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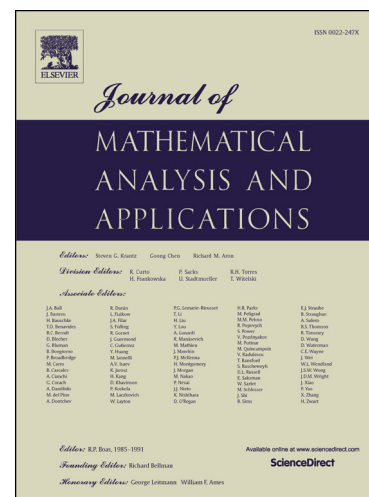
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Conditions on Unimodality and Logconcavity for Densities of Coherent Systems with an Application to Bernstein Operators

Mariusz Bieniek* Marco Burkschat[†] Tomasz Rychlik[‡]

Abstract

In this note, the distribution of the lifetime of a coherent system with independent and identically distributed component lifetimes is considered. Conditions yielding unimodality or logconcavity of the density function of the system lifetime are obtained. In the conditions, only assumptions on the density function of the components and on the signature of the system are imposed. The results are illustrated with several examples. Additionally, a problem on preservation of logconcavity under the classical Bernstein operator is solved.

Keywords: unimodality, logconcavity, signature, coherent system, variation diminishing property, Bernstein polynomial

2010 AMS Subject Classification: Primary: 62N05, Secondary: 60E05

1 Introduction

In the setting of coherent systems with independent and identically distributed (iid) component lifetimes, conditions for unimodality and logconcavity of the density function of the system lifetime distribution have been studied. Alam (1972) has obtained conditions for unimodality of the lifetime distribution of k -out-of- n systems. An extension to coherent system is given in Sabnis and Nair (1997). Recently, Bieniek and Burkschat (2018) derived conditions on the signature of a coherent system which yield unimodality or bimodality in the case of uniformly distributed component lifetimes. Logconcavity of the density of the system lifetime has been examined in Barlow and Proschan (1966), Huang and Ghosh (1982) and Franco et al. (2003). Moreover, further results on logconcavity can be found, e.g., in Barlow and Proschan (1981), Finner and Roters (1993), An (1998), Sengupta and Nanda (1999), Bagnoli and Bergstrom (2005), Marshall and Olkin (2007) and Alimohammadi et al. (2016). Unimodality and related properties are also central topics of the monograph by Dharmadhikari and Joag-dev (1988). Recently, the problem of preservation of logconcavity under the classical Bernstein operator has been studied by Badía (2009) and Badía and Sangüesa (2014).

It is well-known that the distribution function of the system lifetime T of a coherent system consisting of n iid components with underlying continuous distribution function F possesses the

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representation (cf. Samaniego 2007)

$$F_T(t) = \sum_{r=1}^n s_r F_{X_{r:n}}(t), \quad t \in \mathbb{R}, \quad (1.1)$$

where $X_{1:n} \leq \dots \leq X_{n:n}$ denote the order statistics of the component lifetimes X_1, \dots, X_n , and $F_{X_{r:n}}$ denotes the distribution function of the r th order statistic and $\mathbf{s} = (s_1, \dots, s_n)$ the signature of the system. In the following, we always impose that the distribution function F is absolutely continuous with a density function f . The density f is assumed to be positive only on the interval (a, b) with parameters a, b such that $0 \leq a < b \leq \infty$ (in fact, the following results are also valid for $-\infty \leq a < b \leq \infty$). The signature has the form $\mathbf{s} = (0, \dots, 0, s_p, \dots, s_q, 0, \dots, 0)$ with $s_p, \dots, s_q > 0$ for some p, q with $1 \leq p \leq q \leq n$. Moreover, we will put for convenience $s_0 = 0$ and $s_{n+1} = 0$. Imposing such structure of the signature represents no restriction, because it is known from Ross et al. (1980) (see also D'Andrea and De Sanctis 2015) that the signature of every coherent system has this shape. However, we point out that our results remain also valid for mixed systems (see Boland and Samaniego 2004) with a signature of the above form.

