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Fine properties of fractional Brownian motions on Wiener space

Jiawei Li *, Zhongmin Qian

Mathematical Institute, University of Oxford, Oxford OX2 6GG, United Kingdom

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ABSTRACT

We study several important fine properties for the family of fractional Brownian motions with Hurst parameter H under the (p, r)-capacity on classical Wiener space introduced by Malliavin. We regard fractional Brownian motions as Wiener functionals via the integral representation discovered by Decreusefond and Üstünel, and show non-differentiability, modulus of continuity, law of the iterated logarithm (LIL) and self-avoiding properties of fractional Brownian motion sample paths using Malliavin calculus as well as the tools developed in the previous work by Fukushima, Takeda and etc. for Brownian motion case.

1. Introduction

Fractional Brownian motions (fBMs for simplicity), as archetypical examples of Gaussian processes have attracted researchers in recent years. The stochastic calculus and sample path properties for them are mainly studied in the setting of Gaussian measures (the Malliavin calculus for example) and Gaussian processes. In this article, we explore the fine properties of fBMs as measurable functions on the Wiener space. By fine properties here we mean those sample properties which are measured uniformly by the capacities associated with the classical Wiener space.

Recall that an fBM $(B_t)_{t\geq 0}$ with Hurst parameter $H \in (0,1)$ is, by definition, a centred Gaussian process with its co-variance function given by

$$R(t,s) = \mathbb{E}\left[B_t B_s\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

for $s, t \geq 0$. FBMs were firstly introduced by Kolmogorov [18] in early 1940s, which were named as fractional Brownian motions by Mandelbrot and Van Ness [26] in 1968. An integral representation for fBM with Hurst parameter H was discovered in [26], which is given by

* Corresponding author.
E-mail addresses: jiawei.li@maths.ox.ac.uk (J. Li), zhongmin.qian@maths.ox.ac.uk (Z. Qian).

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$$B_t = \frac{1}{\sqrt{C(H)}} \left\{ \int_{-\infty}^{0} \left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right] dW_s + \int_0^t (t-s)^{H-\frac{1}{2}} dW_s \right\},$$

where $(W_t)_{t\geq 0}$ is a standard two-sided Brownian motion, and

$$C(H) = \int_{-\infty}^{0} \left[(1-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right]^2 ds + \frac{1}{2H}.$$

The sample path properties of fBMs, like other aspects of their laws, depend crucially on the Hurst parameter H. FBM with Hurst parameter $H = \frac{1}{2}$ is just a standard Brownian motion. The study of sample paths of Brownian motion has been one of the primary components in stochastic analysis, see e.g. Itô–McKean [13], Karatzas–Shreve [16], Revuz–Yor [30] and other excellent references there-in. FBMs have stationary increments, unlike Brownian motion however, the increments of fBMs are no longer independent in the case where $H \neq \frac{1}{2}$. If $H > \frac{1}{2}$, the increments over different time intervals are positively correlated, while for $H < \frac{1}{2}$, the increments are negatively correlated. FBMs are self-similar Gaussian processes with long time memory if $H \neq \frac{1}{2}$, which are neither Markov processes nor semi-martingales. Decreusefond and Üstünel [5] identified the Cameron–Martin spaces of fBMs, and deduced another form of representations for fBMs in terms of Wiener integrals with respect to Brownian motion, and thus realised fBMs as measurable functionals of Brownian motion. FBMs are examples of Wiener functionals which are not solutions to Itô's stochastic differential equations. The advantage of considering fBMs as Wiener functionals lies in the fact that one may derive results for fBMs with different Hurst parameters in terms of concepts defined by Brownian motion, such as capacities. In this paper we derive several sample path properties of fBMs with respect to the capacities defined on the classical Wiener space by the standard Brownian motion, rather than on different Gaussian spaces induced by fBMs with different Hurst parameters. We prove a few interesting fine properties for the family of fBMs with respect to the (p, r)-capacity defined in the sense of Malliavin [24] on the classical Wiener space. To be more specific, we will study non-differentiability, modulus of continuity, law of the iterated logarithm and self-intersection of fBMs measured by capacities on the classical Wiener space. These sample path properties have been investigated over past few decades, for both Brownian motion and fBMs, even for general Gaussian processes, under both probability and (p, r)-capacity, see for example [3,16,30]. There is a huge amount of literature on this aspect. Paley, Wiener and Zygmund [29] showed the almost everywhere non-differentiability of Brownian motion sample paths (see also the argument by Dvoretzky, Erdős and Kakutani in [7]), and Mandelbrot and Van Ness [26] proved that fBM sample paths are also non-differentiable almost surely. For the modulus of continuity, Lévy [20] established the result on Hölder continuity for Brownian motion. In [5], it was shown by Decreusefond and Üstünel that sample paths of fBM with Hurst parameter H are almost surely Hölder continuous only of order less than H. Khintchine [17] extended the law of the iterated logarithm from the case of random walk to Brownian motion. In [4], Coutin [4] mentioned the following result on the law of the iterated logarithm for fBM

$$\limsup_{\varepsilon \to 0^+} \frac{B_{t+\varepsilon} - B_t}{\sqrt{2\varepsilon^{2H} \log\log(1/\varepsilon)}} = 1, \quad \text{a.s.} \tag{1.1}$$

while, to the best knowledge of the present authors, a written proof doesn't exist for the case that $H < \frac{1}{2}$, but see e.g. [1] for the functional version of the law of the iterated logarithm for Gaussian processes. For the case that $H \in (0, \frac{1}{2}]$, this was established in Cohen and Istas [3]. Whether a sample path of one stochastic process intersects itself has been an appealing problem due to its connection with statistical field theory (see e.g. Itzykson–Drouffe [14]). It dates back to 1944 when Kakutani [15] answered this question for Brownian motion. He demonstrated that d-dimensional Brownian motion is self-avoiding when $d \geq 5$, and his solution was accomplished in his joint work with Dvoretzky and Erdős [6] showing that $d = 4$ is the optimal dimensional for this property. One can show that, when $d > \frac{2}{H}$, with probability one $(B_t)_{t\geq 0}$ has no double point almost surely by using the classical argument, see e.g. Kakutani [15]. There is little information on the optimal dimension of self-avoiding property for fractional Brownian motion case due to the loss of potential theory. In early 1980s, Fukushima [8] introduced the capacity defined via Dirichlet forms, which is equivalent to (p, r)-capacity given by Malliavin [24] with $r = 1$ and $p = 2$, and proved many sample path properties for Brownian motion with respect to this capacity. Malliavin [24] introduced the (p, r)-capacity defined via Malliavin derivatives for subsets of the Wiener space, and Takeda [32] extended Fukushima's result for Brownian motion to the case of (p, r)-capacity. Fukushima [9] also showed the absence of double points under $(2, 1)$-capacity for d-dimensional Brownian motion when $d \geq 7$, and later Lyons [21] determined the critical dimension $d = 6$ for the absence of double points, by using potential theory of Brownian motion. Inspired by the argument in [9] and [32], we will derive similar results for the family of fBMs with different Hurst parameters H with respect to one uniform capacity on classical Wiener space. These results describe better the behaviour of sample paths for fBM as they remain true when H varies.

The quasi-sure analysis, initiated and created mainly by Malliavin (see e.g. [22–25]), Fukushima, Watanabe and etc. [8–10,33], is the research area whose main feature is to study various Wiener functionals whose laws are typically mutually singular such as Brownian motion and Brownian bridge. In the past, the majority of Wiener functionals considered in literature are solutions to Itô's stochastic differential equations, for which Itô's stochastic calculus and the potential theory for diffusion processes may be utilised to study their fine properties. In this article, we take the point-view that fBMs are typical Wiener functionals, i.e. measurable functionals of Brownian motion, for which traditional tools such as Markovian or/and Itô's calculus are no longer applicable. In order to derive sample path properties of fBMs in terms of capacities of Brownian motion, we employ basic techniques developed by Malliavin, Fukushima, Takeda and etc. during last decades and adopted their fundamental ideas to our study, while we have to overcome several difficulties, which were mainly achieved by carefully controlling the Malliavin derivatives of fBMs.

The paper is organised as the following. In Section 2, we introduce definitions and notations related to classical Wiener capacities and fractional Brownian motions. In section 3, we establish the modulus of continuity result following the argument by Fukushima [9], and hence deduce the quasi-surely Hölder continuity of fBMs regarded as Wiener functionals. This allows us to take continuous modifications of fBMs and prove non-differentiability in section 4 based on the argument by Dvoretzky, Erdős and Kakutani in [7], as well as the law of the iterated logarithm (LIL) when $p = 2$ and $r = 1$ with restriction $H \leq \frac{1}{2}$ in section 5. Finally, in section 6, we prove the self-avoiding property of d-dimensional fBMs under $c_{2,1}$ when $d > \frac{2}{H} + 2$ and $H \leq \frac{1}{2}$.

2. Wiener functionals

The Wiener measure is by definition the distribution of Brownian motion, which defines in turn the Wiener space, a convenient framework for the study of Wiener functionals (see e.g. Chapter V Section 8, Ikeda and Watanabe [12]). Let $\boldsymbol{W}_0^d$ denote the space of all continuous paths in the Euclidean space $\mathbb{R}^d$, started at the origin. $\boldsymbol{W}_0^d$ is a complete separable Banach space under the norm

$$\|\omega\| = \sum_{n=1}^{\infty} 2^{-n} \max_{0\leq t\leq n} |\omega(t)| \,,$$

which induces the topology of uniform convergence over every compact subset of $[0, \infty)$. The Borel σ-algebra on $\boldsymbol{W}_0^d$ is denoted by $\mathscr{B}(\boldsymbol{W}_0^d)$ or by $\mathscr{B}$ if no confusion may arise. Following Itô and McKean [13], we will use ω to denote a general element, so that $\omega(t)$ is the value of a sample path ω at $t \geq 0$, the t-th coordinate of a sample point $\omega \in \boldsymbol{W}_0^d$. The same notation $\omega(t)$ denotes also the coordinate mapping $\omega \to \omega(t)$, and the parametrised family $\{\omega(t) : t \geq 0\}$ is the coordinate process on $\boldsymbol{W}_0^d$. The coordinate mapping $\omega(t)$ may be denoted by ω_t (for $t \geq 0$) too. Then the Borel σ-algebra $\mathscr{B}(\boldsymbol{W}_0^d)$ is the smallest σ-algebra on $\boldsymbol{W}_0^d$ with which all coordinate functions $\omega(t)$ (for $t \geq 0$) are measurable (for a proof, see e.g. Stroock and Varadhan [31]). The Wiener measure P^W is the unique probability on $(\boldsymbol{W}_0^d, \mathscr{B})$ such that the coordinate process $(\omega(t))_{t\geq 0}$ of $\boldsymbol{W}_0^d$ is a standard Brownian motion in $\mathbb{R}^d$. To complete the definition of the classical Wiener space, one should identify the Cameron–Martin space of the Wiener measure P^W. To this end, it is better to identify the Wiener measure P^W as a Gaussian measure on $\boldsymbol{W}_0^d$. For simplicity, $\boldsymbol{W}_0^d$ and P^W will be denoted by $\boldsymbol{W}$ and P respectively, if no confusion is possible.

Let $\mathcal{H}$ be the space of all $h \in \boldsymbol{W}$ such that $t \to h(t)$ is absolutely continuous and its generalized derivative $\dot{h}$ is square-integrable on $[0, \infty)$. $\mathcal{H}$ is a Hilbert space under the norm $\|h\|_{\mathcal{H}} = \sqrt{\int_0^\infty |\dot{h}(t)|^2 dt}$, and the dual space $\boldsymbol{W}^\star$ of all continuous linear functionals on $\boldsymbol{W}$ can be identified as a subset of $\mathcal{H}$, so that we have the continuous densely embedding $\boldsymbol{W}^\star \hookrightarrow \mathcal{H} \hookrightarrow \boldsymbol{W}$ with respect to their corresponding norms.

P is the unique measure on $(\boldsymbol{W}, \mathscr{B})$ such that every continuous linear functional $\gamma \in \boldsymbol{W}^\star$ has a normal distribution with mean zero and variance $\|\gamma\|_{\mathcal{H}}^2$. In other words, P is the unique probability measure on $\boldsymbol{W}$ such that

$$\int_{\boldsymbol{W}} e^{i\gamma(\omega)} P(d\omega) = \exp\left[-\frac{1}{2}\|\gamma\|_{\mathcal{H}}^2\right]$$

for every $\gamma \in \boldsymbol{W}^\star$. Therefore, every $h \in \mathcal{H}$ corresponds (unique up to almost surely) to a random variable on $\boldsymbol{W}$, still denoted by h, which has a normal distribution $N(0, \|h\|_{\mathcal{H}}^2)$. In fact, for every $h \in \mathcal{H}$, the corresponding Gaussian variable h can be identified with the Itô integral, denoted by $[h]$, $\int_0^\infty \dot{h} d\omega$ of $\dot{h}$ against the Brownian motion $(\omega(t))_{t\geq 0}$, which is defined in probability sense. Under this sense, the triple $(\boldsymbol{W}, \mathcal{H}, P)$ is an example of abstract Wiener spaces, a concept introduced by L. Gross [11], called the classical Wiener space. The completion of the Borel σ-algebra $\mathscr{B}$ is denoted by $\mathscr{F}$.

An $\mathscr{F}$-measurable (valued in a separable Hilbert space) function on $\boldsymbol{W}$ is called, according to the convention in literature, a Wiener functional.

2.1. Malliavin derivative and capacity

A differential structure on the Wiener space $(\boldsymbol{W}, \mathcal{H}, P)$ compatible to the Wiener measure was introduced by Malliavin [22], [23]. The Malliavin derivative for smooth random variables of form

$$F = f([h_1], \cdots, [h_n]), \quad h_i \in \mathcal{H},$$

can be defined formally by differentiating F, as long as $f \in C_p^\infty(\mathbb{R}^n)$, a function whose partial derivatives have polynomial growth. The Malliavin derivative of F is an $\mathcal{H}$-valued random variable defined by

$$DF = \sum_{i=1}^n \partial_i f([h_1], \cdots, [h_n]) h_i,$$

where $\partial_i f(x_1, \cdots, x_n)$ is the partial derivative of f in the i-th component. The high order Malliavin derivatives $D^k F$ for all $k \geq 1$ may be defined inductively. The collection of all such smooth random variables F is denoted by $\mathcal{S}$. For $r \in \mathbb{N}$ and $1 < p < \infty$, let $\mathbb{D}_r^p$ be the completion of $\mathcal{S}$ with respect to the Sobolev norm

$$\|F\|_{\mathbb{D}_r^p} = \left(\mathbb{E}\left[|F|^p\right] + \sum_{k=1}^r \mathbb{E}\left[\left|\|D^k F\|_{\mathcal{H}^{\otimes k}}\right|^p\right]\right)^{1/p}.$$

The (p,r)-capacity of an open subset O of $\boldsymbol{W}$ is defined by (see e.g. [25]):

$$c_{p,r}(O) = \inf\left\{\|\varphi\|_{\mathbb{D}_r^p} : \varphi \in \mathbb{D}_r^p,\ \varphi \geq 1 \text{ a.e. on } O,\ \varphi \geq 0 \text{ a.e. on } \boldsymbol{W}\right\},$$

and for an arbitrary subset A of $\boldsymbol{W}$, its (p,r)-capacity is

$$c_{p,r}(A) = \inf\left\{c_{p,r}(O) : A \subset O,\ O \ \text{ is open}\right\}.$$

$A \subset \boldsymbol{W}$ is said to be slim if $c_{p,r}(A) = 0$ for all $r \in \mathbb{N}$ and $1 < p < \infty$. A property π defined over $\boldsymbol{W}$ is said to hold quasi-surely (q.s.) if the set on which this property is not satisfied is slim.

