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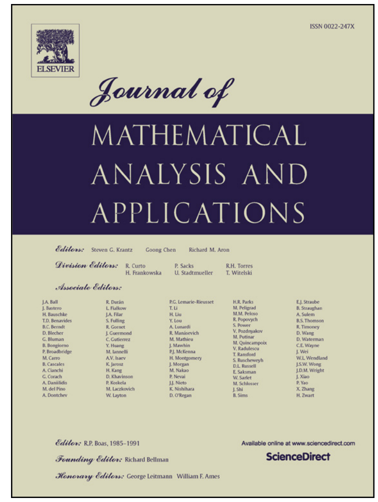

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SIGN-CHANGING STATIONARY SOLUTIONS AND BLOWUP FOR THE TWO POWER NONLINEAR HEAT EQUATION IN A BALL

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Abstract. Consider the nonlinear heat equation

$$u_t = \Delta u + |u|^{p-1}u - |u|^{q-1}u, \tag{0.1}$$

where $t \geq 0$ and $x \in \Omega$, the unit ball of $\mathbb{R}^N$, $N \geq 3$, with Dirichlet boundary conditions. Let h be a radially symmetric, sign-changing stationary solution of (0.1). We prove that the solution of (0.1) with initial value λh blows up in finite time if $|\lambda - 1| > 0$ is sufficiently small and if $1 < q < p < p_S = \frac{N+2}{N-2}$ and p sufficiently close to p_S. This proves that the set of initial data for which the solution is global is not star-shaped around 0.

1. Introduction

This paper studies finite-time blowup of sign-changing, regular solutions of the initial value problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u - |u|^{q-1}u, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{1.1}$$

Here, $u = u(t,x) \in \mathbb{R},\ t \geq 0,\ x \in \Omega$, and

$$\Omega = B_1, \tag{1.2}$$

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is the (open) unit ball of $\mathbb{R}^N$,

$$N \geq 3. \tag{1.3}$$

Furthermore, we consider

$$1 < q < p < p_S, \tag{1.4}$$

where

$$p_S = \frac{N+2}{N-2}. \tag{1.5}$$

It is well known that the initial value problem (1.1) is locally well-posed in $C_0(\Omega)$, where $C_0(\Omega)$ is the Banach space of continuous functions on $\overline{\Omega}$ that vanish on $\partial\Omega$, with the sup norm. More precisely, given $u_0 \in C_0(\Omega)$, there exists a maximal time $0 < T_{u_0} \leq \infty$ and a unique function $u \in C([0, T_{u_0}), C_0(\Omega)) \cap C\left((0, T_{u_0}), C^2(\overline{\Omega})\right) \cap C^1((0, T_{u_0}), C_0(\Omega))$ which is a classical solution of (1.1) on $(0, T_{u_0})$ and such that $u(0) = u_0$. Furthermore if $T_{u_0} < \infty$, then $\lim\limits_{t \uparrow T_{u_0}} \|u(t)\|_\infty = \infty$, and we say that u blows up in finite time. In addition, if $v \in C([0, T), C_0(\Omega)) \cap C\left((0, T), C^2(\overline{\Omega})\right) \cap C^1((0, T), C_0(\Omega))$ is a supersolution of (1.1), i.e $v_t - \Delta v \geq |v|^{p-1}v - |v|^{q-1}v$, $v_{|\partial\Omega} \geq 0$ and $v(0) \geq u_0$, then $v(t) \geq u(t)$ as long as both u and v are defined. The notion of subsolution is defined with reversed inequalities, yielding the analogous conclusion. See, for example Proposition 52.6 in [17].

We define the set $\mathcal{G}$ by

$$\mathcal{G} = \{u_0 \in C_0(\Omega), T_{u_0} = \infty\}.$$

It is interesting to study the geometrical properties of the set $\mathcal{G}$. First of all we note that every solution h of

$$\begin{cases} -\Delta h = |h|^{p-1}h - |h|^{q-1}h, \\ h_{|\partial\Omega} = 0, \end{cases} \tag{1.6}$$

is a stationary, hence global, solution of (1.1), whose initial value is of course $u_0 = h$, and so is in $\mathcal{G}$. Since the nonlinearity $|s|^{p-1}s - |s|^{q-1}s$ satisfies the properties of [3, Theorem 1.1, p. 15], it follows that the set $\mathcal{G}$ is not convex. As $u(t) = 0$ is a solution of (1.1) one can ask if $\mathcal{G}$ has the weaker property of being star-shaped around 0. The aim of this paper is to prove that $\mathcal{G}$ is not star-shaped.

This result is already well-known in the case of a single power nonlinearity

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{1.7}$$

In particular, it is proved in [2] that if h is a radially symmetric, sign-changing stationary solution of the problem (1.7), with $\Omega = B_1$, then the solution of (1.7) with initial value λh blows up in finite time if $|\lambda - 1| > 0$ is sufficiently small and if p is subcritical and sufficiently close to $p_S = \frac{N+2}{N-2}$. More precisely, there exists $1 < \underline{p} < p_S = \frac{N+2}{N-2}$ such that if $\underline{p} < p < p_S$

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and if $h \in C_0(\Omega)$ is a radially symmetric, sign-changing stationary solution of (1.7), then there exists $\varepsilon > 0$ such that if $0 < |\lambda - 1| < \epsilon$, then the classical solution of (1.7) with the initial condition $u(0) = \lambda h$ blows up in finite time. In particular, $\mathcal{G}$, for the problem (1.7), is not star-shaped.

The fact that h changes sign is fundamental in this affirmation. In fact in the case where $h > 0$ it follows from the comparison principle of the heat equation that if $0 < \lambda \leq 1$, then the solution is global and if $\lambda > 1$, then u blows up in finite time. For an elementary proof of the case $\lambda > 1$, see Theorem 17.8 in [17]. We remark, as was done in [2], that if h changes sign, then h and λh are not comparable if $\lambda \neq 1$.

In addition to the result in [2], it is known that $\mathcal{G}$ for the problem (1.7) is not star-shaped in several other circumstances :

- $N = 3$, $\Omega = B_1$ and $p > 1$ sufficiently near to 1, see [4] ;
- $N \geq 3$, Ω is a general domain and $p < p_S$ sufficiently near to p_S or $p = p_S$, see [14, 15] ;
- $N = 2$, $\Omega = B_1$ or Ω is a general domain and p sufficiently large, see [8, 9].

See [5, 6, 7, 10, 11, 12, 16] for other properties of the set $\mathcal{G}$ for the problem (1.7).