Furthermore, recall that a univariate continuous distribution function G is called unimodal with a mode at $m \in \mathbb{R}$ if G is convex on $(-\infty, m)$ and concave on (m, ∞) . If G has a density function g , which is positive on the interval (a, b) , where $-\infty \leq a < b \leq \infty$, and zero otherwise, then G (or the density g) is unimodal with mode m if there is some $m \in [a, b]$ such that g is increasing on (a, m) and decreasing on (m, b) . The density g is called logconcave if

$$g(\alpha x + (1 - \alpha)y) \geq (g(x))^\alpha (g(y))^{1-\alpha}$$

for all $x, y \in \mathbb{R}$ and $\alpha \in (0, 1)$. Clearly, this condition is equivalent to the logarithm of g being concave on the interval (a, b) . If g is logconcave, then G is unimodal (see, e.g., Marshall and Olkin 2007, Proposition B.2., p. 99). In fact, logconcavity of the density is connected to the notion of strong unimodality defined in Ibragimov (1956). A univariate distribution is called strongly unimodal if its convolution with any unimodal distribution is again unimodal. Strong unimodality can be characterized by the property of having a logconcave density function.

In this note, we obtain signature-based conditions for unimodality and strong unimodality of the lifetime distribution given in (1.1) for a coherent system with iid component lifetimes. At first, we study the situation of uniformly distributed component lifetimes in Section 2. In particular, we can answer affirmatively a question posed in Badía and Sangüesa (2014) on the preservation of logconcavity under the classical Bernstein operator. Based on the findings in Section 2, we give in Section 3 conditions on the signature \mathbf{s} and the density f which yield that the system lifetime density is unimodal or logconcave. Our results are illustrated with several examples.

2 The case of uniformly distributed component lifetimes

The density function of the system lifetime given in (1.1) for the case of iid component lifetimes following the standard uniform distribution is given by

$$g_{\mathbf{s}}(u) = \sum_{r=1}^n s_r f_r(u), \quad u \in (0, 1), \quad (2.1)$$

where

$$f_r(u) = f_{U_{r:n}}(u) = \frac{n!}{(r-1)!(n-r)!} u^{r-1} (1-u)^{n-r}, \quad r = 1, \dots, n, \quad (2.2)$$

represents the density function of the r th order statistic $U_{r:n}$ from iid random variables U_1, \dots, U_n with uniform distribution on the interval $[0, 1]$. In the next theorem, we present a condition on the signature which ensures logconcavity of the preceding density function (2.1).

Theorem 2.1. *If the function*

$$h_s(r) = \frac{s_{r+1} - s_r}{s_r}(n - r) + \frac{s_r - s_{r-1}}{s_r}(r - 1), \quad r \in \{p, \dots, q\},$$

is decreasing, then the density g_s is logconcave.

Proof. Let g'_s denote the derivative of g_s . In order to prove logconcavity of g_s , we show that g'_s/g_s is a decreasing function on the interval $(0, 1)$. This property is equivalent to the condition that for every $c \in \mathbb{R}$ the function

$$\frac{g'_s(u)}{g_s(u)} - c, \quad u \in (0, 1),$$

changes the sign at most once and, if there is a sign change, it is from $+$ to $-$. Since $g_s > 0$, it is sufficient to study the sign of $g'_s - cg_s$. It can be shown that (cf. Bieniek and Burkschat 2018)

$$g'_s(u) = \frac{1}{1-u} \sum_{r=1}^{n-1} (s_{r+1} - s_r)(n-r)f_r(u).$$

Moreover, note that

$$(n-r) \cdot \frac{f_r(u)}{1-u} = n \cdot \frac{(n-1)!}{(r-1)!(n-r-1)!} u^{r-1}(1-u)^{n-r-1} = n f^{U_{r:n-1}}(u)$$

and, by applying the well-known triangle rule for order statistics,

$$n f^{U_{r:n-1}}(u) = (n-r)f^{U_{r:n}}(u) + r f^{U_{r+1:n}}(u).$$

Consequently, we obtain

$$\begin{aligned} g'_s(u) &= \sum_{r=1}^{n-1} (s_{r+1} - s_r) ((n-r)f^{U_{r:n}}(u) + r f^{U_{r+1:n}}(u)) \\ &= \sum_{r=1}^n [(s_{r+1} - s_r)(n-r) + (s_r - s_{r-1})(r-1)] f^{U_{r:n}}(u) \end{aligned}$$

where $s_0 = 0, s_{n+1} = 0$. Thus,

$$g'_s(u) - c g_s(u) = \sum_{r=1}^n [(s_{r+1} - s_r)(n-r) + (s_r - s_{r-1})(r-1) - c \cdot s_r] f^{U_{r:n}}(u).$$

