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Global existence of weak solution to compressible Navier-Stokes/Allen-Cahn system in three dimensions

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In this paper, we construct a global weak solution to the compressible Navier-Stokes/Allen-Cahn system with possibly large initial data. The result holds in three dimensions subject to the conditions that the initial energy is finite and that the adiabatic constant $\gamma > 2$.

1. Introduction

In this paper, we study a diffuse interface model for two-phase flow in a bounded domain $\Omega \subset \mathbb{R}^3$. The model is governed by compressible Navier-Stokes equations coupled with Allen-Cahn equations, i.e.,

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \nu \Delta u + (\nu + \lambda)\nabla \operatorname{div} u - \iota \operatorname{div}(\nabla\chi \otimes \nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I}), \\ (\rho\chi)_t + \operatorname{div}(\rho\chi u) = -\mu, \\ \rho\mu = -\iota\Delta\chi + \rho\frac{\partial f(\rho,\chi)}{\partial\chi}, \end{cases} \tag{1.1}$$

which is complemented by the initial-boundary conditions

$$\rho(x,0) = \rho_0(x), \quad (\rho u)(x,0) = m_0(x), \quad \chi(x,0) = \chi_0(x), \tag{1.2}$$

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$$u|_{\partial\Omega} = 0, \quad \nabla\chi \cdot n|_{\partial\Omega} = 0. \tag{1.3}$$

Here ρ, u, χ, P denote the total fluid density, the mean velocity of the fluid mixture, the concentration of one selected constituent, and the pressure, respectively. $\mathbb{I}$ is a 3×3 identity matrix, and μ is the chemical potential and n is a normal unit vector. The viscosity coefficients ν, λ satisfy $\nu > 0$, $2\nu + 3\lambda \geq 0$. The thermodynamic pressure $P = \rho^2 \frac{\partial f(\rho,\chi)}{\partial \rho}$ where $f(\rho, \chi)$ is the potential energy density. As in [4], we take the specific free energy f as below

$$f(\rho, \chi) = \frac{\rho^{\gamma-1}}{\gamma - 1} + \frac{1}{\iota}\Big(\frac{\chi^4}{4} - \frac{\chi^2}{2}\Big), \tag{1.4}$$

where the adiabatic constant $\gamma > 1$ and the constant $\sqrt{\iota}$ denotes the thickness of the interfacial region. In this context, we get

$$P = \rho^\gamma, \ \rho\frac{\partial f(\rho, \chi)}{\partial \chi} = \frac{\rho}{\iota}(\chi^3 - \chi).$$

For simplicity, we assume that $\iota = 1$ throughout the rest of the paper.

The model (1.1) proposed by Blesgen [2] describes the behavior of gas phases in a flowing liquid. In the last several years, significant mathematical progress for Navier-Stokes/Allen-Cahn system has been achieved by many mathematicians. When the density ρ is a positive constant, (1.1) corresponds to the incompressible Navier-Stokes/Allen-Cahn system. In this case, Xu, Zhao, Liu in [16] investigated the axisymmetric solutions in 3D and obtained the global regularity of the finite energy solutions with large viscosity and small initial data. With the specific free energy f as (1.4), Zhao, Guo, Huang [17] proved that the solutions of Navier-Stokes/Allen-Cahn system converge to that of the Euler/Allen-Cahn system in a proper small time. The existence of the trajectory attractor was obtained by Gal and Grasselli [10] by using the trajectory approach. Li, Ding, Huang in [15] obtained the existence and uniqueness of local strong solution as well as a blow-up criterion.

For the compressible case, it is much more complicated and mathematically difficult to study. The first mathematical work on this model is obtained by Feireisl, Petzeltov́, Rocca, Schimperna in [8] where the existence of global weak solution in three dimensions was obtained with initial density bounded and away from vacuum. Note that the model in [8] is slightly different from (1.1) in the sense that there are some singular terms in the system due to the different choice of f and that the viscosity coefficients depend on χ. One of the key estimates in [8] is the equi-integrability of the pressure, where the Bogovskii operator $\mathcal{B} : [W^{1,\frac{6\gamma}{6+\gamma}}]^* \to L^{\frac{6\gamma}{5\gamma-6}}$ (see Geissert, Heck, Hieber [11]) and the imbedding $L^1(\Omega) \hookrightarrow [W^{1,\frac{6\gamma}{6+\gamma}}]^*$ are used, provided $\frac{6\gamma}{6+\gamma} > 3$ which implies that $\gamma > 6$. For the steady Navier-Stokes/Allen-Cahn system, this constraint can be relaxed to $\gamma > 3$, see Axmann and Mucha' work [1]. For the non-isothermal model, we refer the readers to [13] where local existence and uniqueness of strong solutions with arbitrary initial data are obtained.

Motivated by [16,17], Ding, Li, and Luo [4] considered the compressible model (1.1) in one dimension with f as in (1.4), and obtained the existence and uniqueness of global strong/classical solutions as well as the global existence of weak solution provided that $\gamma > 1$ and that the initial density is bounded and away from zero. Very recently, Chen and Guo [3] extended Ding-Li-Luo's results to the more general case that $\rho_0 \geq 0$. In view of (1.4), the model (1.1) enjoys similar structures as compressible Navier-Stokes equations for isentropic flow although it contains some terms with strong nonlinear couplings. In this paper, we will adapt Lions-Feireisl's approach for compressible Navier-Stokes equations to extend the one-dimensional case for weak solution as in [4] to the three-dimensional case with the condition of γ as close as possible to those for compressible Navier-Stokes equations.[1]

Before stating our main result, we derive the energy equality provided that the solutions are smooth enough.

Multiplying $(1.1)_2$ by u, integrating over Ω, and using the boundary condition (1.3), we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho u^2 + \frac{1}{\gamma-1}\rho^\gamma\right)dx + \int_\Omega \left(\nu|\nabla u|^2 + (\nu+\lambda)|\mathrm{div} u|^2\right)dx \\ &= -\int_\Omega \mathrm{div}(\nabla\chi\otimes\nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I})\cdot u\, dx. \end{aligned} \tag{1.5}$$

Integrating by parts, the last term of (1.5) can be rewritten as

$$-\int_\Omega \mathrm{div}(\nabla\chi\otimes\nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I})\cdot u\, dx = \int_\Omega \left(\nabla\chi\otimes\nabla\chi : \nabla u - \frac{|\nabla\chi|^2}{2}\mathrm{div} u\right)dx. \tag{1.6}$$

Multiplying $(1.1)_3$ by μ, integrating over Ω, and using $(1.1)_1$, one obtains

$$\int_\Omega \rho\mu(\partial_t\chi + \nabla\chi\cdot u)dx + \int_\Omega \mu^2\, dx = 0. \tag{1.7}$$

Taking $(1.1)_4$ into (1.7) and integrating by parts, we get

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \left(\frac{1}{2}|\nabla\chi|^2 + \frac{1}{4}\rho(\chi^2-1)^2\right)dx + \int_\Omega \mu^2\, dx \\ &= -\int_\Omega \left(\nabla\chi\otimes\nabla\chi : \nabla u - \frac{|\nabla\chi|^2}{2}\mathrm{div} u\right)dx. \end{aligned} \tag{1.8}$$