We now turn to problem (1.1), and we recall the following explosion criterion, see [2, Proposition B.1, p. 447].

Proposition 1.1 ([2, Proposition B.1, p. 447])**.** *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Let $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfy $g(0) = 0$,*

$$s^2 g'(s) \geq (1 + \epsilon) s g(s), \tag{1.8}$$

and

$$|g(s)| \leq C(1 + |s|^\beta), \tag{1.9}$$

for all $s \in \mathbb{R}$, where $\epsilon > 0$ and $1 \leq \beta < \frac{N+2}{N-2}$. Let $\psi \in C_0(\Omega)$ be a solution of the equation

$$\begin{cases} -\Delta \psi = g(\psi), \\ \psi_{|\partial\Omega} = 0. \end{cases} \tag{1.10}$$

Let $u_0 \in C_0(\Omega)$ and let $u \in C\left([0, T_{u_0}); C_0(\Omega)\right)$ be the maximal solution of

$$\begin{cases} u_t = \Delta u + g(u), \\ u_{|\partial\Omega} = 0, \end{cases} \tag{1.11}$$

with the initial condition $u(0) = u_0$. If $\psi^+ \neq 0$ and $u_0 \geq \psi$, $u_0 \neq \psi$, then u blows up in finite time. Similarly if $\psi^- \neq 0$ and $u_0 \leq \psi$, $u_0 \neq \psi$, then u blows up in finite time.

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Remark 1.2. *Note that if* $1 < q < p < p_S$*, then* $g(s) = |s|^{p-1}s - |s|^{q-1}s$ *satisfies (1.8) with* $\epsilon = q - 1$ *and (1.9) with* C *sufficiently large and* $\beta = p$*.*

It is immediate that if h is a positive solution of (1.6) with $1 < q < p < p_S$, and if u is the solution of (1.1) with initial value $u(0) = \lambda h$, then for $0 < \lambda \leq 1$, u is global (by the comparison principle) and if $\lambda > 1$, then u blows up in finite time (by Proposition 1.1).

The question remains as to whether or not the result in [2], cited above, concerning sign-changing solutions to (1.7) also carries over to sign-changing solutions of (1.1).

The point of view in this paper is to fix a value of q with

$$1 < q < p_S, \tag{1.12}$$

and then consider all p with

$$q < p < p_S. \tag{1.13}$$

In fact we will ultimately consider what happens as $p \to p_S$. The main purpose of this paper is to establish the following result.

Theorem 1. *Assume (1.2)-(1.3). Given* $1 < q < p_S = \frac{N+2}{N-2}$*. It follows that there exists* $1 < q < \underline{p} < p_S$ *with the following property. If* $\underline{p} < p < p_S$ *and if* $h_p \in C_0(\Omega)$ *is a radially symmetric stationary solution of (1.1) which takes both positive and negative values, then there exist* $0 < \underline{\lambda} < 1 < \overline{\lambda}$ *such that if* $\underline{\lambda} < \lambda < \overline{\lambda}$ *and* $\lambda \neq 1$*, then the classical solution of (1.1) with the initial condition* $u(0) = \lambda h$ *blows up in finite time.*

The first observation is that there does exists a radially symmetric, sign-changing stationary solution of (1.1), since the nonlinearity $|s|^{p-1}s - |s|^{q-1}s$ satisfies the hypothesis of [13, Theorem 2 , p. 376]. More precisely, if we consider the problem:

$$\begin{cases} h'' + \frac{N-1}{r}h' + |h|^{p-1}h - |h|^{q-1}h = 0, \\ h(0) = a > 0, \quad h'(0) = 0. \end{cases} \tag{1.14}$$

It is well-known by [13] that (1.14) admits a unique solution $h \in C^2([0, \infty), \mathbb{R})$, which we denote sometimes by $h_p(r, a)$ to emphasize the dependence on a. Recall that we are fixing a value of q satisfying (1.12) and letting p vary in the interval (1.13). Under these conditions, by Theorem 2 in [13] for all integer $m \geq 0$, there exists $a_{p,m}$ such that

a) $h_p(1, a_{p,m}) = 0$,
b) $h_p(r, a_{p,m})$ has precisely m zeros in $(0, 1)$.

In particular, $h_p(\cdot, a_{p,m})$, considered as a function on $\Omega = B_1$, is a radially symmetric solution of (1.6) which changes sign precisely m times.

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Now, let h_p be any nontrivial solution of (1.6) and consider the linearized operator F_p on $L^2(\Omega)$ defined by

$$\begin{cases} D(F_p) = H^2(\Omega) \cap H_0^1(\Omega), \\ F_p u = -\Delta u - \left(p|h_p|^{p-1} - q|h_p|^{q-1}\right) u, \qquad u \in D(F_p). \end{cases} \tag{1.15}$$

We recall the following result from [3].

Theorem 2 ([3, Corollary 2.5, p. 18])**.** *Let $h_p \in C_0(\Omega)$ be a sign-changing solution of (1.6). Let φ_p be a positive eigenvector of the self-adjoint operator F_p given by (1.15), corresponding to the first eigenvalue. Suppose that*

$$\int_\Omega h_p \varphi_p \neq 0.$$

It follows that there exists $\epsilon > 0$ such that if $0 < |1-\lambda| < \epsilon$, then the solution of (1.1) with the initial value $u_0 = \lambda h_p$ blows up in finite time.

To prove Theorem 1, it thus suffices to establish the following.

Theorem 3. *Assume (1.2)-(1.3). Given $1 < q < p_S = \frac{N+2}{N-2}$. It follows that there exists $1 < q < \underline{p} < p_S$ with the following property. If $\underline{p} < p < p_S$ and if $h_p \in C_0(\Omega)$ is a radially symmetric stationary solution of (1.1) which takes both positive and negative values, then*

$$\int_\Omega h_p \varphi_p \neq 0.$$

Where φ_p is a positive eigenvector of the self-adjoint operator F_p given by (1.15), corresponding to the first eigenvalue.

The proof of Theorem 3 is based on rescaling argument. Contrary to the case of single power nonlinearity, a rescaled function v_p defined by (2.4) below in terms of h_p, where h_p is a radially symmetric stationary solution of (1.1) doesn't satisfy the same differential equation satisfied by h_p, which make the situation more difficult. Also, unlike the case of the single power nonlinearity, there exist some solutions $v_p(r)$ of the problem (2.5) below which do not tend to zero as $r \to \infty$.

The rest of the paper is devoted to proving Theorem 3, which as already noted, implies Theorem 1 when combined with Theorem 2. Our basic approach follows that in [2]. However because of the differences just noted between the single power and the two power cases, many of the arguments in [2] do not immediately apply for the current situation.

