_{k+1}} \right|^G \right\} \\
&= |x_1 \oplus y_1|^G \oplus \sup_k \{ |x_k \ominus x_{k+1}|^G \oplus |y_k \ominus y_{k+1}|^G \} \\
&= |x_1 \oplus y_1|^G \oplus \sup_k \{ |\Delta_G x|^G \oplus |\Delta_G y|^G \} \\
&\leq |x_1|^G \oplus |y_1|^G \oplus \sup_k \{ |\Delta_G x|^G \} \oplus \sup_k \{ |\Delta_G y|^G \} \\
&= \left[|x_1|^G \oplus \sup_k \{ |\Delta_G x|^G \} \right] \oplus \left[|y_1|^G \oplus \sup_k \{ |\Delta_G y|^G \} \right] \\
&= \|x\|_{\Delta_G}^G \oplus \|y\|_{\Delta_G}^G.
\end{aligned}$$

N4.

$$\begin{aligned}
\|\alpha \odot x\|_{\Delta_G}^G &= |\alpha \odot x_1|^G \oplus \|\Delta_G(\alpha \odot x)\|_{\infty}^G, \quad \alpha \in \mathbb{C}(G) \\
&= |\alpha| \odot |x_1|^G \oplus \|\alpha \odot x_k \ominus \alpha \odot x_{k+1}\|_{\infty}^G \\
&= |\alpha| \odot |x_1|^G \oplus \|\alpha \odot (x_k \ominus x_{k+1})\|_{\infty}^G \\
&= |\alpha| \odot |x_1|^G \oplus |\alpha| \odot \|x_k \ominus x_{k+1}\|_{\infty}^G \\
&= |\alpha| \odot [|x_1|^G \oplus \|\Delta_G x\|_{\infty}^G] \\
&= |\alpha| \odot \|x\|_{\Delta_G}^G.
\end{aligned}$$

Thus $\|\cdot\|_{\Delta_G}^G$ is a norm on $\mathbb{R}(G)$. \square

Theorem 3.2. The space $\ell_{\infty}^G(\Delta_G)$ is a Banach space with respect to the norm $\|\cdot\|_{\Delta_G}^G$.

Proof. Let (x_n) be a Cauchy sequence in $\ell_{\infty}^G(\Delta_G)$, where

$$x_n = \left(x_k^{(n)} \right) = \left(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \dots \right)$$

$\forall n \in \mathbb{N}$, $x_k^{(n)}$ is the k^{th} coordinate of x_n . Then

$$\begin{aligned}
\|x_n \ominus x_m\|_{\Delta_G}^G &= \left| x_1^{(n)} \ominus x_1^{(m)} \right|^G \oplus \|\Delta_G x_n \ominus \Delta_G x_m\|_{\infty}^G \rightarrow 1 \text{ as } m, n \rightarrow \infty \\
&= \left| x_1^{(n)} \ominus x_1^{(m)} \right|^G \oplus \left\| (x_k^{(n)} \ominus x_{k+1}^{(n)}) \ominus (x_k^{(m)} \ominus x_{k+1}^{(m)}) \right\|_{\infty}^G \rightarrow 1 \\
&= \left| x_1^{(n)} \ominus x_1^{(m)} \right|^G \oplus \left\| (x_k^{(n)} \ominus x_k^{(m)}) \ominus (x_{k+1}^{(n)} \ominus x_{k+1}^{(m)}) \right\|_{\infty}^G \rightarrow 1 \\
&= \left| x_1^{(n)} \ominus x_1^{(m)} \right|^G \oplus \sup_k \left| (x_k^{(n)} \ominus x_k^{(m)}) \ominus (x_{k+1}^{(n)} \ominus x_{k+1}^{(m)}) \right|^G \rightarrow 1 \text{ as } m, n \rightarrow \infty.
\end{aligned}$$

This implies that $\left| x_k^{(n)} \ominus x_k^{(m)} \right|^G \rightarrow 1$ as $n, m \rightarrow \infty$, $\forall k \in \mathbb{N}$, since $\left| x_k^{(n)} \ominus x_k^{(m)} \right|^G \geq 1$.

Therefore for fixed k , k th coordinates of all sequences form a Cauchy sequence in $\mathbb{R}(G)$ i.e. $x_k^{(n)} = (x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, x_k^{(4)}, \dots)$ is a Cauchy sequence. Then by the completeness of $\mathbb{R}(G)$, $(x_k^{(n)})$ converges to x_k (say) as follows:

$$\begin{array}{cccccc}
x_1 & = & (& x_1^{(1)}, & x_2^{(1)}, & x_3^{(1)}, & \cdots, & x_k^{(1)}, & \cdots) \\
x_2 & = & (& x_1^{(2)}, & x_2^{(2)}, & x_3^{(2)}, & \cdots, & x_k^{(2)}, & \cdots) \\
x_3 & = & (& x_1^{(3)}, & x_2^{(3)}, & x_3^{(3)}, & \cdots, & x_k^{(3)}, & \cdots) \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots & \\
x_m & = & (& x_1^{(m)}, & x_2^{(m)}, & x_3^{(m)}, & \cdots, & x_k^{(m)}, & \cdots) \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots & \\
x_n & = & (& x_1^{(n)}, & x_2^{(n)}, & x_3^{(n)}, & \cdots, & x_k^{(n)}, & \cdots) \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots & \\
\downarrow & & & \downarrow & \downarrow & \downarrow & & \downarrow & \\
x & = & (& x_1, & x_2, & x_3, & \cdots, & x_k, & \cdots).
\end{array}$$

That is

$${}_G \lim_{n \rightarrow \infty} x_k^{(n)} = x_k, \forall k \in \mathbb{N}.$$

Further for each $\varepsilon > 1$, $\exists N = N(\varepsilon)$ such that for all $n, m \geq N$, we have

$$|x_1^{(n)} \ominus x_1^{(m)}|^G < \varepsilon, |x_{k+1}^{(n)} \ominus x_{k+1}^{(m)} \ominus (x_k^{(n)} \ominus x_k^{(m)})|^G < \varepsilon$$

and

$${}_G \lim_{m \rightarrow \infty} |x_1^{(n)} \ominus x_1^{(m)}|^G = |x_1^{(n)} \ominus x_1|^G < \varepsilon.$$

This implies

$${}_G \lim_{m \rightarrow \infty} |(x_{k+1}^{(n)} \ominus x_{k+1}^{(m)}) \ominus (x_k^{(n)} \ominus x_k^{(m)})|^G = |(x_{k+1}^{(n)} \ominus x_{k+1}) \ominus (x_k^{(n)} \ominus x_k)|^G < \varepsilon, \forall n \geq N.$$

Since ε is independent of k ,

$$\begin{aligned}
&\sup_k |(x_{k+1}^{(n)} \ominus x_{k+1}) \ominus (x_k^{(n)} \ominus x_k)|^G < \varepsilon \\
&\Rightarrow \sup_k |(x_{k+1}^{(n)} \ominus x_k^{(n)}) \ominus (x_{k+1} \ominus x_k)|^G = \|\Delta_G x_n \ominus \Delta_G x\|_{\infty}^G < \varepsilon.
\end{aligned}$$

Consequently we have $\|x_n \ominus x\|_{\Delta_G}^G = |x_1^{(n)} \ominus x_1|^G \oplus \|\Delta_G x_n \ominus \Delta_G x\|_{\infty}^G < \varepsilon^2, \forall n \geq N$.

