



Stochastic differential switching game in infinite horizon

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ABSTRACT

We study a zero-sum stochastic differential switching game in infinite horizon. We prove the existence of the value of the game and characterize it as the unique viscosity solution of the associated system of quasi-variational inequalities with bilateral obstacles. We also obtain a verification theorem which provides an optimal strategy of the game. Finally, some numerical examples with two regimes are given.

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1. Introduction

Differential game theory involves multiple persons (also called players or individuals) decision making in the context of dynamic system. The study of the two-person zero-sum differential games could be traced back to the pioneering work by Isaacs [9], which inspired further research in this area.

In 1989, Fleming and Souganidis [6], for the first time, studied two-person zerosum stochastic differential games by the PDE tools. They adopted the definitions of upper and lower value functions introduced by Elliott and Kalton [4] and proved that the two value functions satisfy the dynamic programming principle and they are the unique viscosity solutions to the associated Hamilton–Jacobi–Bellman–Isaacs partial differential equations (HJBI PDEs). Their work generalized that of Evans and Souganidis [5] from the deterministic framework to the stochastic one and is now seen as one of the outstanding results in the field of stochastic differential games. In 1995, Hamadène and Lepeltier [7] introduced the backward stochastic differential equation (BSDE) theory to study stochastic differential games, later there were several other works using this BSDEs approach, such as Hamadène, Lepeltier, and Peng [8]. Subsequently, Buckdahn and Li [1] extended the findings presented in [7,8], and generalized the framework introduced in [6].

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In this paper we consider the state process of the stochastic differential game, defined as the solution of the following stochastic equation:

$$X_s = x + \int_0^s b(r, X_r, a_r, b_r) dr + \int_0^s \sigma(r, X_r, a_r, b_r) dW_r \quad s \geq t,$$

with $X_{0-} = x$. Here W is a d -dimensional Wiener process, while

$$a_t = \sum_{m \geq 1} \xi_m \mathbb{1}_{(\tau_m \leq t < \tau_{m+1}]}(t) \quad \text{and} \quad b_t = \sum_{n \geq 1} \eta_n \mathbb{1}_{(\rho_n \leq t < \rho_{n+1}]}(t),$$

with the cost functional

$$\mathbb{E} \left[\int_0^\infty e^{-rs} f(X_s^{a_s, b_s}, a_s, b_s) ds - \sum_{m \geq 1} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) + \sum_{n \geq 1} e^{-r\rho_n} \chi(\eta_{n-1}, \eta_n) \right]. \quad (1.1)$$

The first player chooses the control a from a given finite set \mathcal{I} to maximize the payoff (1.1), and each of his action is related with one cost C , while the second player chooses the control b from \mathcal{I} to minimize the payoff (1.1), and each of his actions is associated with the other cost χ . The zero-sum stochastic differential games problems we will investigate is to show that the upper and lower value functions coincide and the game admits a value. The Isaacs system of quasi-variational inequalities for this switching game is the following: for any $i, j \in \mathcal{I}$, and $x \in \mathbb{R}^m$,

$$\max \left\{ \min \left[rV^{ij}(x) - \mathcal{A}^{ij}V^{ij}(x) - f(x, i, j); \right. \right. \\ \left. \left. V^{ij}(x) - M^{ij}[V](x) \right]; V^{ij}(x) - N^{ij}[V](x) \right\} = 0, \quad (1.2)$$

and

$$\min \left\{ \max \left[rV^{ij}(x) - \mathcal{A}^{ij}V^{ij}(x) - f(x, i, j); \right. \right. \\ \left. \left. V^{ij}(x) - N^{ij}[V](x) \right]; V^{ij}(x) - M^{ij}[V](x) \right\} = 0, \quad (1.3)$$

where,

$$M^{ij}[V](x) = \max_{k \neq i} \{V^{kj}(x) - C(i, k)\}, \quad N^{ij}[V](x) = \min_{l \neq j} \{V^{il}(x) + \chi(j, l)\}.$$

In the finite horizon framework, the switching game have been studied by several authors. The most recent works discussing this topic include the papers by Djehiche et al. [3] (2017), Tang and Hou [13] (2007).

The objective of this work is to establish existence and uniqueness of a continuous viscosity solution of (1.2) and (1.3). The second contribution of our paper is the Verification Theorem which provides an optimal strategy of the game. To the best of our knowledge, this issue have not been addressed in the literature yet.

This paper is organized as follows: in Section 2 we list all the notations, state the full set of assumptions, and define viscosity sub- and supersolutions along with equivalent characterizations. In Section 3, we shall introduce the stochastic differential game problem and give some preliminary results of the lower and the upper value functions of stochastic differential game. In Section 4, by the dynamic programming principle we prove that the lower and upper value functions of the game satisfy the Isaacs system of quasi-variational inequalities in viscosity solution sense. In Section 5, we show that the solution of Isaacs system of quasi-variational inequalities is unique. Furthermore, the upper and the lower value functions coincide and the

game admits a value. In Section 6, we present a verification theorem which gives an optimal strategy of the switching game, finally in Section 7 we will present some numerical treatment.

2. Assumptions and problem formulation

Throughout this paper m and d are two integers. Let $\mathcal{I} = \{1, \dots, q\}$ the finite set of regimes, and assume the following assumptions:

[H1] $b : \mathbb{R}^m \times \mathcal{I} \times \mathcal{I} \rightarrow \mathbb{R}^m$ and $\sigma : \mathbb{R}^m \times \mathcal{I} \times \mathcal{I} \rightarrow \mathbb{R}^{m \times d}$ be two continuous functions for which there exists a constant $C > 0$ such that for any $x, x' \in \mathbb{R}^m$ and $i, j \in \mathcal{I}$

$$|\sigma(x, i, j) - \sigma(x', i, j)| + |b(x, i, j) - b(x', i, j)| \leq C|x - x'|. \quad (2.1)$$

Thus they are also of linear growth. i.e., there exists a constant C such that for any $x \in \mathbb{R}^m$ and $i, j \in \mathcal{I}$

$$|\sigma(x, i, j)| + |b(x, i, j)| \leq C(1 + |x|). \quad (2.2)$$

[H2] $f : \mathbb{R}^m \times \mathcal{I} \times \mathcal{I} \rightarrow \mathbb{R}$ is a continuous function for which there exists a constant C such that for each $i, j \in \mathcal{I}$, $x, x' \in \mathbb{R}^m$:

$$|f(x, i, j)| \leq C(1 + |x|) \quad \text{and} \quad (2.3)$$

$$|f(x, i, j) - f(x', i, j)| \leq C|x - x'|. \quad (2.4)$$

[H3] For any $i, j \in \mathcal{I}$, the switching costs $C(i, j)$ and $\chi(i, j)$ are constants, and we assume the triangular condition:

$$C(i, k) < C(i, j) + C(j, k), \quad j \neq i, k. \quad (2.5)$$

$$\chi(i, k) < \chi(i, j) + \chi(j, k), \quad j \neq i, k, \quad (2.6)$$

which means that it is less expensive to switch directly in one step from regime i to k than in two steps via an intermediate regime j . Notice that a switching costs $C(i, j)$ and $\chi(i, j)$ may be negative, and conditions (2.5) and (2.6) for $i = k$ prevents an arbitrage by simply switching back and forth, i.e.

$$0 < C(i, j) + C(j, i). \quad (2.7)$$

$$0 < \chi(i, j) + \chi(j, i). \quad (2.8)$$

We set $b^{ij}(\cdot) = b(\cdot, i, j)$, $\sigma^{ij}(\cdot) = \sigma(\cdot, i, j)$ and $f^{ij}(\cdot) = f(\cdot, i, j)$.

We now consider the following Isaacs system of quasi-variational inequalities: for any $i, j \in \mathcal{I}$, and $x \in \mathbb{R}^m$,

$$\max \left\{ \min \left[rV^{ij}(x) - \mathcal{A}^{ij}V^{ij}(x) - f^{ij}(x); \right. \right. \\ \left. \left. V^{ij}(x) - M^{ij}[V](x) \right]; V^{ij}(x) - N^{ij}[V](x) \right\} = 0, \quad (2.9)$$

and

$$\min \left\{ \max \left[rV^{ij}(x) - \mathcal{A}^{ij}V^{ij}(x) - f^{ij}(x); \right. \right. \\ \left. \left. V^{ij}(x) - N^{ij}[V](x) \right]; V^{ij}(x) - M^{ij}[V](x) \right\} = 0, \quad (2.10)$$

where

(i) r is a positive discount factor and \mathcal{A}^{ij} is the following infinitesimal generator:

$$\mathcal{A}^{ij}V^{ij} = \langle b^{ij}, \nabla_x V^{ij} \rangle + \frac{1}{2} \text{tr}[\sigma^{ij}(\sigma^{ij})^* \nabla_x^2 V^{ij}].$$

(ii) For any $x \in \mathbb{R}^m$ and $i, j \in \mathcal{I}$,

$$M^{ij}[V](x) = \max_{k \neq i} \{V^{kj}(x) - C(i, k)\}, \quad N^{ij}[V](x) = \min_{l \neq j} \{V^{il}(x) + \chi(j, l)\}.$$

We now define the notion of viscosity solution of the system (2.9). We can similarly define the notion for (2.10).