The notion of slim sets on the classical Wiener space $(\boldsymbol{W}, \mathcal{H}, P)$ can be studied via the Ornstein–Uhlenbeck operator, which gives rise to a different but equivalent approach to (p,r)-capacity. For a given $p \in [1,\infty]$, let $(T_t)_{t\geq 0}$ denote the Ornstein–Uhlenbeck semigroup on $L^p(\boldsymbol{W}, P)$, which is the one-parameter semigroup of contractions on $L^p(\boldsymbol{W}, P)$ given by

$$T_t u(x) = \int_{\boldsymbol{W}} u\left(e^{-t}x + \sqrt{1-e^{-2t}}\omega\right) P(d\omega).$$

Let L be the generator of the semigroup (T_t), that is,

$$\mathcal{D}(L) = \left\{u \in L^p(\boldsymbol{W}, P) : \lim_{t\downarrow 0} \frac{T_t u - u}{t} \text{ exists in } L^p\text{-space}\right\}$$

and

$$Lu = \lim_{t\downarrow 0} \frac{T_t u - u}{t} \text{ for } u \in \mathcal{D}(L).$$

For each $r > 0$, $(I-L)^{-\frac{r}{2}}$ is again a contraction on $L^p(\boldsymbol{W}, P)$, and is given by the following integral

$$(I-L)^{-\frac{r}{2}} = \frac{1}{\Gamma(r/2)} \int_0^\infty t^{\frac{r}{2}-1} e^{-t} T_t dt$$

(defined in the sense of Bochner's integrals). The corresponding Sobolev norm $\|\cdot\|_{r,p}$ (where $1 < p < \infty$) is then defined by

$$\|u\|_{r,p} = \|(I-L)^{-\frac{r}{2}} u\|_p.$$

The corresponding (p,r)-capacity $C_{r,p}$, following Fukushima's convention in [10], can be defined in a similar manner as before, namely, for an open subset O of $\boldsymbol{W}$,

$$C_{r,p}(O) = \inf\left\{\|\phi\|_p^p :\ (I-L)^{-\frac{r}{2}}\phi \geq 1 \text{ a.e. on } O,\ (I-L)^{-\frac{r}{2}}\phi \geq 0 \text{ a.e. on } \boldsymbol{W}\right\}$$

(with convention that $\inf \emptyset = \infty$) and

$$C_{r,p}(A) = \inf\left\{\mathrm{C}_{r,p}(O) : A \subset O,\ O \ \text{ is open}\right\}$$

for an arbitrary subset A of $\boldsymbol{W}$. It was Meyer [27] who proved that norms $\|\cdot\|_{\mathbb{D}_r^p}$ and $\|\cdot\|_{r,p}$ are equivalent, and it follows that there exists a constant $\alpha_{r,p} > 0$ such that

$$\frac{1}{\alpha_{r,p}} C_{r,p}(A) \leq [c_{p,r}(A)]^p \leq \alpha_{r,p} C_{r,p}(A) \tag{2.1}$$

for every $A \subset \boldsymbol{W}$. For further details about the norms $\|\cdot\|_{r,p}$ and the corresponding capacity, one should refer to [10], [32] and [33].

The important properties about (p,r)-capacity are stated below, which will be used in the following text. Firstly capacities $c_{p,r}$ and $C_{r,p}$ are outer measures in the sense that $c_{p,r}$ and $C_{r,p}$ are monotonic and sub-additive, that is, $c_{p,r}(A) \leq c_{p,r}(B)$ for any $A \subseteq B$, and $c_{p,r}(A) \leq \sum_n c_{p,r}(A_n)$ if $A \subset \cup_n A_n$. These properties hold for $C_{r,p}$ as well. Let us point out that the sub-additivity of $c_{p,r}$ follows from the localization of $\|\cdot\|_{\mathbb{D}_r^p}$, while the sub-additivity of $C_{r,p}$ follows from the triangle inequality for norms. It follows that the first Borel–Cantelli lemma applies to these capacities (see e.g. Corollary 1.2.4, Chapter IV, [25]). More precisely, if $\{A_n\}_{n=1}^\infty$ is a sequence of subsets of $\boldsymbol{W}$ such that $\sum_{n=1}^\infty c_{p,r}(A_n) < \infty$, then $c_{p,r}(\limsup_{n\to\infty} A_n) = 0$. The capacity version of the Borel–Cantelli lemma, together with the concept of the Malliavin derivative, are the major tools in our arguments in this work. In fact, the definition of the capacity $c_{p,r}$ implies the following Chebyshev's inequality (see e.g. Corollary 1.2.5, Chapter IV, [25]). If $\varphi \in \mathbb{D}_r^p$ and φ is lower-semi continuous, then

$$c_{p,r}(\varphi > \lambda) \leq \lambda^{-1} \|\varphi\|_{\mathbb{D}_r^p}$$

for every $\lambda > 0$.

Lemma 1.1 in [10] with Meyer's inequality implies a stronger version of the sub-additivity for $c_{p,r}$, which says that

$$[c_{p,r}(A)]^p \leq M_{p,r} \sum_{n=1}^\infty [c_{p,r}(A_n)]^p \tag{2.2}$$

for some constant $M_{p,r}$ depending only on p and r, and for any $A \subset \bigcup_n A_n$.

$c_{p,r}$ is lower continuous (see e.g. Theorem 5.1, Chapter IV, [25]) in the sense that for an increasing sequence of sets $\{A_n\}_{n=1}^\infty$,

$$c_{p,r}\left(\bigcup_{n=1}^\infty A_n\right) = \lim_{n\to\infty} c_{p,r}(A_n). \tag{2.3}$$

2.2. Fractional Brownian motion

In this sub-section we consider a class of Wiener functionals, fractional Brownian motions (fBM) with Hurst parameter H, which are defined as singular Itô's integrals with respect to Brownian motion. FBMs are measurable functions on the Wiener space $(\boldsymbol{W}, \mathcal{H}, P)$ which are smooth in the sense of Malliavin differentiation.

An fBM $(B_t)_{t\geq 0}$ (of dimension one) with Hurst parameter $H \in (0,1)$ is a centred Gaussian process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose covariance function is given by

$$R(t,s) = \mathbb{E}[B_t B_s] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

An fBM has stationary increments, i.e. $B_t - B_s$ and B_{t-s} have the same distribution. FBMs are known as examples of self-similar processes, i.e. for any $\alpha > 0$, $\{B_t : t \geq 0\} = \left\{\alpha^{-H} B_{\alpha t} : t \geq 0\right\}$ in distribution.

In this paper, fBMs will be realised as Wiener functionals on the classical Wiener space $(\boldsymbol{W}, \mathcal{H}, P)$, in terms of the following integral representation (see e.g. [5]):

$$B_t = \int_0^t K(t,s)d\omega(s), \tag{2.4}$$

where the integrals on the right-hand side have to be interpreted as Itô integrals against Brownian motion $\{\omega(t) : t \geq 0\}$ under the Wiener measure P. Here, for each pair $t > s \geq 0$ define K to be the reproducing kernel

$$K(t,s) = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}} s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

if $H > \frac{1}{2}$, and for $H < \frac{1}{2}$,

$$K(t,s) = \sqrt{\frac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})}} \cdot \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right],$$

and we define $K = 1$ when $H = \frac{1}{2}$, so that our results are compatible with the classical results for Brownian motion. We notice that K is a non-negative but singular kernel and it satisfies that

$$\int_0^{t\wedge u} K(t,s)K(u,s)ds = R(t,u).$$

For further details on the above integral representation and reproducing kernel K, one may refer to [5] and Chapter 5 in [28]. B_t (for $t > 0$) are Gaussian random variables on Wiener space $(\boldsymbol{W}, P)$, and $\mathbb{E}|B_t - B_s|^2 = |t-s|^{2H}$. By choosing proper modifications of B_t we may assume that $t \to B_t$ are continuous.

For every $t \geq 0$, B_t defined by the previous integral representation is smooth in Malliavin's sense, that is, it belongs to Sobolev space $\mathbb{D}_r^p$ for any $r \in \mathbb{N}$ and $p \in (1, \infty)$. In what follows, we will work with this version of fBM only. For example, the Malliavin derivative of B_t as a function on $\boldsymbol{W}$ can be calculated as in the following lemma, which will be used in our main arguments.

Lemma 2.1. *Let $H \in (0,1)$, $r \in \mathbb{N}$ and $p \in (1,\infty)$. Then $B_t \in \mathbb{D}_r^p$ (for every $t > 0$) and its first order Malliavin derivative is given by*

$$DB_t(s) = \int_0^{s\wedge t} K(t,u)du. \tag{2.5}$$

The higher-order derivatives of B_t vanish (which reflects the fact that B_t is an integral of a deterministic function against Brownian motion).

This lemma is a consequence of the transfer principle provided in Proposition 5.2.1, page 288, [28]. We provide an elementary proof slightly different from that in [28] in the appendix for completeness.

Remark 2.2. As a consequence, according to Malliavin (Theorem 2.3.3, page 97, [25]), given a pair $r \in \mathbb{N}$ and $p \geq 1$, for every $\varepsilon > 0$, there is an open subset $\mathcal{O}_\varepsilon \subseteq \boldsymbol{W}$ with $c_{p,r}(\mathcal{O}_\varepsilon) < \varepsilon$, and there is a family of continuous functions $\tilde{B}_t$ (for $t > 0$) on $\boldsymbol{W}$ such that $B_t = \tilde{B}_t$ (for all $t > 0$) P-a.e., and $\tilde{B}_t$ are continuous on $\boldsymbol{W} \setminus \mathcal{O}_\varepsilon$ for all $t > 0$.

3. Several technical facts

In this section, we shall prove several technical facts about fBMs which will be used in proving our main results. The first one is the following inequality, which is similar to the result due to Fukushima in [9], however the proof of our case is more subtle.

Lemma 3.1. *For all $H \in (0,1)$,*

$$c_{p,r}(|B_t - B_s| > \eta) \le \sqrt[p]{2\left[\sum_{l=0}^{r}\left(\frac{\eta}{p(t-s)^H}\right)^{lp}\right]} e^{-\frac{\eta^2}{2p(t-s)^{2H}}}$$

for any $r \in \mathbb{N}$, $1 < p < \infty$, $\eta > 0$, and $0 \le s < t$.

Proof. Let $M_{s,t} = B_t - B_s$ with $0 \le s < t$. Then by the definition of Malliavin derivative, we obtain that

$$DM_{s,t}(u) = \int_0^u K(t,r)\mathbb{1}_{[0,t]}(r) - K(s,r)\mathbb{1}_{[0,s]}(r)dr \in \mathcal{H}$$

and higher order derivatives of $M_{s,t}$ all vanish. We show that for $\alpha \ge 0$, $e^{\frac{\alpha}{p}M_{s,t}} \in \mathbb{D}^p_r$, and

$$D^l e^{\frac{\alpha}{p}M_{s,t}} = \left(\frac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}} DM_{s,t} \otimes \cdots \otimes DM_{s,t} \in L^p(\boldsymbol{W};\mathcal{H}^{\otimes l})$$

for all $1 \le l \le r$.

Set $f(x) = e^{\frac{\alpha}{p}x}$. For each $N \in \mathbb{N}$, let $\psi_N \in C_0^\infty(\mathbb{R})$ be a cut-off function taking values in $[0,1]$ such that

$$\psi_N(x) = \begin{cases} 1, & |x| \le N \\ 0 & |x| \ge N+1, \end{cases}$$

and $\sup_{x,N} |\psi_N^{(k)}(x)| = C < \infty$ for all $1 \le k \le r$. Set $f_N(x) = f(x) \cdot \psi_N(x)$. For convenience, write $F_N = f_N(M_{s,t})$, then $F_N \in \mathcal{S}$ as $f_N \in C_0^\infty(\mathbb{R})$, and by using the chain rule for Malliavin derivatives, we have

$$D^l F_N = f_N^{(l)}(M_{s,t}) DM_{s,t} \otimes \cdots \otimes DM_{s,t}$$

for $1 \le l \le r$. Hence,

$$\begin{aligned}
&\mathbb{E}\left[\left|\left\|D^l F_N - \left(\tfrac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}} DM_{s,t} \otimes \cdots \otimes DM_{s,t}\right\|_{\mathcal{H}^{\otimes l}}\right|^p\right] \\
&= \mathbb{E}\left[\left|f_N^{(l)}(M_{s,t}) - \left(\tfrac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}}\right|^p \left\|DM_{s,t}\right\|_{\mathcal{H}}^{lp}\right] \\
&= \mathbb{E}\left[\left|\sum_{j=0}^{l} \tbinom{l}{j} f^{(j)}(M_{s,t})\psi_N^{(l-j)}(M_{s,t}) - \left(\tfrac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}}\right|^p\right] \left\|DM_{s,t}\right\|_{\mathcal{H}}^{lp}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\Big|\sum_{j=0}^{l-1}\tbinom{l}{j}f^{(j)}(M_{s,t})\psi_N^{(l-j)}(M_{s,t}) + \left(\tfrac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}}\psi_N(M_{s,t}) - \left(\tfrac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}}\Big|^p\right]\|DM_{s,t}\|_{\mathcal{H}}^{lp} \\
&\le l^{p-1}\mathbb{E}\left[\sum_{j=0}^{l-1}\Big|\tbinom{l}{j}f^{(j)}(M_{s,t})\psi_N^{(l-j)}(M_{s,t})\Big|^p + \Big|\left(\tfrac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}}\left(\psi_N(M_{s,t})-1\right)\Big|^p\right]\|DM_{s,t}\|_{\mathcal{H}}^{lp} \\
&\le l^{p-1}\mathbb{E}\left[\sum_{j=0}^{l-1}\Big|\tbinom{l}{j}\left(\tfrac{\alpha}{p}\right)^j e^{\frac{\alpha}{p}M_{s,t}}M\cdot \mathbb{1}_{\{|M_{s,t}|\ge N\}}\Big|^p + \Big|\left(\tfrac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}}\mathbb{1}_{\{|M_{s,t}|\ge N\}}\Big|^p\right]\|DM_{s,t}\|_{\mathcal{H}}^{lp},
\end{aligned}
$$

which tends to zero as $N\to\infty$ by the dominated convergence theorem. Since $F_N \to e^{\frac{\alpha}{p}M_{s,t}}$ as N goes to infinity in $L^p(\boldsymbol{W})$, and according to the previous estimate, we get that

$$
D^l F_N \to \left(\frac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}} DM_{s,t}\otimes\cdots\otimes DM_{s,t}
$$

in $L^p(\boldsymbol{W};\mathcal{H}^{\otimes l})$. Since D^l is closable, together with the definition of $\mathbb{D}_l^p$, we deduce that

$$
D^l F = \left(\frac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}} DM_{s,t}\otimes\cdots\otimes DM_{s,t}
$$

for each $1\le l\le r$ and $e^{\frac{\alpha}{p}M_{s,t}}\in\mathbb{D}_r^p$.