Hence we obtain $x_n \rightarrow x$ as $n \rightarrow \infty$.

Now we must show that $x \in \ell_{\infty}^G(\Delta_G)$. We have

$$\begin{aligned} |x_k \ominus x_{k+1}|^G &= |x_k \ominus x_k^N \oplus x_k^N \ominus x_{k+1}^N \oplus x_{k+1}^N \ominus x_{k+1}|^G \\ &\leq |x_k^N \ominus x_{k+1}^N|^G \oplus \|x^N \ominus x\|_{\Delta_G}^G = O(e). \end{aligned}$$

This implies $x = (x_k) \in \ell_{\infty}^G(\Delta_G)$. \square

Furthermore since $\ell_{\infty}^G(\Delta_G)$ is a Banach space with continuous coordinates (that is $\|x_n \ominus x\|_{\Delta_G}^{\infty} \rightarrow 1$ implies $|x_k^{(n)} \ominus x_k|^G \rightarrow 1$ for each $k \in \mathbb{N}$, as $n \rightarrow \infty$), it is a BK-space.

Remark 3.3. The spaces

- (a) $c^G(\Delta_G) = \{(x_k) \in \omega(G) : \Delta_G x_k \in c^G\}$
- (b) $c_0^G(\Delta_G) = \{(x_k) \in \omega(G) : \Delta_G x_k \in c_0^G\}$

are Banach spaces with respect to the norm $\|\cdot\|_{\Delta_G}^G$. Also these spaces are BK-space.

Now we define $s : \ell_{\infty}^G(\Delta_G) \rightarrow \ell_{\infty}^G(\Delta_G), x \rightarrow sx = y = (1, x_2, x_3, \dots)$. It is clear that s is a bounded linear operator on $\ell_{\infty}^G(\Delta_G)$ and $\|s\|_{\infty}^G = e$. Also

$$s[\ell_{\infty}^G(\Delta_G)] = s\ell_{\infty}^G(\Delta_G) = \{x = (x_k) : x \in \ell_{\infty}^G(\Delta_G), x_1 = 1\} \subset \ell_{\infty}^G(\Delta_G)$$

is a subspace of $\ell_{\infty}^G(\Delta_G)$ and as $|x_1|^G = 1$ for $x_1 = 1$ we have

$$\|x\|_{\Delta_G}^G = \|\Delta_G x\|_{\infty}^G \quad \text{in } s\ell_{\infty}^G(\Delta_G).$$

On the other hand we can show that

$$\begin{aligned} \Delta_G : s\ell_{\infty}^G(\Delta_G) &\rightarrow \ell_{\infty}^G \\ x = (x_k) &\rightarrow y = (y_k) = (x_k \ominus x_{k+1}) \end{aligned} \tag{3.1}$$

is a linear homomorphism. So $s\ell_{\infty}^G(\Delta_G)$ and ℓ_{∞}^G are equivalent as topological space. Δ_G and Δ_G^{-1} are norm preserving and $\|\Delta_G\|_{\infty}^G = \|\Delta_G^{-1}\|_{\infty}^G = e$.

Let $[s\ell_{\infty}^G(\Delta_G)]^*$ and $[\ell_{\infty}^G]^*$ denote the continuous duals of $s\ell_{\infty}^G(\Delta_G)$ and ℓ_{∞}^G , respectively.

We can prove that

$$T : [s\ell_{\infty}^G(\Delta_G)]^* \rightarrow [\ell_{\infty}^G]^*, f_{\Delta_G} \rightarrow f = f_{\Delta_G} \circ \Delta_G^{-1}$$

is a linear isometry. Thus $[s\ell_{\infty}^G(\Delta_G)]^*$ is equivalent to $[\ell_{\infty}^G]^*$. In the same way we can show that $sc^G(\Delta_G)$ and c^G , $sc_0^G(\Delta_G)$ and c_0^G are equivalent as topological spaces and

$$[sc^G(\Delta_G)]^* = [sc_0^G(\Delta_G)]^* = \ell_1^G(\ell_1^G, \text{ the space of geometric absolutely convergent series}).$$

4. Dual spaces of $\ell_{\infty}^G(\Delta_G)$

Lemma 4.1. *The following statements are equivalent:*

- (a) $\sup_k |x_k \ominus x_{k+1}|^G < \infty$ i.e. $\sup_k |\Delta_G x_k|^G < \infty$;
- (b)(i) $\sup_k e^{k-1} \odot |x_k|^G < \infty$ and
- (ii) $\sup_k |x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1}|^G < \infty$.

Proof. Let (a) be true i.e. $\sup_k |x_k \ominus x_{k+1}|^G < \infty$.

Now

$$\begin{aligned} |x_1 \ominus x_{k+1}|^G &= \left| \sum_{v=1}^k (x_v \ominus x_{v+1}) \right|^G \\ &= \left| \sum_{v=1}^k \Delta_G x_v \right|^G \\ &\leq \sum_{v=1}^k |\Delta_G x_v|^G = O(e^k) \end{aligned}$$

and

$$\begin{aligned} |x_k|^G &= |x_1 \ominus x_1 \oplus x_{k+1} \oplus x_k \ominus x_{k+1}|^G \\ &\leq |x_1|^G \oplus |x_1 \ominus x_{k+1}|^G \oplus |x_k \ominus x_{k+1}|^G = O(e^k). \end{aligned}$$

This implies that $\sup_k e^{k-1} \odot |x_k|^G < \infty$. This completes the proof of Part (i) of (b).

Again

$$\begin{aligned} \sup_k \left| x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1} \right|^G &= \left| \left\{ e^{(k+1)} \odot e^{(k+1)^{-1}} \right\} \odot x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1} \right|^G \\ &= \left| \left\{ (e^k \oplus e) \odot e^{(k+1)^{-1}} \right\} \odot x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1} \right|^G \\ &= \left| \left\{ e^{k(k+1)^{-1}} \odot x_k \oplus e^{(k+1)^{-1}} \odot x_k \right\} \ominus e^{k(k+1)^{-1}} \odot x_{k+1} \right|^G \\ &= \left| \left\{ e^{k(k+1)^{-1}} \odot (x_k \ominus x_{k+1}) \right\} \oplus \left\{ e^{(k+1)^{-1}} \odot x_k \right\} \right|^G \\ &\leq e^{k(k+1)^{-1}} \odot |x_k \ominus x_{k+1}|^G \oplus e^{(k+1)^{-1}} \odot |x_k|^G = O(e). \end{aligned}$$

Therefore $\sup_k |x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1}|^G < \infty$. This completes the proof of Part (ii) of (b).

