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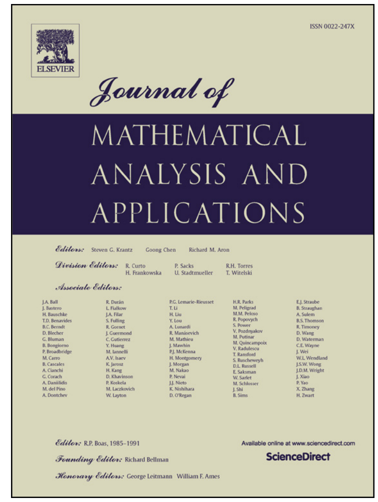

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Approximate amenability of tensor products of Banach algebras

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Abstract

Examples constructed by the first author and Charles Read make it clear that many of the hereditary properties of amenability no longer hold for approximate amenability. These and earlier results of the authors also show that the presence of a bounded approximate identity often entails positive results. Here we show that the tensor product of approximately amenable algebras need not be approximately amenable, and investigate conditions under which A and B being approximately amenable implies, or is implied by, $A\widehat{\otimes}B$ or $A^{\#}\widehat{\otimes}B^{\#}$ being approximately amenable. Once again, the role of having a bounded approximate identity comes to the fore. Our methods also enable us to prove that if $A\widehat{\otimes}B$ is amenable, then so are A and B, a result proved by Barry Johnson in 1996 under an additional assumption.

Keywords: Approximately amenable Banach algebra, amenable Banach algebra, tensor product, approximate identity
2010 MSC: 46H25

In memoriam
Charles John Read – mathematician, gentleman and friend

1. Introduction

The concept of amenability for a Banach algebra, introduced by Johnson in [17], has proved to be of enormous importance in Banach algebra theory. In [10], and subsequently in [14], several modifications of this notion were introduced, in

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particular that of approximate amenability; and much work has been done in the last 10 years or so, [8, 5, 4, 13, 7, 11, 12], for example. See also [21] for a recent survey. In this paper the focus is on these newer notions for tensor products. In particular, we investigate relations between the approximate amenability of A and B and that of $A\widehat{\otimes}B$ or $A^{\#}\widehat{\otimes}B^{\#}$.

As a by-product of our investigations, we show in Theorem 4.9 that amenability of $A\widehat{\otimes}B$ implies amenability of A and of B. This is a new result, although it was known in many special cases by previous results of Johnson [18, Section 3].

Let A be an algebra, and let X be an A-bimodule. A *derivation* is a linear map $D : A \to X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A)\,.$$

For $x \in X$, set $ad_x : a \mapsto a \cdot x - x \cdot a,\ \ A \to X$. Then ad_x is a derivation; these are the *inner* derivations.

Let A be a Banach algebra, and let X be a Banach A-bimodule. A continuous derivation $D : A \to X$ is *approximately inner* if there is a net (x_α) in X such that

$$D(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in A)\,,$$

so that $D = \lim_\alpha ad_{x_\alpha}$ in the strong-operator topology of $\mathcal{B}(A, X)$.

Definition 1.1. [10, 14] Let A be a Banach algebra. Then A is *approximately amenable* (resp. *approximately contractible*) if, for each Banach A-bimodule X, every continuous derivation $D : A \to X^*$ (resp. $D : A \to X$), is approximately inner. If it is always possible to choose the approximating net (ad_{x_α}) to be bounded (with the bound dependent only on D) then A is *boundedly* approximately amenable (resp. *boundedly* approximately contractible).

Of course A is *amenable* (resp. *contractible*) if every continuous derivation $D : A \to X^*$ (resp. $D : A \to X$), is inner, for every Banach A-bimodule X.

Of these various notions, amenability, contractibility, approximate amenability, bounded approximate amenability and bounded approximate contractibility are all distinct, while approximate contractibility and approximate amenability coincide, [14, 11, 12]. Requiring the approximating net of derivations to converge weak* is the same as approximate amenability [14]. This latter notion will arise naturally below. If, instead of requiring the nets of approximating inner derivations to be bounded in bounded approximate amenability and bounded approximate contractibility, one requires boundedness on the nets of implementing *elements*, then one recovers amenability, [15].

2. Some observations

Recall the result of Johnson [17, Proposition 5.4]:

Proposition 2.1. *Let A and B be amenable Banach algebras. Then $A\widehat{\otimes}B$ is amenable.* □

A version of this for the approximately amenable case was stated in [10, Proposition 2.3], but the argument there is incomplete. The matter is clarified in [4, Proposition 4.1], from which we state:

Theorem 2.2. *Suppose that A is a boundedly approximately amenable Banach algebra with a bounded approximate identity, and that B is an amenable Banach algebra. Then $A\widehat{\otimes}B$ is boundedly approximately amenable.* □

In [4] the question is raised whether the tensor product of two (boundedly) approximately amenable Banach algebras is itself (boundedly) approximately amenable. We begin by answering this question in the negative. Note that A is boundedly approximately amenable if and only $A^{\#}$ is boundedly approximately amenable, [14, Lemma 5.9].