Definition 1. Let $\vec{V} = (V^{kl}(x))_{k,l \in \mathcal{I}} : \mathbb{R}^m \rightarrow \mathbb{R}$ such that for any $(i, j) \in \mathcal{I} \times \mathcal{I}$, V^{ij} is continuous, is called:

(i) A viscosity supersolution to (2.9) if for any $i, j \in \mathcal{I}$, for any $x_0 \in \mathbb{R}^m$ and any function $\phi^{ij} \in C^2(\mathbb{R}^m)$ such that $\phi^{ij}(x_0) = V^{ij}(x_0)$ and x_0 is a local maximum of $\phi^{ij} - V^{ij}$, we have:

$$\max \left\{ \min \left[r\phi^{ij}(x_0) - \mathcal{A}^{ij}\phi^{ij}(x_0) - f^{ij}(x_0), \right. \right. \\ \left. \left. V^{ij}(x_0) - M^{ij}[V](x_0) \right], V^{ij}(x_0) - N^{ij}[V](x_0) \right\} \geq 0. \quad (2.11)$$

(ii) A viscosity subsolution to (2.9) if for any $i, j \in \mathcal{I}$, for any $x_0 \in \mathbb{R}^m$ and any function $\phi^{ij} \in C^2(\mathbb{R}^m)$ such that $\phi^{ij}(x_0) = V^{ij}(x_0)$ and x_0 is a local minimum of $\phi^{ij} - V^{ij}$, we have:

$$\max \left\{ \min \left[r\phi^{ij}(x_0) - \mathcal{A}^{ij}\phi^{ij}(x_0) - f^{ij}(x_0), \right. \right. \\ \left. \left. V^{ij}(x_0) - M^{ij}[V](x_0) \right], V^{ij}(x_0) - N^{ij}[V](x_0) \right\} \leq 0. \quad (2.12)$$

(iii) A viscosity solution if it is both a viscosity supersolution and subsolution. \square

There is an equivalent formulation of this definition (see e.g. [2]) which we give since it will be useful later. So firstly we define the notions of superjet and subjet of a continuous function V .

Definition 2. Let $V \in C(\mathbb{R}^m)$, x an element of \mathbb{R}^m and finally \mathbb{S}_m the set of $m \times m$ symmetric matrices. We denote by $J^{2,+}V(x)$ (resp. $J^{2,-}V(x)$), the superjets (resp. the subjets) of $V(x)$, the set of pairs $(q, X) \in \mathbb{R}^m \times \mathbb{S}_m$ such that:

$$V(y) \leq V(x) + \langle q, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|y - x|^2) \\ (\text{resp. } V(y) \geq V(x) + \langle q, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|y - x|^2)). \quad \square$$

Note that if $\phi - V$ has a local maximum (resp. minimum) at x , then we obviously have:

$$(D_x \phi(x), D_{xx}^2 \phi(x)) \in J^{2,-}V(x) \text{ (resp. } J^{2,+}V(x)). \quad \square$$

We now give an equivalent definition of a viscosity solution of the system (2.9).

Definition 3. Let $\vec{V} = (V^{kl}(x))_{k,l \in \mathcal{I}} : \mathbb{R}^m \rightarrow \mathbb{R}$ such that for any $(i, j) \in \mathcal{I} \times \mathcal{I}$, V^{ij} continuous, is called a viscosity supersolution (resp. a viscosity subsolution) to (2.9) if for any $i, j \in \mathcal{I}$, for any $x \in \mathbb{R}^m$ and any $(q, X) \in J^{2,-}V^{ij}(x)$ (resp. $J^{2,+}V^{ij}(x)$),

$$\max \left\{ \min \left[rV^{ij} - \frac{1}{2} \text{Tr}[(\sigma^{ij})^* X \sigma^{ij}] - \langle b^{ij}, q \rangle - f^{ij}(x); \right. \right. \\ \left. \left. V^{ij}(x) - M^{ij}[V](x); V^{ij}(x) - N^{ij}[V](x) \right\} \geq 0 \quad (\text{resp. } \leq 0). \quad (2.13)$$

It is called a viscosity solution if it is both a viscosity subsolution and supersolution. \square

As pointed out previously we will show that system (2.9) and (2.9) has a unique solution in viscosity sense. The systems are the deterministic version of the zero-sum switching game which we will describe briefly in the next section.

3. The zero-sum switching game

3.1. Setting of the problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space on which is defined a standard d -dimensional Brownian motion $W = (W_t)_{t \leq T}$, whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{W_s; s \leq t\})_{0 \leq t \leq T}$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ the completed filtration of $(\mathcal{F}_t^0)_{t \leq T}$ with the \mathbb{P} -null sets of \mathcal{F} . We begin by description of the zero-sum switching game.

Definition 4. Assume we have two players I and II who intervene on a system (e.g. the production of energy from several sources such as oil, coal, hydro-electric, etc.) with the help of switching strategy. An admissible switching strategies for I (resp. II) is a sequence $\delta := (\tau_m, \xi_m)_{m \geq 0}$ (resp. $\nu := (\rho_n, \eta_n)_{n \geq 0}$) where:

- (i) $(\tau_m)_m$ (resp. $(\rho_n)_n$), the action times, is a sequence of \mathcal{F} -stopping times such that $\tau_m < \tau_{m+1}$ (resp. $\rho_n < \rho_{n+1}$) and $\tau_n \rightarrow \infty$ (resp. $\rho_n \rightarrow \infty$).
- (ii) $(\xi_m)_m$ (resp. $(\eta_n)_n$), the actions, is a sequence of \mathcal{I} -valued random variables, where each ξ_m (resp. η_n) is \mathcal{F}_{τ_m} -measurable (resp. \mathcal{F}_{ρ_n} -measurable).

Next let $(i, j) \in \mathcal{I} \times \mathcal{I}$ be fixed. We say that the admissible switching strategy $\delta := (\tau_m, \xi_m)_{m \geq 0}$ of I (resp. $\nu := (\rho_n, \eta_n)_{n \geq 0}$) (resp. II) belongs to \mathcal{A}^i (resp. \mathcal{B}^j) if

- $\tau_0 = 0, \xi_0 = i$ (resp. $\rho_0 = 0, \eta_0 = j$).
- The family of random variables $\{C_n^\delta\}_{n \geq 1}$ (resp. $\{\chi_n^\nu\}_{n \geq 1}$), where $\{C_n^\delta\}_{n \geq 1}$ (resp. $\{\chi_n^\nu\}_{n \geq 1}$), is the total cost of the first n switches

$$C_n^\delta = \sum_{m=1}^n e^{-r\tau_m} C(\xi_{m-1}, \xi_m) \quad (\text{resp. } \chi_n^\nu = \sum_{l=1}^n e^{-r\rho_l} \chi(\eta_{l-1}, \eta_l))$$

converges \mathbb{P} -a.s. and satisfies

$$\lim_{n \rightarrow \infty} C_n^\delta \in \mathbb{L}^1 \quad (\text{resp. } \lim_{n \rightarrow \infty} \chi_n^\nu \in \mathbb{L}^1). \quad (3.1)$$

Given an admissible strategy δ (resp. ν) of I (resp. II), one associates a stochastic process $(a_t)_{t \geq 0}$ (resp. $(b_t)_{t \geq 0}$) which indicates along with time the current mode of I (resp. II) and which is defined by: $\forall t \geq 0$

$$a_t = \xi_0 \mathbb{1}_{\{\tau_0\}}(t) + \sum_{m \geq 1} \xi_{m-1} \mathbb{1}_{(\tau_{m-1}, \tau_m]}(t) \quad (\text{resp. } b_t = \eta_0 \mathbb{1}_{\{\rho_0\}}(t) + \sum_{n \geq 1} \eta_{n-1} \mathbb{1}_{(\rho_{n-1}, \rho_n]}(t)). \quad (3.2)$$

The evolution of the state of the game is described by the following stochastic differential equation:

$$\begin{cases} dX_t = b^{a_t b_t}(X_t) dt + \sigma^{a_t b_t}(X_t) dW_t, & t \geq 0, \\ X_0 = x. \end{cases} \quad (3.3)$$

Note that the Assumption **(H1)** ensure, for any $x \in \mathbb{R}^n$, the existence and uniqueness of a solution to (3.3). Let now $\delta := (\tau_m, \xi_m)_{m \geq 0}$ (resp. $\nu := (\rho_n, \eta_n)_{n \geq 0}$) be an admissible strategy for I (resp. II) which belongs to \mathcal{A}^i (resp. \mathcal{B}^j). The interventions of the players are not free and generate a payoff which is a reward (resp. cost) for I (resp. II) and whose expression is given by

$$J(x, \delta, \nu) = \mathbb{E} \left[\int_0^\infty e^{-rs} f^{a_s b_s}(X_s^{x, \delta, \nu}) ds - \sum_{m \geq 1} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) + \sum_{n \geq 1} e^{-r\rho_n} \chi(\eta_{n-1}, \eta_n) \right]. \quad (3.4)$$

Following Buckdahn and Li [1], we define nonanticipative strategies as follows.

Definition 5. A nonanticipative strategy for player I is a mapping

$$\alpha : \cup_{j \in \mathcal{I}} \mathcal{B}^j \rightarrow \mathcal{A}^i$$

such that for any stopping time τ and any $b, b' \in \mathcal{B}^j$, with $b \equiv b'$ on $\llbracket 0, \tau \rrbracket$, it holds that $\alpha(b) \equiv \alpha(b')$ on $\llbracket 0, \tau \rrbracket$ (with the notation $\llbracket 0, \tau \rrbracket = \{(s, \omega) \in [0, \infty) \times \Omega, 0 \leq s \leq \tau(\omega)\}$).

A nonanticipative strategy for player II

$$\beta : \cup_{i \in \mathcal{I}} \mathcal{A}^i \rightarrow \mathcal{B}^j$$

is defined similarly. The set of all nonanticipative strategies α (resp. β) for player I (resp. II) is denoted by Γ^i (resp. Δ^j), for all $i, j \in \mathcal{I}$.

We define the upper (resp. lower) value of the game by

$$\overline{V}^{ij}(x) = \inf_{\beta \in \Delta^j} \sup_{\delta \in \mathcal{A}^i} J(x, \delta, \beta(\delta)) \quad (3.5)$$

(resp.

$$\underline{V}^{ij}(x) = \sup_{\alpha \in \Gamma^i} \inf_{\nu \in \mathcal{B}^j} J(x, \alpha(\nu), \nu)). \quad (3.6)$$

The game has a value if $\underline{V}^{ij} = \overline{V}^{ij}$.

3.2. Preliminary results

In this section we present some properties of the lower and upper value functions of our switching game.