Observe that

$$f^{U_{r:n}}(u) = n \frac{(n-1)!}{(r-1)!(n-r)!} u^{r-1}(1-u)^{n-r} = n \cdot b_{r-1, n-1}(u),$$

where

$$b_{k,n}(u) = \binom{n}{k} u^k (1-u)^{n-k}, \quad k = 0, \dots, n, \quad (2.3)$$

denote the Bernstein polynomials of order n . Hence, the assertion follows from the variation diminishing property of the Bernstein polynomials, if for every $c \in \mathbb{R}$ there is at most one sign change in the sequence

$$(s_{r+1} - s_r)(n - r) + (s_r - s_{r-1})(r - 1) - c \cdot s_r, \quad r = 1, \dots, n,$$

and, if there is a sign change, it is a change from $+$ to $-$. Due to the assumed structure of the signature, we observe that, if $r = p - 1 \geq 1$, then

$$(s_{r+1} - s_r)(n - r) + (s_r - s_{r-1})(r - 1) - c \cdot s_r = s_p(n - (p - 1)) > 0.$$

Also, if $r = q + 1 \leq n$, then

$$(s_{r+1} - s_r)(n - r) + (s_r - s_{r-1})(r - 1) - c \cdot s_r = -s_q q < 0.$$

Hence, we have already shown the result for the case $p = q$. Now, let $p < q$. In particular, we assume $n \geq 2$. Then, the desired sign change behavior holds iff the function

$$h_s(r) = \frac{s_{r+1} - s_r}{s_r}(n - r) + \frac{s_r - s_{r-1}}{s_r}(r - 1)$$

is decreasing on the set $\{p, \dots, q\}$. This yields the assertion. \square

Theorem 2.1 confirms an obvious fact that the density functions (2.1) of all k -out-of- n systems are logconcave. Decreasing monotonicity of the sequence h_s is not a necessary condition for logconcavity of g_s which is seen from the following example.

Example 2.2. *The coherent system with 5 components and minimal path sets*

$$P_1 = \{2, 3, 4, 5\}, P_2 = \{1, 2\}, P_3 = \{1, 3\}, P_4 = \{1, 4\}, P_5 = \{1, 5\},$$

has signature $s = (0, \frac{2}{5}, \frac{1}{5}, \frac{2}{5}, 0)$ (cf. Navarro and Rubio 2009, Table 2, system 101). Here $p = 2$, $q = 4$, and sequence $h_s(2) = -\frac{1}{2}$, $h_s(3) = 0$, $h_s(4) = \frac{1}{2}$ is increasing. Nevertheless, we can prove that the logarithm of

$$g_s(u) = 8u(1 - u)^3 + 6u^2(1 - u)^2 + 8u^3(1 - u) = 2u(1 - u)(5u^2 - 5u + 4)$$

is concave. The polynomial has the derivative

$$g'_s(u) = 8(1 - u)^3 - 12u(1 - u)^2 + 12u^2(1 - u) - 8u^3 = 4(1 - 2u)(5u^2 - 5u + 2).$$

We prove the claim by checking that the ratio

$$\frac{g'_s(u)}{g_s(u)} = 2 \frac{1 - 2u}{u(1 - u)} \frac{5u^2 - 5u + 2}{5u^2 - 5u + 4}$$

is decreasing on $(0, 1)$. By simple analysis we show that the first fraction is antisymmetric about $\frac{1}{2}$, and decreasing from $+\infty$ at 0 to $-\infty$ at 1, and it vanishes at $\frac{1}{2}$. The second one is positive on $[0, 1]$, symmetric about $\frac{1}{2}$, decreasing on $(0, \frac{1}{2})$, and increasing on $(\frac{1}{2}, 1)$. The product is positive on $(0, \frac{1}{2})$, and decreasing from $+\infty$ to 0 there. Since it is antisymmetric about $\frac{1}{2}$ on $(0, 1)$, it further decreases from 0 at $\frac{1}{2}$ to $-\infty$ at 1. This ends the proof of logconcavity.