Theorem 2.3. *The tensor product of two boundedly approximately amenable Banach algebras need not be approximately amenable.*

Proof. Let A be the Banach algebra constructed in [11] such that A is boundedly approximately amenable yet $A \oplus A^{\mathrm{op}}$ is not approximately amenable. For convenience, set $B = A^{\mathrm{op}}$. Adjoin identities 1_A to A and 1_B to B, and set $\mathcal{A} = A^{\#}\widehat{\otimes}B^{\#}$. Then we have the decomposition into closed subspaces:

$$\mathcal{A} = (\mathbb{C}1_A \otimes 1_B) + (1_A \otimes B) + (A \otimes 1_B) + (A\widehat{\otimes}B)\,.$$

Now $A\widehat{\otimes}B$ is a closed two-sided ideal in $\mathcal{A}$, and the quotient algebra $\mathcal{A}/A\widehat{\otimes}B$ is isomorphic to $(A \oplus B)^{\#}$ via the map

$$(\lambda 1_A \otimes 1_B) + (1_A \otimes b) \oplus (a \otimes 1_B) + c \otimes d \mapsto \lambda 1_{A\oplus B} + (a \oplus b)\,.$$

Thus by [10, Proposition 2.4] $(A \oplus B)^{\#}$ is approximately amenable, whence so is $A \oplus B$. But this contradicts the specific choice of A and B. Thus $\mathcal{A}$ cannot be approximately amenable. □

Note that the argument sheds no light on whether in this case the subalgebra $A\widehat{\otimes}B$ is approximately amenable.

Remark 2.4. The same example from [11] also answers Question 1 raised in [10, §9]. Namely $A \oplus B$ is not approximately amenable, yet the ideal A is boundedly approximately amenable, as is the quotient B.

We now build on this example to give a slightly sharper result.

Lemma 2.5. *For Banach algebras C and D, there is a natural epimorphism $(C \oplus D)\widehat{\otimes}(C \oplus D) \to C\widehat{\otimes}D$.*

Proof. We have

$$(C \oplus D)\widehat{\otimes}(C \oplus D) = (C\widehat{\otimes}C) \oplus (D\widehat{\otimes}D) \oplus (D\widehat{\otimes}C) \oplus (C\widehat{\otimes}D)\,,$$

and

$$I = (C\widehat{\otimes}C) \oplus (D\widehat{\otimes}D) \oplus (D\widehat{\otimes}C)$$

is a closed two-sided ideal in $(C \oplus D)\widehat{\otimes}(C \oplus D)$ with quotient $C\widehat{\otimes}D$.

□

Corollary 2.6. *There exists a unital boundedly approximately amenable Banach algebra $\mathcal{A}$ such that $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is not approximately amenable.*

Proof. Let A and B be the boundedly approximately amenable algebras as above, and set $\mathcal{A} = A^{\#} \oplus B^{\#}$, boundedly approximately amenable by [14, Proposition 6.1]. By Lemma 2.5, $A^{\#}\widehat{\otimes}B^{\#}$ is a quotient, and this latter is not approximately amenable from Theorem 2.3. Thus by [10, Proposition 2.4] $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is not approximately amenable. □

In comparison, note that since boundedly approximately contractible algebras have bounded approximate identities [5, Theorem 2.5], the direct sum of boundedly approximately contractible algebras is boundedly approximately contractible by a variant of [10, Proposition 2.7].

There is a special situation when approximate amenability of the tensor product comes for free.

Proposition 2.7. *Let A and B be Banach function algebras on their respective carrier spaces, and suppose that A and B have bounded approximate identities consisting of elements of finite support. Then $A\widehat{\otimes}B$ is approximately amenable.*

Proof. That A and B are approximately amenable follows from [13, Proposition 4.2]. Now A and B have the bounded approximation property, so by [19, §3.2.18] $A\widehat{\otimes}B$ is semisimple, and so is again a Banach function algebra. It also has a bounded approximate identity of elements of finite support, built from those of A and B, and once more [13, Proposition 4.2] applies. □

The same style of argument as above using compositions can also give some positive results.

Theorem 2.8. *Suppose that $A^{\#}\widehat{\otimes}B^{\#}$ is approximately amenable. Then A, B and $A \oplus B$ are approximately amenable.*

Proof. The algebra $A^{\#}$ admits a character φ, and $a\widehat{\otimes}b \mapsto \varphi(a)b$ defines an epimorphism $A^{\#}\widehat{\otimes}B^{\#} \to B^{\#}$ so that $B^{\#}$, and hence B, is approximately amenable. Similarly for A.

We have the decomposition into closed subalgebras,

$$A^{\#}\widehat{\otimes}B^{\#} = (\mathbb{C}1_A \otimes 1_B) + (1_A \otimes B) + (A \otimes 1_B) + (A\widehat{\otimes}B)\,.$$

Here $A\widehat{\otimes}B$ is a closed ideal, with approximately amenable quotient $A^{\#}\widehat{\otimes}B^{\#}/A\widehat{\otimes}B$ given by

$$(\mathbb{C}1_A \otimes 1_B) + (1_A \otimes A) \oplus (B \otimes 1_B)$$

having zero product between the second and third summands. But this latter is isomorphic to the unitization of $A \oplus B$. □

The obvious omission here is whether $A\widehat{\otimes}B$ is approximately amenable. This is certainly the case under an additional hypothesis.