Remark 1.3. *The results in this paper are equally valid for*

$$u_t = \Delta u + |u|^{p-1}u - c|u|^{q-1}u,$$

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for any $c > 0$. The case where $c < 0$ is not as clear, since in that case, the proof of Proposition 2.1 below is no longer valid.

2. Stationary solutions

The proof of Theorem 3 exploits strongly the radial symmetry of the stationary solutions. By abuse of notation we will use the same letter, for example h, to denote a radially symmetric function $h : \mathbb{R}^N \to \mathbb{R}$, and the corresponding function $h : [0, \infty) \to \mathbb{R}$ such that, $h(x) = h(|x|)$, $\forall x \in \mathbb{R}^N$. Throughout this paper, we will use this convention without further comment.

Any radially symmetric solution $h_p \in C_0(\Omega)$ of (1.6) satisfies the ODE

$$\begin{cases} h_p'' + \frac{N-1}{r} h_p' + |h_p|^{p-1} h_p - |h_p|^{q-1} h_p = 0, \\ h_p'(0) = h_p(1) = 0. \end{cases} \tag{2.1}$$

Since $h_p \neq 0$, it follows by uniqueness for the ODE (2.1) that $h_p(0) \neq 0$. Therefore, since if u satisfies (1.1) then $-u$ satisfies the same problem, it suffice to prove Theorem 3 under the additional assumption

$$h_p(0) > 0. \tag{2.2}$$

In the rest of this paper we set

$$h_p(0) = a_p > 0.$$

Clearly $h_p(r) = h_p(r, a_p)$, where $h_p(., a_p)$ is the solution of (1.14) with $a = a_p$. We let $\lambda_p > 0$ be such that

$$\lambda_p^{\frac{2}{p-1}} = a_p, \tag{2.3}$$

also we define

$$v_p(r) = \lambda_p^{-\frac{2}{p-1}} h_p\left(\frac{r}{\lambda_p}, \lambda_p^{\frac{2}{p-1}}\right). \tag{2.4}$$

A simple calculation shows that v_p satisfies

$$\begin{cases} v'' + \frac{N-1}{r} v' + |v|^{p-1} v - \lambda_p^{-\frac{2}{p-1}(p-q)} |v|^{q-1} v = 0, \\ v(0) = 1, \quad v'(0) = 0. \end{cases} \tag{2.5}$$

As such, v_p may be considered as a function $[0, \infty) \to \mathbb{R}$. It is known by [13, Lemma 1, p. 371] that $a_p \geq 1$. In fact, if $0 < a_p \leq \left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}}$ then $h_p(r, a_p) > 0$ for all $r > 0$. Thus,

$$\lambda_p \geq 1. \tag{2.6}$$

We have also

$$v_p(\lambda_p) = 0. \tag{2.7}$$

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Proposition 2.1. *Let λ_p defined in (2.3), then*

$$\lambda_p \underset{p\to p_S}{\longrightarrow} \infty. \tag{2.8}$$

Proof. Suppose to the contrary that $\lambda_p \nrightarrow \infty$ as $p \to p_S$. It follows that there exists a subsequence (p_k) such that $p_k \underset{k\to\infty}{\longrightarrow} p_S$ and

$$\lambda_{p_k} \underset{k\to\infty}{\longrightarrow} \overline{\lambda}, \tag{2.9}$$

where $1 \leq \overline{\lambda} < \infty$, by (2.6). By continuous dependence it follows that

$$v_{p_k} \underset{k\to\infty}{\longrightarrow} \overline{v}, \tag{2.10}$$

uniformly on all compact intervals $[0, M] \subset [0, \infty)$, where $\overline{v}$ satisfies

$$\begin{cases} \overline{v}'' + \frac{N-1}{r}\overline{v}' + |\overline{v}|^{p_S-1}\overline{v} - \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}|\overline{v}|^{q-1}\overline{v} = 0, \\ \overline{v}(0) = 1, \quad \overline{v}'(0) = 0. \end{cases} \tag{2.11}$$

It follows from (2.7), (2.9) and (2.10) that

$$\overline{v}(\overline{\lambda}) = 0. \tag{2.12}$$

And so $\overline{v}$ satisfies the equation

$$\begin{cases} -\Delta\overline{v} = |\overline{v}|^{p_S-1}\overline{v} - \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}|\overline{v}|^{q-1}\overline{v}, \\ \overline{v}_{|\partial B(0,\overline{\lambda})} = 0. \end{cases} \tag{2.13}$$

If we apply the Pohozaev identity as was done in [1, Remark 1.2, p. 442], and if we set $g(u) = |u|^{p_S-1}u - \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}|u|^{q-1}u$ and $G(u) = \frac{|u|^{p_S+1}}{p_S+1} - \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}\frac{|u|^{q+1}}{q+1}$, we obtain

$$\begin{aligned} \frac{2-N}{2}\int_{B(0,\overline{\lambda})} g(\overline{v})\overline{v} + N\int_{B(0,\overline{\lambda})} G(\overline{v}) &= \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}\left[\frac{N-2}{2} - \frac{N}{q+1}\right]\int_{B(0,\overline{\lambda})} |\overline{v}|^{q+1} \\ &= \frac{1}{2}\int_{\partial B(0,\overline{\lambda})} (x.\nu)\left(\frac{\partial\overline{v}}{\partial\nu}\right)^2 \geq 0. \end{aligned} \tag{2.14}$$

From (2.14), one can conclude that

$$0 \leq \left(\frac{N-2}{2} - \frac{N}{q+1}\right)\overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}\|\overline{v}\|^{q+1}_{L^{q+1}(B(0,\overline{\lambda}))}. \tag{2.15}$$

Since $q < p_S$ inequality (2.15) is possible only if $\overline{v} = 0$, which contradicts $\overline{v}(0) = 1$. ◻

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Let now w_p be the solution of

$$\begin{cases} w'' + \frac{N-1}{r}w' + |w|^{p-1}w = 0, \\ w(0) = 1, \quad w'(0) = 0. \end{cases} \tag{2.16}$$

It is well-known and easy to verify that w_{p_S} given by

$$w_{p_S}(r) = \left(1 + \frac{1}{N(N-2)}r^2\right)^{-\frac{N-2}{2}} \tag{2.17}$$

is the solution of (2.16) with $p = p_S$.

Proposition 2.2. *Let v_p defined by (2.4) and w_{p_S} by (2.17), then*

$$v_p \underset{p \to p_S}{\longrightarrow} w_{p_S}, \tag{2.18}$$

uniformly on bounded sets of $[0, \infty)$.