For every system with signature $\mathbf{s} = (s_1, \dots, s_n)$ and density function $g_{\mathbf{s}}(u)$, $0 < u < 1$, the respective dual system has the signature $\mathbf{s}^d = (s_n, \dots, s_1)$ (cf. Samaniego 2007, Theorem 3.3) and density function $g_{\mathbf{s}^d}(u) = g_{\mathbf{s}}(1 - u)$, $0 < u < 1$. Logconcavity of $g_{\mathbf{s}}$ implies logconcavity of $g_{\mathbf{s}^d}$. Observe that decrease of $h_{\mathbf{s}}$ is also equivalent with decrease of $h_{\mathbf{s}^d}$, because the following relation holds

$$h_{\mathbf{s}^d}(r) = -h_{\mathbf{s}}(n + 1 - r), \quad r \in \{n + 1 - q, \dots, n + 1 - p\}.$$

Accordingly, for a symmetric signature satisfying $s_{n+1-r} = s_r$, $r = 1, \dots, n$, the sequence $h_{\mathbf{s}}$ is antisymmetric about $(n + 1)/2$.

The following lemma establishes the connection between the condition in the previous theorem and the well-known notion of logconcavity for a discrete distribution.

Lemma 2.3. *If the signature is logconcave, i.e.*

$$s_i^2 \geq s_{i-1}s_{i+1}, \quad i = 1, \dots, n,$$

then the condition in Theorem 2.1 is satisfied.

Obviously, if \mathbf{s} is logconcave, so is \mathbf{s}^d .

Proof. In order to show that the function $h_{\mathbf{s}}$ is decreasing on the set $\{p, \dots, q\}$, consider for $p \leq r < q$:

$$\begin{aligned} & s_r s_{r+1} (h_{\mathbf{s}}(r) - h_{\mathbf{s}}(r + 1)) \\ &= (s_{r+1}^2 - s_r s_{r+1})(n - r) + (s_r s_{r+1} - s_{r-1} s_{r+1})(r - 1) \\ &\quad - ((s_r s_{r+2} - s_r s_{r+1})(n - r - 1) + (s_r s_{r+1} - s_r^2)r) \\ &= (s_{r+1}^2 - s_r s_{r+2})(n - r - 1) + s_{r+1}^2 - s_r s_{r+1} + (s_r^2 - s_{r-1} s_{r+1})(r - 1) - (s_r s_{r+1} - s_r^2) \\ &= (s_{r+1}^2 - s_r s_{r+2})(n - r - 1) + (s_r^2 - s_{r-1} s_{r+1})(r - 1) + (s_{r+1} - s_r)^2. \end{aligned}$$

This last expression is non-negative, since the signature is logconcave. \square

Example 2.4. *The signature (s_1, \dots, s_{2n}) of a k -out-of- $n:F$ system with systemwise redundancy (cf. Samaniego 2007, Theorem 4.7) is given by*

$$s_{2k+r} = \frac{\binom{n-1}{k-1} \binom{n}{k+r}}{\binom{2n-1}{2k+r-1}}, \quad r \in \{0, \dots, n - k\},$$

and all other entries are zero. It can be shown that

$$\frac{s_{2k+r}}{s_{2k+r+1}} = \frac{k+r+1}{2k+r} \cdot \left(\frac{n-k}{n-k-r} + 1 \right), \quad r \in \{0, \dots, n - k - 1\},$$

which is a product of increasing functions in r . Consequently, the signature is logconcave. Furthermore, the signature (s_1, \dots, s_{2n}) of a k -out-of- $n:F$ system with componentwise redundancy (cf. Samaniego 2007, Theorem 4.8) is given by

$$s_{2k+r} = \frac{\binom{n-1}{k-1} \binom{n-k}{r}}{\binom{2n-1}{2k+r-1}} \cdot 2^r, \quad r \in \{0, \dots, n - k\},$$

and all other entries are again zero. Here, we get

$$\frac{s_{2k+r}}{s_{2k+r+1}} = \frac{1}{2} \cdot \frac{r+1}{2k+r} \cdot \left(\frac{n-k}{n-k-r} + 1 \right), \quad r \in \{0, \dots, n - k - 1\},$$

and therefore this signature is also logconcave.

The next example shows that the condition in Theorem 2.1 is essentially weaker than log-concavity of the signature.