By Chebyshev's inequality for (p,r)-capacity, it follows that

$$
\begin{aligned}
\left[c_{p,r}\left(M_{s,t}-\frac{\alpha}{2}(t-s)^{2H} > \beta\right)\right]^p &= \left[c_{p,r}\left(\frac{\alpha}{p}M_{s,t}-\frac{\alpha^2}{2p}(t-s)^{2H} > \frac{\alpha\beta}{p}\right)\right]^p \\
&= \left[c_{p,r}\left(\exp\left(\frac{\alpha}{p}M_{s,t}\right) > \exp\left(\frac{\alpha^2}{2p}(t-s)^{2H}+\frac{\alpha\beta}{p}\right)\right)\right]^p \qquad (3.1)\\
&\le \exp\left(-\frac{\alpha^2}{2}(t-s)^{2H}-\alpha\beta\right)\left\|e^{\frac{\alpha}{p}M_{s,t}}\right\|_{\mathbb{D}_r^p}^p,
\end{aligned}
$$

for any $\alpha,\beta>0$. It is clear that

$$
\begin{aligned}
\langle DM_{s,t}, DM_{s,t}\rangle_{\mathcal{H}} &= \int_0^\infty \left[K(t,u)\mathbb{1}_{[0,t]}(u) - K(s,u)\mathbb{1}_{[0,s]}(u)\right]^2 du \\
&= R(t,t) - 2R(s,t) + R(s,s) \\
&= (t-s)^{2H}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\langle D^l e^{\frac{\alpha}{p}M_{s,t}}, D^l e^{\frac{\alpha}{p}M_{s,t}}\rangle_{\mathcal{H}^{\otimes l}} &= \left(\frac{\alpha}{p}\right)^{2l} e^{\frac{2\alpha}{p}M_{s,t}}\left(\langle DM_{s,t}, DM_{s,t}\rangle_{\mathcal{H}}\right)^l \\
&= \left(\frac{\alpha}{p}\right)^{2l} e^{\frac{2\alpha}{p}M_{s,t}}(t-s)^{2lH},
\end{aligned}
$$

which implies that

$$\mathbb{E}\left[\left|\|D^l e^{\frac{\alpha}{p}M_{s,t}}\|_{\mathcal{H}^{\otimes l}}\right|^p\right] = \mathbb{E}\left[\left(\frac{\alpha}{p}\right)^{lp} e^{\alpha M_{s,t}}(t-s)^{lHp}\right]$$
$$= \left(\frac{\alpha}{p}\right)^{lp}(t-s)^{lHp}e^{\frac{\alpha^2}{2}(t-s)^{2H}},$$

where we have used that $M_{s,t} \sim N(0,(t-s)^{2H})$. Hence,

$$\|e^{\frac{\alpha}{p}M_{s,t}}\|^p_{\mathbb{D}^p_r} = \mathbb{E}\left[\left|e^{\frac{\alpha}{p}M_{s,t}}\right|^p\right] + \sum_{l=1}^r \mathbb{E}\left[\left|\|D^l e^{\frac{\alpha}{p}M_{s,t}}\|_{\mathcal{H}^{\otimes l}}\right|^p\right]$$
$$= \mathbb{E}\left[e^{\alpha M_{s,t}}\right] + \sum_{l=1}^r \left(\frac{\alpha}{p}\right)^{lp}(t-s)^{lHp}e^{\frac{\alpha^2}{2}(t-s)^{2H}}$$
$$= \left[\sum_{l=0}^r \left(\frac{\alpha}{p}\right)^{lp}(t-s)^{lHp}\right]e^{\frac{\alpha^2}{2}(t-s)^{2H}}.$$

Now by (3.1), we obtain that

$$\left[c_{p,r}\left(M_{s,t} - \frac{\alpha}{2}(t-s)^{2H} > \beta\right)\right]^p \le \left[\sum_{l=0}^r \left(\frac{\alpha}{p}\right)^{lp}(t-s)^{lHp}\right]e^{-\alpha\beta}.$$

For any positive η, optimise the above inequality by setting $\alpha = \frac{\eta}{(t-s)^{2H}}$ and $\beta = \frac{\eta}{2}$, and conclude that

$$[c_{p,r}(M_{s,t} > \eta)]^p \le \left[\sum_{l=0}^r \left(\frac{\eta}{p(t-s)^{2H}}\right)^{lp}(t-s)^{lHp}\right]e^{-\frac{\eta^2}{2(t-s)^{2H}}}$$
$$= \sum_{l=0}^r \left(\frac{\eta}{p(t-s)^H}\right)^{lp} e^{-\frac{\eta^2}{2(t-s)^{2H}}}.$$

By replacing B with $-B$, we may conclude that

$$[c_{p,r}(|M_{s,t}| > \eta)]^p \le 2\left[\sum_{l=0}^r \left(\frac{\eta}{p(t-s)^H}\right)^{lp}\right]e^{-\frac{\eta^2}{2(t-s)^{2H}}}. \quad \square$$

Let $0 \le u < r < s < t \le T$. Set

$$X = \frac{B_t - B_s}{(t-s)^H}, \qquad Y = \frac{B_r - B_u}{(r-u)^H},$$

so that $X, Y \sim N(0,1)$. Moreover,

$$\mathbb{E}[XY] = (t-s)^{-H}(r-u)^{-H}\left(R(t,r) - R(t,u) - R(s,r) + R(s,u)\right)$$
$$= \frac{1}{2(t-s)^H(r-u)^H}\left[(t-u)^{2H} - (t-r)^{2H} - \left((s-u)^{2H} - (s-r)^{2H}\right)\right], \tag{3.2}$$

which is non-negative when $H \in \left[\frac{1}{2}, 1\right)$, and non-positive when $H \in \left(0, \frac{1}{2}\right]$. We need the following simple observation. By (2.5),

$$\begin{aligned} \langle DX, DY \rangle_{\mathcal{H}} &= (t-s)^{-H}(r-u)^{-H} \langle DB_t - DB_s, DB_r - DB_u \rangle_{\mathcal{H}} \\ &= (t-s)^{-H}(r-u)^{-H} \int_0^\infty \big(K(t,v)\mathbb{1}_{[0,t]}(v) - K(s,v)\mathbb{1}_{[0,s]}(v) \big) \\ &\quad \cdot \big(K(r,v)\mathbb{1}_{[0,r]}(v) - K(u,v)\mathbb{1}_{[0,u]}(v) \big)\, dv \\ &= \mathbb{E}[XY]. \end{aligned} \tag{3.3}$$

Next technical lemma contains results similar to Proposition 1 in Fukushima [9] and Proposition 2 in Takeda [32].

Lemma 3.2. *For all $H \in (0,1)$ and each $N \in \mathbb{N}$, let $0 \le t_0 < t_1 < \cdots < t_N$ with $|t_i - t_{i-1}| = L$, $1 \le i \le N$. Take $-\infty < a_i < b_i < \infty$, $c_i > 0$, $1 \le i \le N$. Then it holds that*

$$\left[c_{p,r} \left(\bigcap_{i=1}^N \{a_i < X_i < b_i\} \right) \right]^p \le \left[\sum_{l=0}^r N^{lp} C_H^{lp/2} \left(\frac{M_r}{c} \right)^{lp} \right] P \left(\bigcap_{i=1}^N \{a_i - c_i < X_i < b_i + c_i\} \right)$$

for all $r \in \mathbb{N}$ and $p \in (1, \infty)$, where

$$X_i = \frac{B_{t_i} - B_{t_{i-1}}}{L^H} \sim N(0,1), \tag{3.4}$$

$c = \min_{1 \le i \le N} c_i$, M_r is a constant depending only on r, and

$$C_H = \max\left\{ 2^{2H-1} - 1, 1 \right\} \le 1$$

is some constant depending only on H.

Proof. The proof is a modification of Takeda's argument in [32]. For $i = 1, 2, \cdots, N$, let $f_i \in C_c^\infty(\mathbb{R})$ be the cut-off functions valued in $[0,1]$ such that

$$f_i(x) = \begin{cases} 1, & x \in (a_i, b_i), \\ 0, & x \in (-\infty, a_i - c_i) \cup (b_i + c_i, \infty), \end{cases}$$

and

$$\left| \frac{d^l f_i}{dx^l} \right| \le \frac{M_r}{c_i^l}$$

for all $l \le r$, where $M_r \ge 1$ is a constant depending on r. Set $F(x_1, \cdots, x_N) = \prod_{i=1}^N f_i(x_i)$, then according to the above conditions, we have that

$$\left| \partial^l_{n_1, \cdots, n_l} F(x_1, \cdots, x_N) \right| \le \left(\frac{M_r}{c} \right)^l \mathbb{1}_{\prod_{i=1}^N (a_i - c_i, b_i + c_i)}(x_1, \cdots, x_N) \tag{3.5}$$

for each $l \le r$, where $c = \min_{1 \le i \le N} c_i$. For simplicity, write $Y = F(X_1, \cdots, X_N)$, where X_i's are defined as in (3.4). Then $Y \in \mathbb{D}_l^p$, and since all Malliavin derivatives of X_i with order higher than 2 vanish, it holds that

$$D^l Y = \sum_{1 \le n_1, \cdots, n_l \le N} \partial^l_{n_1, \cdots, n_l} F(X_1, \cdots, X_N) DX_{n_1} \otimes \cdots \otimes DX_{n_l}.$$

Moreover, $D^l F \in \mathcal{H}^{\otimes l}$ and

$$\|D^l Y\|^2_{\mathcal{H}^{\otimes l}} = \sum_{\substack{1\le n_1,\cdots,n_l\le N\\1\le m_1,\cdots,m_l\le N}} \Bigg(\partial^l_{n_1,\cdots,n_l} F(X_1,\cdots,X_N)\partial^l_{m_1,\cdots,m_l}F(X_1,\cdots,X_N) \cdot \prod_{i=1}^{l} \langle DX_{n_i}, DX_{m_i}\rangle_{\mathcal{H}} \Bigg). \tag{3.6}$$

Our next step is to find an upper bound for $|\langle DX_j, DX_k\rangle_{\mathcal{H}}|$ for all $1 \le j,k \le N$. When $1 \le j = k \le N$, $\langle DX_j, DX_k\rangle_{\mathcal{H}} = 1$; when $1 \le j < k \le N$, by (3.2) and (3.3),

$$\begin{aligned}\langle DX_i, DX_j\rangle_{\mathcal{H}} &= \mathbb{E}[X_j X_k] \\ &= \frac{1}{2}\left[(k-j+1)^{2H} + (k-j-1)^{2H} - 2(k-j)^{2H}\right].\end{aligned}$$

Set $g(x) = \frac{1}{2}\left[(x+1)^{2H} + (x-1)^{2H} - 2x^{2H}\right]$. Observe that when $H < \frac{1}{2}$, x^{2H} is concave, so $g(x) \le 0$, and similarly when $H > \frac{1}{2}$, $g(x) \ge 0$. The derivative of g is given by

$$g'(x) = H\left[\left((x+1)^{2H-1} - x^{2H-1}\right) - \left(x^{2H-1} - (x-1)^{2H-1}\right)\right].$$

Using the fact that the function x^{2H-1} is convex if $H \in \left(0,\frac{1}{2}\right)$, we deduce that when $H \in \left(0,\frac{1}{2}\right)$, $g'(x) \ge 0$. As $k - j \in \{1,2,\cdots,N-1\}$, it follows that

$$|\langle DX_j, DX_k\rangle_{\mathcal{H}}| \le 2^{2H-1} - 1.$$

When $H \in \left(\frac{1}{2},1\right)$, $g'(x) \le 0$ and thus $|\langle DX_j, DX_k\rangle_{\mathcal{H}}| \le 2^{2H-1} - 1$. Set $C_H = \max\left\{2^{2H-1}-1, 1\right\}$, then $|\langle DX_j, DX_k\rangle_{\mathcal{H}}| \le C_H$ for all $1 \le j,k \le N$. Moreover, as H takes values in $(0,1)$, $C_H \le 1$.

Therefore, by (3.6), together with (3.5), it follows that

$$\|D^l Y\|^2_{\mathcal{H}^{\otimes l}} \le N^{2l}\left(\frac{M_r}{c}\right)^{2l} \mathbb{1}_{\prod_{i=1}^N (a_i-c_i,b_i+c_i)}(X_1,\cdots,X_N) C_H^l$$

for all $l \le r$. Hence

$$\left|\|D^l Y\|_{\mathcal{H}^{\otimes l}}\right|^p \le N^{lp} C_H^{lp/2}\left(\frac{M_r}{c}\right)^{lp} \mathbb{1}_{\prod_{i=1}^N (a_i-c_i,b_i+c_i)}(X_1,\cdots,X_N).$$

By the definition of (p,r)-capacity,

$$\begin{aligned}&\left[c_{p,r}\left(\bigcap_{i=1}^N \{a_i < X_i < b_i\}\right)\right]^p \\ &\le \|Y\|^p_{\mathbb{D}^p_r} \\ &= \mathbb{E}\left[|Y|^p\right] + \sum_{l=1}^r \mathbb{E}\left[\left|\|D^l Y\|_{\mathcal{H}^{\otimes l}}\right|^p\right] \\ &\le P\left(\bigcap_{i=1}^N \{a_i - c_i < X_i < b_i + c_i\}\right)\end{aligned}$$

$$+\sum_{l=1}^{r}\left(N^{lp}C_H^{lp/2}\left(\frac{M_r}{c}\right)^{lp}\right)P\left(\bigcap_{i=1}^{N}\{a_i-c_i<X_i<b_i+c_i\}\right)$$
$$=\left[\sum_{l=0}^{r}N^{lp}C_H^{lp/2}\left(\frac{M_r}{c}\right)^{lp}\right]P\left(\bigcap_{i=1}^{N}\{a_i-c_i<X_i<b_i+c_i\}\right). \quad \square$$

In what follows, as in the proof of the previous lemma, $X_\cdot$ denotes a normalised increment of fBM, though it may refer to increment over time interval of different length, which has a standard Gaussian distribution.