Conversely let (b) be true. Then

$$\begin{aligned} \left| x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1} \right|^G &= \left| e^{(k+1)(k+1)^{-1}} \odot x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1} \right|^G \\ &\geq e^{k(k+1)^{-1}} \odot |x_k \ominus x_{k+1}|^G \ominus e^{(k+1)^{-1}} \odot |x_k|^G \end{aligned}$$

$$\text{i.e. } e^{k(k+1)^{-1}} \odot |x_k \ominus x_{k+1}|^G \leq e^{(k+1)^{-1}} \odot |x_k|^G \oplus \left| x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1} \right|^G.$$

Thus $\sup_k |x_k \ominus x_{k+1}|^G < \infty$ as Parts (i) and (ii) of (b) hold. \square

Geometric form of Abel's partial summation formula Abel's partial summation formula states that if (a_k) and (b_k) are sequences, then

$$\sum_{k=1}^n a_k b_k = \sum_{k=1}^n S_k (b_k - b_{k+1}) + S_n b_{n+1},$$

where $S_k = \sum_{i=1}^k a_i$. Then

$$\begin{aligned} \sum_{k=1}^{\infty} a_k b_k &= \sum_{k=1}^{\infty} S_k (b_k - b_{k+1}) + \lim_{n \rightarrow \infty} S_n b_{n+1} \\ \sum_{k=1}^{\infty} a_k b_k &= \sum_{k=1}^{\infty} S_k (b_k - b_{k+1}), \text{ if } (b_k) \text{ monotonically decreases to zero.} \end{aligned}$$

Similarly as \odot is distributive over \oplus we have

$${}_G \sum_{k=1}^{\infty} a_k \odot b_k = {}_G \sum_{k=1}^{\infty} S_k \odot (b_k \ominus b_{k+1}), \text{ where } S_k = {}_G \sum_{i=1}^k a_i.$$

In particular, if $(b_k) = (e^{-k})$, then (b_k) monotonically decreases to zero. Then

$$\begin{aligned} {}_G \sum_{k=1}^{\infty} a_k \odot e^{-k} &= {}_G \sum_{k=1}^{\infty} S_k \odot (e^{-k} \ominus e^{-(k+1)}) \\ &= {}_G \sum_{k=1}^{\infty} S_k \odot e = {}_G \sum_{k=1}^{\infty} S_k. \end{aligned}$$

Let (p_n) be a sequence of geometric positive numbers monotonically increasing to infinity. Then $(\frac{e}{p_n})_G$ is a sequence monotonically decreasing to zero (i.e. to 1).

Lemma 4.2.

$$\text{If } \sup_n \left| {}_G \sum_{v=1}^n c_v \right|^G \leq \infty, \text{ then } \sup_n \left(p_n \odot \left| {}_G \sum_{k=1}^{\infty} \frac{c_{n+k-1}}{p_{n+k}} \right|^G \right) < \infty.$$

Proof. Using this Abel's partial summation formula to (c_v) and $(\frac{e}{p_n})_G$ we get

$${}_G \sum_{k=1}^{\infty} \frac{c_{n+k-1}}{p_{n+k}} = {}_G \sum_{k=1}^{\infty} \left({}_G \sum_{v=1}^k c_{n+v-1} \right) \odot \left(\frac{e}{p_{n+k}} \ominus \frac{e}{p_{n+k+1}} \right) \quad (4.1)$$

and

$$p_n \odot \left| {}_G \sum_{k=1}^{\infty} \frac{c_{n+k-1}}{p_{n+k}} \right|^G = O(e). \quad \square$$

Lemma 4.3. If the series ${}_G \sum_{k=1}^{\infty} c_k$ is convergent, then

$$\lim_n \left(p_n \odot {}_G \sum_{k=1}^{\infty} \frac{c_{n+k-1}}{p_{n+k}} \right) = 1.$$

Proof. Since

$$\left| \sum_{v=1}^k c_{n+v-1} \right|_G^G = \left| \sum_{v=n}^{n+k-1} c_v \right|_G^G = O(e) \text{ for every } k \in \mathbb{N}.$$

Using (4.1) we get

$$p_n \odot \left| \sum_{k=1}^{\infty} \frac{c_{n+k-1}}{p_{n+k}} \right|_G^G = O(e). \quad \square$$

Corollary 4.4. Let (p_n) be monotonically increasing. If

$$\sup_n \left| \sum_{v=1}^n p_v \odot a_v \right|_G^G < \infty, \text{ then } \sup_n \left| p_n \odot \sum_{k=n+1}^{\infty} a_k \right|_G^G < \infty.$$

Proof. We put $p_{k+1} \odot a_{k+1}$ instead of c_k in Lemma 4.2, we get

$$\begin{aligned} p_n \odot \sum_{k=1}^{\infty} \frac{c_{n+k-1}}{p_{n+k}} &= p_n \odot \sum_{k=1}^{\infty} \frac{p_{n+k} \odot a_{n+k}}{p_{n+k}} \\ &= p_n \odot \sum_{k=1}^{\infty} a_{n+k} \\ &= p_n \odot \sum_{k=n+1}^{\infty} a_k = O(e). \quad \square \end{aligned}$$

Corollary 4.5.

$$\text{If } \sum_{k=1}^{\infty} p_k \odot a_k \text{ is convergent, then } \lim_n p_n \odot \sum_{k=n+1}^{\infty} a_k = 1.$$

Proof. We put $p_{k+1} \odot a_{k+1}$ instead of c_k in Lemma 4.3. \square

Corollary 4.6.

$$\begin{aligned} \sum_{k=1}^{\infty} e^k \odot a_k \text{ is convergent iff } \sum_{k=1}^{\infty} R_k \text{ is convergent with } e^n \odot R_n = O(e), \text{ where} \\ R_n = \sum_{k=n+1}^{\infty} a_k. \end{aligned}$$

Proof. Let $p_n = e^n$. Then it is monotonically increasing to infinity. Therefore we have

$$\begin{aligned} \sum_{k=1}^n e^k \odot a_{k+1} &= e \odot a_2 \oplus e^2 \odot a_3 \oplus e^3 \odot a_4 \oplus \dots \oplus e^n \odot a_{n+1} \\ &= (a_2 \oplus a_3 \oplus \dots \oplus a_{n+1}) \oplus (a_3 \oplus a_4 \oplus \dots \oplus a_{n+1}) \\ &\quad \oplus \dots \oplus (a_n \oplus a_{n+1}) \oplus (a_{n+1}) \\ &= (R_1 \ominus R_{n+1}) \oplus (R_2 \ominus R_{n+1}) \oplus \dots \oplus (R_{n-1} \ominus R_{n+1}) \oplus (R_n \ominus R_{n+1}) \end{aligned}$$

$$= {}_G \sum_{k=1}^n R_k \odot \{e^n \odot R_{n+1}\}.$$

Therefore, as $e^n \odot R_n = O(e)$, so $e^n \odot R_{n+1} = O(e)$. This implies

$${}_G \sum_{k=1}^n e^k \odot a_{k+1} \text{ is convergent if } {}_G \sum_{k=1}^n R_k \text{ is convergent and vice versa.} \quad \square$$