Proof. By Proposition 2.1 $\lambda_p \to \infty$ as $p \to p_S$, and so by continuous dependence we can conclude that

$$v_p \underset{p \to p_S}{\longrightarrow} w_{p_S},$$

uniformly on bounded sets of $[0, \infty)$. □

Proposition 2.3. *Given $1 < q < p_S$ and $0 < \eta < p_S - q$. There exist $M, C > 0$ such that for all $p \in [q + \eta, p_S)$ and $r \geq 0$,*

$$|v_p(r)| \leq M \text{ and } |v_p'(r)| \leq C. \tag{2.19}$$

Proof. Let $1 < q < p_S$ and $0 < \eta < p_S - q$. Note first that by (2.5)

$$\left[\frac{1}{2}v_p'(r)^2 + \frac{1}{p+1}|v_p(r)|^{p+1} - \frac{1}{q+1}\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p(r)|^{q+1}\right]' = -\frac{N-1}{r}|v_p'(r)|^2, \tag{2.20}$$

so that

$$\begin{aligned} \frac{1}{2}v_p'(r)^2 + \frac{1}{p+1}|v_p(r)|^{p+1} - \frac{1}{q+1}\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p(r)|^{q+1} &\leq \frac{1}{p+1} - \frac{1}{q+1}\lambda_p^{-\frac{2}{p-1}(p-q)} \\ &\leq \frac{1}{p+1}. \end{aligned} \tag{2.21}$$

Now since λ_p satisfies (2.6), it follows from (2.21) that

$$\frac{1}{p+1}|v_p(r)|^{p+1} - \frac{1}{q+1}|v_p(r)|^{q+1} \leq \frac{1}{p+1}. \tag{2.22}$$

Suppose by contradiction that, there exist $(p_n) \subset [q + \eta, p_S)$ and $(r_n) \subset [0, \infty)$ such that

$$|v_{p_n}(r_n)| \underset{n \to \infty}{\longrightarrow} \infty.$$

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Since (p_n) is bounded we can suppose that $p_n \to p_* \in [q+\eta, p_S]$, we apply now inequality (2.22), which we note as

$$|v_p(r)|^{p+1}\left(\frac{1}{p+1} - \frac{1}{q+1}|v_p(r)|^{q-p}\right) \leq \frac{1}{p+1},$$

with $p = p_n$, $r = r_n$. By letting $n \to \infty$, it follows that

$$\infty \leq \frac{1}{p_* + 1},$$

which is absurd. It follows so that there exists $M > 0$, such that for all $p \in [q+\eta, p_S)$ and $r \geq 0$,

$$|v_p(r)| \leq M. \tag{2.23}$$

We turn now to prove the second assertion. It follows from (2.21), $\lambda_p \geq 1$, (2.23) and $p > q$ that

$$\begin{aligned} \frac{1}{2}v_p'(r)^2 &\leq \frac{1}{p+1} + \frac{1}{q+1}|v_p(r)|^{q+1} \\ &\leq \frac{1}{q+1} + \frac{1}{q+1}M^{q+1},\ \forall p \in [q+\eta, p_S),\ \forall r \geq 0, \end{aligned}$$

so that

$$|v_p'(r)| \leq \sqrt{\frac{2}{q+1}}\sqrt{1+M^{q+1}},\ \forall p \in [q+\eta, p_S),\ \forall r \geq 0.$$

□

The following lemma is one of the key points which differ from the calculations in [2]. Compare Lemma 3.3 in [2]. Indeed, Lemma 3.3 in [2] cannot be true in the present context since not all solutions v_p of (2.5) tend to 0 as $r \to \infty$. We do obtain, however, a similar estimate, valid only for $r \leq \lambda_p$.

Lemma 2.4. *Given $1 < q < p_S$ and $0 < \eta < p_S - q$. There exists a constant $\gamma = \gamma(N,q)$ such that*

$$\frac{1}{2}|v_p'(r)|^2 + \frac{1}{p+1}|v_p(r)|^{p+1} \leq \gamma\left[\frac{1}{r+1} + \frac{1}{(r+1)^{\frac{2}{p_S-1}\eta}}\right], \tag{2.24}$$

for all $p \in [q+\eta, p_S)$ and for all $0 \leq r \leq \lambda_p$.

Proof. Fix $1 < q < p_S$ and $0 < \eta < p_S - q$. Let r such that $1 \leq r \leq \lambda_p$ and $p \in [q+\eta, p_S)$. Define now

$$F(r) = \frac{1}{2}v_p'(r)^2 + \frac{1}{p+1}|v_p(r)|^{p+1} - \frac{1}{q+1}\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p(r)|^{q+1} + \frac{1}{r}v_p(r)v_p'(r). \tag{2.25}$$

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It follows from (2.20) and (2.5) that

$$\begin{aligned} F'(r) &= -\frac{N-1}{r}v_p'(r)^2 - \frac{1}{r^2}v_p(r)v_p'(r) + \frac{1}{r}v_p'(r)^2 + \frac{1}{r}v_p(r)v_p''(r) \\ &= -\frac{N-2}{r}v_p'(r)^2 - \frac{1}{r^2}v_p(r)v_p'(r) + \frac{1}{r}v_p(r)v_p''(r) \\ &= -\frac{N-2}{r}v_p'(r)^2 - \frac{1}{r^2}v_p(r)v_p'(r) + \\ &\quad \frac{1}{r}v_p(r)\left[-\frac{N-1}{r}v_p'(r) - |v_p(r)|^{p-1}v_p(r) + \lambda_p^{-\frac{2}{p-1}(p-q)}|v_p(r)|^{q-1}v_p(r)\right]. \end{aligned}$$