Lemma 1. (see e.g. [11]) *The process X^x satisfies the following estimate: There exists a constant ρ such that*

$$E[|X_t^x|] \leq e^{\rho t}(1 + |x|) \quad \forall t \geq 0. \quad (3.7)$$

Proposition 1. *Under the standing assumptions **(H1)**, **(H2)** and **(H3)**, there exists some positive constant ρ such that for $r \geq \rho$, the lower and upper value function are satisfying a linear growth condition: for all $i, j \in \mathcal{I}$ and $x \in \mathbb{R}^m$ there exists a constant C such that*

$$|\overline{V}^{ij}(x)|, |\underline{V}^{ij}(x)| \leq C(1 + |x|). \quad (3.8)$$

Proof. We make the proof only for the lower value function \bar{V}^{ij} , the other case being analogous. First, the switching cost for player I is dominated by

$$-\sum_{m=1}^N e^{-r\tau_m} C(\xi_{m-1}, \xi_m) \leq \max_{k \in \mathcal{I}} (-C(i, k)) \quad \text{a.s.} \quad (3.9)$$

for any $N \geq 1$, which can be proved by induction as in [14]. Indeed, (3.9) obviously holds for $N = 1$. Suppose now that it is also verified for some $N - 1$ and let us show that it is valid for N . Let $A := \{\omega \in \Omega \mid C(\xi_{N-1}, \xi_N)(\omega) < 0\}$ and A^c be its complement. We have

$$\begin{aligned} & -\sum_{m=1}^N e^{-r\tau_m} C(\xi_{m-1}, \xi_m) \\ &= \left(-\sum_{m=1}^{N-1} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) - e^{-r\tau_N} C(\xi_{N-1}, \xi_N) \right) \mathbf{1}_{A^c} \\ & \quad + \left(-\sum_{m=1}^{N-2} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) - (e^{-r\tau_{N-1}} C(\xi_{N-2}, \xi_{N-1}) + e^{-r\tau_N} C(\xi_{N-1}, \xi_N)) \right) \mathbf{1}_A \\ &\leq \left(-\sum_{m=1}^{N-1} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) \right) \mathbf{1}_{A^c} \\ & \quad + \left(-\sum_{m=1}^{N-2} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) - e^{-r\tau_{N-1}} (C(\xi_{N-2}, \xi_{N-1}) + C(\xi_{N-1}, \xi_N)) \right) \mathbf{1}_A \\ &\leq \left(-\sum_{m=1}^{N-1} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) \right) \mathbf{1}_{A^c} \\ & \quad + \left(\sum_{l \in \mathcal{I}} \left[-\sum_{m=1}^{N-2} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) - e^{-r\tau_{N-1}} C(\xi_{N-2}, l) \right] \mathbf{1}_{[\xi_N=l]} \right) \mathbf{1}_A \\ &\leq \left(-\sum_{m=1}^{N-1} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) \right) \mathbf{1}_{A^c} \\ & \quad + \left(\sum_{l \in \mathcal{I}} \left[-\sum_{m=1}^{N-1} e^{-r\tau_m} C(\tilde{\xi}_{m-1}, \tilde{\xi}_m) \right] \mathbf{1}_{[\xi_N=l]} \right) \mathbf{1}_A \leq \max_{k \in \mathcal{I}} (-C(i, k)), \end{aligned}$$

where $\tilde{\xi}_m = \xi_m$ for $m = 1, \dots, N - 2$, and $\tilde{\xi}_{N-1} = l$.

Now, using the latter inequality and consider the particular strategy for player II: $\tilde{\beta} := (\tilde{\rho}_n, \tilde{\eta}_n)$ given by $\tilde{\rho}_n = \infty, \tilde{\eta}_n = j$ for all $n \geq 1$ and by admissibility condition (3.1) for every $\delta \in \mathcal{A}^i$ we have

$$J(x, \delta, \tilde{\beta}(\delta)) \leq \mathbb{E} \left[\int_0^\infty e^{-rs} f^{a_s b_s}(X_s^{x, \delta, \tilde{\beta}(\delta)}) ds + \max_{k \in \mathcal{I}} (-C(i, k)) \right].$$

By the estimate (3.7) and the polynomial growth condition of f^{ij} in (H2), there exists C and ρ such that

$$J(x, \delta, \tilde{\beta}(\delta)) \leq \frac{C}{r} + C(1 + |x|) \int_0^\infty e^{(\rho-r)s} ds + \max_{k \in \mathcal{I}} (-C(i, k)).$$

Therefore from the arbitrariness of $\delta \in \mathcal{A}^i$ and if $r > \rho$ we have

$$\overline{V}^{ij}(x) \leq C(1 + |x|). \quad (3.10)$$

On the other hand, we have

$$\sum_{n=1}^N e^{-r\rho_n} \chi(\eta_{n-1}, \eta_n) \geq \min_{l \in \mathcal{I}} (\chi(j, l)) \quad \text{a.s.} \quad (3.11)$$

for all $N \geq 1$. By considering the particular strategy $\tilde{\delta} := (\tilde{\tau}_m, \tilde{\xi}_m)$ given by $\tilde{\tau}_m = \infty, \tilde{\xi}_m = i$ for all $m \geq 1$, and by admissibility condition (3.1) for every $\beta \in \Delta^j$ we have

$$J(x, \tilde{\delta}, \beta(\tilde{\delta})) \geq \mathbb{E} \left[\int_0^\infty e^{-rs} f^{a_s b_s}(X_s^{x, \tilde{\delta}, \beta(\tilde{\delta})}) ds + \min_{l \in \mathcal{I}} (\chi(j, l)) \right].$$

As a consequence, there exists C and ρ such that

$$J(x, \tilde{\delta}, \beta(\tilde{\delta})) \geq -\frac{C}{r} - C(1 + |x|) \int_0^\infty e^{(\rho-r)s} ds + \min_{l \in \mathcal{I}} (\chi(j, l)).$$

Therefore from the arbitrariness of $\beta \in \Delta^j$ and if $r > \rho$, we have

$$\overline{V}^{ij}(x) \geq -C(1 + |x|),$$

from which we deduce the claim. \square

Lemma 2. *There exists some positive constant ρ such that for $r > \rho, x, x' \in \mathbb{R}^m, \delta \in \mathcal{A}^i$ and $\nu \in \mathcal{B}^j$ we have*

$$|J(x, \delta, \nu) - J(x', \delta, \nu)| + |\overline{V}^{ij}(x) - \overline{V}^{ij}(x')| + |\underline{V}^{ij}(x) - \underline{V}^{ij}(x')| \leq C|x - x'|,$$

for some positive constant C .

Proof. It is enough to show that the conclusion holds for the gain functional J .

For every $x, x' \in \mathbb{R}^m, \delta \in \mathcal{A}^i, \nu \in \mathcal{B}^j$, we have

$$|J(x, \delta, \nu) - J(x', \delta, \nu)| \leq \mathbb{E} \left[\int_0^\infty e^{-rs} |f^{a_s b_s}(X_s^{x, \delta, \nu}) - f^{a_s b_s}(X_s^{x', \delta, \nu})| ds \right].$$

By a standard estimate for the SDE applying Itô's formula to $|X^{x, \delta, \nu} - X^{x', \delta, \nu}|^2$ and using Gronwall's lemma, we then obtain from the Lipschitz condition on b^{ij}, σ^{ij} , the following inequality

$$E[|X_t^{x, \delta, \nu} - X_t^{x', \delta, \nu}|] \leq e^{\rho t} |x - x'| \quad \forall x, x' \in \mathbb{R}^m, t \geq 0.$$

From the Lipschitz condition on f^{ij} , we deduce

$$\begin{aligned} |J(x, \delta, \nu) - J(x', \delta, \nu)| &\leq C \mathbb{E} \left[\int_0^\infty e^{-rs} |X_s^{x, \delta, \nu} - X_s^{x', \delta, \nu}| ds \right] \\ &\leq C \mathbb{E} \left[\int_0^\infty e^{-rs} e^{\rho s} |x - x'| ds \right] \leq C |x - x'|, \end{aligned}$$

for $r > \rho$. This ends the proof. \square

In the sequel, we shall assume that r is large enough, which ensures that the value functions are Lipschitz continuous. We now present the dynamic programming principle of the zero-sum switching game.

Theorem 1. For any $x \in \mathbb{R}^m$ and $i, j \in \mathcal{I}$, we have

$$\begin{aligned} \bar{V}^{ij}(x) = \inf_{\beta \in \Delta^j} \sup_{\delta \in \mathcal{A}^i} \mathbb{E} \left[\int_0^\theta e^{-rs} f^{a_s b_s}(X_s^{x, \delta, \beta(\delta)}) ds - \sum_{\tau_m \leq \theta} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) \right. \\ \left. + \sum_{\rho_n \leq \theta} e^{-r\rho_n} \chi(\eta_{n-1}, \eta_n) + e^{-r\theta} \bar{V}^{a_\theta b_\theta}(X_\theta^{x, \delta, \beta(\delta)}) \right]. \end{aligned} \quad (3.12)$$

$$\begin{aligned} \underline{V}^{ij}(x) = \sup_{\alpha \in \Gamma^i} \inf_{\nu \in \mathcal{B}^j} \mathbb{E} \left[\int_0^\theta e^{-rs} f^{a_s b_s}(X_s^{x, \alpha(\nu), \nu}) ds - \sum_{\tau_m \leq \theta} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) \right. \\ \left. + \sum_{\rho_n \leq \theta} e^{-r\rho_n} \chi(\eta_{n-1}, \eta_n) + e^{-r\theta} \underline{V}^{a_\theta b_\theta}(X_\theta^{x, \alpha(\nu), \nu}) \right], \end{aligned} \quad (3.13)$$

where θ is any stopping time.

Proof. We make the proof only for the lower value function \bar{V}^{ij} , the other case being analogous. Some proof ideas come from [1] and [6].