Theorem 2.9. *Suppose that $A^{\#}\widehat{\otimes}B^{\#}$ is (boundedly) approximately amenable and that A and B have bounded approximate identities. Then $A\widehat{\otimes}B$ is (boundedly) approximately amenable.*

Proof. The argument of [9, Proposition 2.5] will show that since $A\widehat{\otimes}B$ has a bounded approximate identity, it suffices to show that for every neo-unital Banach $A\widehat{\otimes}B$-bimodule X, continuous derivations from $A\widehat{\otimes}B$ into X^* are approximately inner.

Let $D : A\widehat{\otimes}B \to X^*$ be a continuous derivation. Then D extends uniquely to a derivation $\widehat{D} : \Delta(A\widehat{\otimes}B) \to X^*$, where $\Delta(A\widehat{\otimes}B)$ is the double centralizer algebra of $A\widehat{\otimes}B$, [16, 17]. Then restrict $\widehat{D}$ to $A^{\#}\widehat{\otimes}B^{\#}$. By hypothesis this restriction is approximately inner, a fortiori, so is D.

□

Remark 2.10. An alternate proof would be to use the argument of [10, Corollary 2.3].

Remark 2.11. 1. A possibly related question is whether $c_0(A)$ is approximately amenable given that A is approximately amenable. The argument of [10, Example 6.1] shows that $c_0(A^{\#})$ will be approximately amenable. For the algebra $\mathcal{A}$ of [11], $c_0(\mathcal{A})$ is again of the specified form of [11, Theorem 3.1], and so is approximately amenable. The more general question as to whether $c_0(A_n)$ is approximately amenable, where the (A_n) are approximately amenable, has a negative answer in general, as shown by the example $\mathcal{A} \oplus \mathcal{A}^{\mathrm{op}}$ of [11].

2. Note that $c_0\widehat{\otimes}A$ will be approximately amenable if A is boundedly approximately amenable and has a bounded approximate identity (Proposition 2.2). For more general A the question is open. Of course there is a natural homomorphism $c_0\widehat{\otimes}A \to c_0(A)$ determined by $(\alpha_n) \otimes x \mapsto (\alpha_n x)$. Since elements of the range are summable sequences of elements of A, the homomorphism has properly dense range. Supposing that $c_0\widehat{\otimes}A$ is approximately amenable it is not known whether $c_0(A)$ must be approximately amenable. However the epimorphism $c_0\widehat{\otimes}A \to A \oplus A$ determined by $(\alpha_n) \otimes x \mapsto (\alpha_1 x, \alpha_2 x)$ shows that $A \oplus A$ would be.

3. Semi-inner derivations

We first introduce a new notion which will arise in later arguments of §4. The concept itself is not new, but the variant we wish to use seems to be.

Definition 3.1. Let A be a algebra, X an A-bimodule. A derivation $D : A \to X$ is *semi-inner*[2] if there are $m, n \in X$ such that

$$D(a) = \mathrm{ad}_{m,n}(a) = a \cdot m - n \cdot a \qquad (a \in A)\,.$$

[2]Such maps, without the derivation condition, are called generalized inner, or 'generalized inner derivations' in the literature [3, 2, 6]. We require the approximate version, and view 'approximately generalized' as an oxymoron, and so will use 'semi-inner', but only for derivations.

More generally, for A a Banach algebra, X a Banach A-bimodule, a continuous derivation $D : A \to X$ is *approximately semi-inner* if there are nets $(m_i), (n_i)$ in X with

$$D(a) = \lim_i (a \cdot m_i - n_i \cdot a) \qquad (a \in A)\,,$$

that is, D is the limit in the strong operator topology of the net (ad_{m_i,n_i}).

Here m and n may be distinct but are highly constrained. The derivation identity shows that if $\mathrm{ad}_{m,n}$ is a semi-inner derivation then

$$a \cdot (m-n) \cdot b = 0 \qquad (a, b \in A)\,. \tag{1}$$

Conversely, if $m, n \in X$ satisfy (1) then it is immediate that $\mathrm{ad}_{m,n}$ is a derivation.

Example 3.2. To see that semi-inner is indeed a strictly weaker notion than inner, consider the following simple example. Suppose that A is commutative and $A^3 = 0$. By commutativity the only inner derivation on A is the zero map, but multiplication by any element is a semi-inner derivation since $A^3 = 0$, and some such will be non-zero provided $A^2 \neq \{0\}$.

For an specific example, take $A = \mathbb{C}^4$ with product

$$(a_1, a_2, a_3, a_4)(b_1, b_2, b_3, b_4) = (0, 0, a_1 b_1, a_2 b_2)\,.$$

However, in many situations of interest the notions coincide.

Proposition 3.3. *Let A be a Banach algebra, X a Banach A-bimodule, $D : A \to X$ (resp $D : A \to X^*$) a derivation. Suppose that D is semi-inner: $D = \mathrm{ad}_{m,n}$. Then in each of the following cases D is inner:*

(i) A has left and right approximate identities for X;
(ii) $D : A \to X^*$ and X is neo-unital;
(iii) $D : A \to X^*$ and A has a bounded approximate identity;

Proof. (i). Immediate from (1).