From (2.19), (1.3), the fact that $1 \le r \le \lambda_p$, $1 < q < p$, Young's inequality (applied twice) and denoting $\alpha := \frac{p-q}{p+1} + \frac{2}{p-1}(p-q)$, one can find the estimate

$$\begin{aligned} F'(r) + \frac{1}{r}F(r) &= -\frac{2N-5}{2r}v_p'(r)^2 - \frac{p}{(p+1)r}|v_p(r)|^{p+1} - \frac{N-1}{r^2}v_p(r)v_p'(r) \\ &\quad + \lambda_p^{-\frac{2}{p-1}(p-q)}\frac{q}{(q+1)r}|v_p(r)|^{q+1} \\ &\le -\frac{2N-5}{2r}v_p'(r)^2 - \frac{p}{(p+1)r}|v_p(r)|^{p+1} + \frac{1}{2}\left[\frac{(N-1)^2}{r^3}v_p(r)^2 + \frac{1}{r}v_p'(r)^2\right] \\ &\quad + \frac{q}{q+1}|v_p(r)|^{q+1}r^{-\frac{q+1}{p+1}}r^{-\alpha} \\ &\le \frac{(N-1)^2}{2r^3}v_p(r)^2 - \frac{p}{(p+1)r}|v_p(r)|^{p+1} \\ &\quad + \frac{q}{(p+1)r}|v_p(r)|^{p+1} + \frac{q(p-q)}{(q+1)(p+1)}r^{-\alpha\frac{p+1}{p-q}} \\ &\le \frac{(N-1)^2}{2r^3}M^2 + \frac{q(p_S-q)}{(q+1)^2}r^{-\alpha\frac{p+1}{p-q}}. \end{aligned}$$

Now since $\alpha\frac{p+1}{p-q} = 1 + 2\frac{p+1}{p-1} \ge 3$, we obtain that for $1 \le r \le \lambda_p$

$$F'(r) + \frac{1}{r}F(r) \le Ar^{-3}.$$

One can conclude now for all $s \in [1, \lambda_p]$, for all $p \in [q+\eta, p_S)$ that

$$\frac{d}{ds}\left(sF(s)\right) = sF'(s) + F(s) \le As^{-2}. \tag{2.26}$$

Integration of (2.26) on $[1, r]$ gives

$$rF(r) - F(1) \le A\left(-\frac{1}{r} + 1\right).$$

We can affirm for $r \in [1, \lambda_p]$ that

$$F(r) \le B\frac{1}{r}. \tag{2.27}$$

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Using also (2.25), (2.27), (2.19), $p \in [q+\eta, p_S)$ and the fact that $1 \le r \le \lambda_p$, it follows that

$$\begin{aligned}\frac{1}{2}v_p'(r)^2 + \frac{1}{p+1}|v_p(r)|^{p+1} &\le \frac{1}{q+1}\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p(r)|^{q+1} - \frac{1}{r}v_p(r)v_p'(r) + B\frac{1}{r}\\ &\le \frac{1}{q+1}\frac{1}{r^{\frac{2}{p_S-1}\eta}}M^{q+1} + \frac{M.C}{r} + B\frac{1}{r}.\end{aligned}$$

Finally, using (2.19) one can conclude that there exists $\gamma > 0$ such that

$$\frac{1}{2}v_p'(r)^2 + \frac{1}{p+1}|v_p(r)|^{p+1} \le \gamma\left[\frac{1}{r+1} + \frac{1}{(r+1)^{\frac{2}{p_S-1}\eta}}\right],$$

for all $0 \le r \le \lambda_p$. □

We set

$$\widetilde{v}_p(r) = \begin{cases} v_p(r) \text{ if } 0 \le r \le \lambda_p,\\ 0 \quad\quad \text{ if } r > \lambda_p.\end{cases} \tag{2.28}$$

Corollary 2.5. *Given* $1 < q < p_S$ *and* $0 < \eta < p_S - q$. *There exists a decreasing function* $j : [0,\infty) \to [0,\infty)$ *satisfying* $j(r) \underset{r\to\infty}{\longrightarrow} 0$ *such that*

$$|\widetilde{v}_p(r)| \le j(r), \ \forall r \ge 0, \ \forall p \in [q+\eta, p_S). \tag{2.29}$$

Proposition 2.6. $\|\widetilde{v}_p - w_{p_S}\|_{L^\infty(\mathbb{R}^N)} \to 0$, *as* $p \to p_S$.

Proof. Fix $1 < q < p_S$, $0 < \eta < p_S - q$ and $R \ge 0$. Let $p \in [q+\eta, p_S)$, on the one hand it follows from (2.29) and (2.17) that

$$\begin{aligned}|\widetilde{v}_p(r) - w_{p_S}(r)| &\le |\widetilde{v}_p(r)| + w_{p_S}(r)\\ &\le j(r) + w_{p_S}(r)\\ &\le j(R) + w_{p_S}(R), \ \forall r \ge R.\end{aligned}$$

It follows that

$$\sup_{r\ge R}|\widetilde{v}_p(r) - w_{p_S}(r)| \le j(R) + w_{p_S}(R) \underset{R\to\infty}{\longrightarrow} 0.$$

Thus, there exists $R_0 \ge 0$ such that

$$\sup_{r\ge R_0}|\widetilde{v}_p(r) - w_{p_S}(r)| \le \frac{\varepsilon}{2}. \tag{2.30}$$

On the other hand, since $\lambda_p \underset{p\to p_S}{\longrightarrow} \infty$, by choosing p_0 sufficiently close to p_S, we can assume that $R_0 \le \lambda_p$ for $p_0 \le p < p_S$. It follows from (2.18) that there exists $p_0 \le \overline{p} < p_S$ such that if $\overline{p} < p < p_S$ then

$$\sup_{r\in[0,R_0]}|\widetilde{v}_p(r) - w_{p_S}(r)| = \sup_{r\in[0,R_0]}|v_p(r) - w_{p_S}(r)| \le \frac{\varepsilon}{2}. \tag{2.31}$$

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One can conclude from (2.30) and(2.31). □

3. The linearized operator

We consider now the self-adjoint operator F_p defined on $L^2(\Omega)$ by

$$\begin{cases} D(F_p) = H^2(\Omega) \cap H_0^1(\Omega), \\ F_p u = -\Delta u - \left(p|h_p|^{p-1} - q|h_p|^{q-1}\right) u, \quad \forall u \in D(H_p). \end{cases} \tag{3.1}$$

We denote by

$$\theta_p = \theta_p\left(F_p\right), \tag{3.2}$$

its first eigenvalue and by φ_p the corresponding eigenvector, i.e.

$$F_p\varphi_p = -\Delta\varphi_p - (p|h_p|^{p-1} - q|h_p|^{q-1})\varphi_p = \theta_p\varphi_p, \tag{3.3}$$

where we require

$$\varphi_p > 0, \quad \|\varphi_p\|_{L^2(\Omega)} = 1. \tag{3.4}$$

Since φ_p is radially symmetric, it satisfies the ODE

$$\varphi_p'' + \frac{N-1}{r}\varphi_p' + \left(p|h_p|^{p-1} - q|h_p|^{q-1}\right)\varphi_p + \theta_p\varphi_p = 0. \tag{3.5}$$