Let x be fixed, and let $W(x)$ be the right-hand side of (3.12). We fix $\epsilon > 0$. Then, there exists β^1 such that for any $\delta \in \mathcal{A}^i$, we have

$$\begin{aligned} W(x) \geq \mathbb{E} \left[\int_0^\theta e^{-rs} f^{a_s b_s}(X_s^{x, \delta, \beta^1(\delta)}) ds - \sum_{\tau_m \leq \theta} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) \right. \\ \left. + \sum_{\rho_n \leq \theta} e^{-r\rho_n} \chi(\eta_{n-1}, \eta_n) + e^{-r\theta} \bar{V}^{a_\theta b_\theta}(X_\theta^{x, \delta, \beta^1(\delta)}) \right] - \epsilon. \end{aligned} \quad (3.14)$$

By the definition of $\bar{V}^{ij}(x)$ and the properties of \bar{V}^{ij} and J , it's not difficult to deduce that there exists β^2 such that

$$\mathbb{E}[\bar{V}^{a_\theta b_\theta}(X_\theta^{x, \delta, \beta^1(\delta)})] \geq \mathbb{E}[J(X_\theta^{x, \delta, \beta^1(\delta)}, \delta, \beta^2(\delta))] - \epsilon. \quad (3.15)$$

For the proof of issue (3.15), is the same to the one in ([6, Proposition 1.10]). Consequently, by (3.14) and (3.15), we get

$$\begin{aligned} W(x) \geq \mathbb{E} \left[\int_0^\theta e^{-rs} f^{a_s b_s}(X_s^{x, \delta, \beta^1(\delta)}) ds - \sum_{\tau_m \leq \theta} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) \right. \\ \left. + \sum_{\rho_n \leq \theta} e^{-r\rho_n} \chi(\eta_{n-1}, \eta_n) + e^{-r\theta} J(X_\theta^{x, \delta, \beta^1(\delta)}, \delta, \beta^2(\delta)) \right] - 2\epsilon. \end{aligned} \quad (3.16)$$

Finally, we define a new strategy β , equal to β^1 up to θ and to β^2 from θ . Then it follows from (3.16) that

$$W(x) \geq \mathbb{E} \left[\int_0^\infty e^{-rs} f^{a_s b_s}(X_s^{x, \delta, \beta(\delta)}) ds - \sum_{m \geq 1} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) + \sum_{n \geq 1} e^{-r\rho_n} \chi(\eta_{n-1}, \eta_n) \right] - 2\epsilon.$$

By the arbitrariness of δ and ϵ , we obtain $W(x) \geq \bar{V}^{ij}(x)$. In a similar way we can prove the reverse inequality, hence deducing the claim. \square

4. Isaacs system of quasi-variational inequalities

In the present section we prove that the two value functions are viscosity solutions to (2.9) and (2.10). But we make the proof only for (2.9), the other case is analogous. We begin with the following lemma.

Lemma 3. *For any $i, j \in \mathcal{I}$, the lower and upper value functions satisfy the following properties: for all $x \in \mathbb{R}^m$,*

$$V^{ij}(x) \leq N^{ij}[V](x) \quad (4.1)$$

$$M^{ij}[V](x) \leq V^{ij}(x). \quad (4.2)$$

Proof. We make the proof only for the lower value function \bar{V}^{ij} , the other case being analogous. For every $j, \tilde{j} \in \mathcal{I}, j \neq \tilde{j}$ and $\tilde{\nu} \in \mathcal{B}^{\tilde{j}}$ we define $\nu \in \mathcal{B}^j$ by

$$\begin{cases} \rho_0 = 0, \eta_0 = j. \\ \tilde{\rho}_{l-1} = \rho_l, \tilde{\eta}_{l-1} = \eta_l, \quad \forall l \geq 1. \end{cases}$$

Note that $\rho_0 = \rho_1 = 0$. Let $\tilde{\nu} := \{\tilde{\rho}_l, \tilde{\eta}_l\}$ and $\nu := \{\rho_l, \eta_l\}$, then

$$\bar{V}^{ij}(x) \leq \sup_{\delta \in \mathcal{A}^i} J(x, \delta, \nu) = \sup_{\delta \in \mathcal{A}^i} [J(x, \delta, \tilde{\nu}) + \chi(j, \tilde{j})],$$

from which we deduce that the following inequality holds:

$$\bar{V}^{ij}(x) \leq \bar{V}^{i\tilde{j}}(x) + \chi(j, \tilde{j}).$$

Therefore

$$\bar{V}^{ij}(x) \leq \min_{\tilde{j} \neq j} \{\bar{V}^{i\tilde{j}}(x) + \chi(j, \tilde{j})\}.$$

Next, let us show (4.2). For every $i, \tilde{i} \in \mathcal{I}, i \neq \tilde{i}$ and $\tilde{\delta} \in \mathcal{A}^{\tilde{i}}$ we define δ by

$$\begin{cases} \tau_0 = 0, \xi_0 = i. \\ \tilde{\tau}_{m-1} = \tau_m, \tilde{\xi}_{m-1} = \xi_m, \quad \forall m \geq 1. \end{cases}$$

Then we have

$$\bar{V}^{ij}(x) \geq \bar{V}^{i\tilde{j}}(x) - C(i, \tilde{i}).$$

Therefore

$$\bar{V}^{ij}(x) \geq \max_{\tilde{i} \neq i} \{\bar{V}^{\tilde{i}j}(x) - C(i, \tilde{i})\},$$

from which we deduce the claim. \square

Theorem 2. For any $i, j \in \mathcal{I}$, the lower value function \bar{V}^{ij} and the upper value function \underline{V}^{ij} are viscosity solutions to (2.9).

Proof. We state the proof only for lower value function, the other case is analogous. First, we prove the supersolution property. Fix $i, j \in \mathcal{I}$ and let $\bar{x} \in \mathbb{R}^m$, $\varphi^{ij} \in C^2(\mathbb{R}^m)$ such that \bar{x} is a minimum of $\bar{V}^{ij} - \varphi^{ij}$ with $\bar{V}^{ij}(\bar{x}) = \varphi^{ij}(\bar{x})$. We have, by Lemma 3

$$\bar{V}^{ij}(\bar{x}) - N^{ij}[\bar{V}](\bar{x}) \leq 0.$$

If

$$\bar{V}^{ij}(\bar{x}) - N^{ij}[\bar{V}](\bar{x}) = 0,$$

then we are done. Now suppose

$$\bar{V}^{ij}(\bar{x}) - N^{ij}[\bar{V}](\bar{x}) \leq -2\epsilon < 0,$$

we prove by contradiction that

$$r\varphi^{ij}(\bar{x}) - \mathcal{A}^{ij}\varphi^{ij}(\bar{x}) - f^{ij}(\bar{x}) \geq 0.$$

Suppose otherwise, i.e., $r\varphi^{ij}(\bar{x}) - \mathcal{A}^{ij}\varphi^{ij}(\bar{x}) - f^{ij}(\bar{x}) < 0$. Then without loss of generality we can assume that $r\varphi^{ij}(x) - \mathcal{A}^{ij}\varphi^{ij}(x) - f^{ij}(x) < 0$ and $\bar{V}^{ij}(x) - N^{ij}[\bar{V}](x) < -\epsilon < 0$ on $B(\bar{x}, \theta)$.

Define the stopping time τ by

$$\tau = \inf\{t \geq 0 : X_t \notin B(\bar{x}, \theta)\}.$$

Let $\gamma = \tau \wedge h$ with $h > 0$ and let $\epsilon_1 > 0$. By taking the no-switching control for player I, i.e. $\tau_n = \infty, n \geq 1$ and by using the dynamic programming principle, we deduce the existence of a strategy $\beta \in \Delta^j$ such that

$$\begin{aligned} \bar{V}^{ij}(\bar{x}) &\geq \mathbb{E} \left[\int_0^{\gamma \wedge \rho_1} e^{-rs} f^{ij}(X_s^{\bar{x}, \delta, \beta(\delta)}) ds + \mathbb{1}_{[\rho_1 \leq \gamma]} e^{-r\rho_1} (\chi(j, \eta_1) + \bar{V}^{i\eta_1}(X_{\rho_1}^{\bar{x}, \delta, \beta(\delta)})) \right] \\ &\quad + \mathbb{E} \left[\mathbb{1}_{[\rho_1 > \gamma]} e^{-r\gamma} \bar{V}^{ij}(X_\gamma^{\bar{x}, \delta, \beta(\delta)}) \right] - \epsilon_1 \\ &\geq \mathbb{E} \left[\int_0^{\gamma \wedge \rho_1} e^{-rs} f^{ij}(X_s^{\bar{x}, \delta, \beta(\delta)}) ds + \mathbb{1}_{[\rho_1 \leq \gamma]} e^{-r\rho_1} N^{ij}[\bar{V}](X_{\rho_1}^{\bar{x}, \delta, \beta(\delta)}) \right] \\ &\quad + \mathbb{E} \left[\mathbb{1}_{[\rho_1 > \gamma]} e^{-r\gamma} \bar{V}^{ij}(X_\gamma^{\bar{x}, \delta, \beta(\delta)}) \right] - \epsilon_1 \\ &\geq \mathbb{E} \left[\int_0^{\gamma \wedge \rho_1} e^{-rs} f^{ij}(X_s^{\bar{x}, \delta, \beta(\delta)}) ds + e^{-r(\rho_1 \wedge \gamma)} \bar{V}^{ij}(X_{\rho_1 \wedge \gamma}^{\bar{x}, \delta, \beta(\delta)}) \right] - \epsilon_1. \end{aligned} \quad (4.3)$$

Therefore, by the last inequality, without loss of generality, we only need to consider $\beta \in \Delta^j$ such that $\rho_1 > \gamma$. Then

$$\begin{aligned}\varphi^{ij}(\bar{x}) &= \bar{V}^{ij}(\bar{x}) \geq \mathbb{E} \left[\int_0^\gamma e^{-rs} f^{ij}(X_s^{\bar{x}, \delta, \beta(\delta)}) ds + e^{-r\gamma} \bar{V}^{ij}(X_\gamma^{\bar{x}, \delta, \beta(\delta)}) \right] - \epsilon_1 \\ &\geq \mathbb{E} \left[\int_0^\gamma e^{-rs} f^{ij}(X_s^{\bar{x}, \delta, \beta(\delta)}) ds + e^{-r\gamma} \varphi^{ij}(X_\gamma^{\bar{x}, \delta, \beta(\delta)}) \right] - \epsilon_1.\end{aligned}$$

By applying Itô's formula to $e^{-rt} \varphi^{ij}(X_t^{\bar{x}, \delta, \beta(\delta)})$ and plugging into the last inequality, we obtain

$$0 \leq \frac{1}{h} \mathbb{E} \left[\int_0^{\tau \wedge h} e^{-rs} (r\varphi^{ij} - \mathcal{A}^{ij} \varphi^{ij} - f^{ij})(X_s^{\bar{x}, \delta, \beta(\delta)}) ds \right] + \epsilon_1,$$

by sending h and ϵ_1 to zero we obtain the contradiction. Therefore, we must have $0 \leq (r\varphi^{ij} - \mathcal{A}^{ij} \varphi^{ij} - f^{ij})(\bar{x})$.