Combining (1.5), (1.6), (1.8), one has

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho u^2 + \frac{1}{\gamma-1}\rho^\gamma + \frac{1}{2}|\nabla\chi|^2 + \frac{1}{4}\rho(\chi^2-1)^2\right)dx \\ &+ \int_\Omega \nu|\nabla u|^2 + (\nu+\lambda)|\mathrm{div} u|^2 + \mu^2\, dx = 0, \end{aligned} \tag{1.9}$$

which yields

$$\begin{aligned} &\rho|u|^2 \in L^\infty(0,T;L^1(\Omega)); \\ &u \in L^2(0,T;H_0^1(\Omega)); \\ &\chi \in L^\infty(0,T;H^1(\Omega)); \\ &\rho \in L^\infty(0,T;L^\gamma(\Omega)); \\ &\mu \in L^2((0,T)\times\Omega), \end{aligned} \tag{1.10}$$

[1] See the celebrated works by Lions [14] ($\gamma \geq \frac{9}{5}$ for 3d), Feireisl-Novotny-Petzeltova [7] ($\gamma > \frac{3}{2}$ for 3d), Jiang-Zhang [12] ($\gamma > 1$ for spherically symmetric solution).

under some suitable assumptions on the initial data. Moreover, assuming that $\chi \in L^\infty((0,T)\times\Omega)$, we can apply the regularity theorem of elliptic equation to $(1.1)_4$ to obtain

$$\chi \in L^2(0,T;W^{2,\frac{2\gamma}{\gamma+2}}(\Omega)),$$

where the indices of the Sobolev spaces are due to the last two estimates of (1.10).

Throughout this paper, we will use $\mathcal{D}$ to denote C_0^∞, $\mathcal{D}'$ to denote the sense of distributions and C to denote a genetic positive constant. Following the strategy in [7], [14], we define the finite energy weak solution (ρ,u,χ,μ) to (1.1)-(1.3) as below.

Definition 1.1. For any $0<T<+\infty$, (ρ,u,χ,μ) is the finite energy weak solution to (1.1)-(1.3) if

- $\rho \geq 0$, $\rho \in L^\infty([0,T];L^\gamma(\Omega))$, $u \in L^2([0,T];H_0^1(\Omega))$, $\mu \in L^2(0,T;L^2(\Omega))$, $\chi \in [0,1]$, $\chi \in L^\infty([0,T];H^1(\Omega)) \cap L^2(0,T;W^{2,\frac{2\gamma}{\gamma+2}}(\Omega))$, with $(\rho,\rho u,\chi)(0,x) = (\rho_0(x), m_0(x), \chi_0(x))$ for a.e. $x \in \Omega$;
- The equations (1.1) hold in $\mathcal{D}'((0,T)\times\Omega)$ and $(1.1)_1$ holds in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ provided ρ, u are prolonged to zero in $\mathbb{R}^3/\Omega$;
- the equation $(1.1)_1$ is satisfied in the sense of renormalized solution [5], i.e.

$$\partial_t b(\rho) + \mathrm{div}(b(\rho)u) + [b'(\rho)\rho - b(\rho)]\mathrm{div}u = 0$$

  holds in $\mathcal{D}'((0,T)\times\Omega)$, for any $b \in C^1(\mathbb{R})$ such that $b'(z) \equiv 0$ for all $z \in \mathbb{R}$ large enough;
- The energy inequality

$$E(t) + \int_0^T \int_\Omega \left(\nu|\nabla u|^2 + (\nu+\lambda)|\mathrm{div}u|^2 + \mu^2\right) dxdt \leq E(0)$$

  holds for a.e. $t \in [0,T]$, where

$$E(t) = \int_\Omega \left(\frac{1}{2}\rho|u|^2 + \frac{1}{\gamma-1}\rho^\gamma + \frac{1}{2}|\nabla\chi|^2 + \frac{1}{4}\rho(\chi^2-1)^2\right) dx \tag{1.11}$$

  and

$$E(0) = \int_\Omega \left(\frac{1}{2}\frac{|m_0|^2}{\rho_0} + \frac{1}{\gamma-1}\rho_0^\gamma + \frac{1}{2}|\nabla\chi_0|^2 + \frac{1}{4}\rho_0(\chi_0^2-1)^2\right) dx.$$

Now we are in the position to state our main result.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\kappa}$, $\kappa > 0$, and the initial data satisfy*

$$\begin{cases} \rho_0 \geq 0, \quad \rho_0 \in L^\gamma(\Omega), \\ m_0 = 0 \quad \text{if} \quad \rho_0 = 0, \quad \dfrac{(m_0)^2}{\rho_0} \in L^1(\Omega), \\ \chi_0 \in [0,1], \quad \chi_0 \in H^1(\Omega). \end{cases}$$

Then there exists a global weak solution $\{\rho,u,\chi,\mu\}$ with finite energy over $(0,\infty)\times\Omega$ to (1.1)-(1.3) for any given $\gamma > 2$.

2. Faedo-Galerkin approximation

In this section, we will construct an approximation system for (1.1) and obtain the global existence.

2.1. Local existence of the approximation

For any given $\epsilon > 0, \delta > 0$, we construct the following approximation system for (1.1)-(1.3)

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = \epsilon\Delta\rho, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla\rho^\gamma + \delta\nabla(\rho^2 + \rho^\beta) + \epsilon\nabla\rho\cdot\nabla u \\ = \nu\Delta u + (\nu+\lambda)\nabla\operatorname{div}u - \operatorname{div}(\nabla\chi\otimes\nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I}), \\ (\rho\chi)_t + \operatorname{div}(\rho\chi u) = -\mu + \epsilon\chi\Delta\rho, \\ \rho\mu = -\Delta\chi + \rho(\chi^3 - \chi), \end{cases} \tag{2.1}$$

which is complemented by the initial and boundary conditions

$$(\rho, \rho u, \chi)|_{t=0} = (\rho_{0,\delta}, m_{0,\delta}, \chi_{0,\delta}) \quad \text{on} \quad \bar{\Omega}, \tag{2.2}$$

$$\left(\frac{\partial\rho}{\partial n},\ u,\ \frac{\partial\chi}{\partial n}\right)\Big|_{\partial\Omega} = 0, \tag{2.3}$$

where $\delta \in (0, 1/4)$, $m_{0,\delta} = \rho_{0,\delta}u_{0,\delta}$, $\rho_{0,\delta}, \chi_{0,\delta} \in C^3(\Omega), u_{0,\delta} \in C^3(\bar{\Omega})$ satisfying

$$\begin{cases} 0 < \delta \le \rho_{0,\delta} \le \delta^{-\frac{1}{\beta}}, \quad \frac{\partial\rho_{0,\delta}}{\partial n}\Big|_{\partial\Omega} = 0, \\ \rho_{0,\delta} \to \rho_0 \quad \text{in } L^\gamma(\Omega) \quad \text{as} \quad \delta \to 0, \\ u_{0,\delta} = \frac{\varphi_\delta}{\sqrt{\rho_{0,\delta}}}\eta_\delta * \left(\frac{m_0}{\sqrt{\rho_0}}\right), \\ \sqrt{\rho_{0,\delta}}u_{0,\delta} \to \frac{m_0}{\sqrt{\rho_0}} \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \delta \to 0, \\ m_{0,\delta} \to m_0 \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad \delta \to 0, \\ 0 \le \chi_{0,\delta} \le 1, \quad \frac{\partial\chi_{0,\delta}}{\partial n}\Big|_{\partial\Omega} = 0, \\ \chi_{0,\delta} \to \chi_0 \quad \text{in} \quad L^\infty(\Omega)\cap H^1(\Omega) \quad \text{as} \quad \delta \to 0, \end{cases} \tag{2.4}$$

where $\beta > 0$ is sufficiently large, η_δ is the standard mollifier, $\varphi_\delta \in C_0^\infty(\Omega)$, $0 \le \varphi_\delta \le 1$ on $\bar{\Omega}$ and $\varphi_\delta \equiv 1$ on $\{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \delta\}$. The first line of (2.4) will enable the density to be bounded and away from zero, see Proposition 2.1.