5. α -, β -, γ -duals

Definition 5.1. [8,15–17] If X is a sequence space, we define

- (i) $X^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for each } x \in X\};$
- (ii) $X^\beta = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for each } x \in X\};$
- (iii) $X^\gamma = \{a = (a_k) : \sup_n \left| \sum_{k=1}^n a_k x_k \right| < \infty, \text{ for each } x \in X\}.$

X^α, X^β , and X^γ are called α - (or Köthe–Toeplitz), β - (or generalized Köthe–Toeplitz), and γ -dual spaces of X . We can show that $X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$, then $Y^\dagger \subset X^\dagger$, for $\dagger = \alpha, \beta$ or γ .

Theorem 5.2.

$$(i) \quad \text{If } D_1 = \left\{ a = (a_k) : {}_G \sum_{k=1}^{\infty} e^k \odot |a_k|^G < \infty \right\}, \text{ then } (s\ell_{\infty}^G(\Delta_G))^\alpha = D_1.$$

$$(ii) \quad \text{If } D_2 = \left\{ a = (a_k) : {}_G \sum_{k=1}^{\infty} e^k \odot a_k \text{ is convergent with } {}_G \sum_{k=1}^{\infty} |R_k|^G < \infty \right\},$$

then $(s\ell_{\infty}^G(\Delta_G))^\beta = D_2$.

$$(iii) \quad \text{Define the set } D_3 \text{ by } D_3 = \left\{ a = (a_k) : \sup_n \left| {}_G \sum_{k=1}^n e^k \odot a_k \right|^G < \infty, {}_G \sum_{k=1}^{\infty} |R_k|^G < \infty \right\}.$$

Then $(s\ell_{\infty}^G(\Delta_G))^\gamma = D_3$.

Proof. (i) Let $a \in D_1$. Then for each $x \in s\ell_{\infty}^G(\Delta_G)$ we have

$${}_G \sum_{k=1}^{\infty} |a_k \odot x_k|^G = {}_G \sum_{k=1}^{\infty} (e^k \odot |a_k|^G) \odot (e^{k^{-1}} \odot |x_k|^G) < \infty \quad \text{by using Lemma 4.1.}$$

This implies that $a \in (s\ell_{\infty}^G(\Delta_G))^\alpha$. Therefore

$$D_1 \subseteq (s\ell_{\infty}^G(\Delta_G))^\alpha. \quad (5.1)$$

Again let $a \in (s\ell_{\infty}^G(\Delta_G))^\alpha$. Then ${}_G \sum_{k=1}^{\infty} |a_k \odot x_k|^G < \infty$ (by definition of α -dual) for each $x \in s\ell_{\infty}^G(\Delta_G)$. Let us take $x = (x_k) \in s\ell_{\infty}^G(\Delta_G)$ defined by

$$x_k = \begin{cases} 1, & \text{if } k = 1, \\ e^k, & \text{if } k \geq 2. \end{cases}$$

Therefore

$$\begin{aligned} {}_G \sum_{k=1}^{\infty} e^k \odot |a_k|^G &= |a_1|^G \oplus {}_G \sum_{k=2}^{\infty} e^k \odot |a_k|^G \\ &= |a_1|^G \oplus {}_G \sum_{k=1}^{\infty} |a_k \odot x_k|^G < \infty \text{ as } a_1 \odot x_1 = 1. \end{aligned}$$

Therefore $a \in D_1$. This implies that

$$(s\ell_{\infty}^G(\Delta_G))^{\alpha} \subseteq D_1. \quad (5.2)$$

Therefore from (5.1) and (5.2) we get

$$(s\ell_{\infty}^G(\Delta_G))^{\alpha} = D_1.$$

(ii) Let $a \in D_2$. If $x \in s\ell_{\infty}^G(\Delta_G)$ then there exists one and only one $y = (y_k) \in \ell_{\infty}^G$ such that (see equation (3.1))

$$x_k = \ominus_G \sum_{v=1}^k y_{v-1}, \quad y_0 = 1.$$

Therefore

$$\begin{aligned} x_1 &= \ominus_G \sum_{v=1}^1 y_{v-1} = \ominus y_0 = 1 \\ x_2 &= \ominus_G \sum_{v=1}^2 y_{v-1} = \ominus y_1 \\ x_3 &= \ominus_G \sum_{v=1}^3 y_{v-1} = \ominus y_1 \ominus y_2 \\ x_4 &= \ominus_G \sum_{v=1}^4 y_{v-1} = \ominus y_1 \ominus y_2 \ominus y_3 \\ &\dots\dots\dots \\ &\dots\dots\dots \end{aligned}$$

Then

$$\begin{aligned}
{}_G \sum_{k=1}^n a_k \odot x_k &= a_1 \odot x_1 \oplus a_2 \odot x_2 \oplus a_3 \odot x_3 \oplus \dots \oplus a_n \odot x_n \\
&= a_1 \odot 1 \ominus a_2 \odot y_1 \ominus a_3 \odot (y_1 \oplus y_2) \ominus a_4 \odot (y_1 \oplus y_2 \oplus y_3) \ominus \\
&\quad \dots \ominus a_n \odot (y_1 \oplus y_2 \oplus \dots \oplus y_{n-1}) \\
&= \ominus (a_2 \oplus a_3 \oplus \dots \oplus a_n) \odot y_1 \\
&\quad \ominus (a_3 \oplus a_4 \oplus \dots \oplus a_n) \odot y_2 \ominus \dots \ominus a_n \odot y_{n-1} \\
&= (\ominus R_1 \odot y_1 \oplus R_n \odot y_1) \oplus (\ominus R_2 \odot y_2 \oplus R_n \odot y_2) \oplus \dots \\
&\quad \dots \oplus (\ominus R_{n-1} \odot y_{n-1} \oplus R_n \odot y_{n-1}) \\
&= \ominus {}_G \sum_{k=1}^{n-1} R_k \odot y_k \oplus R_n \odot {}_G \sum_{k=1}^{n-1} y_k, \\
{}_G \sum_{k=1}^n a_k \odot x_k &= \ominus {}_G \sum_{k=1}^{n-1} R_k \odot y_k \oplus R_n \odot {}_G \sum_{k=1}^{n-1} y_k. \tag{5.3}
\end{aligned}$$