Thanks to Lemma 3, we have

$$M^{ij}[\bar{V}](\bar{x}) \leq \bar{V}^{ij}(\bar{x}).$$

Then

$$\begin{aligned} \max \left\{ \min \left[r\varphi^{ij}(\bar{x}) - \mathcal{A}^{ij} \varphi^{ij}(\bar{x}) - f^{ij}(\bar{x}); \right. \right. \\ \left. \left. \bar{V}^{ij}(\bar{x}) - M^{ij}[\bar{V}](\bar{x}) \right]; \bar{V}^{ij}(\bar{x}) - N^{ij}[\bar{V}](\bar{x}) \right\} \geq 0. \end{aligned} \quad (4.4)$$

Next, suppose $\bar{V}^{ij} - \varphi^{ij}$ achieves a local maximum in $B(\bar{x}, \theta)$ with $\bar{V}^{ij}(\bar{x}) = \varphi^{ij}(\bar{x})$. By Lemma 3 we have, $M^{ij}[\bar{V}](\bar{x}) \leq \bar{V}^{ij}(\bar{x})$. Then if $M^{ij}[\bar{V}](\bar{x}) - \bar{V}^{ij}(\bar{x}) = 0$, we already have the desired inequality. Now suppose $\bar{V}^{ij}(x) - M^{ij}[\bar{V}](x) \geq \epsilon > 0$ and $r\varphi^{ij}(x) - \mathcal{A}^{ij} \varphi^{ij}(x) - f^{ij}(x) > 0$ on $B(\bar{x}, \theta)$.

Define the stopping time τ as before, i.e.,

$$\tau = \inf \{ t \geq 0 : X_t \notin B(\bar{x}, \theta) \}.$$

Let $\gamma = \tau \wedge h$ with $h > 0$ and let $\epsilon_1 > 0$. By taking the no-switching strategy for player II i.e. $\rho_n = \infty, n \geq 1$, and by using the dynamic programming principle, we deduce the existence of a control $\delta \in \mathcal{A}^i$ such that

$$\begin{aligned}\bar{V}^{ij}(\bar{x}) &\leq \mathbb{E} \left[\int_0^{\gamma \wedge \tau_1} e^{-rs} f^{ij}(X_s^{\bar{x}, \delta, \beta(\delta)}) ds + \mathbb{1}_{[\tau_1 \leq \gamma]} e^{-r\tau_1} (\bar{V}^{\xi_1 j}(X_{\tau_1}^{\bar{x}, \delta, \beta(\delta)}) - C(i, \xi_1)) \right] \\ &\quad + \mathbb{E} \left[\mathbb{1}_{[\tau_1 > \gamma]} e^{-r\gamma} \bar{V}^{ij}(X_\gamma^{\bar{x}, \delta, \beta(\delta)}) \right] + \epsilon_1 \\ &\leq \mathbb{E} \left[\int_0^{\gamma \wedge \tau_1} e^{-rs} f^{ij}(X_s^{\bar{x}, \delta, \beta(\delta)}) ds + \mathbb{1}_{[\tau_1 \leq \gamma]} e^{-r\tau_1} M^{ij}[\bar{V}](X_{\tau_1}^{\bar{x}, \delta, \beta(\delta)}) \right] \\ &\quad + \mathbb{E} \left[\mathbb{1}_{[\tau_1 > \gamma]} e^{-r\gamma} \bar{V}^{ij}(X_\gamma^{\bar{x}, \delta, \beta(\delta)}) \right] + \epsilon_1\end{aligned}$$

$$\leq \mathbb{E} \left[\int_0^{\gamma \wedge \tau_1} e^{-rs} f^{ij}(X_s^{\bar{x}, \delta, \beta(\delta)}) ds + e^{-r(\tau_1 \wedge \gamma)} \bar{V}^{ij}(X_{\tau_1 \wedge \gamma}^{\bar{x}, \delta, \beta(\delta)}) \right] + \epsilon_1. \quad (4.5)$$

Therefore, without loss of generality, we only need to consider $\delta \in \mathcal{A}^i$ such that $\tau_1 > \gamma$. Then

$$\begin{aligned} \varphi^{ij}(\bar{x}) = \bar{V}^{ij}(\bar{x}) &\leq \mathbb{E} \left[\int_0^\gamma e^{-rs} f^{ij}(X_s^{\bar{x}, \delta, \beta(\delta)}) ds + e^{-r\gamma} \bar{V}^{ij}(X_\gamma^{\bar{x}, \delta, \beta(\delta)}) \right] + \epsilon_1 \\ &\leq \mathbb{E} \left[\int_0^\gamma e^{-rs} f^{ij}(X_s^{\bar{x}, \delta, \beta(\delta)}) ds + e^{-r\gamma} \varphi^{ij}(X_\gamma^{\bar{x}, \delta, \beta(\delta)}) \right] + \epsilon_1. \end{aligned}$$

By applying Itô's formula to $e^{-rt} \varphi^{ij}(X_t^{\bar{x}, \delta, \beta(\delta)})$ and plugging into the last inequality, we obtain

$$0 \geq \frac{1}{h} \mathbb{E} \left[\int_0^{\tau \wedge h} e^{-rs} (r\varphi^{ij} - \mathcal{A}^{ij} \varphi^{ij} - f^{ij})(X_s^{\bar{x}, \delta, \beta(\delta)}) ds \right] - \epsilon_1,$$

by sending h and ϵ_1 to zero we obtain the contradiction. Therefore, we must have $0 \geq (r\varphi^{ij} - \mathcal{A}^{ij} \varphi^{ij} - f^{ij})(\bar{x})$.

Thanks to Lemma 3, we have

$$N^{ij}[\bar{V}](\bar{x}) \geq \bar{V}^{ij}(\bar{x}).$$

Then

$$\begin{aligned} \max \left\{ \min \left[r\varphi^{ij}(\bar{x}) - \mathcal{A}^{ij} \varphi^{ij}(\bar{x}) - f^{ij}(\bar{x}); \right. \right. \\ \left. \left. \bar{V}^{ij}(\bar{x}) - M^{ij}[\bar{V}](\bar{x}) \right]; \bar{V}^{ij}(\bar{x}) - N^{ij}[\bar{V}](\bar{x}) \right\} \leq 0. \end{aligned} \quad \square \quad (4.6)$$

5. Uniqueness of the solution of Isaacs system of quasi-variational inequalities

We now prove that (2.9) admits a unique viscosity solution. We need to make an additional assumption on the switching costs.

[H5] (The no free loop property)

For any sequence of pairs $\{i_p, j_p\}_{p=1}^{N+1} \subset \mathcal{I} \times \mathcal{I}$ such that $(i_{N+1}, j_{N+1}) = (i_1, j_1)$, and for $1 \leq p \leq N$, either $i_{p+1} = i_p$ or $j_{p+1} = j_p$, we have

$$\sum_{p=1}^N \chi(j_p, j_{p+1}) - \sum_{p=1}^N C(i_p, i_{p+1}) \neq 0.$$

We begin with the technical lemma.

Lemma 4. Let $\vec{U} := (U^{ij})_{(i,j) \in \mathcal{I} \times \mathcal{I}}$ and $\vec{V} := (V^{ij})_{(i,j) \in \mathcal{I} \times \mathcal{I}}$ be a viscosity subsolution and a viscosity supersolution to the equation (2.9), respectively. Let $\bar{x} \in \mathbb{R}^m$ be fixed and $(\bar{i}, \bar{j}) \in \mathcal{I} \times \mathcal{I}$ such that

$$U^{\bar{i}\bar{j}}(\bar{x}) - V^{\bar{i}\bar{j}}(\bar{x}) = \max_{i,j \in \mathcal{I}} \{U^{ij}(\bar{x}) - V^{ij}(\bar{x})\}$$

and

$$U^{\bar{i}\bar{j}}(\bar{x}) \leq M^{\bar{i}\bar{j}}[U](\bar{x}) \quad \text{or} \quad V^{\bar{i}\bar{j}}(\bar{x}) \geq N^{\bar{i}\bar{j}}[V](\bar{x}). \quad (5.1)$$

Then there exists $(i_0, j_0) \in \mathcal{I} \times \mathcal{I}$ such that

$$U^{\bar{i}\bar{j}}(\bar{x}) - V^{\bar{i}\bar{j}}(\bar{x}) = U^{i_0 j_0}(\bar{x}) - V^{i_0 j_0}(\bar{x}) \quad (5.2)$$

and

$$U^{i_0 j_0}(\bar{x}) > M^{i_0 j_0}[U](\bar{x}) \quad \text{and} \quad V^{i_0 j_0}(\bar{x}) < N^{i_0 j_0}[V](\bar{x}). \quad (5.3)$$

Proof. We divide the proof into three steps.