We consider a sequence of finite dimensional space

$$X_n = \operatorname{span}\{\Phi_1, \ldots, \Phi_n\},\ n \in \mathbb{N},$$

where Φ_i is the i-th eigenfunction of the Dirichlet problem of Laplacian equation, corresponding the i-th eigenvalue λ_i

$$\begin{cases} -\Delta\Phi_i = \lambda_i\Phi_i \quad \text{on} \quad \Omega, \\ \Phi_i|_{\partial\Omega} = 0. \end{cases}$$

For any given $\varepsilon, \delta > 0$, we look for a function $u_n \in C([0,T]; X_n)$ such that

$$\begin{aligned} &\int_\Omega \rho_n u_n \cdot \Phi dx - \int_\Omega m_{0,\delta} \cdot \Phi dx \\ &= \int_0^t \int_\Omega [\nu \Delta u_n + (\nu + \lambda)\nabla \mathrm{div} u_n - \mathrm{div}(\rho_n u_n \otimes u_n)] \cdot \Phi dx ds \\ &- \int_0^t \int_\Omega \left\{ \nabla[\rho_n^\gamma + \delta(\rho_n^2 + \rho_n^\beta)] + \epsilon \nabla \rho_n \cdot \nabla u_n + \mathrm{div}(\nabla \chi_n \otimes \nabla \chi_n - \frac{|\nabla \chi_n|^2}{2}) \right\} \cdot \Phi dx ds \end{aligned} \tag{2.5}$$

for any $\Phi \in X_n,\ t \in [0,T]$, where $\rho_n = \rho_n(u_n)$ satisfying

$$\begin{cases} \partial_t \rho_n + \mathrm{div}(\rho_n u_n) = \epsilon \Delta \rho_n, \\ \rho_n|_{t=0} = \rho_{0,\delta}, \\ \left.\dfrac{\partial \rho_n}{\partial n}\right|_{\partial\Omega} = 0, \end{cases} \tag{2.6}$$

and $\chi_n = \chi_n(u_n)$ satisfying

$$\begin{cases} \partial_t \chi_n + u_n \cdot \nabla \chi_n = \dfrac{1}{\rho_n^2} \Delta \chi_n - \dfrac{1}{\rho_n}(\chi_n^3 - \chi_n), \\ \chi_n|_{t=0} = \chi_{0,\delta}, \\ \left.\dfrac{\partial \chi_n}{\partial n}\right|_{\partial\Omega} = 0. \end{cases} \tag{2.7}$$

Here we recall the proposition which is given in Chapter 7, [9].

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\eta}$, $\eta > 0$. Assume that the initial function $\rho_{0,\delta}$ is positive and belongs to class $C^{2+\eta}(\overline{\Omega})$, with boundary condition $\frac{\partial \rho_n}{\partial n}|_{\partial\Omega} = 0$. Let*

$$u_n \longmapsto \rho_n[u_n]$$

be the solution mapping for any $u_n \in C([0,T]; C_0^2(\overline{\Omega}; \mathbb{R}^3))$ where ρ_n is the unique solution of (2.6). Then

$$(\inf_{x\in\Omega} \rho_{0,\delta}) \exp\left(-\int_0^t \|\mathrm{div} u_n\|_{L^\infty(\Omega)}\, ds \right) \le \rho_n(x,t) \le (\sup_{x\in\Omega} \rho_{0,\delta}) \exp\left(\int_0^t \|\mathrm{div} u_n\|_{L^\infty(\Omega)}\, ds \right) \tag{2.8}$$

for all $t \in [0,T],\ x \in \Omega$.

Moreover, the mapping takes bounded sets in the space $C([0,T]; C_0^2(\overline{\Omega}; \mathbb{R}^3))$ into bounded sets in the space

$$W \equiv \left\{ \begin{array}{l} \partial_t \rho_n \in C([0,T]; C^\nu(\overline{\Omega})) \\ \rho_n \in C([0,T]; C^{2+\nu}(\overline{\Omega})) \end{array} \right\}$$

and

$$u_n \in C([0,T]; C_0^2(\overline{\Omega})) \longmapsto \rho_n[u_n] \in C^1(\overline{(0,T) \times \Omega})$$

is continuous.

Given ρ_n and u_n, the system (2.7) is a standard parabolic system. Then we get the following lemma which is about the local existence and uniqueness of χ_n.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\alpha}$, $0 \leq \alpha \leq \frac{1}{4}$. Suppose that $u_n \in C([0,T]; C_0^2(\overline{\Omega}; \mathbb{R}^3))$, $\rho_n = \rho_n[u_n]$ with $\rho_{0,\delta}$ strictly positive. Finally, assume that $\chi_{0,\delta} \in C^3(\Omega)$ satisfying $\frac{\partial \chi_{0,\delta}}{\partial n}|_{\partial\Omega} = 0$. Then there exists a unique classical solution χ_n to (2.7) such that*

$$\chi_n \in L^\infty([0,T]; H^1(\Omega)) \cap L^2([0,T]; H^2(\Omega)).$$

Due to (2.5), Proposition 2.1 and Lemma 2.1, the approximate problem (2.1)-(2.3) can be solved on a short time interval $[0, T_n]$ by means of the standard fixed point theorem on a special Banach space $C([0,T_n]; X_n)$, where $T_n \leq T$. Noting that the obtained solution is only local in time, we will extend it to a global one by showing $T_n = T$ in the next subsection.

2.2. Global existence of the approximation

In order to show $T_n = T$, we need an estimate of u_n uniformly for t from the following *global energy estimate*. We test $(2.7)_1$ by χ_n to obtain

$$\rho_n^2 \partial_t(\chi_n^2 - 1) - \Delta(\chi_n^2 - 1) + \rho_n^2 u_n \cdot \nabla(\chi_n^2 - 1) + 2\rho_n(\chi_n^2 - 1) = -2\rho_n(\chi_n^2 - 1)^2 - 2|\nabla\chi_n|^2 \leq 0,$$

which yields

$$\chi_n^2 - 1 \leq 0 \tag{2.9}$$

by using the maximum principle for parabolic equation. On the other hand, from (2.7) and the standard maximum principle for parabolic equations, one has

$$\chi_n \geq 0. \tag{2.10}$$

Therefore, we have

$$0 \leq \chi_n \leq 1. \tag{2.11}$$

Thanks to $(2.1)_1$, we can test testing $(2.1)_3$ by μ_n to obtain

$$\int_\Omega \rho_n \mu_n \partial_t \chi_n \, dx + \int_\Omega \rho_n \mu_n \nabla\chi_n \cdot u_n \, dx + \int_\Omega \mu_n^2 \, dx = 0. \tag{2.12}$$