The third technical lemma we need is a $(2,1)$-capacity estimate on the supremum process for fBM with Hurst parameter $H \in \left(0, \frac{1}{2}\right)$, whose proof is quite technical due to lack of suitable tools such as Doob's maximal inequality for martingales. We overcome the difficulties by carefully applying Slepian's lemma for related Gaussian processes.

Lemma 3.3. *Let $0 \le s < t$. For $H \in (0,1)$ and $\eta > 0$,*

$$c_{2,1}\left(\sup_{s\le u\le t}(B_u-B_s)>\eta\right)\le C_{s,t,\eta,H}\cdot\exp\left(-\frac{\eta^2}{4\left[\gamma_H(t-s)^{2H}+(t-s)\right]}\right), \tag{3.7}$$

and

$$c_{2,1}\left(\sup_{s\le u\le t}|B_u-B_s|>\eta\right)\le \sqrt{2}C_{s,t,\eta,H}\cdot\exp\left(-\frac{\eta^2}{4\left[\gamma_H(t-s)^{2H}+(t-s)\right]}\right), \tag{3.8}$$

$$c_{2,1}\left(\sup_{s\le u\le t}|B_t-B_u|>\eta\right)\le \sqrt{2}C_{s,t,\eta,H}\cdot\exp\left(-\frac{\eta^2}{4\left[\gamma_H(t-s)^{2H}+(t-s)\right]}\right), \tag{3.9}$$

where

$$\gamma_H=\begin{cases}1, & H\le \frac{1}{2},\\ \frac{3}{2}, & H>\frac{1}{2},\end{cases} \tag{3.10}$$

and

$$C_{s,t,\eta,H}=\sqrt{\frac{\eta^2(t-s)^{2H}}{2\left[\gamma_H(t-s)^{2H}+(t-s)\right]^2}+2}.$$

Proof. We shall follow the same ideas as for the proof of Proposition 2 and 3 in [9], while we have to overcome several difficulties arising from the fact that the distribution of supremum process is not known for fBM. When $H=\frac{1}{2}$, the above inequality is covered by the result due to Fukushima in [9].

We prove (3.7) and (3.8) first. For simplicity, define $M^*_{s,t}=\sup_{s\le u\le t}(B_u-B_s)$ for any $0\le s<t$. Following Fukushima's notation in [9], for $s<t_1<\cdots<t_n\le t$, let us define $B_{s;t_1,\cdots,t_n}=(B_{t_1}-B_s,\cdots,B_{t_n}-B_s)$, and let $g(x_1,\cdots,x_n)=x_1\vee\cdots\vee x_n$, and define

$$M_{s;t_1,\cdots,t_n}=g(B_{s;t_1,\cdots,t_n})=\max_{1\le i\le n}(B_{t_i}-B_s).$$

We proceed in 4 steps.

Step 1. In this step, only the law of fBM will be involved, so the argument is applicable to various Gaussian processes. As t_i's are fixed in the first two steps, we simplify our notations by writing $B^{(n)}_{s,t}=B_{s;t_1,\cdots,t_n}$ and $M^{(n)}_{s,t}=M_{s;t_1,\cdots,t_n}$ for the moment. In this step, we establish an upper bound for $\mathbb{E}\left[e^{\alpha M^{(n)}_{s,t}}\right]$, where $\alpha>0$.

Consider the following correlation:

$$\mathbb{E}\left[\left(B_{t_i}-B_s\right)\left(B_{t_j}-B_s\right)\right]=\mathbb{E}\left[\left(B_{t_i}-B_s\right)^2+\left(B_{t_i}-B_s\right)\left(B_{t_j}-B_{t_i}\right)\right]. \tag{3.11}$$

When $H<\frac{1}{2}$, for any $1\le i\le j\le n$, the increments of $(B_t)_{t\ge 0}$ over different time intervals are negatively correlated, which leads to

$$\begin{aligned}\mathbb{E}\left[\left(B_{t_i}-B_s\right)\left(B_{t_j}-B_s\right)\right]&\le\mathbb{E}\left[\left(B_{t_i}-B_s\right)^2\right]\\&=(t_i-s)^{2H}\\&\le(t-s)^{2H}.\end{aligned} \tag{3.12}$$

When $H>\frac{1}{2}$, we seek for an upper bound of

$$\mathbb{E}\left[\left(B_{t_i}-B_s\right)\left(B_{t_j}-B_{t_i}\right)\right].$$

We compute that

$$\mathbb{E}\left[\left(B_{t_i}-B_s\right)\left(B_{t_j}-B_{t_i}\right)\right]=\frac{1}{2}\left[(t_j-s)^{2H}-(t_j-t_i)^{2H}-(t_i-s)^{2H}\right]\le\frac{1}{2}(t-s)^{2H},$$

where $0\le s<t_i<t_j\le t$. Combining with (3.11), we have

$$\mathbb{E}\left[\left(B_{t_i}-B_s\right)\left(B_{t_j}-B_s\right)\right]\le\frac{3}{2}(t-s)^{2H}.$$

Therefore, for all $H\in(0,1)$,

$$\mathbb{E}\left[\left(B_{t_i}-B_s\right)\left(B_{t_j}-B_s\right)\right]\le\gamma_H(t-s)^{2H},$$

where γ_H is defined as in (3.10).

For convenience, set $Z_i=B_{t_i}-B_s\sim N(0,(t_i-s)^{2H})$, and by the above estimate, correlations between any two Z_i's are bounded by $\gamma_H(t-s)^{2H}$. We want to apply Slepian's lemma (see [19]) to overcome the difficulties in finding the distribution of supremum process of fBM, so we take a random variable $\xi_{s,t}\sim N(0,\gamma_H(t-s)^{2H})$ independent of the standard Brownian motion $(\omega_t)_{t\ge 0}$ on $(\boldsymbol{W},\mathcal{H},P)$, so that $\xi_{s,t}$ and $\omega_{t_i}-\omega_s$ are independent for all $i\in\{1,2,\cdots,n\}$. Define $Y_i=\omega_{t_i}-\omega_s+\xi_{s,t}$, $1\le i\le n$ and let

$$N_{s,t}^{(n)}=\max_{1\le i\le n}Y_i=\max_{1\le i\le n}(\omega_{t_i}-\omega_s)+\xi_{s,t}.$$

Then by independence,

$$\begin{aligned}\mathbb{E}\left[Y_iY_j\right]&=\mathbb{E}\left[(\omega_{t_i}-\omega_s+\xi_{s,t})(\omega_{t_j}-\omega_s+\xi_{s,t})\right]\\&=\mathbb{E}\left[(\omega_{t_i}-\omega_s)(\omega_{t_j}-\omega_s)\right]+\mathbb{E}\left[\xi_{s,t}^2\right]\\&=t_i-s+\gamma_H(t-s)^{2H}\\&\ge\gamma_H(t-s)^{2H},\end{aligned}$$

for $1\le i\le j\le n$, and hence by (3.12),

$$\mathbb{E}[Z_iZ_j]\le\mathbb{E}[Y_iY_j]$$

for all $1 \le i, j \le n$. Since both exponential function and maximum function are convex, their composition is also convex, and hence according to Theorem 3.11 in Ledoux and Talagrand [19], Slepian's lemma, we obtain that

$$\mathbb{E}\left[e^{\alpha M_{s,t}^{(n)}}\right] = \mathbb{E}\left[e^{\alpha \max_{1\le i\le n} Z_i}\right] \le \mathbb{E}\left[e^{\alpha \max_{1\le i\le n} Y_i}\right] = \mathbb{E}\left[e^{\alpha N_{s,t}^{(n)}}\right],$$

for all $\alpha > 0$. Due to independence and the fact that $\max_{1\le i\le n}(\omega_{t_i} - \omega_s) \le \sup_{s\le u\le t}(\omega_u - \omega_s)$,

$$\begin{aligned}\mathbb{E}\left[e^{\alpha N_{s,t}^{(n)}}\right] &= \mathbb{E}\left[e^{\alpha \xi_{s,t}}\right]\mathbb{E}\left[\exp\left(\alpha \max_{1\le i\le n}(\omega_{t_i} - \omega_s)\right)\right] \\ &\le \exp\left(\frac{\alpha^2}{2}\gamma_H (t-s)^{2H}\right)\mathbb{E}\left[\exp\left(\alpha \sup_{s\le u\le t}(\omega_u - \omega_s)\right)\right].\end{aligned}$$

Using the distribution of supremum of standard Brownian motion, we obtain that

$$\mathbb{E}\left[e^{\alpha M_{s,t}^{(n)}}\right] = \mathbb{E}\left[\exp\left(\alpha M_{s;t_1,\cdots,t_n}\right)\right] \le 2\exp\left(\frac{\alpha^2}{2}\left[\gamma_H (t-s)^{2H} + (t-s)\right]\right). \tag{3.13}$$

Step 2. The difference from classical approach will be demonstrated in this step since we use only the Brownian motion capacity. In this step, we show that $e^{\frac{\alpha}{2}M_{s,t}^{(n)}} \in \mathbb{D}_1^2$ and

$$De^{\frac{\alpha}{2}M_{s,t}^{(n)}} = \frac{\alpha}{2}\exp\left(\frac{\alpha}{2}M_{s;t_1,\cdots,t_n}\right) DM_{s,t}^{(n)}.$$

Observe that g is Lipschitz, so by Proposition 1.2.4 in Nualart [28], $M_{s,t}^{(n)} = g(B_{s,t}^{(n)}) \in \mathbb{D}_1^2$, and the chain rule applies, which is

$$\begin{aligned}DM_{s,t}^{(n)}(u) &= \sum_{i=1}^{n} \mathbb{1}_{\{M_{s,t}^{(n)} = B_{t_i} - B_s\}}(B_{s,t}^{(n)}) D(B_{t_i} - B_s) \\ &= \sum_{i=1}^{n} \mathbb{1}_{\{M_{s,t}^{(n)} = B_{t_i} - B_s\}}(B_{s,t}^{(n)}) \left[K(t_i, u)\mathbb{1}_{[0,t_i]}(u) - K(s,u)\mathbb{1}_{[0,s]}(u)\right].\end{aligned}$$

Therefore, we have

$$\begin{aligned}\langle DM_{s,t}^{(n)}, DM_{s,t}^{(n)}\rangle_{\mathcal{H}} &= \sum_{i=1}^{n} \mathbb{1}_{\{M_{s,t}^{(n)} = B_{t_i} - B_s\}}(B_{s,t}^{(n)}) \\ &\quad \cdot \int_0^\infty \left[K(t_i, u)\mathbb{1}_{[0,t_i]}(u) - K(s,u)\mathbb{1}_{[0,s]}(u)\right]^2 du \\ &= \sum_{i=1}^{n} \mathbb{1}_{\{M_{s,t}^{(n)} = B_{t_i} - B_s\}}(B_{s,t}^{(n)})(t_i - s)^{2H} \\ &\le (t-s)^{2H} \sum_{i=1}^{n} \mathbb{1}_{\{M_{s,t}^{(n)} = B_{t_i} - B_s\}}.\end{aligned} \tag{3.14}$$

Similar to the argument in Lemma 3.1, we set $f(x) = e^{\frac{\alpha}{2}x}$, and $\psi_N(x)$ as in Lemma 3.1, and set $f_N = f \cdot \psi_N$. For simplicity, denote $F = e^{\frac{\alpha}{2}M_{s,t}^{(n)}}$, and $F_N = f_N(M_{s,t}^{(n)})$. Then since $f_N \in C_0^\infty(\mathbb{R})$, the chain rule applies, and

$$DF_N = f_N'(M_{s,t}^{(n)}) DM_{s,t}^{(n)}.$$

Similarly, we have that

$$\begin{aligned}
&\mathbb{E}\left[\|DF_N - \frac{\alpha}{2} e^{\frac{\alpha}{2} M_{s,t}^{(n)}} DM_{s,t}^{(n)}\|_{\mathcal{H}}^2\right] \\
&= \int_X \left| f'(M_{s,t}^{(n)})\psi_N(M_{s,t}^{(n)}) + f(M_{s,t}^{(n)})\psi_N'(M_{s,t}^{(n)}) - \frac{\alpha}{2} e^{\frac{\alpha}{2} M_{s,t}^{(n)}} \right|^2 \|DM_{s,t}^{(n)}\|_{\mathcal{H}}^2 dP \\
&\le \int_X \left| \frac{\alpha}{2} e^{\frac{\alpha}{2} M_{s,t}^{(n)}} \left(\psi_N(M_{s,t}^{(n)}) - 1\right) + e^{\frac{\alpha}{2} M_{s,t}^{(n)}} \psi_N'(M_{s,t}^{(n)}) \right|^2 \left(\sum_{i=1}^n \mathbb{1}_{\{M_{s,t}^{(n)} = B_{t_i} - B_s\}} \right) (t-s)^{2H} dP \\
&\le 2\mathbb{E}\left[\left| \frac{\alpha}{2} e^{\frac{\alpha}{2} M_{s,t}^{(n)}} \cdot \mathbb{1}_{\{|M_{s,t}^{(n)}| \ge N\}} \right|^2 + \left| e^{\frac{\alpha}{2} M_{s,t}^{(n)}} C \cdot \mathbb{1}_{\{|M_{s,t}^{(n)}| \ge N\}} \right|^2 \right] (t-s)^{2H},
\end{aligned}$$

which tends to zero as $N \to \infty$, where C is defined as in Lemma 3.1. Therefore, since $F_N \to F$ in $L^2(\boldsymbol{W})$, $DF_N \to \frac{\alpha}{2} e^{\frac{\alpha}{2} M_{s,t}^{(n)}} DM_{s,t}^{(n)}$ in $L^2(\boldsymbol{W}; \mathcal{H})$ and D is closable from $L^2(\boldsymbol{W})$ to $L^2(\boldsymbol{W}; \mathcal{H})$, it follows that

$$DF = \frac{\alpha}{2} e^{\frac{\alpha}{2} M_{s,t}^{(n)}} DM_{s,t}^{(n)} \tag{3.15}$$

and $F \in \mathbb{D}_1^2$.