(ii) The hypothesis implies that for $x \in X$,

$$0 = \langle a \cdot (m-n) \cdot b, x \rangle = \langle m-n, b \cdot x \cdot a \rangle\,,$$

so that $m - n$ vanishes on X_{ess}. Thus $m = n$ if X is neo-unital, so D is inner.

(iii). Let (e_i) be a bounded approximate identity of A. Let E be a limit point in the weak*-operator topology of the left multiplication operators on X^* by the elements e_i, F similarly for right multiplication. Then E and F are commuting A-bimodule morphisms of X^* into itself, and give a decomposition

$$X^* = EFX^* \oplus E(I-F)X^* \oplus (I-E)X^*\,. \tag{2}$$

Correspondingly, set

$$D_1 = EFD, D_2 = E(I-F)D, D_3 = (I-E)D\,.$$

These are easily seen to be derivations into the corresponding summands in (2). Since A has trivial action on the right of $E(I-F)X^*$, and has a bounded approximate identity, we conclude that D_2 is approximately inner, with a bounded net of implementing elements, whence D_2 is inner. Similarly for D_3. Now for $a \in A$,

$$\begin{aligned} D_1(a) &= EF\mathrm{ad}_{m,n}(a) = EF(a \cdot m - n \cdot a) \\ &= \mathrm{weak}^* - \lim_i \lim_j [e_i a \cdot m \cdot e_j - e_i \cdot n \cdot a e_j] . \end{aligned}$$

But from (1), $e_i \cdot n \cdot e_j a = e_i \cdot m \cdot e_j a$, so we have

$$D_1(a) = EF(a \cdot m - m \cdot a) = a \cdot EF(m) - EF(m) \cdot a \qquad (a \in A) .$$

□

Remark 3.4. In (i) and (ii) it is clear that $m = n$. This is not clear in (iii) and indeed need not be the case. For suppose there is a non-zero $p \in X$, satisfying $Ap = 0$, take $k \in X$ and set $m = k + p$. Then $\mathrm{ad}_{m,k} = \mathrm{ad}_k$ is a semi-inner (in fact inner) derivation with $m \neq k$.

Corollary 3.5. *Let A be a Banach algebra and suppose that for any Banach A-bimodule X, any continuous derivation $D : A \to X^*$ is semi-inner. Then A is amenable (and such D are inner).*

Proof. The standard argument, [17, Proposition 1.6], showing that amenable algebras have a bounded approximate identity, uses bimodules with trivial action on one side, in which case semi-inner means the same as inner. Thus the argument to applies to A, and it follows that A has a bounded approximate identity, so that Proposition 3.3(iii) applies. □

Remark 3.6. The property of A in the hypothesis of Corollary 3.5 could be taken to define *semi-amenability*, but as just shown there is no use for this term.

On the other hand, we can define a Banach algebra A to be *approximately semi-amenable* if for any Banach A-bimodule X, any continuous derivation $D : A \to X^*$ is approximately semi-inner. This is a strictly weaker notion than approximate amenability, as the following examples show.

Example 3.7. For $1 \leqslant p < \infty$, the algebra ℓ^p under pointwise operations is not approximately amenable, [8, 4]. However, derivations from ℓ^p are always approximately semi-inner. For let $D : \ell^p \to X$ be a continuous derivation into a Banach ℓ^p- bimodule. Set (E_n) to be the standard (unbounded) approximate identity of ℓ^p. Then $D_n = D|E_n\ell^p : E_n\ell^p \to X$ is a derivation from a finite-dimensional semisimple algebra and hence is inner, say implemented by $\xi_n \in X$. Thus for $a \in \ell^p$,

$$\begin{aligned} D(a) = \lim_n D(E_n a) &= \lim_n (E_n a \cdot \xi_n - \xi_n \cdot E_n a) \\ &= \lim \big(a \cdot (E_n \cdot \xi_n) - (\xi_n \cdot E_n) \cdot a\big) , \end{aligned}$$

and so D is approximately semi-inner.

Example 3.8. Let G be an amenable SIN group. Then any non-trivial Segal subalgebra of $L^1(G)$ is approximately semi-amenable. The proof of this is nontrivial, and will appear elsewhere. No such algebras are approximately amenable, [1]; see also [7, §4], [4, §3].

4. Approximately amenable tensor products

Theorem 4.1. *Let A and B be Banach algebras. Suppose that $A\widehat{\otimes}B$ is*

(i) approximately amenable, or

(ii) boundedly approximately amenable, or

(iii) boundedly approximately contractible.