Step 1: Let us first assume that

$$V^{\bar{i}\bar{j}}(\bar{x}) \geq N^{\bar{i}\bar{j}}[V](\bar{x}), \quad (5.4)$$

then there exists $l \neq \bar{j}$ such that

$$V^{\bar{i}\bar{j}}(\bar{x}) \geq V^{\bar{i}l}(\bar{x}) + \chi(\bar{j}, l). \quad (5.5)$$

Since \vec{U} is a subsolution, it satisfies

$$U^{\bar{i}\bar{j}}(\bar{x}) \leq U^{\bar{i}l}(\bar{x}) + \chi(\bar{j}, l), \quad (5.6)$$

which implies that

$$U^{\bar{i}\bar{j}}(\bar{x}) - V^{\bar{i}\bar{j}}(\bar{x}) \leq U^{\bar{i}l}(\bar{x}) - V^{\bar{i}l}(\bar{x}).$$

Hence

$$U^{\bar{i}\bar{j}}(\bar{x}) - V^{\bar{i}\bar{j}}(\bar{x}) = U^{\bar{i}l}(\bar{x}) - V^{\bar{i}l}(\bar{x}). \quad (5.7)$$

Then (5.5), (5.6) and (5.7) imply

$$U^{\bar{i}\bar{j}}(\bar{x}) - U^{\bar{i}l}(\bar{x}) = V^{\bar{i}\bar{j}}(\bar{x}) - V^{\bar{i}l}(\bar{x}) = \chi(\bar{j}, l).$$

Step 2: Now if (5.4) does not hold, then $V^{\bar{i}\bar{j}}(\bar{x}) < N^{\bar{i}\bar{j}}[V](\bar{x})$, and $U^{\bar{i}\bar{j}}(\bar{x}) \leq M^{\bar{i}\bar{j}}[U](\bar{x})$. Then there exists $k \neq \bar{i}$, such that

$$U^{\bar{i}\bar{j}}(\bar{x}) \leq U^{k\bar{j}}(\bar{x}) - C(\bar{i}, k). \quad (5.8)$$

Since \vec{V} is a supersolution, it satisfies

$$V^{\bar{i}\bar{j}}(\bar{x}) \geq V^{k\bar{j}}(\bar{x}) - C(\bar{i}, k), \quad (5.9)$$

which implies that

$$U^{\bar{i}\bar{j}}(\bar{x}) - V^{\bar{i}\bar{j}}(\bar{x}) \leq U^{k\bar{j}}(\bar{x}) - V^{k\bar{j}}(\bar{x}).$$

Hence

$$U^{\bar{i}\bar{j}}(\bar{x}) - V^{\bar{i}\bar{j}}(\bar{x}) = U^{k\bar{j}}(\bar{x}) - V^{k\bar{j}}(\bar{x}). \quad (5.10)$$

Then (5.8), (5.9) and (5.10) imply

$$U^{\bar{i}\bar{j}}(\bar{x}) - U^{k\bar{j}}(\bar{x}) = V^{\bar{i}\bar{j}}(\bar{x}) - V^{k\bar{j}}(\bar{x}) = -C(\bar{i}, k).$$

Step 3: Now if the new index (k or l) verify (5.1), we repeat this reasoning. If this still to occur, we find a loop $(i_1, j_1), \dots, (i_N, j_N), (i_{N+1}, j_{N+1}) = (i_1, j_1)$ such that

$$\sum_{q=1}^N \chi(j_q, j_{q+1}) - \sum_{q=1}^N C(i_q, i_{q+1}) = 0.$$

Hence we obtain a contradiction to the Assumption (H5). Thus (5.2) and (5.3) holds for some (i_0, j_0) . \square

Theorem 3. Let $\vec{U} = (U^{ij})_{(i,j) \in \mathcal{I} \times \mathcal{I}}$ (resp. $\vec{V} = (V^{ij})_{(i,j) \in \mathcal{I} \times \mathcal{I}}$), be family of continuous viscosity subsolutions (resp. supersolutions) to (2.9), and satisfying a linear growth condition. Then, $U^{ij} \leq V^{ij}$ for all $i, j \in \mathcal{I}$.

Proof. Let us proceed by contradiction. Suppose there exists $\eta_0 > 0$ and $x_0 \in \mathbb{R}^m$ such that

$$\max_{i,j} ((U^{ij} - V^{ij})(x_0)) = \eta_0. \quad (5.11)$$

We can also assume, without loss of generality, that for every $i, j \in \mathcal{I}$

$$\lim_{|x| \rightarrow \infty} (U^{ij} - V^{ij})(x) = -\infty. \quad (5.12)$$

Indeed, if this is not the case, one may replace V^{ij} with W^{ij} defined by

$$W^{ij}(x) = V^{ij}(x) + \theta C(1 + |x|^2),$$

which is still an supersolution of (2.9) for $\theta > 0$ and $C \geq C_0$ which satisfies (5.12) (a proof of the supersolution property can be found in e.g. Pham [10, p. 112]). Therefore it is enough to show that $U^{ij} \leq W^{ij}$. Since in taking the limit as $\theta \rightarrow 0$ we obtain the desired result.

Then, there exists $R > 0$ such that

$$\max_{i,j} \max_{x \in \mathbb{R}^m} (U^{ij} - V^{ij})(x) = \max_{x \in B(0,R)} (U^{\bar{i}\bar{j}} - V^{\bar{i}\bar{j}})(x) \quad (5.13)$$

$$= U^{\bar{i}\bar{j}}(\bar{x}) - V^{\bar{i}\bar{j}}(\bar{x}) = \eta \geq \eta_0, \quad (5.14)$$

where $\bar{x} \in B(0, R)$.

The remaining of the proof is obtained in two steps:

Step 1. Using Lemma 4 we derive the existence of i_0 and j_0 such that

$$(U^{i_0 j_0} - V^{i_0 j_0})(\bar{x}) = \eta \quad (5.15)$$

and

$$V^{i_0 j_0}(\bar{x}) < N^{i_0 j_0}[V](\bar{x}) \quad \text{and} \quad U^{i_0 j_0}(\bar{x}) > M^{i_0 j_0}[U](\bar{x}). \quad (5.16)$$

For a small $\epsilon > 0$, let Φ_ϵ be the function defined as follows.

$$\Phi_\epsilon(x, y) = U^{i_0 j_0}(x) - V^{i_0 j_0}(y) - \frac{1}{2\epsilon}|x - y|^2 \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^m. \quad (5.17)$$

By the growth assumption on $U^{i_0j_0}$ and $V^{i_0j_0}$, there exists $(x_\epsilon, y_\epsilon) \in \overline{B(0, R)}^2$ be such that

$$\Phi_\epsilon(x_\epsilon, y_\epsilon) = \max_{(x, y) \in \overline{B(0, R)}^2} \Phi_\epsilon(x, y). \quad (5.18)$$

On the other hand, from $2\Phi_\epsilon(x_\epsilon, y_\epsilon) \geq \Phi_\epsilon(x_\epsilon, x_\epsilon) + \Phi_\epsilon(y_\epsilon, y_\epsilon)$ we have

$$\frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2 \leq (U^{i_0j_0}(x_\epsilon) - U^{i_0j_0}(y_\epsilon)) + (V^{i_0j_0}(x_\epsilon) - V^{i_0j_0}(y_\epsilon)), \quad (5.19)$$

and consequently $\frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2$ is bounded, and as $\epsilon \rightarrow 0$, $|x_\epsilon - y_\epsilon| \rightarrow 0$. Since $U^{i_0j_0}$ and $V^{i_0j_0}$ are uniformly continuous, then $\frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^2 \rightarrow 0$ as $\epsilon \rightarrow 0$.

Next, recalling that $U^{i_0j_0}$ and $V^{i_0j_0}$ are continuous, then, for ϵ small enough and at least for a subsequence which we still index by ϵ , we obtain

$$V^{i_0j_0}(y_\epsilon) < N^{i_0j_0}[V](y_\epsilon) \quad \text{and} \quad U^{i_0j_0}(x_\epsilon) > M^{i_0j_0}[U](x_\epsilon). \quad (5.20)$$

Step 2. Let us denote

$$\varphi_\epsilon(x, y) = \frac{1}{2\epsilon}|x - y|^2. \quad (5.21)$$

Then we have:

$$\begin{cases} D_x \varphi_\epsilon(x, y) = \frac{1}{\epsilon}(x - y), \\ D_y \varphi_\epsilon(x, y) = -\frac{1}{\epsilon}(x - y), \\ B(x, y) = D_{x, y}^2 \varphi_\epsilon(x, y) = \frac{1}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \end{cases} \quad (5.22)$$

Then applying the result by Crandall et al. ([2, Theorem 3.2]) to the function

$$U^{i_0j_0}(x) - V^{i_0j_0}(y) - \varphi_\epsilon(x, y)$$

at the point (x_ϵ, y_ϵ) , for any $\epsilon_1 > 0$, we can find $X, Y \in \mathbb{S}_m$, such that:

$$\begin{cases} \left(\frac{1}{\epsilon}(x_\epsilon - y_\epsilon), X\right) \in J^{2,+}(U^{i_0j_0}(x_\epsilon)), \\ \left(\frac{1}{\epsilon}(x_\epsilon - y_\epsilon), Y\right) \in J^{2,-}(V^{i_0j_0}(y_\epsilon)), \\ -\left(\frac{1}{\epsilon_1} + \|B(x_\epsilon, y_\epsilon)\|\right)I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq B(x_\epsilon, y_\epsilon) + \epsilon_1 B(x_\epsilon, y_\epsilon)^2. \end{cases} \quad (5.23)$$

Then by definition of viscosity solution, we get:

$$\begin{aligned} rU^{i_0j_0}(x_\epsilon) - \left\langle \frac{1}{\epsilon}(x_\epsilon - y_\epsilon), \right. \\ \left. b^{i_0j_0}(x_\epsilon) \right\rangle - \frac{1}{2} \text{tr}[(\sigma^{i_0j_0})^*(x_\epsilon)X\sigma^{i_0j_0}(x_\epsilon)] - f^{i_0j_0}(x_\epsilon) \leq 0, \end{aligned} \quad (5.24)$$

and

$$rV^{i_0j_0}(y_\epsilon) - \left\langle \frac{1}{\epsilon}(x_\epsilon - y_\epsilon), b^{i_0j_0}(y_\epsilon) \right\rangle - \frac{1}{2} \text{tr}[(\sigma^{i_0j_0})^*(y_\epsilon)Y\sigma^{i_0j_0}(y_\epsilon)] - f^{i_0j_0}(y_\epsilon) \geq 0, \quad (5.25)$$

which implies that:

$$\begin{aligned} & rU^{i_0j_0}(x_\epsilon) - rV^{i_0j_0}(y_\epsilon) \\ & \leq \left\langle \frac{1}{\epsilon}(x_\epsilon - y_\epsilon), b^{i_0j_0}(x_\epsilon) - b^{i_0j_0}(y_\epsilon) \right\rangle + \frac{1}{2} \text{tr}[(\sigma^{i_0j_0})^*(x_\epsilon)X\sigma^{i_0j_0}(x_\epsilon) - (\sigma^{i_0j_0})^*(y_\epsilon)Y\sigma^{i_0j_0}(y_\epsilon)] \\ & \quad + f^{i_0j_0}(x_\epsilon) - f^{i_0j_0}(y_\epsilon), \end{aligned} \quad (5.26)$$

we have:

$$B + \epsilon_1 B^2 \leq \frac{\epsilon + \epsilon_1}{\epsilon^2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad (5.27)$$

where C is a constant which hereafter may change from line to line. Choosing now $\epsilon_1 = \epsilon$, yields the relation

$$B + \epsilon_1 B^2 \leq \frac{2}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \quad (5.28)$$