Step 3. In this step, we find an upper bound for $\mathbb{E}\left[e^{\alpha M_{s,t}^*}\right]$ for any $\alpha > 0$, then we prove that $e^{\alpha M_{s,t}^*} \in \mathbb{D}_1^2$ and find an upper bound for $\|e^{\alpha M_{s,t}^*}\|_{\mathbb{D}_1^2}$. As $M_{s;t_1,\cdots,t_n}$ increases to $M_{s,t}^*$ when we refine the partition and let n go to infinity, the monotone convergence theorem and (3.13) imply that

$$\mathbb{E}\left[e^{\alpha M_{s,t}^*}\right] \le 2\exp\left(\frac{\alpha^2}{2} \left[\gamma_H (t-s)^{2H} + (t-s)\right] \right).$$

We have already proved that $e^{\alpha M_{s;t_1,\cdots,t_n}} \in \mathbb{D}_1^2$ in last step, by (3.14) and (3.15),

$$\begin{aligned}
\langle D\exp\left(\frac{\alpha}{2} M_{s;t_1,\cdots,t_n}\right), D\exp\left(\frac{\alpha}{2} M_{s;t_1,\cdots,t_n}\right)\rangle_{\mathcal{H}} &\le \frac{\alpha^2}{4}(t-s)^{2H} \exp\left(\alpha M_{s;t_1,\cdots,t_n}\right) \\
&\quad \cdot \sum_{i=1}^n \mathbb{1}_{\{M_{s;t_1,\cdots,t_n} = B_{t_i} - B_s\}}.
\end{aligned}$$

Therefore, by (3.13), we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\langle D\exp\left(\frac{\alpha}{2} M_{s;t_1,\cdots,t_n}\right), D\exp\left(\frac{\alpha}{2} M_{s;t_1,\cdots,t_n}\right)\rangle_{\mathcal{H}}\right] \\
&\le \frac{\alpha^2}{4}(t-s)^{2H} \sum_{i=1}^n \int_{\{M_{s;t_1,\cdots,t_n} = B_{t_i} - B_s\}} \exp\left(\alpha M_{s;t_1,\cdots,t_n}\right) dP \\
&= \frac{\alpha^2}{4}(t-s)^{2H} \mathbb{E}\left[\exp\left(\alpha M_{s;t_1,\cdots,t_n}\right)\right] \\
&\le \frac{\alpha^2}{2}(t-s)^{2H} \exp\left(\frac{\alpha^2}{2} \left[\gamma_H (t-s)^{2H} + (t-s)\right] \right),
\end{aligned}$$

which implies that

$$\sup_n \mathbb{E}\left[\|D\exp\left(\frac{\alpha}{2} M_{s;t_1,\cdots,t_n}\right)\|_{\mathcal{H}}^2\right] < \infty.$$

Applying Lemma 1.2.3 in [28], we deduce that $e^{\frac{\alpha}{2}M^*_{s,t}} \in \mathbb{D}^2_1$ and

$$D\exp\left(\frac{\alpha}{2}M_{s;t_1,\cdots,t_n}\right) \to De^{\frac{\alpha}{2}M^*_{s,t}}$$

weakly in $L^2(\boldsymbol{W};\mathcal{H})$. As a consequence, we have

$$\begin{aligned}\mathbb{E}\left[\langle De^{\frac{\alpha}{2}M^*_{s,t}}, De^{\frac{\alpha}{2}M^*_{s,t}}\rangle_{\mathcal{H}}\right] &\leq \liminf_{n\to\infty}\mathbb{E}\left[\langle D\exp\left(\frac{\alpha}{2}M_{s;t_1,\cdots,t_n}\right), D\exp\left(\frac{\alpha}{2}M_{s;t_1,\cdots,t_n}\right)\rangle_{\mathcal{H}}\right]\\ &\leq \frac{\alpha^2}{4}(t-s)^{2H}\mathbb{E}\left[e^{\alpha M^*_{s,t}}\right]\\ &\leq \frac{\alpha^2}{2}(t-s)^{2H}\exp\left(\frac{\alpha^2}{2}\left[\gamma_H(t-s)^{2H}+(t-s)\right]\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|e^{\frac{\alpha}{2}M^*_{s,t}}\right\|^2_{\mathbb{D}^2_1} &= \mathbb{E}\left[e^{\alpha M^*_{s,t}}\right] + \mathbb{E}\left[\langle De^{\frac{\alpha}{2}M^*_{s,t}}, De^{\frac{\alpha}{2}M^*_{s,t}}\rangle_{\mathcal{H}}\right]\\ &\leq \left(\frac{\alpha^2}{2}(t-s)^{2H}+2\right)\exp\left(\frac{\alpha^2}{2}\left[\gamma_H(t-s)^{2H}+(t-s)\right]\right).\end{aligned} \tag{3.16}$$

Step 4. By Chebyshev's inequality for capacity and (3.16), we thus have

$$\begin{aligned}&\left[c_{2,1}\left(M^*_{s,t}-\frac{\alpha}{2}(t-s)^{2H}>\beta\right)\right]^2\\ &= \left[c_{2,1}\left(\frac{\alpha}{2}M^*_{s,t}-\frac{\alpha^2}{4}(t-s)^{2H}>\frac{\alpha\beta}{2}\right)\right]^2\\ &= \left[c_{2,1}\left(\exp\left(\frac{\alpha}{2}M^*_{s,t}\right)>\exp\left(\frac{\alpha\beta}{2}+\frac{\alpha^2}{4}(t-s)^{2H}\right)\right)\right]^2\\ &\leq \exp\left(-\alpha\beta-\frac{\alpha^2}{2}(t-s)^{2H}\right)\left\|e^{\frac{\alpha}{2}M^*_{s,t}}\right\|^2_{\mathbb{D}^2_1}\\ &\leq \left(\frac{\alpha^2}{2}(t-s)^{2H}+2\right)\exp\left(-\alpha\beta+\frac{\alpha^2}{2}\left[(\gamma_H-1)(t-s)^{2H}+(t-s)\right]\right)\end{aligned} \tag{3.17}$$

for any positive constants α and β.

Notice that the exponential function is the dominating part in the last term of (3.17), so we optimise the above quantity by minimising the exponent and setting

$$\alpha = \frac{\eta}{\gamma_H(t-s)^{2H}+(t-s)},$$

and

$$\beta = \eta - \frac{\alpha}{2}(t-s)^{2H}.$$

Therefore, we get that

$$\left[c_{2,1}\left(M^*_{s,t}>\eta\right)\right]^2 \leq C^2_{s,t,\eta,H}\exp\left(-\frac{\eta^2}{2\left[\gamma_H(t-s)^{2H}+(t-s)\right]}\right),$$

where

$$C_{s,t,\eta,H} = \sqrt{\frac{\eta^2(t-s)^{2H}}{2\left[\gamma_H(t-s)^{2H}+(t-s)\right]^2}+2}.$$

Moreover, by replacing B with $-B$, it follows that

$$\left[c_{2,1}\left(\sup_{s\le u\le t}|B_u-B_s|>\eta\right)\right]^2 \le 2C^2_{s,t,\eta,H}\exp\left(-\frac{\eta^2}{2\left[\gamma_H(t-s)^{2H}+(t-s)\right]}\right).$$

Finally, (3.9) may be established directly following the same argument with slight modification in the definition of $M^*_{s,t}$. $\square$

Remark 3.4. The results in the previous lemma can be considered as the maximal inequality for fBMs but with respect to Brownian motion capacity. For a similar result when $H=\frac{1}{2}$, one may refer to Fukushima [9], or Takeda [32] for any $r\in\mathbb{N}$ and $p\in(1,\infty)$. Though we establish the inequalities for all $H\in(0,1)$, when considering a sufficiently small time interval $[s,t]$, the result looks weaker when $H>\frac{1}{2}$ due to the appearance of $(t-s)$ in the exponent. In fact, when $H>\frac{1}{2}$, $(t-s)$ will be the dominating part rather than $(t-s)^{2H}$. However, the factor $(t-s)$ appears necessary for small time intervals.

4. Modulus of continuity

In this part, we shall show the result on modulus of continuity for fractional Brownian motion with respect to the (p,r)-capacity defined on classical Wiener space. We shall adopt the arguments in Fukushima's work [9] and the original proof by Lévy [20], who proved the modulus of continuity of Brownian motion in probability sense.

Theorem 4.1. *Let $(B_t)_{t\ge0}$ be an fBM with Hurst parameter H. Then it holds that*

$$\limsup_{\delta\downarrow0}\frac{1}{\sqrt{2\delta^{2H}\log(1/\delta)}}\max_{\substack{0\le s<t\le1\\ t-s\le\delta}}|B_t-B_s|\le1,\quad q.s. \tag{4.1}$$

when $H\in(0,1)$ and

$$\limsup_{\delta\downarrow0}\frac{1}{\sqrt{2\delta^{2H}\log(1/\delta)}}\max_{\substack{0\le s<t\le1\\ t-s\le\delta}}|B_t-B_s|\ge1,\quad q.s. \tag{4.2}$$

when $H\in(0,\frac{1}{2}]$.

Proof. Let us prove (4.2) first. For any $r\in\mathbb{N}$ and $p\in(1,\infty)$, we want to show that

$$c_{p,r}\left(\limsup_{\delta\downarrow0}\frac{1}{g(\delta)}\max_{\substack{0\le s<t\le1\\ t-s\le\delta}}|B_t-B_s|<1\right)=0,$$

where $g(\delta)=\sqrt{2\delta^{2H}\log(1/\delta)}$.

By Lemma 3.2, we have

$$\begin{aligned}
&c_{p,r}\left(\max_{1\le j\le 2^n}\left|B_{\frac{j}{2^n}}-B_{\frac{j-1}{2^n}}\right|\le(1-\theta)g(2^{-n})\right)\\
&=c_{p,r}\left(\max_{1\le j\le 2^n}2^{nH}\left|B_{\frac{j}{2^n}}-B_{\frac{j-1}{2^n}}\right|\le(1-\theta)\sqrt{2\log(2^n)}\right)
\end{aligned}$$

$$\leq \left[\sum_{k=0}^{r} N^{kp} C_H^{kp/2} \left(\frac{M_r}{c}\right)^{kp}\right]^{1/p} \left[P\left(\max_{1\leq j\leq 2^n} 2^{nH}\left|B_{\frac{j}{2^n}} - B_{\frac{j-1}{2^n}}\right| \leq (1-\theta)\sqrt{2\log(2^n)} + c\right)\right]^{1/p}$$

for $\theta \in (0,1)$, where c is some small constant such that $c < \theta\sqrt{2\log 2}$. Set

$$X_j = 2^{nH}\left(B_{\frac{j}{2^n}} - B_{\frac{j-1}{2^n}}\right) \sim N(0,1),$$

then $X_j \sim N(0,1)$ and when $H \leq \frac{1}{2}$, $\mathbb{E}[X_j X_k] \leq 0$ for $j \neq k$. Take a sequence of independent standard Gaussian random variables Y_j's so that $\mathbb{E}[X_j X_k] \leq 0 = \mathbb{E}[Y_j Y_k]$ for $j \neq k$. Let $c' = \frac{c}{\sqrt{2log2}}$ so that $\theta - c' > 0$ and hence $0 < 1-\theta+c' < 1$. Slepian's lemma (see Corollary 3.12, [19]) implies that

$$\begin{aligned}
&P\left(\bigcap_{1\leq j\leq 2^n}\left\{|X_j| \leq (1-\theta+c')\sqrt{2\log(2^n)}\right\}\right)\\
&\leq P\left(\bigcap_{1\leq j\leq 2^n}\left\{|X_j| \leq (1-\theta+c')^{1/2}\sqrt{2\log(2^n)}\right\}\right)\\
&\leq P\left(\bigcap_{1\leq j\leq 2^n}\left\{X_j \leq (1-\theta+c')^{1/2}\sqrt{2\log(2^n)}\right\}\right)\\
&\leq P\left(\bigcap_{1\leq j\leq 2^n}\left\{Y_j \leq (1-\theta+c')^{1/2}\sqrt{2\log(2^n)}\right\}\right)\\
&= \prod_{1\leq j\leq 2^n} P\left(Y_j \leq (1-\theta+c')^{1/2}\sqrt{2\log(2^n)}\right)\\
&= \left[1 - P\left(Y_j > (1-\theta+c')^{1/2}\sqrt{2\log(2^n)}\right)\right]^{2^n}\\
&\leq \exp\left(-\xi 2^n\right),
\end{aligned}$$

where

$$\begin{aligned}
\xi &= P\left(Y_j > (1-\theta+c')^{1/2}\sqrt{2\log(2^n)}\right)\\
&\geq \frac{(1-\theta+c')^{1/2}\sqrt{2\log(2^n)}}{1+2(1-\theta+c')\log(2^n)}\exp\left(-(1-\theta+c')\log(2^n)\right)\\
&\geq C2^{-n(1-\theta+c')}
\end{aligned}$$

for n sufficiently large, hence it follows that

$$P\left(\bigcap_{1\leq j\leq 2^n}\left\{|X_j| \leq (1-\theta+c')\sqrt{2\log(2^n)}\right\}\right) \leq \exp\left(-C2^{n(\theta-c')}\right).$$

The right-hand side is a term of a convergent series, and hence by the first Borel–Cantelli Lemma for (p,r)-capacity, (4.2) follows immediately.

For the upper bound, we first notice that $g(k2^{-n}) = (k2^{-n})^H\sqrt{2\log(\frac{2^n}{k})}$. For any $\varepsilon > 0$, applying Lemma 3.1 with $\eta = (1+\varepsilon)g(k2^{-n})$, we get that

$$
\begin{aligned}
I_n^p &= \left[c_{p,r} \left(\max_{\substack{0<k=j-i\leq 2^{n\theta} \\ 0\leq i<j\leq 2^n}} \frac{|B_{j2^{-n}} - B_{i2^{-n}}|}{g(k2^{-n})} \geq 1+\varepsilon \right) \right]^p \\
&\leq M_{p,r} \sum_{\substack{0<k=j-i\leq 2^{n\theta} \\ 0\leq i<j\leq 2^n}} \left[c_{p,r} \left(\frac{|B_{j2^{-n}} - B_{i2^{-n}}|}{g(k2^{-n})} \geq 1+\varepsilon \right) \right]^p \\
&\leq M_{p,r} 2^n \sum_{1\leq k\leq 2^{n\theta}} \left[2 \sum_{l=0}^{r} \left(\frac{(1+\varepsilon) g(k2^{-n})}{p(k2^{-n})^H} \right)^{lp} \right] (k2^{-n})^{(1+\varepsilon)^2} \\
&= M_{p,r} 2^n \sum_{1\leq k\leq 2^{n\theta}} \left[2 \sum_{l=0}^{r} \left(\frac{(1+\varepsilon)}{p} \sqrt{2\log\left(\frac{2^n}{k}\right)} \right)^{lp} \right] (k2^{-n})^{(1+\varepsilon)^2} \\
&\leq M_{p,r} 2^{n(1+\theta)} \left[2 \sum_{l=0}^{r} \left(\frac{(1+\varepsilon)}{p} \sqrt{2n\log 2} \right)^{lp} \right] 2^{-n(1-\theta)(1+\varepsilon)^2},
\end{aligned}
$$

where the first inequality follows from (2.2). Now we only need to pick up suitable θ such that $\sum_n I_n < \infty$. To this end, we want $1+\theta < (1-\theta)(1+\varepsilon)^2$. In fact, any

$$
\theta \in \left(0, \frac{(1+\varepsilon)^2 - 1}{(1+\varepsilon)^2 + 1} \right)
$$

will do. The proof is complete by applying the first Borel–Cantelli lemma for (p,r)-capacity and letting $\varepsilon \to 0$. $\square$

The upper bound (4.1) implies the following result:

Corollary 4.2. *$(B_t)_{t\geq 0}$ is α-Hölder-continuous for $\alpha < H$ quasi-surely with respect to the Brownian motion capacity.*

Remark 4.3. We regard $(B_t)_{t\geq 0}$ as a family of measurable functions on $(\boldsymbol{W}, \mathscr{F})$ with parameter $t \geq 0$. What we proved previously is that apart from a slim set, $t \to B_t(\omega)$ is continuous. Therefore, we can modify $(B_t)_{t\geq 0}$ on the slim set K by for example setting $B_t(\omega) = 0$ for all $\omega \in K$ such that the modified process is continuous, and $K \in \mathscr{F}$ with $P(K) = 0$ as $c_{p,r}$ is increasing in p and r. From now on, we always refer $(B_t)_{t\geq 0}$ to its continuous modification.