Then any continuous derivation D from A or B

- *into any bimodule is approximately semi-inner in clause (i),*
- *into any dual bimodule is boundedly approximately semi-inner in clause (ii),*
- *into any bimodule is boundedly approximately semi-inner in clause (iii).*

Proof. Given a Banach A-bimodule X, we make $X\widehat{\otimes}B$ into a Banach $A\widehat{\otimes}B$-bimodule as follows: for $a \in A, b_1 \in B, b_2 \in B, x \in X$,

$$(a \otimes b_1)\cdot(x \otimes b_2) = a\cdot x \otimes b_1 b_2\,, \qquad (x \otimes b_2)\cdot(a \otimes b_1) = x\cdot a \otimes b_2 b_1\,.$$

Given a continuous derivation $D : A \to X$, we define $\Delta : A\widehat{\otimes}B \to X\widehat{\otimes}B$ by setting

$$\Delta(a \otimes b) = D(a) \otimes b \qquad (a \in A, b \in B)\,.$$

Then

$$\begin{aligned}
\Delta((a_1 \otimes b_1)(a_2 \otimes b_2)) &= \Delta(a_1a_2 \otimes b_1b_2)\\
&= \big(D(a_1)\cdot a_2 + a_1\cdot D(a_2)\big) \otimes (b_1b_2)\\
&= \big((D(a_1)\cdot a_2) \otimes b_1b_2\big) + \big((a_1\cdot D(a_2)) \otimes b_1b_2\big)\\
&= \big((D(a_1) \otimes b_1)\cdot(a_2 \otimes b_2)\big) + \big((a_1 \otimes b_1)\cdot(D(a_2) \otimes b_2\big)\,,
\end{aligned}$$

so that Δ is a derivation.

In clause (i), since approximate amenability and approximate contractibility coincide, [14, Proposition 2.1], there is a net (m_i) in $X\widehat{\otimes}B$ such that for all $a \in A, b \in B$,

$$\Delta(a \otimes b) = \lim_i \Big((a \otimes b)\cdot m_i - m_i\cdot(a \otimes b)\Big)\,. \tag{3}$$

Let

$$m_i = \sum_{k=1}^{\infty} x_{k,i} \otimes b_{k,i}\,,$$

where $x_{k,i} \in X, b_{k,i} \in B$. Then

$$\begin{aligned} D(a) \otimes b &= \Delta(a \otimes b) \\ &= \lim_i \left(\sum_k (a \cdot x_{k,i}) \otimes b b_{k,i} - \sum_k (x_{k,i} \cdot a) \otimes b_{k,i} b \right) . \end{aligned} \tag{4}$$

Fix $b_0 \in B$ non-zero, and take $b_0^* \in B^*$ with $\langle b_0^*, b_0 \rangle = 1$. Applying the operator $T : X \widehat{\otimes} B \to X$ specified by $T(x \otimes b) = \langle b_0^*, b \rangle x$ to both sides of (4) yields

$$D(a) = \lim_i (a \cdot m_i' - n_i' \cdot a) , \tag{5}$$

where $m_i' = \sum_k \langle b_0^*, b_0 b_{k,i} \rangle x_{k,i}$, $n_i' = \sum_k \langle b_0^*, b_{k,i} b_0 \rangle x_{k,i}$.

In clause (iii), the same argument with the extra condition that

$$\|(a \otimes b) \cdot m_i - m_i \cdot (a \otimes b)\| \leqslant K \|a\| \|b\|,$$

yields

$$\|a \cdot m_i' - n_i' \cdot a\| \leqslant K' \|a\| .$$

For clause (ii), let $D : A \to X^*$ be a continuous derivation into a dual bimodule. Since $X^* \widehat{\otimes} B$ is unlikely to be a dual space, let alone a dual module, view the derivation Δ as mapping into $(X^* \widehat{\otimes} B)^{**}$. Then there is a net (m_i) in $(X^* \widehat{\otimes} B)^{**}$ and a constant $K > 0$ such that for $a \in A, b \in B$,

$$D(a) \otimes b = \Delta(a \otimes b) = \lim_i \Big((a \otimes b) \cdot m_i - m_i \cdot (a \otimes b) \Big) , \tag{6}$$

and

$$\|(a \otimes b) \cdot m_i - m_i \cdot (a \otimes b)\| \leqslant K \|a\| \; \|b\| . \tag{7}$$

Fix $b_0 \in B$ of unit norm and take $b_0^* \in B^*$ with $b_0^*(b_0) = 1$. Let $S : X \to (X^* \widehat{\otimes} B)^*$ be specified by

$$\langle S(x), x^* \otimes b \rangle = \langle x^*, x \rangle \langle b_0^*, b \rangle , \qquad (x \in X, x^* \in X^*, b \in B) ,$$

and set $T = S^* : (X^* \widehat{\otimes} B)^{**} \to X^*$. Now take $m \in (X^* \widehat{\otimes} B)^{**}, a \in A, b \in B$, and $x \in X$. Then

$$\langle T((a \otimes b) \cdot m), x \rangle = \langle (a \otimes b) \cdot m, S(x) \rangle = \langle m, S(x) \cdot (a \otimes b) \rangle .$$

For $x^* \in X^*$ and $c \in B$,

$$\langle S(x) \cdot (a \otimes b_0), x^* \otimes c \rangle = \langle S(x), (a \otimes b_0) \cdot (x^* \otimes c) \rangle \tag{8}$$

$$= \langle S(x), a \cdot x^* \otimes b_0 c \rangle = \langle a \cdot x^*, x \rangle \langle b_0^*, b_0 c \rangle . \tag{9}$$