By Corollary 4.6, ${}_G \sum_{k=1}^{\infty} R_k \odot y_k$ is absolutely convergent and $R_n \odot {}_G \sum_{k=1}^{n-1} y_k \rightarrow 1$ as $n \rightarrow \infty$, the series ${}_G \sum_{k=1}^n a_k \odot x_k$ is convergent for each $x \in s\ell_{\infty}^G(\Delta_G)$. This yields $a \in (s\ell_{\infty}^G(\Delta_G))^{\beta}$.
Therefore

$$D_2 \subseteq (s\ell_{\infty}^G(\Delta_G))^{\beta}.$$

Again let $a \in (s\ell_{\infty}^G(\Delta_G))^{\beta}$, then ${}_G \sum_{k=1}^{\infty} a_k \odot x_k$ is convergent for each $x \in s\ell_{\infty}^G(\Delta_G)$. We take

$$x_k = \begin{cases} 1, & \text{if } k = 1; \\ e^k, & \text{if } k \geq 2. \end{cases}$$

Thus ${}_G \sum_{k=1}^{\infty} e^k \odot x_k$ is convergent. This implies $e^n \odot R_n = O(e)$ by Corollary 4.6. Using (5.3) we get

$${}_G \sum_{k=1}^{\infty} a_k \odot x_k = \ominus {}_G \sum_{k=1}^{\infty} R_k \odot y_k$$

converges for all $y \in \ell_{\infty}^G$. So we have

$${}_G \sum_{k=1}^{\infty} |R_k|^G < \infty \text{ and } a \in D_2.$$

Therefore $(s\ell_{\infty}^G(\Delta_G))^{\beta} = D_2$.

(iii) The proof of this part is same as above. \square

6. Some applications of geometric difference

In this section we find the Geometric Newton–Gregory interpolation formulae and solve some numerical problems using these new formulae.

Geometric factorial Let us define geometric factorial notation $!_G$ as

$$n!_G = e^n \odot e^{n-1} \odot e^{n-2} \odot \cdots \odot e^2 \odot e = e^{n!}.$$

For example,

$$\begin{aligned} 0!_G &= e^{0!} = e^0 = 1 \\ 1!_G &= e^{1!} = e = 2.71828 \\ 2!_G &= e^{2!} = e^2 = 7.38906 \\ 3!_G &= e^{3!} = e^6 = 4.03429 \times 10^2 \\ 4!_G &= e^{4!} = e^{24} = 2.64891 \times 10^{10} \\ 5!_G &= e^{5!} = e^{120} = 1.30418 \times 10^{52} \quad \text{etc.} \end{aligned}$$

Generalized geometric forward difference operator Let

$$\begin{aligned} \Delta_G f(a) &= f(a \oplus h) \ominus f(a). \\ \Delta_G^2 f(a) &= \Delta_G f(a \oplus h) \ominus \Delta_G f(a) \\ &= \{f(a \oplus e^2 \odot h) \ominus f(a \oplus h)\} \ominus \{f(a \oplus h) \ominus f(a)\} \\ &= f(a \oplus e^2 \odot h) \ominus e^2 \odot f(a \oplus h) \oplus f(a). \\ \Delta_G^3 f(a) &= \Delta_G^2 f(a \oplus h) \ominus \Delta_G^2 f(a) \\ &= \{f(a \oplus e^3 \odot h) \ominus e^2 \odot f(a \oplus e^2 \odot h) \oplus f(a \oplus h)\} \\ &\quad \ominus \{f(a \oplus e^2 \odot h) \ominus e^2 \odot f(a \oplus h) \oplus f(a)\} \\ &= f(a \oplus e^3 \odot h) \ominus e^3 \odot f(a \oplus e^2 \odot h) \oplus e^3 \odot f(a \oplus h) \ominus f(a). \end{aligned}$$

Thus, n th geometric forward difference is

$$\Delta_G^n f(a) = \sum_{k=0}^n (\ominus e)^{k_G} \odot e^{\binom{n}{k}} \odot f(a \oplus e^{n-k} \odot h), \text{ with } (\ominus e)^{0_G} = e.$$

Generalized geometric backward difference operator Let

$$\begin{aligned} \nabla_G f(a) &= f(a) \ominus f(a \ominus h). \\ \nabla_G^2 f(a) &= \nabla_G f(a) \ominus \nabla_G f(a \ominus h) \\ &= \{f(a) \ominus f(a \ominus h)\} \ominus \{f(a \ominus h) \ominus f(a \ominus e^2 \odot h)\} \\ &= f(a) \ominus e^2 \odot f(a \ominus h) \oplus f(a \ominus e^2 \odot h). \\ \nabla_G^3 f(a) &= \nabla_G^2 f(a) \ominus \nabla_G^2 f(a \ominus h) \\ &= \{f(a) \ominus e^2 \odot f(a \ominus h) \oplus f(a \ominus e^2 \odot h)\} \\ &\quad \ominus \{f(a \ominus h) \ominus e^2 \odot f(a \ominus e^2 \odot h) \oplus f(a \ominus e^3 \odot h)\} \\ &= f(a) \ominus e^3 \odot f(a \ominus h) \oplus e^3 \odot f(a \ominus e^2 \odot h) \ominus f(a \ominus e^3 \odot h). \end{aligned}$$

Thus, n th geometric backward difference is

$$\nabla_G^n f(a) = {}_G \sum_{k=0}^n (\ominus e)^{k_G} \odot e^{(n)}_k \odot f(a \ominus e^k \odot h).$$

Factorial function The product of n consecutive factors each at a constant geometric difference, h , the first factor being x is called a factorial function of degree n and is denoted by $x^{(n_G)}$. Thus

$$x^{(n_G)} = x \odot (x \ominus e \odot h) \odot (x \ominus e^2 \odot h) \odot (x \ominus e^3 \odot h) \odot \cdots \odot (x \ominus e^{n-1} \odot h).$$

In particular, for $h = e$,

$$x^{(n_G)} = x \odot (x \ominus e) \odot (x \ominus e^2) \odot (x \ominus e^3) \odot \cdots \odot (x \ominus e^{n-1}).$$

Geometric Newton–Gregory forward interpolation formula Let $y = f(x)$ be a function which takes the values $f(a), f(a \oplus h), f(a \oplus e^2 \odot h), f(a \oplus e^3 \odot h), \dots, f(a \oplus e^n \odot h)$ for the $n+1$ geometrically equidistant values (which form a Geometric Progression in ordinary sense) $a, a \oplus h, a \oplus e^2 \odot h, a \oplus e^3 \odot h, \dots, a \oplus e^n \odot h$ of the independent variable x and let $P_n(x)$ be a geometric polynomial in x of degree n defined as:

$$\begin{aligned} P_n(x) &= A_0 \oplus A_1 \odot (x \ominus a) \oplus A_2 \odot (x \ominus a) \odot (x \ominus a \ominus h) \\ &\quad \oplus A_3 \odot (x \ominus a) \odot (x \ominus a \ominus h) \odot (x \ominus a \ominus e^2 \odot h) \oplus \cdots \\ &\quad \oplus A_n \odot (x \ominus a) \odot (x \ominus a \ominus h) \odot \cdots \odot (x \ominus a \ominus e^{n-1} \odot h). \end{aligned} \quad (6.1)$$

We choose the coefficients $A_0, A_1, A_2, \dots, A_n$ such that $P_n(a) = f(a), P_n(a \oplus h) = f(a \oplus h), P_n(a \oplus e^2 \odot h) = f(a \oplus e^2 \odot h), \dots, P_n(a \oplus e^n \odot h) = f(a \oplus e^n \odot h)$.