Taking $(2.1)_4$ into account and integrating by parts, one gets

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \left[\frac{1}{2}|\nabla\chi_n|^2 + \frac{1}{4}\rho_n(\chi_n^2 - 1)^2\right] dx + \int_\Omega \mu_n^2 \, dx \\ &+ \int_\Omega \left[\nabla\chi_n \otimes \nabla\chi_n : \nabla u_n - \frac{|\nabla\chi_n|^2}{2}\mathrm{div} u_n\right] dx \\ &= -\varepsilon \int_\Omega \nabla\rho_n \cdot \nabla\chi_n(\chi_n^3 - \chi_n) \, dx. \end{aligned} \tag{2.13}$$

Using u_n as a test function in $(2.1)_2$ and integrating over Ω, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho_n|u_n|^2+\frac{1}{\gamma-1}\rho_n^\gamma+\delta\left(\rho_n^2+\frac{1}{\beta-1}\rho_n^\beta\right)\right)dx\\
&+\int_\Omega \left(\nu|\nabla u_n|^2+(\nu+\lambda)|\mathrm{div}u_n|^2\right)dx+\varepsilon\int_\Omega(\gamma\rho_n^{\gamma-2}+\delta\beta\rho_n^{\beta-2})|\nabla\rho_n|^2\ dx\\
&+2\delta\varepsilon\int_\Omega|\nabla\rho|^2dx-\int_\Omega\left(\nabla\chi_n\otimes\nabla\chi_n:\nabla u_n-\frac{|\nabla\chi_n|^2}{2}\mathrm{div}u\right)dx=0
\end{aligned}
\tag{2.14}
$$

which combined (2.13) and yields

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\left[\frac{1}{2}\rho_n|u_n|^2+\frac{1}{\gamma-1}\rho_n^\gamma+\delta\left(\rho_n^2+\frac{1}{\beta-1}\rho_n^\beta\right)+\frac{1}{2}|\nabla\chi_n|^2+\frac{1}{4}\rho_n(\chi_n^2-1)^2\right]\ dx\\
&+\int_\Omega\mu_n^2\ dx+\int_\Omega\left[\nu|\nabla u_n|^2+(\nu+\lambda)|\mathrm{div}u_n|^2\right]\ dx+\varepsilon\int_\Omega(\gamma\rho_n^{\gamma-2}+\delta\beta\rho_n^{\beta-2})|\nabla\rho_n|^2\ dx\\
&+2\delta\varepsilon\int_\Omega|\nabla\rho_n|^2\ dx=-\varepsilon\int_\Omega\nabla\rho_n\cdot\nabla\chi_n(\chi_n^3-\chi_n)\ dx.
\end{aligned}
\tag{2.15}
$$

By (2.11), one has

$$
\begin{aligned}
\left|-\varepsilon\int_\Omega\nabla\rho_n\cdot\nabla\chi_n(\chi_n^3-\chi_n)\ dx\right|&\le C\varepsilon\|\nabla\rho_n\|_{L^2(\Omega)}\|\nabla\chi_n\|_{L^2(\Omega)}\\
&\le\delta\varepsilon\|\nabla\rho_n\|^2_{L^2(\Omega)}+\frac{C\varepsilon}{\delta}\|\nabla\chi_n\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Integrating (2.15) with respect to t over $[0,T]$ and using the above inequality, we get

$$
\begin{aligned}
&\int_\Omega\left(\frac{1}{2}\rho_n|u_n|^2+\frac{1}{\gamma-1}\rho_n^\gamma+\delta\left(\rho_n^2+\frac{1}{\beta-1}\rho_n^\beta\right)+\frac{1}{2}|\nabla\chi_n|^2+\frac{1}{4}\rho_n(\chi_n^2-1)^2\right)\ dx\\
&+\int_0^T\int_\Omega\mu_n^2\ dxdt+\int_0^T\int_\Omega\left[\nu|\nabla u_n|^2+(\nu+\lambda)|\mathrm{div}u_n|^2\right]\ dxdt\\
&+\varepsilon\int_0^T\int_\Omega(\gamma\rho_n^{\gamma-2}+\delta\beta\rho_n^{\beta-2})|\nabla\rho_n|^2\ dxdt+\delta\varepsilon\int_0^T\int_\Omega|\nabla\rho_n|^2\ dxdt\\
\le&CE_\delta[\rho_0,m_0,\chi_0]\left(1+e^{\frac{C\varepsilon}{\delta}T}\right),
\end{aligned}
\tag{2.16}
$$

where

$$
E_\delta[\rho_0,m_0,\chi_0]=\int_\Omega\left(\frac{1}{2}\frac{|m_{0,\delta}|^2}{\rho_{0,\delta}}+\frac{1}{\gamma-1}\rho_{0,\delta}^\gamma+\frac{1}{2}|\nabla\chi_{0,\delta}|^2+\frac{1}{4}\rho_{0,\delta}(\chi_{0,\delta}^2-1)^2\right)dx.
$$

The energy inequality (2.16) yields

$$
\int_0^T\|\nabla u_n\|^2_{L^2(\Omega)}\ dt\le C(\varepsilon,\delta)<+\infty.
\tag{2.17}
$$

Due to $\dim X_n < \infty$ and (2.8), it is easy to show that the density is bounded and bounded away from below with a positive constant, i.e.

$$0 < \frac{1}{C(n)} \le \rho_n \le C(n), \tag{2.18}$$

for any $t \in [0, T^*)$. On the other hand, the energy inequality also gives

$$\sup_{t\in[0,T^*)} \int_\Omega \rho_n(t)|u_n(t)|^2 \, dx \le C < +\infty, \tag{2.19}$$

which together with (2.18) implies

$$\sup_{t\in[0,T^*)} \int_\Omega \|u_n(t)\|_{L^\infty(\Omega)} \le C(n) < +\infty, \tag{2.20}$$

where T^* is the maximal time for existence of the solution, and we have used the fact that all the norms are equivalence on X_n. Then we can repeat the above analysis to extend the local solution u_n to the whole interval $[0, T]$. The functions (ρ_n, χ_n) also can be extended to the whole interval $[0, T]$ by Proposition 2.1 and Lemma 2.1.

In summary, we have proved the global existence of the following system:

$$\begin{cases} \partial_t \rho_n + \operatorname{div}(\rho_n u_n) = \epsilon \Delta \rho_n, \\ (\rho_n u_n)_t + \operatorname{div}(\rho_n u_n \otimes u_n) + \nabla \rho_n^\gamma + \delta \nabla(\rho_n^2 + \rho_n^\beta) + \epsilon \nabla \rho_n \cdot \nabla u_n \\ = \nu \Delta u_n + (\nu + \lambda) \nabla \operatorname{div} u_n - \operatorname{div}(\nabla \chi_n \otimes \nabla \chi_n - \frac{|\nabla \chi_n|^2}{2} \mathbb{I}), \\ (\rho_n \chi_n)_t + \operatorname{div}(\rho_n \chi_n u_n) = -\mu_n + \epsilon \chi_n \Delta \rho_n, \\ \rho_n \mu_n = -\Delta \chi_n + \rho_n(\chi_n^3 - \chi_n), \end{cases} \tag{2.21}$$

with the initial and boundary conditions

$$(\rho_n, \rho_n u_n, \chi_n)|_{t=0} = (\rho_{0,\delta}, m_{0,\delta}, \chi_{0,\delta}) \quad \text{on} \quad \bar{\Omega}, \tag{2.22}$$

$$\left(\frac{\partial \rho_n}{\partial n}, \ u_n, \ \frac{\partial \chi_n}{\partial n} \right) \Big|_{\partial\Omega} = 0. \tag{2.23}$$