Example 2.5. *The 5-component system with the minimal path sets*

$$P_1 = \{1, 2\}, P_2 = \{1, 3\}, P_3 = \{1, 4\}, P_4 = \{2, 3, 4, 5\}$$

has the signature vector $\mathbf{s} = (0, \frac{2}{5}, \frac{3}{10}, \frac{3}{10}, 0)$ (cf. Navarro and Rubio 2009, Table 2, system 86). This signature is not logconcave (because $s_3^2 < s_2 s_4$), but the function $h_{\mathbf{s}}$ from Theorem 2.1 is decreasing, namely

$$h_{\mathbf{s}}(2) = \frac{1}{4}, h_{\mathbf{s}}(3) = -\frac{2}{3}, h_{\mathbf{s}}(4) = -1.$$

Evidently, the respective dual with the following path sets

$$P_1^d = \{1, 2\}, P_2^d = \{1, 3\}, P_3^d = \{1, 4\}, P_4^d = \{1, 5\}, P_5^d = \{2, 3, 4\}$$

(see Navarro and Rubio 2009, Table 2, system 110) has non-logconcave signature $\mathbf{s}^d = (0, \frac{3}{10}, \frac{3}{10}, \frac{2}{5}, 0)$, and decreasing

$$h_{\mathbf{s}^d}(2) = 1, h_{\mathbf{s}^d}(3) = \frac{2}{3}, h_{\mathbf{s}^d}(4) = -\frac{1}{4}.$$

We can also check that the assumption of Theorem 2.1 does not even imply unimodality of the signature.

Example 2.6. *A coherent system with 6 components and minimal path sets*

$$P_1^1 = \{1, 2\}, P_2^1 = \{1, 3, 4\}, P_3^1 = \{1, 3, 5\}, P_4^1 = \{1, 3, 6\}, P_5^1 = \{2, 3, 4, 5, 6\}$$

has bimodal signature $\mathbf{s}_1 = (0, \frac{2}{5}, \frac{1}{4}, \frac{17}{60}, \frac{1}{15}, 0)$ (cf. the list of systems with 6 components announced by Navarro and Rubio (2009)). We easily check that respective sequence

$$h_{\mathbf{s}_1}(2) = -\frac{1}{2}, h_{\mathbf{s}_1}(3) = -\frac{4}{5}, h_{\mathbf{s}_1}(4) = -\frac{20}{17}, h_{\mathbf{s}_1}(5) = -14$$

is decreasing. Obviously, its dual with minimal path sets

$$P_1^{1,d} = \{1, 2\}, P_2^{1,d} = \{1, 3\}, P_3^{1,d} = \{1, 4\}, P_4^{1,d} = \{1, 5\}, P_5^{1,d} = \{1, 6\}, P_6^{1,d} = \{2, 3\}, \\ P_7^{1,d} = \{2, 4, 5, 6\}$$

shares the properties, because respective signature \mathbf{s}_1^d and sequence $h_{\mathbf{s}_1^d}$ take on the forms $\mathbf{s}_1^d = (0, \frac{1}{15}, \frac{17}{60}, \frac{1}{4}, \frac{2}{5}, 0)$ and

$$h_{\mathbf{s}_1^d}(2) = 14, h_{\mathbf{s}_1^d}(3) = \frac{20}{17}, h_{\mathbf{s}_1^d}(4) = \frac{4}{5}, h_{\mathbf{s}_1^d}(5) = \frac{1}{2},$$

respectively.

Another example confirming our claim is provided by the system with the following minimal path sets

$$P_1^2 = \{1, 2\}, P_2^2 = \{1, 3\}, P_3^2 = \{1, 4\}, P_4^2 = \{1, 5\}, P_5^2 = \{2, 3\}, P_6^2 = \{2, 4, 5, 6\}, \\ P_7^2 = \{3, 4, 5, 6\}.$$

It has the signature $\mathbf{s}_2 = (0, 0, \frac{7}{20}, \frac{19}{60}, \frac{1}{3}, 0)$, and respective $h_{\mathbf{s}_2}$ is given by

$$h_{\mathbf{s}_2}(3) = \frac{12}{7}, h_{\mathbf{s}_2}(4) = -\frac{4}{19}, h_{\mathbf{s}_2}(5) = -\frac{4}{5}.$$

There are 5 other coherent systems with 6 components and identical signature \mathbf{s}_2 and function $h_{\mathbf{s}_2}$. By analogy, we establish that respective 6 dual systems possess a bimodal signature \mathbf{s}_2^d and decreasing $h_{\mathbf{s}_2^d}$ as well.