Putting $x = a, a \oplus h, a \oplus e^2 \odot h, a \oplus e^3 \odot h, \dots, a \oplus e^n \odot h$ in (6.1) and then also putting the values of $P_n(a), P_n(a \oplus h), \dots, P_n(a \oplus e^n \odot h)$, we get

$$\begin{aligned} f(a) &= A_0 \implies A_0 = f(a), \\ f(a \oplus h) &= A_0 \oplus A_1 \odot h \implies A_1 = \frac{f(a \oplus h) \ominus f(a)}{h} {}_G = \frac{\Delta_G f(a)}{h} {}_G, \\ f(a \oplus e^2 \odot h) &= A_0 \oplus e^2 \odot h \odot A_1 \oplus e^2 \odot h \odot h \odot A_2 \\ \implies A_2 &= \frac{f(a \oplus e^2 \odot h) \ominus e^2 \odot [f(a \oplus h) \ominus f(a)] \ominus f(a)}{e^2 \odot h^2 {}_G} {}_G \\ &= \frac{f(a \oplus e^2 \odot h) \ominus e^2 \odot f(a \oplus h) \oplus f(a)}{2!_G \odot h^2 {}_G} {}_G \\ &= \frac{\Delta_G^2 f(a)}{2!_G \odot h^2 {}_G} {}_G. \end{aligned}$$

Similarly

$$\begin{aligned} A_3 &= \frac{\Delta_G^3 f(a)}{3!_G \odot h^3 {}_G} {}_G \\ &\vdots \\ A_n &= \frac{\Delta_G^n f(a)}{n!_G \odot h^n {}_G} {}_G. \end{aligned}$$

Putting the values of $A_0, A_1, A_2, \dots, A_n$ found above in (6.1), we get

$$\begin{aligned}
P_n(x) = & f(a) \oplus \frac{\Delta_G f(a)}{h} {}_G \odot (x \ominus a) \oplus \frac{\Delta_G^2 f(a)}{2!_G \odot h^2_G} {}_G \odot (x \ominus a) \odot (x \ominus a \ominus h) \\
& \oplus \frac{\Delta_G^3 f(a)}{3!_G \odot h^3_G} {}_G \odot (x \ominus a) \odot (x \ominus a \ominus h) \odot (x \ominus a \ominus e^2 \odot h) \oplus \cdots \\
& \oplus \frac{\Delta_G^n f(a)}{n!_G \odot h^n_G} {}_G \odot (x \ominus a) \odot (x \ominus a \ominus h) \odot \cdots \odot (x \ominus a \ominus e^{n-1} \odot h).
\end{aligned}$$

This is the Geometric Newton–Gregory forward interpolation formula.

Putting $\frac{x \ominus a}{h} {}_G = u$ or $x = a \oplus h \odot u$, formula takes the form

$$\begin{aligned}
P_n(x) = & f(a) \oplus u \odot \Delta_G f(a) \oplus \frac{u \odot (u \ominus e)}{2!_G} {}_G \odot \Delta_G^2 f(a) \\
& \oplus \frac{u \odot (u \ominus e) \odot (u \ominus e^2)}{3!_G} {}_G \odot \Delta_G^3 f(a) \oplus \cdots \\
& \oplus \frac{u \odot (u \ominus e) \odot (u \ominus e^2) \odot \cdots \odot (u \ominus e^{n-1})}{n!_G} {}_G \odot \Delta_G^n f(a).
\end{aligned} \tag{6.2}$$

The result (6.2) can be written as

$$\begin{aligned}
P_n(x) = P_n(a \oplus h \odot u) = & f(a) \oplus u^{(1_G)} \odot \Delta_G f(a) \oplus \frac{u^{(2_G)}}{2!_G} {}_G \odot \Delta_G^2 f(a) \\
& \oplus \frac{u^{(3_G)}}{3!_G} {}_G \odot \Delta_G^3 f(a) \oplus \cdots \oplus \frac{u^{(n_G)}}{n!_G} {}_G \odot \Delta_G^n f(a)
\end{aligned}$$

where $u^{(n_G)} = u \odot (u \ominus e) \odot (u \ominus e^2) \odot \cdots \odot (u \ominus e^{n-1})$.

Example 6.1. Given, $f(x) = f(e^t) = \sin(e^t)$. From the following table, find $\sin(e^{1.3})$ using geometric forward interpolation formula.

x	e	$e^{1.2}$	$e^{1.4}$	$e^{1.6}$
$f(x)$	0.0474	0.0579	0.0707	0.0863

Solution The geometric difference table for given data is as follows:

x	$f(x)$	$\Delta_G f(x)$	$\Delta_G^2 f(x)$	$\Delta_G^3 f(x)$
e	0.0474			
		1.2215		
$e^{1.2}$	0.0579		0.9997	
		1.2211		0.9999
$e^{1.4}$	0.0707		0.9996	
		1.3306		
$e^{1.6}$	0.0863			

We have to calculate

$$f(e^{1.3}) = f(a \oplus u \odot h), \text{ say.}$$

$$\therefore a \oplus u \odot h = e^{1.3}$$

$$\Rightarrow e \oplus u \odot e^{0.2} = e^{1.3}, \quad (\text{here } h = e^{1.2} \ominus e = e^{0.2})$$

$$u = \frac{e^{1.3} \ominus e}{e^{0.2}} \circledast$$

$$= (e^{0.3})^{\frac{1}{0.2}} = e^{1.5}$$

By Geometric Newton–Gregory forward interpolation formula we get

$$f(a \oplus u \odot h) = f(a) \oplus u \odot \Delta_G f(a) \oplus \frac{u \odot (u \ominus e)}{e^2} \circledast \Delta_G^2 f(a)$$

$$\oplus \frac{u \odot (u \ominus e) \odot (u \ominus e^2)}{e^6} \circledast \Delta_G^3 f(a)$$

$$f(e^{1.3}) = f(e) \oplus \{e^{1.5} \odot \Delta_G f(e)\} \oplus \left\{ \frac{e^{1.5} \odot (e^{1.5} \ominus e)}{e^2} \circledast \Delta_G^2 f(e) \right\}$$