2.3. The Faedo-Galerkin approximation limit as $n \to +\infty$

Due to the energy inequality (2.16), we observe that the terms related to ρ_n and u_n can be treated similarly to [9]. Here we summarize the convergences and estimates related to ρ_n and u_n, which are useful for showing the convergence in the Allen-Cahn system. Passing the limit $n \to +\infty$, we have

$$\rho_n \to \rho \quad \text{weakly–}(*) \quad \text{in} \quad L^\infty(0, T; L^\beta(\Omega)), \quad \beta > 3; \tag{2.24}$$

$$\rho_n \to \rho \quad \text{strongly} \quad \text{in} \quad L^p((0, T) \times \Omega) \quad \text{for} \quad \text{all} \quad p \in [1, \beta + 1); \tag{2.25}$$

$$u_n \to u \quad \text{weakly in } L^2(0, T; H^1(\Omega)); \tag{2.26}$$

$$\varepsilon \|\nabla \rho_n(t)\|^2_{L^2((0,T)\times\Omega)} \le C E_\delta[\rho_0, m_0, \chi_0]; \tag{2.27}$$

$$\|\operatorname{div}(\rho_n u_n)\|_{L^p((0,T)\times\Omega)} \le C(\varepsilon) E_\delta[\rho_0, m_0, \chi_0] \quad \text{for} \quad \text{some} \quad p > 1; \tag{2.28}$$

$$\|\partial_t \rho_n\|_{L^p((0,T)\times\Omega)} + \|\Delta \rho_n\|_{L^p((0,T\times\Omega))} \le C(\varepsilon) E_\delta[\rho_0, m_0, \chi_0] \quad \text{for} \quad \text{some} \quad p > 1; \tag{2.29}$$

where $C(\varepsilon)$ is dependent on ε but independent on n.

From (2.11) and (2.16), we conclude that

$$\chi_n \to \chi \quad \text{weakly–}(^*) \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)) \cap L^\infty((0,T)\times\Omega), \tag{2.30}$$
$$\mu_n \to \mu \quad \text{weakly} \quad \text{in} \quad L^2((0,T)\times\Omega). \tag{2.31}$$

Moreover, by (2.24), (2.26) and (2.30), one obtains

$$\rho_n\chi_n \to \rho\chi \quad \text{weakly–}(^*) \quad \text{in} \quad L^\infty(0,T;L^\beta(\Omega)), \tag{2.32}$$
$$\rho_n\chi_n u_n \to \rho\chi u \quad \text{weakly} \quad \text{in} \quad L^2(0,T;L^{\frac{6+\beta}{6\beta}}(\Omega)). \tag{2.33}$$

Note that

$$\operatorname{div}(\rho_n\chi_n u_n) = \operatorname{div}(\rho_n u_n)\chi_n + \rho_n u_n \cdot \nabla\chi_n, \tag{2.34}$$

and by virtue of (2.24), (2.26), (2.28) and (2.30), we have

$$\|\operatorname{div}(\rho_n\chi_n u_n)\|_{L^q((0,T)\times\Omega)} \le C(\varepsilon)E_\delta[\rho_0, m_0, \chi_0], \tag{2.35}$$

where $q = \min\{p, \frac{3\beta}{3+2\beta}\}$. Due to (2.29) and (2.31), the right-hand side of $(2.21)_3$ is also bounded in $L^q((0,T)\times\Omega)$, i.e.

$$\|\partial_t(\rho_n\chi_n)\|_{L^q((0,T)\times\Omega)} \le C(\varepsilon)E_\delta[\rho_0, m_0, \chi_0]. \tag{2.36}$$

With the help of (2.24), (2.27), (2.30), one obtains

$$\|\rho_n\chi_n\|_{L^2(0,T;W^{1,\frac{2\beta}{\beta+2}}(\Omega))} \le C(\varepsilon)E_\delta[\rho_0, m_0, \chi_0]. \tag{2.37}$$

Thus applying Aubin-Lions compactness lemma, we have

$$\rho_n\chi_n \to \rho\chi \quad \text{strongly} \quad \text{in} \quad L^q((0,T)\times\Omega). \tag{2.38}$$

Then, we handle the convergence about $(2.21)_4$ and show the strong convergence of χ_n. By virtue of (2.24), (2.30) and (2.31),

$$\rho_n\mu_n \to \rho\mu \quad \text{weakly} \quad \text{in} \quad L^p((0,T)\times\Omega) \quad \text{for} \quad \text{some} \quad p>1, \tag{2.39}$$
$$\rho_n(\chi_n^3 - \chi_n) \to \rho\overline{(\chi^3-\chi)} \quad \text{weakly–}(^*) \quad \text{in} \quad L^\infty(0,T;L^\beta(\Omega)). \tag{2.40}$$

Observing that

$$\partial_t\left[\rho_n(\chi_n^3-\chi_n)\right] = \partial_t\rho_n(-2\chi_n^3) + \partial_t(\rho_n\chi_n)(3\chi_n^2-1), \tag{2.41}$$

and combining with (2.29), (2.30), (2.36), we have

$$\|\partial_t\left[\rho_n(\chi_n^3-\chi_n)\right]\|_{L^q((0,T)\times\Omega)} \le C(\varepsilon)E_\delta[\rho_0, m_0, \chi_0], \tag{2.42}$$

which implies

$$\rho_n(\chi_n^3-\chi_n) \to \rho\overline{(\chi^3-\chi)} \quad \text{in} \quad C([0,T];L^\beta_{weak}(\Omega)), \tag{2.43}$$

$$\rho_n(\chi_n^3-\chi_n)\chi_n \to \rho\overline{(\chi^3-\chi)}\chi \quad \text{weakly} \quad \text{in} \quad L^2((0,T)\times\Omega). \tag{2.44}$$

Next we multiply $(2.21)_4$ by χ_n and integrate by parts:

$$\int_0^T\int_\Omega |\nabla\chi_n|^2\,dxdt = \int_0^T\int_\Omega \rho_n\mu_n\chi_n\,dxdt - \int_0^T\int_\Omega \rho_n(\chi_n^3-\chi_n)\chi_n\,dxdt. \tag{2.45}$$

Passing the limit $n\to+\infty$ in $(2.21)_4$, one gets

$$\rho\mu = -\Delta\chi + \rho\overline{(\chi^3-\chi)}. \tag{2.46}$$

It follows that

$$\int_0^T\int_\Omega |\nabla\chi|^2\,dxdt = \int_0^T\int_\Omega \rho\mu\chi\,dxdt - \int_0^T\int_\Omega \rho\overline{(\chi_n^3-\chi_n)}\chi\,dxdt. \tag{2.47}$$

By (2.30), (2.38) and (2.44), we conclude that

$$\nabla\chi_n \to \nabla\chi \quad \text{strongly} \quad \text{in} \quad L^2((0,T)\times\Omega). \tag{2.48}$$

Then we recall a special Poincaré's inequality in [6] to achieve the strong convergence of χ_n.