Thus, setting $m = \sum_k x_k^* \otimes b_k$, and $\varphi(m) = \sum_k \langle b_0^*, b_0 b_k \rangle x_k^*$,

$$T((a \otimes b_0) \cdot m) = \sum_k \langle b_0^*, b_0 b_k \rangle a \cdot x_k^* = a \cdot \varphi(m) ,$$

where we have the estimate

$$\|\varphi(m)\| \leqslant \|b_0\| \, \|b_0^*\| \, \|m\| .$$

A general $m \in (X^* \widehat{\otimes} B)^{**}$ is the weak*-limit of a net $(\mu_\alpha) \subset X^* \widehat{\otimes} B$, bounded by $\|m\|$, and as an adjoint T is weak*-weak* continuous. It follows that the associated net $(\varphi(\mu_\alpha))$ is bounded and so has a weak* limit point $\xi^* \in X^*$ (depending on m) which satisfies

$$T((a \otimes b_0) \cdot m) = a \cdot \xi^* \qquad (a \in A) . \tag{10}$$

Similarly, there is $\eta^* \in X^*$ with

$$T(m \cdot (a \otimes b_0)) = \eta^* \cdot a \qquad (a \in A) . \tag{11}$$

Applying T to (6) and (7) with $b = b_0$, gives nets (m_i') and (n_i') in X^* with

$$D(a) = \lim_i (a \cdot m_i' - n_i' \cdot a) \qquad (a \in A) , \tag{12}$$

$$\|a \cdot m_i' - n_i' \cdot a\| \leqslant K\|T\| \, \|a\| \qquad (a \in A) . \tag{13}$$

□

To get beyond semi-inner we first observe that if

$$D(a) = \lim_i (a \cdot m_i' - n_i' \cdot a) \qquad (a \in A) , \tag{14}$$

and D is a continuous derivation, then for $a_1, a_2 \in A$,

$$\begin{aligned} D(a_1 a_2) &= D(a_1)a_2 + a_1 D(a_2) \\ &= \lim_i \Big[(a_1 \cdot m_i' - n_i' \cdot a_1)a_2 + a_1(a_2 \cdot m_i' - n_i' \cdot a_2) \Big] \end{aligned} \tag{15}$$

and

$$D(a_1 a_2) = \lim_i (a_1 a_2 \cdot m_i' - n_i' \cdot a_1 a_2) . \tag{16}$$

Comparing (15) and (16) yields

$$\lim_i (a_1 \cdot (m_i' - n_i') \cdot a_2) = 0 . \tag{17}$$

Moreover, in the "bounded" case, we have

$$\|a_1 \cdot (m_i' - n_i') \cdot a_2\| \leqslant 3K\|a_1\| \cdot \|a_2\| . \tag{18}$$

We can now look at conditions that enable us to show that $m_i' = n_i'$, or at least $m_i' - n_i' \to 0$.

Theorem 4.2. *Let A and B be Banach algebras such that $A \widehat{\otimes} B$ is approximately amenable (resp. boundedly approximately amenable, boundedly approximately contractible). If B has an element b_0 with $b_0 \notin \overline{\{b_0 b - b b_0 : b \in B\}}$, then A is approximately amenable (resp. boundedly approximately amenable, boundedly approximately contractible).*

Proof. Choose the functional b_0^* in the proof of Theorem 4.1 to vanish on $\{b_0 b - bb_0 : b \in B\}$. Then the resulting nets (m_i') and (n_i') are the same. Hence the result. $\square$

Omitting 'approximately' the amenable case of Theorem 4.2 is exactly [18, Proposition 3.5]. Natural conditions on B which are sufficient for the above condition are listed in [18]. Note that there is unfortunately no conclusion about approximate amenability of B. Of course in special situations more can be said.

Throughout the next theorem G is a locally compact group and $L^1(G)$ is the usual group algebra of G.

Theorem 4.3. *Let A be a Banach algebra such that $L^1(G)\widehat{\otimes}A$ is approximately amenable (resp. (boundedly) approximately amenable). Then G is amenable and A is approximately amenable (resp. boundedly approximately amenable). Conversely, if G is amenable and A is boundedly approximately amenable with a bounded approximate identity, then $L^1(G)\widehat{\otimes}A$ is boundedly approximately amenable.*

Proof. Let $\Lambda : f \mapsto \int_G f$ be the augmentation character on $L^1(G)$. Then $T : f \otimes a \mapsto \Lambda(f)a$ gives a continuous epimorphism of $L^1(G)\widehat{\otimes}A$ onto A. Thus A is approximately amenable (resp. boundedly approximately amenable).

Let $I_0 = \mathrm{Ker}\Lambda$. Since $I_0\widehat{\otimes}A$ is a complemented ideal in $L^1(G)\widehat{\otimes}A$, by [10, Corollary 2.4] it has a left approximate identity. Hence I_0 has a left approximate identity [9, Theorem 8.2], and so G is amenable by [20, Theorem 5.2].