Finally, we give an example of a coherent systems that has a unimodal, but not strongly unimodal distribution.

Example 2.7. *The signature of the system with 4 components and minimal path sets*

$$P_1 = \{1, 2\}, P_2 = \{1, 3\}, P_3 = \{1, 4\}$$

is given by $\mathbf{s} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$ (cf. Samaniego 2007, Table 3.2, system 5). Note that the function $h_{\mathbf{s}}$ from Theorem 2.1 is not decreasing in this case:

$$h_{\mathbf{s}}(1) = 0, \quad h_{\mathbf{s}}(2) = 2, \quad h_{\mathbf{s}}(3) = 0.$$

For this system, we obtain

$$g_{\mathbf{s}}(u) = (1 - u)^3 + 3u(1 - u)^2 + 6u^2(1 - u), \quad u \in (0, 1),$$

and the second derivative of $\ln g_{\mathbf{s}}$ is given by

$$(\ln g_{\mathbf{s}}(u))'' = (-6) \cdot \frac{8u^4 - 8u^3 + 3u^2 + 4u - 1}{(u - 1)^2(4u^2 + u + 1)^2}, \quad u \in (0, 1).$$

Since the nominator of the fraction is equal to -1 for $u = 0$, it follows that $g_{\mathbf{s}}$ is not logconcave. However, because the signature is unimodal, we can conclude from Bieniek and Burkschat (2018) that $g_{\mathbf{s}}$ is unimodal. We immediately conclude that the corresponding dual system with minimal path sets

$$P_1^d = \{1\}, P_2^d = \{2, 3, 4\}$$

(cf. Samaniego 2007, Table 3.2, system 6) has density function $g_{\mathbf{s}^d}(u) = g_{\mathbf{s}}(1 - u)$ which is unimodal, but not strongly unimodal.

In Badía (2009) and Badía and Sangüesa (2014) preservation properties of Bernstein-type operators are studied. In particular, the classical Bernstein operator

$$B_n(w, x) = \sum_{k=0}^n w\left(\frac{k}{n}\right) b_{k,n}(x), \quad x \in [0, 1], n \in \mathbb{N},$$

with a real-valued function w defined on the interval $[0, 1]$ and the Bernstein polynomials $b_{k,n}$ defined by (2.3) is considered. It is left as an open problem whether the Bernstein operator preserves logconcavity. Our previous arguments can be applied to this problem and they provide a positive answer.

Theorem 2.8. *The Bernstein operator preserves logconcavity, i.e., if w is logconcave on $[0, 1]$, then $B_n(w, \cdot)$ is also logconcave on $[0, 1]$.*

Proof. If $w(k/n) = 0$ for every $k = 0, \dots, n$, then the assertion is trivially correct. Therefore, w.l.o.g. we assume that there is some $k_0 \in \{0, \dots, n\}$ such that $w(k_0/n) > 0$. Define

$$\tilde{s}_{k+1} = \frac{1}{\sigma_w} \cdot w\left(\frac{k}{n}\right), \quad k \in \{0, \dots, n\}, \quad \text{with } \sigma_w = \sum_{i=0}^n w\left(\frac{i}{n}\right).$$

Due to the logconcavity of w , there exist p, q with $1 \leq p \leq q \leq n + 1$ such that $\tilde{s}_p, \dots, \tilde{s}_q > 0$ and $\tilde{s}_1 = \dots = \tilde{s}_{p-1} = \tilde{s}_{q+1} = \dots = \tilde{s}_{n+1} = 0$. Moreover, the sequence $\tilde{s}_1, \dots, \tilde{s}_{n+1}$ is also logconcave, because by logconcavity of w we have

$$w\left(\frac{1}{2} \cdot \frac{k-1}{n} + \frac{1}{2} \cdot \frac{k+1}{n}\right) \geq \left(w\left(\frac{k-1}{n}\right)\right)^{1/2} \left(w\left(\frac{k+1}{n}\right)\right)^{1/2}, \quad k \in \{1, \dots, n-1\}.$$