Lemma 2.2. *Let $\Omega\subset\mathbb{R}^3$ be a bounded regular domain and m and k be two positive real numbers. Assume ρ is a non-negative function such that*

$$0<m=\int_\Omega \rho dx \qquad \text{and} \qquad \int_\Omega \rho^{\frac{5}{3}}dx \le k.$$

Then there exists a constant $c=c(m,k,p)$ such that

$$\left\|\chi - \frac{1}{m}\int_\Omega \rho\chi\,dx\right\|_{L^p(\Omega)} \le c(m,k,p)\|\nabla\chi\|_{L^p(\Omega)}$$

for any $\chi\in W^{1,p}(\Omega)$ if $p>\frac{15}{11}$.

Applying Lemma 2.2, using (2.38) and (2.48), we have

$$\chi_n \to \chi \quad \text{strongly} \quad \text{in} \quad L^2(0,T;H^1(\Omega)). \tag{2.49}$$

And in the view of (2.29) and (2.30), one gets

$$\chi_n \to \chi \quad \text{strongly} \quad \text{in} \quad L^p((0,T)\times\Omega) \quad \text{for all} \quad p\in[1,\infty), \tag{2.50}$$

$$\chi_n^3-\chi_n \to \chi^3-\chi \quad \text{strongly} \quad \text{in} \quad L^p((0,T)\times\Omega) \quad \text{for all} \quad p\in[1,\infty), \tag{2.51}$$

$$\chi_n\Delta\rho_n \to \chi\Delta\rho \quad \text{weakly} \quad \text{in} \quad L^p((0,T)\times\Omega) \quad \text{for some} \quad p>1. \tag{2.52}$$

In summary, we have the following existence result for the problem (2.1)-(2.3).

Proposition 2.2. *Let $\varepsilon > 0$, $\delta > 0$, and $\beta > \max\{3, \gamma\}$ be fixed. Then for any given $T > 0$, there exists at least one solution (ρ, u, χ, μ) to problem (2.1)-(2.3) in the following sense:*

(i) *The density ρ is a non-negative function satisfying*

$$\rho \in L^p([0,T]; W^{2,p}(\Omega)), \quad \partial_t \rho \in L^p(0,T) \times \Omega),$$

for some $p > 1$, the velocity u belongs to the space $L^2([0,T]; H_0^1(\Omega))$, equation $(2.1)_1$ holds almost everywhere in $(0,T) \times \Omega$, and the initial and boundary condition on ρ is satisfied in the sense of traces. Besides, the total mass is conserved, i.e.

$$\int_\Omega \rho(x,t)dx = \int_\Omega \rho_{0,\delta}(x)dx,$$

for all $t \in [0,T]$; and the following estimates hold

$$\delta \int_0^T \int_\Omega \rho^{\beta+1} dxdt \le C_\epsilon,$$

$$\delta\varepsilon \int_0^T \int_\Omega |\nabla \rho|^2 dxdt \le E_\delta[\rho_0, m_0, \chi_0].$$

(ii) *All the quantities appearing in the second equation of (2.1) are locally integrable, and the equation is satisfied in $\mathcal{D}'((0,T) \times \Omega)$. Besides, one has*

$$\rho u \in C([0,T]; L_{weak}^{\frac{2\gamma}{\gamma+1}}(\Omega))$$

and ρu satisfies the initial data.

(iii) *All terms in $(2.1)_3$ and $(2.1)_4$ are locally integrable. The functions χ, μ satisfy the equations $(2.1)_3$, $(2.1)_4$ in the sense of distribution. Moreover, χ satisfies the initial data.*

(iv) *The energy inequality*

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\rho|u|^2 + \frac{1}{\gamma-1}\rho^\gamma + \frac{1}{2}|\nabla \chi|^2 + \frac{1}{4}\rho(\chi^2-1)^2 \right) dx \\
&+ \int_0^T \int_\Omega \left(\nu|\nabla u|^2 + (\nu+\lambda)|\mathrm{div} u|^2 + \mu^2 \right) dxdt \\
&+ \varepsilon \int_0^T \int_\Omega (\gamma\rho^{\gamma-2} + \delta\beta\rho^{\beta-2})|\nabla \rho|^2 \, dxdt + \delta\varepsilon \int_0^T \int_\Omega |\nabla \rho|^2 \, dxdt \\
&\le \int_\Omega \left(\frac{1}{2}\frac{|m_0|^2}{\rho_0} + \frac{1}{\gamma-1}\rho_0^\gamma + \frac{1}{2}|\nabla \chi_0|^2 + \frac{1}{4}\rho_0(\chi_0^2-1)^2 \right) dx
\end{aligned}$$

holds for a.e. $t \in [0,T]$.

In order to complete the proof of Theorem 1.1, we will take the vanishing limits of the artificial viscosity and the artificial pressure in the next two sections.

3. The vanishing artificial viscosity limit

In this section, we study the limit as $\varepsilon \to 0^+$ for the solution $(\rho_\varepsilon, u_\varepsilon, \chi_\varepsilon, \mu_\varepsilon)$ obtained in Proposition 2.2. Since the term $\varepsilon\Delta\rho_\varepsilon$ vanishes as $\varepsilon \to 0^+$, the density ρ_ε will lose some regularity. On the other hand, the energy inequality yields $\rho_\varepsilon \in L^\infty(0,T;L^\beta(\Omega))$ and then the pressure term will only converge to a random measure. It seems not enough to get the desired limits. Thus, we need a higher integrability estimate of the density.

3.1. Higher integrability of the density

The basic idea is to test the pressure term $\nabla(\rho_\varepsilon^\gamma + \delta(\rho_\varepsilon^2 + \rho_\varepsilon^\beta))$ by a suitable function, and then get the desired integrability estimate of the density. The choice of the suitable function relies on a linear integral operator $\mathcal{B}$ which satisfies

$$\mathcal{B} : \left\{ f \in L^p(\Omega) | \int_\Omega f = 0 \right\} \to W_0^{1,p},$$

$$\operatorname{div}\mathcal{B}[f] = f, \quad \mathcal{B}[f]|_{\partial\Omega} = 0,$$

$$\|\mathcal{B}[f]\|_{W^{1,p}(\Omega)} \le c(p)\|f\|_{L^p(\Omega)} \quad \text{for} \quad \text{any} \quad 1 < p < \infty. \tag{3.1}$$

Moreover, for $g \in L^p(\Omega),\ g \cdot n|_{\partial\Omega} = 0$,

$$\|\mathcal{B}[\operatorname{div} g]\|_{L^p(\Omega)} \le c(p)\|g\|_{L^p(\Omega)} \quad \text{for} \quad \text{any } 1 < p < \infty. \tag{3.2}$$

Denote

$$\Phi(t,x) = \psi(t)\mathcal{B}[\rho_\varepsilon - M_0], \quad \psi(t) \in C_0^\infty(0,T), \quad 0 \le \psi(t) \le 1, \quad M_0 = \frac{1}{|\Omega|}\int_\Omega \rho_\varepsilon(t)\, dx$$

as the suitable function where $\psi(t)$ guarantees that the quantity Φ is a test function for the momentum equation.

Now we are in the position to get the higher integrability estimate of the density.