For the partial converse, G amenable implies $L^1(G)$ amenable, now apply Theorem 2.2. $\square$

Note that if $\Lambda(f_0) = 1$ then $L^1(G) \to I_0 : f \mapsto f - \Lambda(f)f_0$ is a bounded projection onto I_0, whence it follows that the norm on $I_0\widehat{\otimes}A$ is equivalent to that inherited from $L^1(G)\widehat{\otimes}A$. Hence the complementation fact.

Theorem 4.4. *Suppose that $A\widehat{\otimes}B$ is boundedly approximately contractible (resp. boundedly approximately amenable). Suppose that one of A or B has an identity. Then A and B are boundedly approximately contractible (resp. boundedly approximately amenable).*

Proof. Suppose that B has an identity e. Then, by Theorem 4.2, A is boundedly approximately contractible (resp. boundedly approximately amenable).

Now let X be a Banach B-bimodule. Then

$$X = e \cdot X \cdot e + (1-e) \cdot X \cdot e + e \cdot X \cdot (1-e) + (1-e) \cdot X \cdot (1-e)$$

is a decomposition into submodules. Given a continuous derivation $D : B \to X$, it decomposes into the sum of 4 derivations into $e \cdot X \cdot e$, $(1-e) \cdot X \cdot e$ etc. The last three of these have trivial module action by B on at least one side, so the corresponding derivations are inner. Thus we may suppose that $e \cdot x = x = x \cdot e$ for $x \in X$.

Let $D : B \to X^*$ be a continuous derivation, and consider the nets given by Theorem 4.1. For the boundedly approximately contractible situation, use clause (iii), for boundedly approximately amenable use clause (ii). Putting $a_1 = a_2 = e$ in (17) we have $m_i - n_i \to 0$, so that (12) and (13) give D is boundedly approximately inner. □

Lemma 4.5. *Let A be a Banach algebra having a bounded approximate identity. Suppose that any continuous derivation from A into the dual of a neo-unital bimodule is boundedly weak*-approximately inner. Then A is boundedly weak*-approximately amenable, and so approximately amenable. If, further, the implementing nets of elements can themselves be chosen to be bounded, then A is amenable.*

Proof. Let X be a general A-bimodule, $D : A \to X^*$ a continuous derivation. Let (e_α) be a bounded approximate identity for A. By Cohen's factorization theorem, $X_{ess} = A \cdot X \cdot A$ is a neo-unital A-bimodule. Referring to the argument used in Proposition 3.3 above we have

$$X^* = EFX^* \oplus E(I-F)X^* \oplus (I-E)X^* \,. \tag{19}$$

where E and F are commuting A-bimodule morphisms on X^*. Correspondingly, set

$$D_1 = EFD, D_2 = E(I-F)D, D_3 = (I-E)D \,.$$

Now let $\varphi \in (X_{ess})^*$, and extend it by Hahn-Banach to $\tilde{\varphi} \in X^*$. Then $\theta(\varphi) = EF\tilde{\varphi}$ is easily seen to be a well-defined A-bimodule monomorphism of $(X_{ess})^*$ into EFX^*. It is surjective since for $x^* \in X^*$, $\theta(x^*|_{X_{ess}}) = EFx^*$. Thus EFX^* is isomorphic to $(X_{ess})^*$, whence D_1 is boundedly weak*-approximately inner. Now this weak*-topology is $\sigma((X_{ess})^*, X_{ess})$, which is clearly weaker than the restriction of $\sigma(X^*, X)$. The unit ball in $(X_{ess})^*$ is compact under both topologies by Banach-Alaoglu, and so the two topologies coincide on bounded sets in $(X_{ess})^*$. Thus D_1 is boundedly weak*-approximately inner considered as mapping into X^*.

The actions of A on the right of $E(I-F)X^*$ and on the left of $(I-E)X^*$ are trivial, and since A has a bounded approximate identity, D_2 and D$_3$ are boundedly approximately inner. It follows that D is boundedly weak*-approximately inner.

That A is approximately amenable now follows from [14, Proposition 2.1].

When the implementing nets of elements are themselves bounded, taking weak* accumulation points shows that the derivations are inner, see [15, Proposition 1]. Hence A is amenable.

□

Remark 4.6. 1. The hypothesis here of the derivations being boundedly weak*-approximately inner is used to get equality of two weak*-topologies. Subsequently, the boundedness is lost with the appeal to [14, Proposition 2.1]. It is not known whether A must be boundedly approximately amenable.

2. In [14, Proposition 2.1], the argument loses control over boundedness as Goldstine is invoked on the implementing elements, which in general will be unbounded. Indeed, since boundedly approximately contractible gives a bounded approximate identity [5, Corollary 3.4], which approximately amenable algebras need not have [11, Corollary 3.2], the implication (2) $\Rightarrow$ (1) of [14, Proposition 2.1] fails with the qualifier 'bounded'. It is not known whether (3) $\Rightarrow$ (2) fails.