Now, from (H1), (5.23) and (5.28) we get:

$$\frac{1}{2} \text{tr}[(\sigma^{i_0j_0})^*(x_\epsilon)X\sigma^{i_0j_0}(x_\epsilon) - (\sigma^{i_0j_0})^*(y_\epsilon)Y\sigma^{i_0j_0}(y_\epsilon)] \leq \frac{C}{\epsilon} |x_\epsilon - y_\epsilon|^2,$$

and

$$\left\langle \frac{1}{\epsilon}(x_\epsilon - y_\epsilon), b^{i_0j_0}(x_\epsilon) - b^{i_0j_0}(y_\epsilon) \right\rangle \leq \frac{C^2}{\epsilon} |x_\epsilon - y_\epsilon|^2.$$

So that by plugging into (5.26) we obtain:

$$\begin{aligned} & rU^{i_0j_0}(x_\epsilon) - rV^{i_0j_0}(y_\epsilon) \\ & \leq \frac{C}{\epsilon} |x_\epsilon - y_\epsilon|^2 + \frac{C^2}{\epsilon} |x_\epsilon - y_\epsilon|^2 + f^{i_0j_0}(x_\epsilon) - f^{i_0j_0}(y_\epsilon). \end{aligned} \quad (5.29)$$

By sending $\epsilon \rightarrow 0$, and taking into account of the continuity of $f^{i_0j_0}$, we obtain $\eta \leq 0$ which is a contradiction. Thus, $U^{ij} \leq V^{ij}$, for any $(i, j) \in \mathcal{I} \times \mathcal{I}$, which is the desired result. \square

Corollary 1. *The lower and upper value functions coincide, and the value function of the stochastic differential game is given by $V^{ij}(x) := \overline{V}^{ij}(x) = \underline{V}^{ij}(x)$ for every $i, j \in \mathcal{I}$ and $x \in \mathbb{R}^m$. As a consequence the two equations (2.9) and (2.10) have the same unique solution in the class of continuous functions of linear growth.*

6. A verification theorem

In this section, we present a verification theorem which gives an optimal strategy of our zero-sum stochastic differential game.

We suppose that a classical solution of (2.9) exists, denoted by $(V^{ij})_{(i,j) \in \mathcal{I} \times \mathcal{I}}$. Let us define the strategies $\delta^* := (\tau_m^*, \xi_m^*)_{m \geq 0}$ (resp. $\nu^* := (\rho_n^*, \eta_n^*)_{n \geq 0}$) as follows:

$$\tau_0^* = 0, \xi_0^* = i \text{ (resp. } \rho_0^* = 0, \eta_0^* = j)$$

and for any $m \geq 1$,

$$\left\{ \begin{array}{l} \tau_m^* = \begin{cases} \inf\{s \geq \tau_{m-1}^*, V^{\xi_{m-1}^* b_s}(X_s^{x, \delta^*, \nu^*}) = \max_{k \neq \xi_{m-1}^*} \{V^k b_s(X_s^{x, \delta^*, \nu^*}) - C(\xi_{m-1}^*, k)\}\} \\ +\infty \text{ if the above set is empty} \end{cases} \\ \text{and} \\ \xi_m^* = \begin{cases} \max\{k \neq \xi_{m-1}^*, V^k b_{\tau_m^*}(X_{\tau_m^*}^{x, \delta^*, \nu^*}) - C(\xi_{m-1}^*, k)\} & \text{if } \tau_m^* < +\infty \\ \xi_{m-1}^* & \text{if } \tau_m^* = +\infty \end{cases} \end{array} \right. \\ \left(\text{resp.} \right. \\ \left. \begin{array}{l} \rho_m^* = \begin{cases} \inf\{s \geq \rho_{m-1}^*, V^{a_s \eta_{m-1}^*}(X_s^{x, \delta^*, \nu^*}) = \min_{l \neq \eta_{m-1}^*} \{V^{a_s l}(X_s^{x, \delta^*, \nu^*}) + \chi(\eta_{m-1}^*, l)\}\} \\ +\infty \text{ if the above set is empty} \end{cases} \\ \text{and} \\ \eta_m^* = \begin{cases} \min\{l \neq \eta_{m-1}^*, V^{a_{\rho_m^*} l}(X_{\rho_m^*}^{x, \delta^*, \nu^*}) + \chi(\eta_{m-1}^*, l)\} & \text{if } \rho_m^* < +\infty \\ \eta_{m-1}^* & \text{if } \rho_m^* = +\infty \end{cases} \end{array} \right)$$

We are now ready to present the verification theorem for our switching game.

Theorem 4. For each $i, j \in \mathcal{I}$, $x \in \mathbb{R}^m$, and assume that $(\delta^*, \nu^*) \in \mathcal{A}^i \times \mathcal{B}^j$. Then we have $V^{ij}(x) = J(x, \delta^*, \nu^*)$.

Proof. First for each $\delta := (\tau_m, \xi_m)_{m \geq 0} \in \mathcal{A}^i$ and $\nu := (\rho_n, \eta_n)_{n \geq 0} \in \mathcal{B}^j$ we define $\theta = (\theta_k)$ an increasing sequence of stopping times by

$$\begin{cases} \theta_0 = \tau_0 \wedge \rho_0 = \tau_{r^0} \wedge \rho_{s^0} \\ \theta_1 = \tau_{r^1} \wedge \rho_{s^1} \\ \dots\dots\dots \\ \theta_k = \tau_{r^k} \wedge \rho_{s^k}, \end{cases}$$

where

$$\begin{array}{ll} r^0 = 0 & s^0 = 0 \\ r^1 = \min\{i : \tau_i \geq \theta_1\} & s^1 = \min\{i : \rho_i \geq \theta_1\} \\ \dots\dots\dots & \dots\dots\dots \\ r^k = \min\{i : \tau_i \geq \theta_{k-1}\} & s^k = \min\{i : \rho_i \geq \theta_{k-1}\}. \end{array}$$

If a minimum is taken over the empty set, then we put suitable r^k or s^k equal plus infinity.

Then the gain functional J is rewritten as:

$$\begin{aligned} J(x, \delta, \nu) = \sum_{k \geq 1} \mathbb{E} \left[\int_{\theta_{k-1}}^{\theta_k} e^{-rs} f^{a_s b_s}(X_s^{x, \delta, \nu}) ds - \sum_{m \geq 1} e^{-r\tau_m} C(\xi_{m-1}, \xi_m) \mathbb{1}_{[\tau_m = \theta_k]} \right. \\ \left. + \sum_{n \geq 1} e^{-r\rho_n} \chi(\eta_{n-1}, \eta_n) \mathbb{1}_{[\rho_n = \theta_k]} \right]. \end{aligned} \quad (6.1)$$

Now, let $(\theta^*)_k$ associated with δ^* and ν^* , then when $\theta_{k-1}^* < t < \theta_k^*$ we have

$$rV^{a_s b_t}(X_t^{x, \delta^*, \nu^*}) - \mathcal{A}^{a_t b_t} V^{a_t b_t}(X_t^{x, \delta^*, \nu^*}) = f^{a_t b_t}(X_t^{x, \delta^*, \nu^*}). \quad (6.2)$$

Then by Itô's formula (see, e.g., Sect. IV.45 of [12]), we obtain

$$\begin{aligned} \mathbb{E} \left[\int_{\theta_{k-1}^*}^{\theta_k^*} e^{-rs} f^{a_s b_s}(X_s^{x, \delta^*, \nu^*}) ds \right] &= \mathbb{E} \left[\int_{\theta_{k-1}^*}^{\theta_k^*} rV^{a_s b_s}(X_s^{x, \delta^*, \nu^*}) - \mathcal{A}^{a_s b_s} V^{a_s b_s}(X_s^{x, \delta^*, \nu^*}) ds \right] \\ &= \mathbb{E} \left[e^{-r\theta_{k-1}^*} V^{a_{\theta_{k-1}^*} b_{\theta_{k-1}^*}}(X_{\theta_{k-1}^*}^{x, \delta^*, \nu^*}) - e^{-r\theta_k^*} V^{a_{\theta_k^*} b_{\theta_k^*}}(X_{\theta_k^*}^{x, \delta^*, \nu^*}) \right] \end{aligned} \quad (6.3)$$

Substituting this into (6.1), we obtain

$$\begin{aligned} J(x, \delta^*, \nu^*) &= \mathbb{E} \left[V^{ij}(x) + \sum_{k \geq 1} \left[\{-V^{a_{\theta_{k-1}^*} b_{\theta_{k-1}^*}}(X_{\theta_{k-1}^*}^{x, \delta^*, \nu^*}) + V^{a_{\theta_k^*} b_{\theta_k^*}}(X_{\theta_k^*}^{x, \delta^*, \nu^*}) \right. \right. \\ &\quad \left. \left. - \sum_{m \geq 1} C(\xi_{m-1}^*, \xi_m^*) \mathbb{1}_{[\tau_m^* = \theta_k^*]} + \sum_{n \geq 1} \chi(\eta_{n-1}^*, \eta_n^*) \mathbb{1}_{[\rho_n^* = \theta_k^*]} \} e^{-r\theta_k^*} \right] \right]. \end{aligned} \quad (6.4)$$

We now estimate the term in the right-hand side of (6.4) for each $k \geq 1$.