Lemma 3.1. *Let $\beta \ge \frac{9}{2}$, $(\rho_\varepsilon, u_\varepsilon, \chi_\varepsilon)$ be the sequences of the problem (2.1), (2.2), (2.3), then*

$$\int_0^T \int_\Omega \left\{ \rho_\varepsilon^{\gamma+1} + \delta(\rho_\varepsilon^3 + \rho_\varepsilon^{\beta+1}) \right\} dxdt \le C, \tag{3.3}$$

where C is independent of ε.

Proof. Testing the momentum equation by $\Phi = \psi(t)\mathcal{B}[\rho_\varepsilon - M_0]$ and integrating over $(0,T)\times\Omega$, we have

$$\begin{aligned}
&\int_0^T \psi \int_\Omega \left\{a\rho_\varepsilon^{\gamma+1} + \delta(\rho_\varepsilon^3 + \rho_\varepsilon^{\beta+1})\right\} dxdt \\
=&M_0 \int_0^T \psi \int_\Omega \left\{\rho_\varepsilon^{\gamma} + \delta(\rho_\varepsilon^2 + \rho_\varepsilon^{\beta})\right\} dxdt + (2\nu+\lambda)\int_0^T \psi \int_\Omega \rho_\varepsilon \mathrm{div} u_\varepsilon \, dxdt \\
&- \int_0^T \psi_t \int_\Omega \rho_\varepsilon u_\varepsilon \cdot \mathcal{B}[\rho_\varepsilon - M_0]\, dxdt - \nu \int_0^T \psi \int_\Omega \nabla u_\varepsilon \cdot \nabla \mathcal{B}[\rho_\varepsilon - M_0]\, dxdt \\
&- \int_0^T \psi \int_\Omega \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon \cdot \nabla\mathcal{B}[\rho_\varepsilon - M_0]\, dxdt - \varepsilon \int_0^T \psi \int_\Omega \rho_\varepsilon u_\varepsilon \cdot \mathcal{B}[\Delta\rho_\varepsilon]\, dxdt \\
&- \int_0^T \psi \int_\Omega \rho_\varepsilon u_\varepsilon \cdot \mathcal{B}[\mathrm{div}(\rho_\varepsilon u_\varepsilon)]\, dxdt + \varepsilon \int_0^T \psi \int_\Omega \nabla u_\varepsilon \cdot \nabla\rho_\varepsilon \cdot \mathcal{B}[\rho_\varepsilon - M_0]\, dxdt \\
&- \int_0^T \psi \int_\Omega \left(\nabla\chi_\varepsilon \otimes \nabla\chi_\varepsilon - \frac{1}{2}|\nabla\chi_\varepsilon|^2\mathbb{I}\right) : \nabla\mathcal{B}[\rho_\varepsilon - M_0]\, dxdt = \sum_{k=1}^{9} I_k.
\end{aligned} \tag{3.4}$$

The boundness of $\sum_{k=1}^{8} I_k$ are treated similarly in [7]. Here we focus on I_9. Applying the regularity theorem of elliptic equation, we have

$$\|\chi_\epsilon\|_{W^{2,\frac{2\beta}{2+\beta}}(\Omega)} \le \|\rho_\epsilon\mu_\epsilon + \rho_\epsilon(\chi_\varepsilon^3 - \chi_\varepsilon)\|_{L^{\frac{2\beta}{2+\beta}}(\Omega)}.$$

It follows that

$$\begin{aligned}
|I_9| =& \left| \int_0^T \psi \int_\Omega \left(\nabla\chi_\varepsilon \otimes \chi_\varepsilon - \frac{1}{2}|\nabla\chi_\varepsilon|^2\mathbb{I}\right) : \nabla\mathcal{B}[\rho_\varepsilon - M_0]\, dxdt \right| \\
\le& C \int_0^T \||\nabla\chi_\varepsilon|^2\|_{L^{\frac{\beta}{\beta-1}}(\Omega)} \|\nabla\mathcal{B}[\rho_\varepsilon - M_0]\|_{L^\beta(\Omega)}\, dt \\
\le& C \int_0^T \||\nabla\chi_\varepsilon|^2\|_{L^{\frac{\beta}{\beta-1}}(\Omega)} \|\rho_\varepsilon\|_{L^\beta(\Omega)}\, dt \\
\le& C \int_0^T \|\nabla\chi_\varepsilon\|^2_{L^{\frac{2\beta}{\beta-1}}(\Omega)} dt \\
\le& C \int_0^T \|\nabla\chi_\varepsilon\|^2_{L^{\frac{6\beta}{6+\beta}}(\Omega)} dt \\
\le& C \int_0^T \|\chi_\varepsilon\|^2_{W^{2,\frac{2\beta}{2+\beta}}(\Omega)} dt
\end{aligned}$$

$$\leq C \int_0^T \|\rho_\varepsilon\|_{L^\beta(\Omega)}^2 (1 + \|\mu_\varepsilon\|_{L^2(\Omega)}^2) dt$$
$$\leq C, \tag{3.5}$$

where we have used the fact that $W^{2,\frac{2\beta}{2+\beta}} \hookrightarrow W^{1,\frac{6\beta}{6+\beta}}$ and $\frac{2\beta}{\beta-1} \leq \frac{6\beta}{6+\beta}$, and that $\rho_\varepsilon \in L^\infty(0,T;L^\beta(\Omega))$ and $\mu_\varepsilon \in L^2((0,T)\times\Omega)$ due to the energy inequality. Note that the inequality $\frac{2\beta}{\beta-1} \leq \frac{6\beta}{6+\beta}$ implies $\beta \geq \frac{9}{2}$. □

3.2. The limit passage

With the help of the previous energy estimate, Lemma 3.1 and (2.11), we have

$$\varepsilon \nabla\rho_\varepsilon \cdot \nabla u_\varepsilon \to 0 \quad \text{in} \quad L^1((0,T)\times\Omega), \tag{3.6}$$
$$\varepsilon \Delta\rho_\varepsilon \to 0 \quad \text{in} \quad L^2(0,T;W^{-1,2}(\Omega)), \tag{3.7}$$
$$\rho_\varepsilon \to \rho \quad \text{weakly in } L^{\beta+1}((0,T)\times\Omega) \quad \text{and} \quad \text{weakly–}(^*) \text{ in } L^\infty(0,T;L^\beta(\Omega)), \tag{3.8}$$
$$\chi_\varepsilon \to \chi \quad \text{weakly–}(^*) \quad \text{in} \quad L^\infty((0,T)\times\Omega) \cap L^\infty(0,T;H^1(\Omega)), \tag{3.9}$$
$$u_\varepsilon \to u \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \tag{3.10}$$
$$\mu_\varepsilon \to \mu \quad \text{weakly in } L^2((0,T)\times\Omega), \tag{3.11}$$
$$\rho_\varepsilon^\gamma + \delta(\rho_\varepsilon^2 + \rho_\varepsilon^\beta) \to \overline{\rho^\gamma + \delta(\rho^2+\rho^\beta)} \quad \text{weakly in } L^{\frac{\beta+1}{\beta}}((0,T)\times\Omega), \tag{3.12}$$

where $\overline{f}$ denotes the limit function of f_ε.