Therefore, by observing that

$$B_n(w, x) = \sum_{k=0}^n w \left(\frac{k}{n} \right) b_{k,n}(x) = \sigma_w \sum_{r=1}^{n+1} \tilde{s}_r b_{r-1,n}(x) = \frac{\sigma_w}{n+1} \sum_{r=1}^{n+1} \tilde{s}_r f^{U_{r:n+1}}(x)$$

can be treated (up to a positive factor) as the density function of a mixed system with $n+1$ components and the signature vector $(\tilde{s}_1, \dots, \tilde{s}_{n+1})$ (see (2.1) and (2.2)), the result follows from Theorem 2.1 and Lemma 2.3. \square

3 Unimodality and logconcavity for coherent systems

Now, we consider the lifetime of a system with iid components which are distributed according to a distribution function F with density f as described in Section 1. In this situation, the density function of the system lifetime is given by

$$f_T(t) = g_s(F(t))f(t) = \sum_{r=1}^n s_r f_r(F(t))f(t), \quad t \in (a, b).$$

The following result can be concluded from Sabnis and Nair (1997) (see also Bieniek and Burkschat 2018, Theorem 2.1) and Theorem 2.1.

Theorem 3.1. *If $1/f$ is convex on (a, b) and the system signature \mathbf{s} satisfies the condition in Theorem 2.1, then the density function f_T is unimodal.*

The condition on the underlying density in the preceding theorem is satisfied for logconcave f . Therefore, by utilizing Lemma 2.3, we obtain the following corollary.

Corollary 3.2. *If f is logconcave on (a, b) and the system signature \mathbf{s} is logconcave, then the density function f_T is unimodal.*

The following example illustrates that the conclusion in the previous corollary cannot be strengthened to logconcavity of the system lifetime density.

Example 3.3. *We consider the 5-component system with minimal path sets*

$$P_1 = \{1, 2, 3\}, P_2 = \{1, 2, 4, 5\}, P_3 = \{1, 3, 4, 5\}, P_4 = \{2, 3, 4, 5\}.$$

Its system signature is given by $\mathbf{s} = (0, \frac{9}{10}, \frac{1}{10}, 0, 0)$ (cf. Navarro and Rubio 2009, Table 2, system 11). In particular, the signature is trivially logconcave. Additionally, we assume that the component lifetimes in the system follow an exponential distribution with mean equal to one. Clearly, the density $f(t) = e^{-t}$ is logconcave on $(0, \infty)$. Moreover, the density function of the system lifetime T is given by

$$f_T(t) = g_s(1 - e^{-t})e^{-t} = (18(1 - e^{-t})e^{-3t} + 3(1 - e^{-t})^2e^{-2t})e^{-t}, \quad t > 0.$$

It can be shown that the second derivative of the logarithm of f_T is given by

$$(\ln f_T(t))'' = -\frac{4e^{-t}(5e^{-2t} + 5e^{-t} - 1)}{(e^{-t} - 1)^2(5e^{-t} + 1)^2}, \quad t > 0.$$

This derivative attains positive values. For example, evaluating the function at $t = \ln(10)$ yields the value $\frac{8}{81}$. Therefore, the density f_T is not logconcave.

In Franco et al. (2003), results on the closure of the ILR property under the formation of coherent systems are given. The ILR property of an absolutely continuous random variable is equivalent to the logconcavity of its density. Therefore, they obtained sufficient conditions on logconcavity of the density of the system lifetime under the assumption that the component lifetimes have logconcave densities. The following rather restrictive, but easy-to-check criteria are proven similarly to Theorem 1 in Franco et al. (2003).

Theorem 3.4. *Let the signature be given by $\mathbf{s} = (0, \dots, 0, s_p, \dots, s_n)$ with $s_p, \dots, s_n > 0$ for some $1 \leq p \leq n$, that is, assume that $q = n$. Moreover, let the signature \mathbf{s} be increasing. If f is logconcave on (a, b) and additionally either*

- (i) *f is decreasing on (a, b) and the signature \mathbf{s} satisfies the condition in Theorem 2.1, or*
- (ii) *the function*

$$h_{\mathbf{s},1}(r) = (r-1) \frac{s_r - s_{r-1}}{s_r}, \quad r \in \{p, \dots, n\},$$

is decreasing,

then the density function f_T is logconcave.