$$\oplus \left\{ \frac{e^{1.5} \odot (e^{1.5} \ominus e) \odot (e^{1.5} \ominus e^2)}{e^6} \circledast \Delta_G^3 f(e) \right\}$$

$$= 0.0474 \oplus \{e^{1.5} \odot 1.2215\} \oplus \left\{ \frac{e^{1.5} \odot e^{0.5}}{e^2} \circledast 0.9997 \right\}$$

$$\oplus \left\{ \frac{e^{1.5} \odot e^{0.5} \odot e^{-0.5}}{e^6} \circledast 0.9999 \right\}$$

$$= 0.0474 \oplus (1.2215)^{1.5} \oplus (0.9997)^{0.325} \oplus (0.9999)^{\frac{1}{0.0625}}$$

$$= 0.0474 \oplus 1.3500 \oplus 0.9999 \oplus 0.9984$$

$$= 0.0474 \times 1.3500 \times 0.9999 \times 0.9984 = 0.0639.$$

Thus $\sin(e^{1.3}) = 0.0639$.

Note It is to be noted that $e^x \odot e^y = e^{xy}$, $e^x \oplus e^y = e^{x+y}$, $x \odot e^y = x^{\frac{1}{y}}$.

Geometric Newton–Gregory backward interpolation formula Let $y = f(x)$ be a function which takes the values $f(a \oplus e^n \odot h)$, $f(a \oplus e^{n-1} \odot h)$, $f(a \oplus e^{n-2} \odot h)$, $f(a \oplus e^{n-3} \odot h)$, ..., $f(a)$ for the $n+1$ geometrically equidistant values $a \oplus e^n \odot h$, $a \oplus e^{n-1} \odot h$, $a \oplus e^{n-2} \odot h$, $a \oplus e^{n-3} \odot h$, ..., a of the independent variable x and let $P_n(x)$ be a geometric polynomial in x of degree n defined as:

$$P_n(x) = A_0 \oplus A_1 \odot (x \ominus a \ominus e^n \odot h) \oplus A_2 \odot (x \ominus a \ominus e^n \odot h) \odot (x \ominus a \ominus e^{n-1} \odot h)$$

$$\oplus A_3 \odot (x \ominus a \ominus e^n \odot h) \odot (x \ominus a \ominus e^{n-1} \odot h) \odot (x \ominus a \ominus e^{n-2} \odot h) \oplus \dots \quad (6.3)$$

$$\oplus A_n \odot (x \ominus a \ominus e^n \odot h) \odot (x \ominus a \ominus e^{n-1} \odot h) \odot \dots \odot (x \ominus a \ominus h),$$

where $A_0, A_1, A_2, \dots, A_n$ are constants which are to be determined so as to make

$$P_n(a \oplus e^n \odot h) = f(a \oplus e^n \odot h), P_n(a \oplus e^{n-1} \odot h) = f(a \oplus e^{n-1} \odot h), \dots, P_n(a) = f(a).$$

Putting $x = a \oplus e^n \odot h$, $a \oplus e^{n-1} \odot h$, ... in (6.3) and also putting $P_n(a \oplus e^n \odot h) = f(a \oplus e^n \odot h)$, ..., we get

$$A_0 = f(a \oplus e^n \odot h)$$

$$A_1 = \frac{\nabla_G f(a \oplus e^n \odot h)}{h} \circledast$$

$$A_2 = \frac{\nabla_G^2 f(a \oplus e^n \odot h)}{2!_G \odot h^2 \circledast} \circledast$$

$$\begin{aligned}
 A_3 &= \frac{\nabla_G^3 f(a \oplus e^n \odot h)}{3!_G \odot h^{3G}}_G \\
 &\vdots \\
 A_n &= \frac{\nabla_G^n f(a \oplus e^n \odot h)}{n!_G \odot h^{nG}}_G
 \end{aligned}$$

Substituting the values of A_0, A_1, A_2, \dots in (6.3), we get

$$\begin{aligned}
 P_n(x) &= f(a \oplus e^n \odot h) \oplus \frac{\nabla_G f(a \oplus e^n \odot h)}{h}_G \odot (x \ominus a \ominus e^n \odot h) \\
 &\oplus \frac{\nabla_G^2 f(a \oplus e^n \odot h)}{2!_G \odot h^{2G}}_G \odot (x \ominus a \ominus e^n \odot h) \odot (x \ominus a \ominus e^{n-1} \odot h) \\
 &\oplus \frac{\nabla_G^3 f(a \oplus e^n \odot h)}{3!_G \odot h^{3G}}_G \odot (x \ominus a \ominus e^n \odot h) \odot (x \ominus a \ominus e^{n-1} \odot h) \odot (x \ominus a \ominus e^{n-2} \odot h) \\
 &\oplus \dots \oplus \frac{\nabla_G^n f(a \oplus e^n \odot h)}{n!_G \odot h^{nG}}_G \odot (x \ominus a \ominus e^n \odot h) \odot (x \ominus a \ominus e^{n-1} \odot h) \odot \dots \odot (x \ominus a \ominus h).
 \end{aligned}$$

This is the Geometric Newton–Gregory backward interpolation formula.

Putting $u = \frac{x \ominus (a \oplus e^n \odot h)}{h}_G$ or $x = a \oplus e^n \odot h \oplus u \odot h$, we get

$$\begin{aligned}
 P_n(x) &= P_n(a \oplus e^n \odot h \oplus u \odot h) = f(a \oplus e^n \odot h) \oplus u \odot \nabla_G f(a \oplus e^n \odot h) \\
 &\oplus \frac{u \odot (u \oplus e)}{2!_G}_G \odot \nabla_G^2 f(a \oplus e^n \odot h) \\
 &\oplus \frac{u \odot (u \oplus e) \odot (u \oplus e^2)}{3!_G}_G \odot \nabla_G^3 f(a \oplus e^n \odot h) \oplus \dots \\
 &\oplus \frac{u \odot (u \oplus e) \odot (u \oplus e^2) \odot \dots \odot (u \oplus e^{n-1})}{n!_G}_G \odot \nabla_G^n f(a \oplus e^n \odot h).
 \end{aligned}$$

Example 6.2. Given, $f(x) = \ln(x)$. From the following table, find $\ln(22)$ using geometric backward interpolation formula.

x	3	6	12	24
$f(x)$	1.0986	1.7918	2.4849	3.1781

Solution The geometric difference table for given data is as follows:

x	$f(x)$	$\nabla_G f(x)$	$\nabla_G^2 f(x)$	$\nabla_G^3 f(x)$
3	1.0986			
		1.6310		
6	1.7918		0.8503	
		1.3868		1.0847
12	2.4849		0.9223	
		1.2790		
24	3.1781			