In order to transform the operator F_p into another operator we introduce $l_p \in \mathbb{R}$ and ψ_p, a positive, spherically symmetric function on Ω_p defined by

$$\theta_p = \lambda_p^2 l_p, \quad \varphi_p(x) = \lambda_p^{\frac{N}{2}}\psi_p(\lambda_p x), \tag{3.6}$$

where

$$\Omega_p = B(0, \lambda_p). \tag{3.7}$$

It follows from (3.5), (2.4) and (3.6) that ψ_p satisfies the equation

$$\begin{cases} -\Delta\psi_p - \left[p|v_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p|^{q-1}\right]\psi_p = l_p\psi_p \quad \text{in } \Omega_p, \\ \psi_p = 0 \quad \text{on } \partial\Omega_p, \end{cases} \tag{3.8}$$

and that

$$\int_\Omega h_p\varphi_p = \lambda_p^{\frac{2}{p-1}-\frac{N}{2}}\int_{\Omega_p} v_p\psi_p, \tag{3.9}$$

and

$$\psi_p > 0, \quad \|\psi_p\|_{L^2(\Omega_p)} = 1. \tag{3.10}$$

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We have also that l_p is the first eigenvalue associated to the eigenvector ψ_p of the self-adjoint operator L_p defined on $L^2(\Omega_p)$ by

$$\left\{\begin{array}{l} D(L_p) = H^2(\Omega_p) \cap H_0^1(\Omega_p), \\ L_p u = -\Delta u - \left[p|v_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p|^{q-1}\right] u, \quad \forall u \in D(L_p). \end{array}\right. \tag{3.11}$$

Given $0 < p < p_S$, we set

$$J_p(w) = \int_{\Omega_p} |\nabla w|^2 - \int_{\Omega_p} \left[p|v_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p|^{q-1}\right] w^2, \tag{3.12}$$

for all $w \in H_0^1(\Omega_p)$, so that

$$l_p = \inf\left\{J_p(u), u \in H_0^1(\Omega_p), \|u\|_{L^2(\Omega_p)} = 1\right\}. \tag{3.13}$$

Also we define the self-adjoint operator L_* on $L^2(\mathbb{R}^N)$ by

$$\left\{\begin{array}{l} D(L_*) = H^2(\mathbb{R}^N), \\ L_* u = -\Delta u - p_S w_{p_S}^{p_S-1} u, \quad \forall u \in D(L_*), \end{array}\right. \tag{3.14}$$

where w_{p_S} is given by (2.17). We set

$$\lambda_* = \inf\left\{J_*(u), u \in H^1(\mathbb{R}^N), \|u\|_{L^2(\mathbb{R}^N)} = 1\right\}, \tag{3.15}$$

where

$$J_*(w) = \int_{\mathbb{R}^N} |\nabla w|^2 - p_S \int_{\mathbb{R}^N} w_{p_S}^{p_S-1} w^2, \tag{3.16}$$

for all $w \in H^1(\mathbb{R}^N)$. We recall now the following proposition from [2].

Proposition 3.1 ([2, Proposition 3.4, p. 439])**.** *If L_* is defined by (3.14) and λ_* is defined by (3.15), then the following properties hold.*

(i) *$\lambda_* < 0$ and λ_* is an eigenvalue of L_*.*

(ii) *There exists a unique eigenvector ψ_* of L_* corresponding to the eigenvalue λ_* which is positive, radially decreasing with $\|\psi_*\|_{L^2(\mathbb{R}^N)} = 1$.*

(iii) *If $(u_n)_{n\geq 1} \subset H^1(\mathbb{R}^N)$ is a minimizing sequence of (3.15) and $u_n \geq 0$, then $u_n \to \psi_*$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$.*

We set

$$\widetilde{\psi}_p(x) = \left\{\begin{array}{ll} \psi_p(x) & \text{if } 0 \leq |x| < \lambda_p, \\ 0 & \text{if } |x| \geq \lambda_p, \end{array}\right. \tag{3.17}$$

for all $1 < p < p_S$, so that

$$\widetilde{\psi}_p \in H^1(\mathbb{R}^N), \ \|\widetilde{\psi}_p\|_{L^2(\mathbb{R}^N)} = 1, \ \widetilde{\psi}_p \geq 0. \tag{3.18}$$

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Lemma 3.2. *Let $\psi \in H^1(\mathbb{R}^N)$ such that $\|\psi\|_{L^2(\mathbb{R}^N)} = 1$. Consider a smooth radial cut-off function $\eta : \mathbb{R}^N \to [0,1]$ such that $\eta(r) = 1$ for $r \leq \frac{1}{2}$ and $\eta(r) = 0$ for $r \geq 1$. Set*

$$k_\lambda(r) = \eta\left(\frac{r}{\lambda}\right)\psi(r), \tag{3.19}$$

and

$$u_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{L^2(\mathbb{R}^N)}}. \tag{3.20}$$

Then $u_\lambda \in H_0^1(\Omega_\lambda)$ and

$$\|u_\lambda - \psi\|_{H^1(\mathbb{R}^N)} \underset{\lambda\to\infty}{\longrightarrow} 0. \tag{3.21}$$

Where $\Omega_\lambda = B(0,\lambda)$.

Proof. This follows by standard arguments, using the observation that $\|k_\lambda\|_{L^2(\mathbb{R}^N)} \underset{\lambda\to\infty}{\longrightarrow} 1$.
□

Lemma 3.3. *Let l_p defined by (3.6), then*

$$l_p \to \lambda_* \quad as \quad p \to p_S.$$

Proof. We first use $\widetilde{\psi}_p$ as a test function in (3.15). It follows from (3.18) that

$$\begin{aligned}
\lambda_* \leq J_*(\widetilde{\psi}_p) &= J_p(\widetilde{\psi}_p) + \int_{\mathbb{R}^N}\left[p|\widetilde{v}_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p|^{q-1} - p_S w_{p_S}^{p_S-1}\right]\widetilde{\psi}_p^2 \\
&= l_p + \int_{\mathbb{R}^N}\left[p|\widetilde{v}_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p|^{q-1} - p_S w_{p_S}^{p_S-1}\right]\widetilde{\psi}_p^2 \\
&= l_p + \int_{\mathbb{R}^N}\left[p|\widetilde{v}_p|^{p-1} - p_S w_{p_S}^{p_S-1}\right]\widetilde{\psi}_p^2 - q\lambda_p^{-\frac{2}{p-1}(p-q)}\int_{\mathbb{R}^N}|\widetilde{v}_p|^{q-1}\widetilde{\psi}_p^2.
\end{aligned} \tag{3.22}$$