Proof. Since f is logconcave and

$$\ln f_T(t) = \ln g_{\mathbf{s}}(F(t)) + \ln f(t),$$

it is sufficient to show that $\ln g_{\mathbf{s}}(F)$ is concave. We will prove that the derivative

$$(\ln g_{\mathbf{s}}(F(t)))' = \frac{g'_{\mathbf{s}}(F(t))}{g_{\mathbf{s}}(F(t))} f(t) \quad (3.1)$$

is a decreasing function under the given conditions. Due to the increasing signature, $g'_{\mathbf{s}} \geq 0$ holds (see Bieniek and Burkschat 2018, Theorem 2.3). Now, assume that the conditions in (i) are valid. Because the condition in Theorem 2.1 is fulfilled, the function $g'_{\mathbf{s}}/g_{\mathbf{s}}$ is decreasing. Then, since f is decreasing, too, we obtain that the derivative in (3.1) is also decreasing. Furthermore, assuming that the condition in (ii) holds, we rewrite (3.1) as

$$(\ln g_{\mathbf{s}}(F(t)))' = \frac{g'_{\mathbf{s}}(F(t))F(t)}{g_{\mathbf{s}}(F(t))} \cdot \frac{f(t)}{F(t)}.$$

It is known that logconcavity of f implies that f/F is decreasing (see, e.g., Sengupta and Nanda 1999). Therefore, in this case, it is sufficient to prove that $\frac{g'_{\mathbf{s}}(u)u}{g_{\mathbf{s}}(u)}$ is a decreasing function. Equivalently, we can show that for every $c \in \mathbb{R}$ the function

$$\frac{g'_{\mathbf{s}}(u)u}{g_{\mathbf{s}}(u)} - c, \quad u \in (0, 1),$$

changes the sign at most once and, if there is a sign change, it is from $+$ to $-$. By arguing similarly to the proof of Theorem 2.1, we obtain

$$g'_{\mathbf{s}}(u)u = \frac{u}{1-u} \sum_{r=1}^{n-1} (s_{r+1} - s_r)(n-r)f_r(u) = \sum_{r=1}^{n-1} (s_{r+1} - s_r)r f^{U_{r+1:n}}(u),$$

and, therefore,

$$g'_{\mathbf{s}}(u)u - c g_{\mathbf{s}}(u) = \sum_{r=1}^n ((s_r - s_{r-1})(r-1) - c \cdot s_r) f^{U_{r:n}}(u)$$

which yields by applying the variation diminishing property and condition (ii) that $\frac{g'_{\mathbf{s}}(u)u}{g_{\mathbf{s}}(u)}$ is decreasing. This proves the assertion. \square

Example 3.5. The 4-component system with minimal path sets

$$P_1 = \{1\}, P_2 = \{2\}, P_3 = \{3, 4\}$$

represents an example of a system with an increasing signature $\mathbf{s} = (0, 0, \frac{1}{2}, \frac{1}{2})$ (cf. Samaniego 2007, Table 3.2, system 17). Since the function $h_{\mathbf{s},1}$ from Theorem 3.4 is given by

$$h_{\mathbf{s},1}(3) = 2, \quad h_{\mathbf{s},1}(4) = 0,$$

the system lifetime has a logconcave density whenever the underlying density f is logconcave.

The following counterpart for decreasing signatures can be shown analogously to Theorem 3.4.

Theorem 3.6. Let the signature be given by $\mathbf{s} = (s_1, \dots, s_q, 0, \dots, 0)$ with $s_1, \dots, s_q > 0$ for some $1 \leq q \leq n$, that is, assume that $p = 1$. Moreover, let the signature \mathbf{s} be decreasing. If f is logconcave on (a, b) and additionally either

- (i) f is increasing on (a, b) and the signature \mathbf{s} satisfies the condition in Theorem 2.1, or
- (ii) the function

$$h_{\mathbf{s},2}(r) = (n - r) \frac{s_{r+1} - s_r}{s_r}, \quad r \in \{1, \dots, q\},$$

is decreasing,

then the density function f_T is logconcave.

Remark 3.7. It can be easily seen that the conditions in Theorem 3.4 (ii) and Theorem 3.6 (ii) imply that the signature is logconcave.

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