3. Note that by Banach-Steinhaus sequentially weak*-approximately inner implies boundedly weak*-approximately inner.

Theorem 4.7. *Let A and B be Banach algebras such that $A\widehat{\otimes}B$ is boundedly approximately amenable, and that A has a bounded approximate identity. Then A is approximately amenable.*

Proof. Let $D : A \to X^*$ be a continuous derivation into the dual of a neo-unital bimodule X. From Theorem 4.1(ii), we have nets (m'_i) and (n'_i) in X^* such that

$$D(a) = \lim_i (a \cdot m'_i - n'_i \cdot a) \qquad (a \in A)\,, \tag{20}$$

and $\|a \cdot m'_i - n'_i \cdot a\| \leqslant K\|a\|$, where from (17) and (18)

$$\lim_i (a_1 \cdot (m'_i - n'_i) \cdot a_2) = 0\,, \qquad \|a_1 \cdot (m'_1 - n'_i) \cdot a_2\| \leqslant 3K\|a_1\| \cdot \|a_2\| \tag{21}$$

for $a_1, a_2 \in A$.

In particular, for a given $x \in X$, and $a_1, a_2 \in A$,

$$\langle m'_i - n'_i, a_2 x a_1\rangle \to 0, \qquad |\langle m'_i - n'_i, a_2 x a_1\rangle| \leqslant 3K\|a_1\| \cdot \|a_2\| \cdot \|x\|\,.$$

Since X is neo-unital, it follows that

$$\langle m'_i - n'_i, x\rangle \to 0\,,$$

and letting a_1, a_2 range over an approximate identity with bound M,

$$\|m'_i - n'_i\| \leqslant 3KM^2\,.$$

Thus for $a \in A$,

$$D(a) = \text{weak}^* - \lim_i (a \cdot m'_i - m'_i \cdot a)\,, \qquad \|a \cdot m'_i - m'_i \cdot a\| \leqslant 4KM^2\|a\|\,. \tag{22}$$

So we have that derivations into duals of neo-unital bimodules are boundedly weak*- approximately inner, and the result follows from Lemma 4.5. □

The unwanted 'bounded' assumption of Lemma 4.5 and Theorem 4.7 can be removed at the expense of a stronger hypothesis on the bounded approximate identity. However, with this assumption comes a bonus to the conclusion of Theorem 4.7.

Theorem 4.8. *Let A and B be Banach algebras such that $A\widehat{\otimes}B$ is approximately amenable, and that one of A or B has a central bounded approximate identity. Then A and B are approximately amenable.*

Proof. Suppose that (e_α) is a central bounded approximate identity in B. Let $D : B \to X^*$ be a continuous derivation into the dual of a bimodule X. From Theorem 4.1(i), we have nets (m_i') and (n_i') in X^* such that

$$D(b) = \lim_i (b \cdot m_i' - n_i' \cdot b) \qquad (b \in B)\,, \tag{23}$$

and

$$\lim_i (b_1 \cdot (m_i' - n_i') \cdot b_2) = 0 \qquad (b_1, b_2 \in B)\,. \tag{24}$$

Now follow Lemma 4.5 to get D_1, D_2 and D_3. Then for $b \in B$,

$$\begin{aligned} D_1(b) &= (w^* - \lim_\alpha)(w^* - \lim_\beta) e_\alpha D(b) e_\beta \\ &= (w^* - \lim_\alpha)(w^* - \lim_\beta) \lim_i \Big(e_\alpha (b \cdot m_i' - n_i' \cdot b) e_\beta \Big)\,. \end{aligned} \tag{25}$$

Then (24) and (25) give, using centrality of the bounded approximate identity,

$$D_1(b) = (w^* - \lim_\alpha)(w^* - \lim_\beta) \lim_i \Big(b \cdot (e_\alpha \cdot m_i' \cdot e_\beta) - (e_\alpha \cdot n_i' \cdot e_\beta) \cdot b \Big)\,.$$

Thus the standard method of considering finite subsets of B and X, gives a net $(x_\gamma^*) \subset X^*$ such that

$$D_1(b) = w^* - \lim_\gamma (b \cdot x_\gamma^* - x_\gamma^* \cdot b)\,, \qquad (b \in B)\,.$$

Since D_2 and D_3 are approximately inner we finally deduce that D is weak*-approximately inner. Thus B is approximately amenable.

That A is approximately amenable is now a consequence of Theorem 4.2. □

Finally, an application of our method that improves on the result [18, Proposition 3.5].

Theorem 4.9. *Let A and B be Banach algebras such that $A\widehat{\otimes}B$ is amenable. Then A and B are amenable.*

Proof. Amenability of $A\widehat{\otimes}B$ implies it has a bounded approximate identity, whence so do A and B, [9, Theorem 8.2]. Now let Y be a neo-unital A-bimodule, $D : A \to Y^*$ a continuous derivation. It suffices to prove that D is inner. Arguing as in Theorem 4.1 with $X = Y^*$ until at (3) and using the necessary part of [15, Proposition 1] we obtain a bounded net (m_i) in $Y^*\widehat{\otimes}B$. Continuing, (5) gives bounded nets (m_i') and (n_i') in Y^* with $D(a) = \lim_i (a \cdot m_i' - n_i' \cdot a)$. Passing to subnets if necessary, we may assume that $m_i' \to m$ and $n_i' \to n$ weak* in Y^*. Thus D is semi-inner, hence inner by Proposition 3.3.

□

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