- If $\theta_k^* = \tau_m^*$ for some $m \geq 1$ we have

$$V^{\xi_m^* b_{\theta_{k-1}^*}}(X_{\theta_{k-1}^*}^{x, \delta^*, \nu^*}) - V^{\xi_{m-1}^* b_{\theta_{k-1}^*}}(X_{\theta_{k-1}^*}^{x, \delta^*, \nu^*}) - C(\xi_{m-1}^*, \xi_m^*) = 0 \quad (6.5)$$

- If $\theta_k^* = \rho_n^*$ for some $n \geq 1$ we have

$$V^{a_{\theta_{k-1}^*} \eta_n^*}(X_{\theta_{k-1}^*}^{x, \delta^*, \nu^*}) - V^{a_{\theta_{k-1}^*} \eta_{n-1}^*}(X_{\theta_{k-1}^*}^{x, \delta^*, \nu^*}) + \chi(\eta_{n-1}^*, \eta_n^*) = 0 \quad (6.6)$$

- If $\theta_k^* = \tau_m^* = \rho_n^*$ for some $m, n \geq 1$ we have

$$V^{\xi_m^* \eta_n^*}(X_{\theta_{k-1}^*}^{x, \delta^*, \nu^*}) - V^{\xi_{m-1}^* \eta_{n-1}^*}(X_{\theta_{k-1}^*}^{x, \delta^*, \nu^*}) + \chi(\eta_{n-1}^*, \eta_n^*) - C(\xi_{m-1}^*, \xi_m^*) = 0. \quad (6.7)$$

By (6.4) and the estimates in (6.5)–(6.7) it follows that

$$J(x, \delta^*, \nu^*) = V^{ij}(x). \quad \square$$

7. Numerical results

In this section, we present the results of some numerical simulations on the game by using MATLAB, here we apply the policy iteration algorithm for solving numerically (2.9).

In particular, for the numerical example we consider a two-regime switching problem where the diffusion is independent of the regime and follows a geometric Brownian motion, i.e. $b^{ij}(x) = bx$ and $\sigma^{ij}(x) = \sigma x$ for some $b \in \mathbb{R}$ and $\sigma > 0$.

We consider the following game problem:

$$\begin{aligned}
 \mathcal{I} &= \{1, 2\} \\
 f^{11} &= 5x, & f^{12} &= x, \\
 f^{21} &= -x, & f^{22} &= -4x, \\
 r &= 0.15, \quad \sigma = 0.2, \quad b = 0.01.
 \end{aligned} \tag{7.1}$$

In the flowing figures, we plot the value functions for different switching costs.

Next, we plot the optimal strategies of our zero-sum stochastic differential game (see Fig. 2). We focus on example (a) in Fig. 1. First let $\mathcal{I}_{ij \rightarrow kl} = \{x \in \mathbb{R} : V^{ij}(x) = V^{kl}(x) - C(i, k) + \chi(j, l)\}$, then by Fig. 1 (example (a)) we get

$$\begin{aligned}
 \mathcal{I}_{11 \rightarrow 21} &= (-\infty, -0.25] & \mathcal{I}_{11 \rightarrow 12} &= [1.46, +\infty) \\
 \mathcal{I}_{12 \rightarrow 21} &= (-\infty, -0.62] & \mathcal{I}_{12 \rightarrow 22} &= [-0.62, -0.25] \\
 \mathcal{I}_{21 \rightarrow 11} &= [0.33, 1.46] & \mathcal{I}_{21 \rightarrow 12} &= [1.46, +\infty) \\
 \mathcal{I}_{22 \rightarrow 21} &= (-\infty, -0.62] & \mathcal{I}_{22 \rightarrow 12} &= [0.33, +\infty).
 \end{aligned}$$

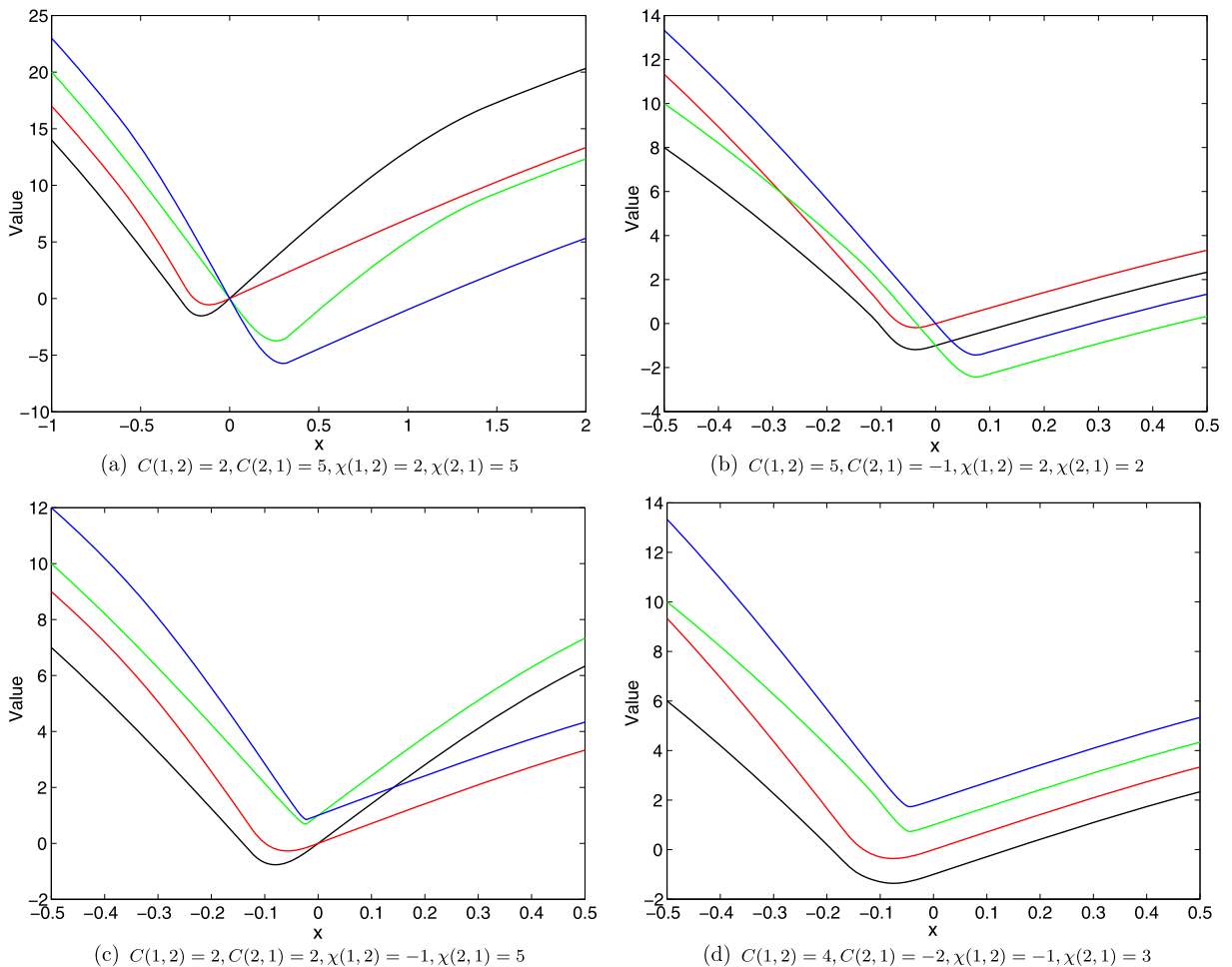


Fig. 1. Value functions: V^{11} (black), V^{12} (red), V^{21} (green), V^{22} (blue). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

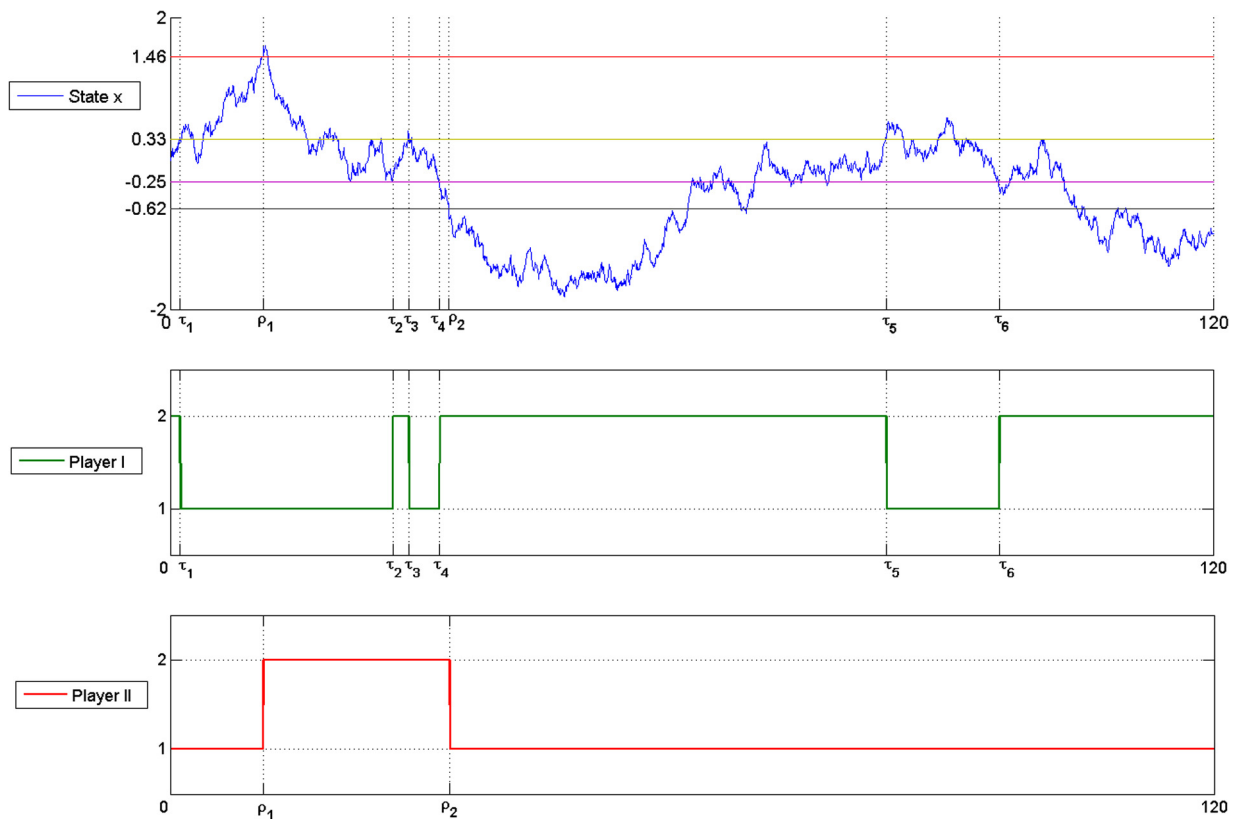


Fig. 2. Optimal strategies and state simulation.

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