It follows from (3.22), Proposition 2.3 and (3.18) that

$$\lambda_* - l_p \leq \left\|p|\widetilde{v}_p|^{p-1} - p_S w_{p_S}^{p_S-1}\right\|_{L^\infty(\mathbb{R}^N)} + qM^{q-1}\lambda_p^{-\frac{2}{p-1}(p-q)}, \ \forall p \in [q+\eta, p_S). \tag{3.23}$$

One can conclude now by applying Proposition 2.6 and Proposition 2.1 that

$$\limsup_{p\to p_S}(\lambda_* - l_p) \leq 0. \tag{3.24}$$

Next, we would like to use ψ_* as a test function in (3.13), but $\psi_* \notin H_0^1(\Omega_p)$. Thus, we need to approximate ψ_* by a sequence in $H_0^1(\Omega_p)$. Consider a smooth radial cut-off function $\eta : \mathbb{R}^N \to [0,1]$ such that $\eta(r) = 1$ for $r \leq \frac{1}{2}$ and $\eta(r) = 0$ for $r \geq 1$. Setting

$$k_p(r) = \eta\left(\frac{r}{\lambda_p}\right)\psi_*(r), \tag{3.25}$$

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and

$$u_p = \frac{k_p}{\|k_p\|_{L^2(\mathbb{R}^N)}}. \tag{3.26}$$

it follows from Lemma 3.2 (since $\lambda_p \to \infty$ as $p \to p_S$) that

$$\|u_p - \psi_*\|_{H^1(\mathbb{R}^N)} \underset{p\to p_S}{\longrightarrow} 0. \tag{3.27}$$

Moreover $u_p \in H_0^1(\Omega_p)$, so that

$$l_p \leq J_p(u_p) = \lambda_* - J_*(\psi_*) + J_*(u_p) - J_*(u_p) + J_p(u_p). \tag{3.28}$$

On the one hand we have by Proposition 2.6 and Proposition 2.1 that

$$\begin{aligned} |J_*(u_p) - J_p(u_p)| &= \left| \int_{\mathbb{R}^N} \left[p|\widetilde{v}_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)} |\widetilde{v}_p|^{q-1} - p_S w_{p_S}^{p_S-1} \right] {u_p}^2 \right| \\ &\leq \int_{\mathbb{R}^N} \left| p|\widetilde{v}_p|^{p-1} - p_S w_{p_S}^{p_S-1} \right| {u_p}^2 + q\lambda_p^{-\frac{2}{p-1}(p-q)} \int_{\mathbb{R}^N} |\widetilde{v}_p|^{q-1} {u_p}^2 \\ &\leq \left\| p|\widetilde{v}_p|^{p-1} - p_S w_{p_S}^{p_S-1} \right\|_{L^\infty(\mathbb{R}^N)} + qM^{q-1} \lambda_p^{-\frac{2}{p-1}(p-q)} \underset{p\to p_S}{\longrightarrow} 0. \end{aligned} \tag{3.29}$$

On the other hand, using the fact that $|J_*(\psi_*) - J_*(u_p)| \leq \left| \|\nabla u_p\|^2_{L^2(\mathbb{R}^N)} - \|\nabla \psi_*\|^2_{L^2(\mathbb{R}^N)} \right| + p_S \displaystyle\int_{\mathbb{R}^N} \left| u_p^2 - \psi_*^2 \right|$, it easily follows from (3.27) and the dominated convergence theorem that

$$|J_*(\psi_*) - J_*(u_p)| \underset{p\to p_S}{\longrightarrow} 0. \tag{3.30}$$

We deduce from (3.29) and (3.30) that

$$-\liminf_{p\to p_S} (\lambda_* - l_p) = \limsup_{p\to p_S} (l_p - \lambda_*) \leq 0. \tag{3.31}$$

We can confirm so by (3.24) and (3.31) that

$$\liminf_{p\to p_S} (\lambda_* - l_p) = \limsup_{p\to p_S} (\lambda_* - l_p) = \lim_{p\to p_S} (\lambda_* - l_p) = 0.$$

The result follows now. $\square$

Lemma 3.4. *Given $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. There exists $C > 0$ such that*

$$|\widetilde{\psi}_p(r)| + |\psi_*(r)| \leq C \frac{1}{r^{\frac{N-1}{2}}} \leq C, \tag{3.32}$$

for all $r \geq 1$ and $q + \eta \leq p < p_S$.

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Proof. Fix $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. We affirm first that

$$l_p < 0 \quad \text{for all} \quad p \in (q, p_S). \tag{3.33}$$

In fact, since l_p satisfies (3.6), it suffice to prove that $\theta_p < 0$. We have on the one hand since $p > q$

$$\begin{aligned} \left(\int_{\Omega_p} h_p^2\right)\theta_p &\leq \int_{\Omega_p} |\nabla h_p|^2 - \int_{\Omega_p} (p|h_p|^{p-1} - q|h_p|^{q-1})h_p^2 \\ &\leq \int_{\Omega_p} |\nabla h_p|^2 - q\int_{\Omega_p} (|h_p|^{p+1} - |h_p|^{q+1}). \end{aligned} \tag{3.34}$$

On the other hand since h_p satisfies (2.1) it follows that

$$\int_{\Omega_p} |\nabla h_p|^2 = \int_{\Omega_p} \left(|h_p|^{p+1} - |h_p|^{q+1}\right). \tag{3.35}$$

It follows from (3.34) and (3.35) since $q > 1$ that $\theta_p < 0$.

We complete now our proof. Since $l_p < 0$, we deduce from (3.13) and Proposition 2.3 that

$$\|\nabla\widetilde{\psi}_p\|^2_{L^2(\mathbb{R}^N)} \leq p_S(M^{q-1} + M^{p_S-1}), \quad \forall p \in [q+\eta, p_S). \tag{3.36}$$

By (3.18), (3.36) and Strauss' radial lemma [18] that

$$|\widetilde{\psi}_p(r)| \leq c\sqrt{1 + p_S\left(M^{q-1} + M^{p_S-1}\right)}\frac{1}{r^{\frac{N-1}{2}}}, \tag{3.37}$$

for all $r \geq 1$.

A similar argument applies to ψ_* which completes the proof. □

Lemma 3.5. *Given $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. There exist R, $C > 0$, $\theta > 0$ and $q + \eta \leq p_0 < p_S$ such that*

$$|\widetilde{\psi}_p(r)| + |\psi_*(r)| \leq Ce^{-\theta r}, \tag{3.38}$$

for all $r \geq R$ and $p_0 \leq p < p_S$.