We have to compute

$$f(22) = f(a \oplus e^n \odot h \oplus u \odot h), \text{ say.}$$

$$\begin{aligned}
\therefore a \oplus e^n \odot h \oplus u \odot h &= 22 \\
\Rightarrow 24 \oplus u \odot h &= 22, \quad (\text{here } h = 6 \ominus 3 = 2) \\
u &= \frac{22 \ominus 24}{2}_G \\
&= (0.9167)^{\frac{1}{\ln 2}} = 0.8820.
\end{aligned}$$

By Geometric Newton–Gregory backward interpolation formula we get

$$\begin{aligned}
f(22) &= f(24) \oplus u \odot \nabla_G f(24) \oplus \frac{u \odot (u \oplus e)}{2!_G} \odot \nabla_G^2 f(24) \\
&\quad \oplus \frac{u \odot (u \oplus e) \odot (u \oplus e^2)}{3!_G} \odot \nabla_G^3 f(24) \\
&= 3.1781 \oplus \{0.8820 \odot 1.2790\} \oplus \left\{ \frac{0.8820 \odot (0.8820 \oplus e)}{e^2} \odot 0.9223 \right\} \\
&\quad \oplus \left\{ \frac{0.8820 \odot (0.8820 \oplus e) \odot (0.8820 \oplus e^2)}{e^6} \odot 1.0847 \right\} \\
&= 3.1781 \oplus 0.9696 \oplus \{0.9466 \odot 0.9223\} \oplus \{0.9663 \odot 1.0847\} \\
&= 3.1781 \oplus 0.9696 \oplus 1.0045 \oplus 0.9972 \\
&= 3.1781 \times 0.9696 \times 1.0045 \times 0.9972 = 3.0867.
\end{aligned}$$

Therefore $\ln(22) = 3.0867$.

Note Since small change in x results large change in e^x . So, for better accuracy, values should be taken up to maximum possible decimal places.

Advantages of geometric interpolation formulae over ordinary interpolation formulae All the ordinary interpolation formulae are based upon the fundamental assumption that the data is expressible or can be expressed as a polynomial function with fair degree of accuracy. But geometric interpolation formulae have no such restriction. Because geometric interpolation formulae are based on geometric polynomials which are not polynomials in ordinary sense. So geometric interpolation formulae can be used to generate transcendental functions, mainly to compute exponential and logarithmic functions. Also geometric forward and backward interpolation formulae are based on the values of the argument that are geometrically equidistant but need not be equidistant like classical interpolation formulae.

7. Conclusion

In this paper, we have defined geometric difference sequence space and obtained the Geometric Newton–Gregory interpolation formulae. Our main aim is to bring up geometric calculus to the attention of researchers in the branch of numerical analysis and to demonstrate its usefulness. We think that geometric calculus may especially be useful as a mathematical tool for economics, management and finance.

Acknowledgments

The authors would like to thank Prof. Steven Krantz, Editor-in-Chief for his comments and the referees for their much encouragement, support, constructive criticism, careful reading and making a useful comment which improved the presentation and the readability of the paper. Also, it is our pleasure to thank Prof.

M. Grossman for his constructive suggestions and inspiring comments regarding the improvement of the geometric-calculus.

References

- [1] A.E. Bashirov, M. Rıza, On complex multiplicative differentiation, *TWMS J. Appl. Eng. Math.* 1 (1) (2011) 75–85.
- [2] A.E. Bashirov, E. Mısırlı, Y. Tandoğdu, A. Özyapıcı, On modeling with multiplicative differential equations, *Appl. Math. J. Chinese Univ.* 26 (4) (2011) 425–438.
- [3] A.E. Bashirov, E.M. Kurpınar, A. Özyapıcı, Multiplicative calculus and its applications, *J. Math. Anal. Appl.* 337 (2008) 36–48.
- [4] A.F. Çakmak, F. Başar, Certain spaces of functions over the field of non-Newtonian complex numbers, *Abstr. Appl. Anal.* 2014 (2014) 236124.
- [5] A.F. Çakmak, F. Başar, On classical sequence spaces and non-Newtonian calculus, *J. Inequal. Appl.* 2012 (2012) 932734.
- [6] A.F. Çakmak, F. Başar, On line and double integrals in the non-Newtonian sense, *AIP Conf. Proc.* 1611 (2014) 415–423.
- [7] A.F. Çakmak, F. Başar, Some sequence spaces and matrix transformations in multiplicative sense, *TWMS J. Pure Appl. Math.* 6 (1) (2015) 27–37.
- [8] D.J.H. Garling, The β - and γ -duality of sequence spaces, *Proc. Cambridge Philos. Soc.* 63 (1967) 963–981.
- [9] M. Grossman, *Bigeometric Calculus: A System with a Scale-Free Derivative*, Archimedes Foundation, Massachusetts, 1983.
- [10] M. Grossman, R. Katz, *Non-Newtonian Calculus*, Lee Press, Piegon Cove, Massachusetts, 1972.
- [11] U. Kadak, H. Efe, Matrix transformation between certain sequence spaces over the non-Newtonian complex field, *Sci. World J.* 2014 (2014) 705818.
- [12] U. Kadak, M. Kirişçi, A.F. Çakmak, On the classical paranormed sequence spaces and related duals over the non-Newtonian complex field, *J. Funct. Spaces Appl.* 2015 (2015) 416906.
- [13] U. Kadak, M. Özlük, Generalized Runge–Kutta method with respect to non-Newtonian calculus, *Abstr. Appl. Anal.* 2015 (2015) 594685.
- [14] H. Kizmaz, On certain sequence spaces, *Canad. Math. Bull.* 24 (2) (1981) 169–176.
- [15] G. Köthe, O. Topf, *Vector Spaces I*, Springer-Verlag, 1969.
- [16] G. Köthe, O. Topf, Linear Räume mit unendlichen koordinaten und Ring unendlichen Matrizen, *J. Reine Angew. Math.* 171 (1934) 193–226.
- [17] I.J. Maddox, *Infinite Matrices of Operators*, Lecture Notes in Mathematics, vol. 786, Springer-Verlag, 1980.
- [18] D. Stanley, A multiplicative calculus, *Primus* IX 4 (1999) 310–326.
- [19] S. Tekin, F. Başar, Certain Sequence spaces over the non-Newtonian complex field, *Abstr. Appl. Anal.* 2013 (2013) 739319.
- [20] C. Türkmen, F. Başar, Some basic results on the sets of sequences with geometric calculus, *Commun. Fac. Sci. Univ. Ank. Sér. A1* 61 (2) (2012) 17–34.
- [21] A. Uzer, Exact solution of conducting half plane problems as a rapidly convergent series and an application of the multiplicative calculus, *Turk. J. Elec. Eng. Comput. Sci.* 23 (5) (2015) 1294–1311.
- [22] A. Uzer, Multiplicative type complex calculus as an alternative to the classical calculus, *Comput. Math. Appl.* 60 (2010) 2725–2737.