5. Non-differentiability

In this part, we will generalize a very standard result based on the argument in [7] (see also [16] page 110), [9] and [32]). In fact, non-differentiability of sample paths is a fact for an fBM quasi-surely with respect to either Brownian motion capacity or the natural capacity defined by its law, for any Hurst parameter H.

Theorem 5.1. *Let $H \in (0,1)$. Then*

$$
\limsup_{h\downarrow 0} \frac{|B_{t+h} - B_t|}{h} = \infty \quad \text{for all } t \in [0,1] \quad q.s.
$$

Proof. Let

$$
A = \left\{ \limsup_{h\downarrow 0} \frac{|B_{t+h} - B_t|}{h} < \infty \text{ for some } t \in [0,1] \right\}.
$$

The goal is to show that A is a slim set. If $\omega \in A$, then there exists a $t \in [0,1]$, positive integers M and k, such that $|B_{t+h}(\omega) - B_t(\omega)| \leq Mh$ for all $0 \leq h \leq \frac{1}{k}$. Therefore, we may consider

$$A^t_{k,M} = \left\{ \sup_{h\in[0,\frac{1}{k}]} \frac{|B_{t+h} - B_t|}{h} \leq M \right\}$$

where M and k are positive integers. Then

$$A = \bigcup_{t\in[0,1]} \bigcup_{M=1}^{\infty} \bigcup_{k=1}^{\infty} A^t_{k,M}.$$

By the sub-additivity property of (p,r)-capacity, it remains to show that

$$c_{p,r}\Big(\bigcup_{t\in[0,1]} A^t_{k,M} \Big) = 0$$

for all $r \in \mathbb{N}$ and $1 < p < \infty$.

Fix r, p, k and M. For $H \in (0,1)$, take N to be the smallest integer such that $\frac{N(1-H)}{p} > 1$, and divide $[0,1]$ into n subintervals with $n \geq (N+1)k$. Then for all $t \in [\frac{i-1}{n}, \frac{i}{n}]$, $1 \leq i \leq n$,

$$\frac{i+N}{n} - t \leq \frac{1}{k},$$

which indicates that for $1 \leq j \leq N$,

$$\frac{i+j-1}{n} - t \leq \frac{i+j}{n} - t \leq \frac{i+N}{n} - t \leq \frac{1}{k}. \tag{5.1}$$

Now if $\omega \in A^t_{k,M}$ with $t \in [\frac{i-1}{n}, \frac{i}{n}]$, then for each $1 \leq j \leq N$, by (5.1),

$$\begin{aligned}\left|B_{\frac{i+j}{n}}(\omega) - B_{\frac{i+j-1}{n}}(\omega)\right| &\leq \left|B_{t+(\frac{i+j}{n}-t)}(\omega) - B_t(\omega)\right| + \left|B_t(\omega) - B_{t+(\frac{i+j-1}{n}-t)}(\omega)\right| \\ &\leq \left[\left(\frac{i+j}{n} - t\right) + \left(\frac{i+j-1}{n} - t\right)\right] M \\ &\leq \frac{(2j+1)M}{n}.\end{aligned}$$

Therefore, if we define

$$C_{i,n} = \bigcap_{j=1}^{N} \left\{ n^H \left| B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}} \right| \leq \frac{(2j+1)M}{n^{1-H}} \right\}, \quad 1 \leq i \leq n$$

for each $n \geq (N+1)k$, then

$$\bigcup_{t\in[0,1]} A^t_{k,M} \subset \bigcup_{i=1}^{n} C_{i,n}.$$

Therefore, it suffices to prove that $\sum_{i=1}^n c_{p,r}(C_{i,n}) \to 0$ as $n \to \infty$.

To this end, we apply Lemma 3.2 to bound $c_{p,r}(C_{i,n})$ from above. For each fixed i, set $X_j = n^H(B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}})$, and $\alpha_j = \frac{(2j+1)M}{n^{1-H}}$, $1 \leq j \leq N$. By Lemma 3.2 with $L = \frac{1}{n}$, it follows that

$$\sum_{i=1}^{n} c_{p,r}(C_{i,n}) \leq \left[\sum_{l=0}^{r}\left(N^{lp}C_H^{lp/2}\left(\frac{M_r}{c}\right)^{lp}\right)\right]^{1/p} \cdot \sum_{i=1}^{n}\left[P\left(\bigcap_{j=1}^{N}\{-\alpha_j - c \leq X_j \leq \alpha_j + c\}\right)\right]^{1/p}, \tag{5.2}$$

where $c > 0$ is a constant, M_r and C_H are as in Lemma 3.2. Note that $(X_1, \cdots, X_N)$ is a centred Gaussian random variable with covariance matrix Σ, determined by

$$\mathbb{E}[X_j X_k] = \frac{1}{2}\left[(k-j+1)^{2H} + (k-j-1)^{2H}\right] - (k-j)^{2H},$$

which depends only on j and k. Σ is an $N \times N$ positive definite matrix independent of n. Therefore, the right-hand side of (5.2) may be computed explicitly as

$$\begin{aligned} P\left(\bigcap_{j=1}^{N}\{|X_j| \leq \alpha_j + c\}\right) &= 2^N \int_0^{\alpha_N + c} \cdots \int_0^{\alpha_1 + c} \frac{1}{\sqrt{2\pi|\Sigma|}} \exp\left(-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}\right) dx_1 \cdots dx_N \\ &\leq 2^N \frac{1}{\sqrt{2\pi|\Sigma|}} \prod_{j=1}^{N}(\alpha_j + c) \\ &= O(n^{-N(1-H)}), \end{aligned}$$

hence it follows that

$$\sum_{i=1}^{n}\left[P\left(\bigcap_{j=1}^{N}\{-\alpha_j - c \leq X_j \leq \alpha_j + c\}\right)\right]^{1/p} \leq O(n \cdot n^{-N(1-H)/p}) \to 0$$

as $n \to \infty$, which completes the proof. □

6. Law of the iterated logarithm

In this section, we establish the result on the law of the iterated logarithm for fBM with Hurst parameter $H \in \left(0, \frac{1}{2}\right]$ with respect to (2, 1)-capacity on classical Wiener space, using the argument from [9] together with the technical lemmas we established in Section 3.

Theorem 6.1. *Let $H \in \left(0, \frac{1}{2}\right]$. Then it holds that*

$$c_{2,1}\left(\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{2t^{2H}\log\log(1/t)}} > 1\right) = 0.$$

Proof. When $H = \frac{1}{2}$, the problem will be reduced to Brownian motion case, which will be the same as in [9] and [32]. The rest of our proof will be similar to the argument in [9]. Let $h(t) = \sqrt{2t^{2H}\log\log(1/t)}$. Fix $\theta, \delta \in (0, 1)$, and set $\eta = (1+\delta)h(\theta^n)$, $s = 0$, $t = \theta^n$ in Lemma 3.3, then it follows that

$$\begin{aligned}
&\left[c_{2,1}\left(\sup_{0\le u\le\theta^n} B_u > (1+\delta)h(\theta^n)\right)\right]^2 \\
&\le \left[\left(\frac{\theta^{2nH}}{\theta^{2nH}+\theta^n}\right)^2 (1+\delta^2)\log\log(\theta^{-n})+2\right]\exp\left(-\frac{\theta^{2nH}}{\theta^{2nH}+\theta^n}(1+\delta)^2\log\log(\theta^{-n})\right) \\
&\le \left[(1+\delta^2)\log\log(\theta^{-n})+2\right]\left(n\log(\theta^{-1})\right)^{-\frac{\theta^{2nH}}{\theta^{2nH}+\theta^n}(1+\delta)^2} \\
&= C_1(\log n + C_2)n^{-\frac{\theta^{2nH}}{\theta^{2nH}+\theta^n}(1+\delta)^2}.
\end{aligned} \tag{6.1}$$

For each θ and δ, as $H<\frac{1}{2}$ and $\theta<1$, there exists some N_0 such that for all $n\ge N_0$,

$$\frac{\theta^{2nH}}{\theta^{2nH}+\theta^n}(1+\delta)^2 > 1,$$

so the right-hand side of (6.1) is a term of a convergent series, and thus by the first Borel–Cantelli lemma for capacity,

$$\sup_{0\le u\le\theta^n} B_u \le (1+\delta)h(\theta^n) \quad \text{eventually}$$

under $(2,1)$-capacity. The rest of proof remains the same as in probability case. □

Theorem 6.2. *Let $(B_t)_{t\ge0}$ be a one-dimensional fBM on $(\boldsymbol{W},\mathcal{H},P)$ with Hurst parameter $H\in\left(0,\frac{1}{2}\right]$. Then it holds that*

$$c_{2,1}\left(\limsup_{t\downarrow0}\frac{B_t}{\sqrt{2t^{2H}\log\log(1/t)}}<1\right)=0.$$

Proof. When $H=\frac{1}{2}$, the problem is reduced to Brownian motion case, so we only need to consider the case when $H\in\left(0,\frac{1}{2}\right)$. Let $h(t)=\sqrt{2t^{2H}\log\log(1/t)}$, and let $\theta\in(0,1)$, define

$$G_n=\left\{B_{\theta^n}-B_{\theta^{n+1}}<(1-\theta^H)h(\theta^n)\right\}.$$

Our next step is to prove that

$$c_{2,1}\left(\liminf_{n\to\infty} G_n\right)=0,$$

from which we may deduce that for sufficiently large n,

$$B_{\theta^n}-B_{\theta^{n+1}}>(1-\theta^H)h(\theta^n)$$

apart from on a $(2,1)$-capacity zero set.

Write

$$X_n=\frac{B_{\theta^n}-B_{\theta^{n+1}}}{(\theta^n-\theta^{n+1})^H}\sim N(0,1),$$

then by definition,

$$G_n=\left\{X_n<\frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})}\right\}.$$

For any integers $l \le N$, take a decreasing sequence of real numbers $\{a_i\}_{i=1}^{\infty}$ such that $a_i \downarrow -\infty$ as $i \to \infty$, due to the continuity of capacity (2.3), we have that

$$\begin{aligned}\left[c_{2,1}\left(\bigcap_{n=l}^{N} G_n\right)\right]^2 &= \left[c_{2,1}\left(\bigcap_{n=l}^{N}\left\{X_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})}\right\}\right)\right]^2 \\ &= \left[c_{2,1}\left(\bigcup_{i=1}^{\infty}\bigcap_{n=l}^{N}\left\{a_i < X_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})}\right\}\right)\right]^2 \\ &= \lim_{i\to\infty}\left[c_{2,1}\left(\bigcap_{n=l}^{N}\left\{a_i < X_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})}\right\}\right)\right]^2.\end{aligned}$$

Then we may apply Lemma 3.2 to control the intersection capacity with probability as the following:

$$\begin{aligned}\left[c_{2,1}\left(\bigcap_{n=l}^{N} G_n\right)\right]^2 &\le \lim_{i\to\infty}\left(1+(N-l)^2 C_H\left(\frac{M_r}{c}\right)^2\right) \\ &\quad \cdot P\left(\bigcap_{n=l}^{N}\left\{a_i - c_n < X_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})}+c_n\right\}\right) \\ &\le \left(1+(N-l)^2 C_H\left(\frac{M_r}{c}\right)^2\right) \\ &\quad \cdot P\left(\bigcap_{n=l}^{N}\left\{X_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})}+c_n\right\}\right).\end{aligned} \tag{6.2}$$

When $H \in \left(0, \frac{1}{2}\right)$, the increments of fBM over different time intervals are negatively correlated, i.e. $\mathbb{E}[X_n X_m] \le 0$. For all $l \le n, m \le N$, we may take a sequence of independent standard Gaussian random variables $\{Y_n\}$, and apply Slepian's lemma to the intersection probability in the last line in (6.2) to obtain that

$$\begin{aligned}&P\left(\bigcap_{n=l}^{N}\left\{X_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})}+c_n\right\}\right) \\ &\le P\left(\bigcap_{n=l}^{N}\left\{Y_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})}+c_n\right\}\right) \\ &= \prod_{n=l}^{N} P\left(Y_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})}+c_n\right) \\ &= \prod_{n=l}^{N}\left[1 - P\left(Y_n \ge \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})}+c_n\right)\right] \\ &\le \exp\left[-\sum_{n=l}^{N} P\left(Y_n \ge \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})}+c_n\right)\right],\end{aligned}$$

where the last inequality follows from the fact that $1-x \le e^{-x}$. We proceed by picking up suitable c_n's such that the left-hand of (6.2) vanishes as N goes to infinity. Notice that for each $n \in [l, N]$, it holds that

$$\begin{aligned} &P\left(Y_n \geq \alpha\sqrt{2\log\log(\theta^{-n})}\right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\alpha\sqrt{2\log\log(\theta^{-n})}}^{\infty} e^{-\frac{x^2}{2}} dx \\ &\geq \frac{1}{\sqrt{2\pi}} \frac{\alpha\sqrt{2\log\log(\theta^{-n})}}{1+2\alpha^2\log\log(\theta^{-n})} \exp\left(-\alpha^2\log\log(\theta^{-n})\right) \\ &\geq \frac{1}{\sqrt{2\pi}} \frac{1}{C_1\sqrt{2\alpha^2\log\log(\theta^{-n})}} \cdot \frac{1}{n^{\alpha^2}(\log(\theta^{-1}))^{\alpha^2}} \\ &\geq \frac{C_2}{n^{\alpha^2}\sqrt{\log n}}, \end{aligned} \tag{6.3}$$

where C_1 and C_2 are positive constants. Choose suitable C and small β such that $\log x < Cx^\beta$ for large x, and set c_n to be small enough such that the quantity

$$\alpha = \frac{c_n}{\sqrt{2\log\log(\theta^{-n})}} + \frac{1-\theta^H}{(1-\theta)^H}$$

satisfies $\gamma = \alpha^2 + \frac{\beta}{2} < 1$. By taking α equal to the above value in (6.3), we conclude that

$$\sum_{n=l}^{N} P\left(Y_n \geq \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})} + c_n\right) \geq \sum_{n=l}^{N} \frac{C_3}{n^\gamma} \geq C_3(N^{1-\gamma} - l^{1-\gamma}),$$

where C_3 is a positive constant. Therefore,

$$\left[c_{2,1}\left(\bigcap_{n=l}^{\infty} G_n\right)\right]^2 \leq \left[c_{2,1}\left(\bigcap_{n=l}^{N} G_n\right)\right]^2 \leq C'(N-l)^2 C_H e^{-C_3(N^{1-\gamma}-l^{1-\gamma})},$$

where C' is some positive constant, and $C_H = \max\left\{2^{2H-1}-1, 1\right\} \leq 1$ as in Lemma 3.2. Since the right-hand side of above inequality vanishes as N goes to infinity, we arrive at

$$c_{2,1}\left(\liminf_{n\to\infty} G_n\right) = 0. \quad \square$$

We are unable to extend the result to the case where $H > \frac{1}{2}$, and we do not believe a similar result is true for this case in fact.