Proof. Fix $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. We start first by showing that there exists R, $C > 0$, $\theta > 0$ and $q + \eta \leq p_0 < p_S$ such that

$$|\psi_p(r)| \leq Ce^{-\theta r}, \tag{3.39}$$

for all $r \geq R$ (with $r \leq \lambda_p$) and $p_0 \leq p < p_S$. It follows from (3.8) that ψ_p satisfies

$$-\psi_p''(r) - \frac{N-1}{r}\psi_p'(r) - \left\{\left[p|\widetilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p(r)|^{q-1}\right] + l_p\right\}\psi_p(r) = 0, \tag{3.40}$$

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for all $0 \leq r < \lambda_p$. We would like to use a method of energy in equation (3.40), but the term $-\Big[p|\widetilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p(r)|^{q-1}\Big] - l_p$ is difficult to handle so we may estimate it. On the one hand, since $|p|\widetilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p(r)|^{q-1}| \leq L(r) \underset{r\to\infty}{\longrightarrow} 0$ by Corollary 2.5 and the fact that $p \in [q+\eta, p_S)$, it follows that there exists $R > 0$, such that for all $r \geq R$

$$-\left[p|\widetilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p(r)|^{q-1}\right] \geq \frac{\lambda_*}{4}. \tag{3.41}$$

On the other hand, since $-l_p \to -\lambda_*$ as $p \to p_S$ by Lemma 3.3, it follows that there exists $p_0 \in [q+\eta, p_S)$ such that for all $p_0 \leq p < p_S$

$$-l_p \geq -\frac{3}{4}\lambda_*. \tag{3.42}$$

Finally one can conclude from (3.41) and (3.42) that there exist $R > 0$ and $q+\eta \leq p_0 < p_S$ such that

$$-\left[p|\widetilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p(r)|^{q-1}\right] - l_p \geq -\frac{\lambda_*}{2} > 0, \tag{3.43}$$

for all $p_0 \leq p < p_S$ and all $r \geq R$. By choosing p_0 possibly larger, we also may assume that $\lambda_p > R$ for $p_0 \leq p < p_S$. Since $\psi_p \geq 0$, we deduce from (3.40) and (3.43) that

$$\psi_p'' + \frac{N-1}{r}\psi_p' \geq -\frac{\lambda_*}{2}\psi_p, \tag{3.44}$$

for all $R \leq r \leq \lambda_p$. We now claim that

$$\psi_p'(r) < 0, \tag{3.45}$$

for all $p_0 \leq p < p_S$ and all $R < r < \lambda_p$. We argue by contradiction and suppose that $\psi_p'(r_p) \geq 0$ for some $p_0 \leq p < p_S$ and some $R < r_p < \lambda_p$. Since $\psi_p(\lambda_p) = 0$, there exists $r_p \leq r_p' < \lambda_p$ such that $\psi_p'(r_p') = 0$ and $\psi_p''(r_p') \leq 0$. This is impossible by (3.44) since $\lambda* < 0$. Multiplying (3.44) by $\psi_p' < 0$, see that

$$\psi_p''\psi_p' + \frac{N-1}{r}\psi_p'\psi_p' \leq -\frac{\lambda_*}{2}\psi_p\psi_p',$$

which implies

$$\left(\psi_p'^2 + \frac{\lambda_*}{2}\psi_p^2\right)' \leq 0, \tag{3.46}$$

for $R \leq r \leq \lambda_p$. It follows from (3.46) that

$$\left[\psi_p'^2 + \frac{\lambda_*}{2}\psi_p^2\right](r) \geq \psi_p'(\lambda_p)^2 \geq 0,$$

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for $R < r < \lambda_p$. Since $\psi_p > 0$ and $\psi_p' < 0$, we obtain that $\psi_p' + \sqrt{-\frac{\lambda_*}{2}}\psi_p \leq 0$ for $R < r < \lambda_p$, so that

$$\psi_p(r) \leq \psi_p(R) e^{\sqrt{-\frac{\lambda_*}{2}}R} e^{-\sqrt{-\frac{\lambda_*}{2}}r},$$

for $R < r < \lambda_p$. By choosing $R \geq 1$ we have $\psi_p(R) \leq C$ by Lemma 3.4 . The exponential decay follows. As remarked in [2], the proof for ψ_* is similar. This completes the proof. □

Lemma 3.6. *$\widetilde{\psi}_p$ and $\psi_* \in L^1(\mathbb{R}^N)$. Moreover, $\|\widetilde{\psi}_p - \psi_*\|_{L^1(\mathbb{R}^N)} \to 0$ as $p \to p_S$.*

Proof. The proof is similar to the proof of Lemma 3.7 in [2]. □

Proof of Theorem 3 . Fix $q \in (1, p_S)$ and $0 < \eta < p_S - q$. Let $h_p \in C_0(\Omega)$ be a radially symmetric, sign-changing stationary solution of (1.1). Let φ_p be the positive eigenvector normalized in $L^2(\Omega)$ of the self-adjoint operator F_p given by (3.1), corresponding to the first eigenvalue. We have from Proposition 2.3

$$\begin{aligned}\left|\int_{\Omega_p} v_p\psi_p - \int_{\mathbb{R}^N} w_{p_S}\psi_*\right| &= \left|\int_{\mathbb{R}^N} \widetilde{v}_p\widetilde{\psi}_p - \int_{\mathbb{R}^N} w_{p_S}\psi_*\right| \\ &\leq \left|\int_{\mathbb{R}^N} \widetilde{v}_p\left(\widetilde{\psi}_p - \psi_*\right)\right| + \left|\int_{\mathbb{R}^N}\left(\widetilde{v}_p - w_{p_S}\right)\psi_*\right| \\ &\leq M\left\|\widetilde{\psi}_p - \psi_*\right\|_{L^1(\mathbb{R}^N)} + \|\psi_*\|_{L^1(\mathbb{R}^N)}\|\widetilde{v}_p - w_{p_S}\|_{L^\infty(\mathbb{R}^N)},\end{aligned}$$

$\forall p \in [q + \eta, p_S)$. It follows so by Lemma 3.6, Proposition 2.6 that

$$\int_\Omega v_p\varphi_p \underset{p \to p_S}{\longrightarrow} \int_{\mathbb{R}^N} w_{p_S}\psi_* > 0.$$

We can now conclude from (3.9)that there exists $1 < q < \underline{p} < p_S$ such that if $\underline{p} < p < p_S$, then

$$\int_\Omega h_p\varphi_p > 0.$$

This finishes the proof. □

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