7. Self-intersection of sample paths

Recall that $\boldsymbol{W}_0^d$ consists of all $\mathbb{R}^d$-valued continuous paths, started at the origin, and $(\boldsymbol{W}_0^d, \mathcal{H}, P)$ is the corresponding classical Wiener space. In this section, a d-dimensional fBM is defined to be the functional on $(\boldsymbol{W}_0^d, \mathcal{H}, P)$ given by the integral

$$B_t = \int_0^t K(t,s) d\omega(s), \tag{7.1}$$

where $\omega \in \boldsymbol{W}_0^d$ is d-dimensional Brownian motion. By definition, a d-dimensional fBM is d copies of independent one-dimensional fBM defined as in (2.4) due to the definition of multi-dimensional Brownian motion. Like in the one-dimensional case, we take a suitable modification of B_t such that it is quasi-surely continuous with respect to classical Wiener capacity.

In this section, we will study the self-avoiding property for d-dimensional fBM and establish a result with respect to $(2,1)$-capacity on $(\boldsymbol{W}_0^d, \mathcal{H}, P)$, following the idea by Kakutani [15] together with several techniques in Fukushima [9] and Takeda [32] to tackle with capacities.

Theorem 7.1. *Let $B = (B_t)_{t\geq 0}$ be the d-dimensional fBM defined in (7.1) with Hurst parameter H. When $H \leq \frac{1}{2}$ and $d > \frac{2}{H} + 2$, B has no double point under $(2,1)$-capacity on classical Wiener space; when $H \geq \frac{1}{2}$ and $d > 6$, B has no double point under $(2,1)$-capacity.*

Proof. When $H = \frac{1}{2}$, the above result is proved in Fukushima [9] and Takeda [32]. It suffices to show that for any two disjoint intervals $I = (s_0, s_1)$ and $J = (t_0, t_1)$ with $s_0 < s_1 < t_0 < t_1$,

$$c_{2,1}\left(B_s = B_t, \text{ for some } s \in I \text{ and some } t \in J\right) = 0. \tag{7.2}$$

By self-similarity property of fBM, we only need to establish the above equality for $0 \leq s_0 < s_1 < t_0 < t_1 \leq 1$. Denote the set in (7.2) by A. Then for any $\eta > 0$, we may write

$$A \subset \bigcap_{i=1}^{d} \left\{\left|B_{s_1}^i - B_{t_0}^i\right| < 2\eta\right\} \cup \bigcup_{i=1}^{d} \left\{\sup_{s\in I} \left|B_{s_1}^i - B_s^i\right| > \eta\right\} \cup \bigcup_{i=1}^{d} \left\{\sup_{t\in J} \left|B_t^i - B_{t_0}^i\right| > \eta\right\},$$

where B^i is the i-th component of B. It thus follows from sub-additivity property of capacity that

$$\begin{aligned} c_{2,1}(A) \leq\ & c_{2,1}\left(\bigcap_{i=1}^{d} \left\{\left|B_{s_1}^i - B_{t_0}^i\right| < 2\eta\right\}\right) \\ & + \sum_{i=1}^{d} c_{2,1}\left(\sup_{s\in I} \left|B_{s_1}^i - B_s^i\right| > \eta\right) \\ & + \sum_{i=1}^{d} c_{2,1}\left(\sup_{t\in J} \left|B_t^i - B_{t_0}^i\right| > \eta\right). \end{aligned}$$

Applying Lemma 3.2 with $c = c_i = \eta$, $i = 1, 2, \cdots, d$, we obtain that

$$\begin{aligned} c_{2,1}\left(\bigcap_{i=1}^{d} \left\{-2\eta < B_{s_1}^i - B_{t_0}^i < 2\eta\right\}\right) \leq & \left(1 + d^2 C_H \left(\frac{M}{\eta}\right)^2\right) \\ & \cdot P\left(\bigcap_{i=1}^{d} \left\{\left|B_{s_1}^i - B_{t_0}^i\right| < 3\eta\right\}\right), \end{aligned}$$

where $C_H = \max\left\{2^{2H-1} - 1, 1\right\} \leq 1$, and M is some positive constant. Therefore,

$$\begin{aligned} & c_{2,1}\left(\bigcap_{i=1}^{d} \left\{\left|B_{s_1}^i - B_{t_0}^i\right| < 2\eta\right\}\right) \\ & \leq \left(1 + d^2 \left(\frac{M}{\eta}\right)^2\right) \prod_{i=1}^{d} P\left(\left|B_{s_1}^i - B_{t_0}^i\right| < 3\eta\right) \end{aligned}$$

$$= \left(1 + d^2 \left(\frac{M}{\eta}\right)^2\right) \left[\frac{1}{\sqrt{2\pi (t_0 - s_1)^{2H}}} \int_{-3\eta}^{3\eta} \exp\left(-\frac{x^2}{2(t_0 - s_1)^{2H}}\right) dx\right]^d$$

$$\leq \left(1 + d^2 \left(\frac{M}{\eta}\right)^2\right) \left(\frac{6\eta}{\sqrt{2\pi \left(d(I,J)\right)^{2H}}}\right)^d,$$

where $d(I,J) = t_0 - s_1$ denotes the distance between these two intervals. Also, by Lemma 3.3, it follows that

$$c_{2,1}\left(\sup_{s\in I} \left|B^i_{s_1} - B^i_s\right| > \eta\right) \leq \sqrt{\frac{\eta^2 (s_1 - s_0)^{2H}}{\left[\gamma_H (s_1 - s_0)^{2H} + (s_1 - s_0)\right]^2} + 4} \cdot \exp\left(-\frac{\eta^2}{4[\gamma_H (s_1 - s_0)^{2H} + (s_1 - s_0)]}\right)$$
$$= \sqrt{\frac{\eta^2 |I|^{2H}}{\left[\gamma_H |I|^{2H} + |I|\right]^2} + 4} \cdot \exp\left(-\frac{\eta^2}{4[\gamma_H |I|^{2H} + |I|]}\right),$$

where $|I| = s_1 - s_0$ denotes the length of I. Accordingly,

$$c_{2,1}\left(\sup_{t\in J} \left|B^i_t - B^i_{t_0}\right| > \eta\right) \leq \sqrt{\frac{\eta^2 |J|^{2H}}{\left[\gamma_H |J|^{2H} + |J|\right]^2} + 4} \exp\left(-\frac{\eta^2}{4[\gamma_H |J|^{2H} + |J|]}\right),$$

with $|J| = t_1 - t_0$, the length of interval J.

Divide I and J into k subintervals evenly, i.e. $I = \bigcup_{m=1}^k I_m$, $J = \bigcup_{l=1}^k J_l$, I_m and J_l are disjoint for all $1 \leq m, l \leq k$ and $|I_m| = |I|/k$, $|J_l| = |J|/k$. By sub-additivity and above,

$$c_{2,1}(A) \leq \sum_{m=1}^k \sum_{l=1}^k c_{2,1}\Big(B_s = B_t, \text{ for some } s \in I_m \text{ and some } t \in J_l\Big)$$
$$\leq \sum_{m=1}^k \sum_{l=1}^k \left[\left(1 + d^2 \left(\frac{M}{\eta}\right)^2\right) \left(\frac{6\eta}{\sqrt{2\pi \left(d(I_m, J_l)\right)^{2H}}}\right)^d \right.$$
$$+ d\sqrt{\frac{\eta^2 |I_m|^{2H}}{\left(\gamma_H |I_m|^{2H} + |I_m|\right)^2} + 4} \exp\left(-\frac{\eta^2}{4\left(\gamma_H |I_m|^{2H} + |I_m|\right)}\right)$$
$$\left. + d\sqrt{\frac{\eta^2 |J_l|^{2H}}{\left(\gamma_H |J_l|^{2H} + |J_l|\right)^2} + 4} \exp\left(-\frac{\eta^2}{4\left(\gamma_H |J_l|^{2H} + |J_l|\right)}\right)\right]$$
$$\leq k^2 \left[\left(1 + d^2 \left(\frac{M}{\eta}\right)^2\right) \left(\frac{6\eta}{\sqrt{2\pi \left(d(I,J)\right)^{2H}}}\right)^d \right.$$
$$+ d\sqrt{\frac{\eta^2 k^{2H}}{\gamma_H^2 |I|^{2H}} + 4} \exp\left(-\frac{\eta^2}{4\left(\gamma_H |I|^{2H} k^{-2H} + |I| k^{-1}\right)}\right)$$
$$\left. + d\sqrt{\frac{\eta^2 k^{2H}}{\gamma_H^2 |J|^{2H}} + 4} \exp\left(-\frac{\eta^2}{4\left(\gamma_H |J|^{2H} k^{-2H} + |J| k^{-1}\right)}\right)\right].$$

Set $\eta = k^{-\sigma}$, then according to the previous estimate, when k is sufficiently large and $H < \frac{1}{2}$, it holds that

$$c_{2,1}(A) \leq C_1 \left(k^{2-\sigma(d-2)} + k^{(H-\sigma)+2} e^{-Ck^{2(H-\sigma)}} \right), \tag{7.3}$$

where C_1 is some constant. Notice that when $\frac{2}{d-2} < \sigma < H$, the expression on the right-hand side of (7.3) vanishes as k tends to infinity. This implies that if such a σ exists, then B has no double point under $(2,1)$-capacity, which only requires $\frac{2}{d-2} < H$, i.e. $d > \frac{2}{H} + 2$.

On the other hand, when $H > \frac{1}{2}$, by setting $\eta = k^{-\sigma}$, we get that

$$c_{2,1}(A) \leq C_2 \left(k^{2-\sigma(d-2)} + k^{(H-\sigma)+2} e^{-Ck^{2(1-\sigma)}} \right)$$

when k is sufficiently large, where C_2 is a constant. Therefore, in order to guarantee that the right-hand side vanishes as k tends to infinity, we require $\frac{2}{d-2} < \sigma < \frac{1}{2}$, which forces $d > 6$. $\square$

Remark 7.2. For d-dimensional Brownian motion, absence of double points under $(2,1)$-capacity was proved by Fukushima in [9]. According to Lyons [21], the critical dimension for such property is $d = 6$. Due to lack of tools such as potential theory, the critical dimension of self-avoiding property for fBM remains, we believe, an open question even in probability context.

Appendix A

In this appendix, we provide a proof for Lemma 2.1 in this section. The proof is a modification of the proof for Proposition 3.1 in Decreusefond and Üstünel [5]. The following elementary estimate, which will be used in the proof, is taken from Theorem 3.2 in [5]: for any $H \in (0,1)$, there exists a constant c_H such that

$$K(t,r) \leq c_H r^{-|H-\frac{1}{2}|}(t-r)^{-(\frac{1}{2}-H)_+} \mathbb{1}_{[0,t]}(r) \tag{A.1}$$

for any $t > r \geq 0$, where $x_+ = \max(x,0)$.

Proof of Lemma 2.1. For each fixed $t > 0$, denote $u_t(s) = K(t,s)\mathbb{1}_{[0,t]}(s)$ for simplicity, and set for $n \in \mathbb{N}$,

$$u_t^{(n)}(s) = \sum_{i=0}^{2^n-1} \frac{2^n}{t} \left(\int_{i2^{-n}t}^{(i+1)2^{-n}t} u_t(r)dr \right) \mathbb{1}_{(i2^{-n}t,(i+1)2^{-n}t]}(s).$$

Then u_t and $u_t^{(n)}$, $n \in \mathbb{N}$, belong to $L^2([0,\infty))$. For convenience, let

$$F_i^{t,(n)} = \frac{2^n}{t} \left(\int_{i2^{-n}t}^{(i+1)2^{-n}t} u_t(r)dr \right), \quad 0 \leq i \leq 2^n - 1.$$

We want to apply the dominated convergence theorem to show that for each $t > 0$, $u_t^{(n)} \to u_t$ in $L^2([0,\infty))$. Our first step is to find a control function of $\{u_t^{(n)}\}$ in $L^2([0,\infty))$. Notice that $u_t^{(n)}(s)$ vanishes outside of $(0,t]$, and it is defined to be a step function inside $(0,t]$, so we only need to check that on each "step", i.e. $s \in (i2^{-n}t, (i+1)2^{-n}t]$, $0 \leq i \leq 2^{n-1}$, $u_t^{(n)}(s)$ is controlled.

When $H > \frac{1}{2}$, for each $s \in (i2^{-n}t, (i+1)2^{-n}t]$, $0 \le i \le 2^{n-1}$, by the estimate in (A.1),

$$\begin{aligned}
\left|u_t^{(n)}(s)\right| &= \frac{2^n}{t} \int_{i2^{-n}t}^{(i+1)2^{-n}t} K(t,r)dr \\
&\le \frac{2^n}{t} \int_{i2^{-n}t}^{(i+1)2^{-n}t} c_H r^{\frac{1}{2}-H} dr \\
&= c'_H \left(\frac{t}{2^n}\right)^{\frac{1}{2}-H} \left[\left(\tfrac{i+1}{2^n}\right)^{\frac{3}{2}-H} - \left(\tfrac{i}{2^n}\right)^{\frac{3}{2}-H}\right] \\
&\le c'_H t^{\frac{1}{2}-H} \left[(i+1)\left(\tfrac{i+1}{2^n}\right)^{\frac{1}{2}-H} - i\left(\tfrac{i+1}{2^n}\right)^{\frac{1}{2}-H}\right] \\
&= c'_H t^{\frac{1}{2}-H} \left(\tfrac{i+1}{2^n}\right)^{\frac{1}{2}-H} \\
&\le c'_H t^{\frac{1}{2}-H} s^{\frac{1}{2}-H},
\end{aligned}$$

where $c'_H = c_H \left(\frac{3}{2} - H\right)^{-1}$. This implies that when $H > \frac{1}{2}$, we may take the control function to be $c'_H t^{\frac{1}{2}-H} s^{\frac{1}{2}-H} \mathbb{1}_{(0,t]}(s)$.

When $H < \frac{1}{2}$, similar to above, we have that by (A.1),

$$\begin{aligned}
\left|u_t^{(n)}(s)\right| &= \frac{2^n}{t} \int_{i2^{-n}t}^{(i+1)2^{-n}t} K(t,r)dr \\
&\le c_H \frac{2^n}{t} \int_{s}^{(i+1)2^{-n}t} r^{H-\frac{1}{2}}(t-r)^{H-\frac{1}{2}} dr + c_H \frac{2^n}{t} \int_{i2^{-n}t}^{s} r^{H-\frac{1}{2}}(t-r)^{H-\frac{1}{2}} dr \\
&\le c_H \frac{2^n}{t} s^{H-\frac{1}{2}} \int_{i2^{-n}t}^{(i+1)2^{-n}t} (t-r)^{H-\frac{1}{2}} dr + c_H \frac{2^n}{t}(t-s)^{H-\frac{1}{2}} \int_{i2^{-n}t}^{(i+1)2^{-n}t} r^{H-\frac{1}{2}} dr \\
&= c'_H \frac{2^n}{t} s^{H-\frac{1}{2}} \left[\left(t - \tfrac{i}{2^n}t\right)^{H+\frac{1}{2}} - \left(t - \tfrac{i+1}{2^n}t\right)^{H+\frac{1}{2}}\right] \\
&\quad + c'_H \frac{2^n}{t}(t-s)^{H-\frac{1}{2}} \left[\left(\tfrac{i+1}{2^n}t\right)^{H+\frac{1}{2}} - \left(\tfrac{i}{2^n}t\right)^{H+\frac{1}{2}}\right] \\
&\le c'_H \frac{2^n}{t} s^{H-\frac{1}{2}} \left[\left(t - \tfrac{i}{2^n}t\right)\left(t - \tfrac{i}{2^n}t\right)^{H-\frac{1}{2}} - \left(t - \tfrac{i+1}{2^n}t\right)\left(t - \tfrac{i}{2^n}t\right)^{H-\frac{1}{2}}\right] \\
&\quad + c'_H \frac{2^n}{t}(t-s)^{H-\frac{1}{2}} \left[\left(\tfrac{i+1}{2^n}t\right)\left(\tfrac{i+1}{2^n}t\right)^{H-\frac{1}{2}} - \left(\tfrac{i}{2^n}t\right)\left(\tfrac{i+1}{2^n}t\right)^{H-\frac{1}{2}}\right] \\
&= c'_H s^{H-\frac{1}{2}} \left(t - \tfrac{i}{2^n}t\right)^{H-\frac{1}{2}} + c'_H (t-s)^{H-\frac{1}{2}} \left(\tfrac{i+1}{2^n}t\right)^{H-\frac{1}{2}} \\
&\le 2c'_H s^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}}.
\end{aligned}$$

Therefore, when $H < \frac{1}{2}$, the control function is $2c'_H s^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} \mathbb{1}_{(0,t]}(s)$, which is an element of $L^2([0,\infty))$.

On the other hand,

$$u_t^{(n)}(s) = \frac{\int_0^{(i+1)2^{-n}t} u_t(r)dr - \int_0^{i2^{-n}t} u_t(r)dr}{2^{-n}t} \to u_t(s)$$

as n tends to infinity due to the continuity of $u_t(s)$ on $(0,t)$. Now we may apply the dominated convergence theorem and conclude that $u_t^{(n)} \to u_t$ in $L^2([0,\infty))$.

For fixed $t \in [0,1]$, set

$$B_t^{(n)}(\omega) = \begin{cases} \sum_{i=0}^{2^n-1} F_i^{t,(n)} \left(\omega_{(i+1)2^{-n}t} - \omega_{i2^{-n}t}\right), & 0 < t \leq 1, \\ 0, & t = 0. \end{cases} \tag{A.2}$$

Let $\mathscr{G} = (\mathscr{G}_n)_{n\geq 0}$, where $\mathscr{G}_n = \sigma\left\{\omega_{i2^{-n}t}, 0 \leq i \leq 2^n\right\}$ is the σ-algebra generated by $\omega_{i2^{-n}t}$'s, $0 \leq i \leq 2^n$. Then $(B_t^{(n)})_{n\in\mathbb{N}}$ is a discrete martingale with respect to this filtration $\mathscr{G}$. This was observed by Decreusefond and Üstünel [5].

We claim that $(B_t^{(n)})_{n\in\mathbb{N}}$ defined in (A.2) is a discrete martingale with respect to $\mathscr{G}$, where $\mathscr{G} = (\mathscr{G}_n)_{n\geq 0}$, the σ-algebra generated by $\omega_{i2^{-n}t}$'s, $0 \leq i \leq 2^n$. The proof of this claim relies on the fact that for a standard Brownian motion ω_t and any $0 \leq t_0 < t_1 < \cdots < t_n$,

$$\mathbb{E}\left[\omega_{t_i} \middle| \omega_{t_0}, \omega_{t_1}, \cdots \omega_{t_{i-1}}, \omega_{t_{i+1}}, \cdots, \omega_{t_n}\right] = \frac{t_{i+1}-t_i}{t_{i+1}-t_{i-1}}\omega_{t_{i-1}} + \frac{t_i - t_{i-1}}{t_{i+1}-t_{i-1}}\omega_{t_{i+1}}. \tag{A.3}$$

To verify (A.3), one only needs to spot that for each i and n,

$$X_i := \omega_{t_i} - \frac{t_{i+1}-t_i}{t_{i+1}-t_{i-1}}\omega_{t_{i-1}} - \frac{t_i - t_{i-1}}{t_{i+1}-t_{i-1}}\omega_{t_{i+1}}$$

is independent of $\sigma(\omega_{t_0}, \omega_{t_1}, \cdots \omega_{t_{i-1}}, \omega_{t_{i+1}}, \cdots, \omega_{t_n})$. Indeed, for any $0 \leq j < i \leq n$,

$$\begin{aligned} \mathbb{E}\left[X_i \omega_{t_j}\right] &= \mathbb{E}\left[\omega_{t_i}\omega_{t_j}\right] - \frac{t_{i+1}-t_i}{t_{i+1}-t_{i-1}}\mathbb{E}\left[\omega_{t_{i-1}}\omega_{t_j}\right] - \frac{t_i - t_{i-1}}{t_{i+1}-t_{i-1}}\mathbb{E}\left[\omega_{t_{i+1}}\omega_{t_j}\right] \\ &= t_j - \frac{t_{i+1}-t_i}{t_{i+1}-t_{i-1}}t_j - \frac{t_i - t_{i-1}}{t_{i+1}-t_{i-1}}t_j \\ &= 0, \end{aligned}$$

and one may verify X_i and ω_{t_j} are independent via similar computation when $0 < i < j \leq n$. Thus ω_{t_i} is independent of all linear combinations of $\omega_{t_0}, \omega_{t_1}, \cdots \omega_{t_{i-1}}, \omega_{t_{i+1}}, \cdots, \omega_{t_n}$, and hence $\sigma(\omega_{t_0}, \omega_{t_1}, \cdots \omega_{t_{i-1}}, \omega_{t_{i+1}}, \cdots, \omega_{t_n})$. Therefore, we get that

$$\begin{aligned} &\mathbb{E}\left[\omega_{t_i} \middle| \omega_{t_0}, \omega_{t_1}, \cdots \omega_{t_{i-1}}, \omega_{t_{i+1}}, \cdots, \omega_{t_n}\right] \\ &= \mathbb{E}\left[X_i + \frac{t_{i+1}-t_i}{t_{i+1}-t_{i-1}}\omega_{t_{i-1}} + \frac{t_i - t_{i-1}}{t_{i+1}-t_{i-1}}\omega_{t_{i+1}} \middle| \omega_{t_0}, \omega_{t_1}, \cdots \omega_{t_{i-1}}, \omega_{t_{i+1}}, \cdots, \omega_{t_n}\right] \\ &= \frac{t_{i+1}-t_i}{t_{i+1}-t_{i-1}}\omega_{t_{i-1}} + \frac{t_i - t_{i-1}}{t_{i+1}-t_{i-1}}\omega_{t_{i+1}}. \end{aligned}$$

For each $1 \leq i \leq 2^n - 1$, if i is odd, then we may write $i = 2k+1$, $0 \leq k \leq 2^{-n+1} - 1$, and thus by (A.3),

$$\begin{aligned}\mathbb{E}\left[\omega_{(i+1)2^{-n}t} - \omega_{i2^{-n}t}|\mathscr{G}_{n-1}\right] &= \mathbb{E}\left[\omega_{(k+1)2^{-n+1}t} - \omega_{(2k+1)2^{-n}t}|\mathscr{G}_{n-1}\right] \\ &= \frac{1}{2}\omega_{(k+1)2^{-n+1}t} - \frac{1}{2}\omega_{k2^{-n+1}t}.\end{aligned}$$

If i is even, write $i = 2k$ for $0 \le k \le 2^{n-1} - 1$, then it holds that

$$\begin{aligned}\mathbb{E}\left[\omega_{(i+1)2^{-n}t} - \omega_{i2^{-n}t}|\mathscr{G}_{n-1}\right] &= \mathbb{E}\left[\omega_{(2k+1)2^{-n}t} - \omega_{k2^{-n+1}t}|\mathscr{G}_{n-1}\right] \\ &= \frac{1}{2}\omega_{(k+1)2^{-n+1}t} - \frac{1}{2}\omega_{k2^{-n+1}t}.\end{aligned}$$

Therefore, by the definition of $F_i^{t,(n)}$, we conclude that

$$\begin{aligned}\mathbb{E}\left[B_t^{(n)}\middle|\mathscr{G}_{n-1}\right] &= \sum_{i=0}^{2^n-1} F_i^{t,(n)}\mathbb{E}\left[\omega_{(i+1)2^{-n}t} - \omega_{i2^{-n}t}\middle|\mathscr{G}_{n-1}\right] \\ &= \sum_{k=0}^{2^{n-1}-1} F_{2k+1}^{t,(n)}\left(\frac{1}{2}\omega_{(k+1)2^{-n+1}t} - \frac{1}{2}\omega_{k2^{-n+1}t}\right) \\ &\quad + \sum_{k=0}^{2^{n-1}-1} F_{2k}^{t,(n)}\left(\frac{1}{2}\omega_{(k+1)2^{-n+1}t} - \frac{1}{2}\omega_{k2^{-n+1}t}\right) \\ &= \sum_{k=0}^{2^{n-1}-1} \frac{2^{-n+1}}{t}\left(\omega_{(k+1)2^{-n+1}t} - \omega_{k2^{-n+1}t}\right) \\ &\quad \cdot \left(\int_{k2^{-n+1}t}^{(2k+1)2^{-n}t} u_t(s)ds + \int_{(2k+1)2^{-n}t}^{(k+1)2^{-n+1}t} u_t(s)ds\right) \\ &= B_t^{(n-1)}.\end{aligned}$$

For $p \in (1, \infty)$, because the increments of ω_t over different time intervals are independent, and $B_t^{(n)}$ is contained in the first Wiener chaos, by (2.3) from Lemma 2.2 in [2] with $N = 1$,

$$\begin{aligned}\left\|B_t^{(n)}\right\|_p &\le 2\sqrt{p-1}\left\|B_t^{(n)}\right\|_2 \\ &= 2\sqrt{p-1}\left[\sum_{i=1}^{2^n-1}\left(\frac{2^n}{t}\right)^2\left(\int_{(i-1)2^{-n}t}^{i2^{-n}t} u_t(s)ds\right)^2 \mathbb{E}\left[\left(\omega_{(i+1)2^{-n}} - \omega_{i2^{-n}t}\right)^2\right]\right]^{\frac{1}{2}} \\ &= 2\sqrt{p-1}\left[\sum_{i=1}^{2^n-1}\frac{2^n}{t}\left(\int_{(i-1)2^{-n}t}^{i2^{-n}t} u_t(s)ds\right)^2\right]^{\frac{1}{2}} \\ &\le 2\sqrt{p-1}\left(\sum_{i=1}^{2^n-1}\int_{(i-1)2^{-n}t}^{i2^{-n}t} u_t^2(s)ds\right)^{\frac{1}{2}} \\ &= 2\sqrt{p-1}t^H,\end{aligned}$$

and hence $\sup_{n\in\mathbb{N}}\mathbb{E}[|B_t^{(n)}|^p]<\infty$. It thus follows from the martingale convergence theorem that $(B_t^{(n)})_{n\in\mathbb{N}}$ converges to B_t in $L^p(\boldsymbol{W})$, and B_t's are Gaussian random variables with mean zero and covariance given by

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left[B_s^{(n)}B_t^{(n)}\right]&=\lim_{n\to\infty}\mathbb{E}\left[\left(\int_0^\infty u_t^{(n)}(r)d\omega_r\right)\left(\int_0^\infty u_s^{(n)}(r)d\omega_r\right)\right]\\&=\lim_{n\to\infty}\int_0^\infty u_t^{(n)}(r)u_s^{(n)}(r)dr\\&=\int_0^{s\wedge t}K(t,r)K(s,r)dr\\&=R(s,t)\end{aligned}$$

for any $s,t>0$. In particular, the variance of B_t is given by $\lim_{n\to\infty}\mathbb{E}[|B_t^{(n)}|^2]=t^{2H}$.

Now by the definition of Malliavin derivative, for $t>0$,

$$DB_t^{(n)}(s)=\int_0^s u_t^{(n)}(v)dv,$$

and higher-order derivatives of $B_t^{(n)}$ all vanish. We have already proved that $B_t^{(n)}\to B_t$ in $L^p(\boldsymbol{W})$ and $u_t^{(n)}\to u_t$ in $L^2([0,\infty))$, so for any $r\in\mathbb{N}$ and $p\in(1,\infty)$, as

$$\begin{aligned}\|B_t^{(n)}-B_t^{(m)}\|_{\mathbb{D}_r^p}&=\left(\mathbb{E}\left[|B_t^{(n)}-B_t^{(m)}|^p\right]+\mathbb{E}\left[\left|\|DB_t^{(n)}-DB_t^{(m)}\|_{\mathcal{H}}\right|^p\right]\right)^{1/p}\\&=\left(\mathbb{E}\left[|B_t^{(n)}-B_t^{(m)}|^p\right]+\mathbb{E}\left[\left|\|u_t^{(n)}-u_t^{(m)}\|_{L^2([0,\infty))}\right|^p\right]\right)^{1/p},\end{aligned}$$

we obtain that $(B_t^{(n)})_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{D}_r^p$. By the completeness of $\mathbb{D}_r^p$, this sequence tends to a limit random variable in $\mathbb{D}_r^p$ as n goes to infinity. Now by the definition of $\|\cdot\|_{\mathbb{D}_r^p}$, this convergence implies convergence in $L^p(\boldsymbol{W})$, and by the uniqueness of limit, this random variable must coincide with B_t. Moreover,

$$DB_t(s)=\int_0^{s\wedge t}K(t,u)du,$$

where $DB_t\in\mathcal{H}$ is the Malliavin derivative of B_t with respect to Brownian motion, and its higher order Malliavin derivatives all vanish.

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