Following the same arguments as in [7], we obtain

$$\rho_\varepsilon \to \rho \quad \text{in } C([0,T];L^\beta_{weak}(\Omega)), \tag{3.13}$$
$$\rho_\varepsilon u_\varepsilon \to \rho u \quad \text{in } C([0,T];L^{\frac{2\beta}{1+\beta}}_{weak}(\Omega)), \tag{3.14}$$
$$\rho_\varepsilon(u_\varepsilon \otimes u_\varepsilon) \to \rho(u\otimes u) \quad \text{in} \quad \mathcal{D}'((0,T)\times\Omega). \tag{3.15}$$

We will proceed to handle the χ_ε terms in the equations. To begin with, we recall the Allen-Cahn system, i.e.,

$$\partial_t(\rho_\varepsilon\chi_\varepsilon) = -\mathrm{div}(\rho_\varepsilon\chi_\varepsilon u_\varepsilon) - \mu_\varepsilon + \varepsilon\Delta\rho_\varepsilon\chi_\varepsilon, \tag{3.16}$$
$$\rho_\varepsilon\mu_\varepsilon = -\Delta\chi_\varepsilon + \rho_\varepsilon(\chi_\varepsilon^3 - \chi_\varepsilon). \tag{3.17}$$

For any $\xi \in C_0^\infty((0,T)\times\Omega)$, one has

$$\varepsilon \int_0^T\int_\Omega \Delta\rho_\varepsilon\chi_\varepsilon\xi \, dxdt$$
$$= \varepsilon \int_0^T\int_\Omega \rho_\varepsilon\chi_\varepsilon\Delta\xi \, dxdt + 2\varepsilon \int_0^T\int_\Omega \rho_\varepsilon\nabla\chi_\varepsilon\cdot\nabla\xi \, dxdt + \varepsilon\int_0^T\int_\Omega \rho_\varepsilon\Delta\chi_\varepsilon\xi \, dxdt, \tag{3.18}$$

where all terms go to 0 as $\varepsilon \to 0^+$. It implies

$$\varepsilon\Delta\rho_\varepsilon\chi_\varepsilon \to 0 \quad \text{in } \mathcal{D}'((0,T)\times\Omega). \tag{3.19}$$

By virtue of (3.8) and (3.9), we have

$$\rho_\varepsilon \chi_\varepsilon \to \rho\chi \quad \text{weakly–}(^*) \quad \text{in} \quad L^\infty(0,T;L^\beta(\Omega)), \tag{3.20}$$

which, in accordance with (3.10) yields

$$\|\rho_\varepsilon \chi_\varepsilon u_\varepsilon\|_{L^2(0,T;L^{\frac{6\beta}{6+\beta}}(\Omega))} \le C. \tag{3.21}$$

It follows that

$$\|\partial_t(\rho_\varepsilon \chi_\varepsilon)\|_{L^2(0,T;H^{-1}(\Omega))} \le C. \tag{3.22}$$

Then applying the Aubin-Lions compactness lemma, one obtains

$$\rho_\varepsilon \chi_\varepsilon \to \rho\chi \quad \text{in} \quad C([0,T];L^\beta_{weak}(\Omega)) \cap C([0,T];H^{-1}(\Omega)), \tag{3.23}$$

which combines (3.10) and leads to

$$\rho_\varepsilon \chi_\varepsilon u_\varepsilon \to \rho\chi u \quad \text{weakly} \quad \text{in} \quad L^2(0,T;L^{\frac{6\beta}{6+\beta}}(\Omega)). \tag{3.24}$$

By virtue of (3.8), (3.9) and (3.11), we have

$$\rho_\varepsilon \mu_\varepsilon \to \overline{\rho\mu} \quad \text{weakly} \quad \text{in} \quad L^2(0,T;L^{\frac{2\beta}{2+\beta}}(\Omega)), \tag{3.25}$$

$$\Delta\chi_\varepsilon \to \Delta\chi \quad \text{weakly} \quad \text{in} \quad L^2(0,T;L^{\frac{2\beta}{2+\beta}}(\Omega)), \tag{3.26}$$

$$\chi_\varepsilon^3 - \chi_\varepsilon \to \overline{\chi^3 - \chi} \quad \text{weakly–}(^*) \quad \text{in } L^\infty((0,T)\times\Omega) \cap L^\infty(0,T;H^1(\Omega)), \tag{3.27}$$

which together with (3.23) yield

$$\rho_\varepsilon \chi_\varepsilon(\chi_\varepsilon^3 - \chi_\varepsilon) \to \rho\chi(\overline{\chi^3 - \chi}) \quad \text{weakly–}(^*) \quad \text{in} \quad L^\infty(0,T;L^\beta(\Omega)). \tag{3.28}$$

Then we test equation (3.17) by χ_ε and its limit equation by χ to arrive at

$$\int_0^T\int_\Omega |\nabla\chi_\varepsilon|^2 \, dxdt = \int_0^T\int_\Omega \rho_\varepsilon\mu_\varepsilon\chi_\varepsilon \, dxdt - \int_0^T\int_\Omega \rho_\varepsilon\chi_\varepsilon(\chi_\varepsilon^3 - \chi_\varepsilon) \, dxdt, \tag{3.29}$$

$$\int_0^T\int_\Omega |\nabla\chi|^2 \, dxdt = \int_0^T\int_\Omega \overline{\rho\mu}\chi \, dxdt - \int_0^T\int_\Omega \rho\chi\overline{(\chi^3 - \chi)} \, dxdt. \tag{3.30}$$

Similar to the analysis in Section 2.3, we are able to prove

$$\chi_\varepsilon \to \chi \quad \text{strongly} \quad \text{in} \quad L^2(0,T;H^1(\Omega)), \tag{3.31}$$

as soon as

$$\begin{aligned} &\lim_{\varepsilon\to 0}\left(\int_0^T\int_\Omega \rho_\varepsilon\mu_\varepsilon\chi_\varepsilon \, dxdt - \int_0^T\int_\Omega \rho_\varepsilon\chi_\varepsilon(\chi_\varepsilon^3 - \chi_\varepsilon) \, dxdt\right) \\ &= \int_0^T\int_\Omega \overline{\rho\mu}\chi \, dxdt - \int_0^T\int_\Omega \rho\chi\overline{(\chi^3 - \chi)} \, dxdt. \end{aligned} \tag{3.32}$$

By employing (3.9), (3.23), we have

$$\rho_\varepsilon \chi_\varepsilon^2 \to \rho\chi^2 \quad \text{weakly–}(^*) \quad \text{in} \quad L^\infty(0,T;L^\beta(\Omega)). \tag{3.33}$$

On the other hand, by (3.9), one gets

$$\chi_\varepsilon^2 \to \overline{\chi^2} \quad \text{weakly–}(^*) \quad \text{in} \quad L^\infty((0,T)\times\Omega)\cap L^\infty(0,T;H^1(\Omega)), \tag{3.34}$$

which together with (3.8), (3.13) gives

$$\rho\chi_\varepsilon^2 \to \rho\overline{\chi^2} \quad \text{weakly–}(^*) \quad \text{in} \quad L^\infty(0,T;L^\beta(\Omega)), \tag{3.35}$$

$$\rho_\varepsilon\chi_\varepsilon^2 \to \rho\overline{\chi^2} \quad \text{weakly–}(^*) \quad \text{in} \quad L^\infty(0,T;L^\beta(\Omega)). \tag{3.36}$$

Noting that

$$\rho\chi_\varepsilon^2 - \rho\chi^2 = (\rho\chi_\varepsilon^2 - \rho\overline{\chi^2}) + (\rho\overline{\chi^2} - \rho_\varepsilon\chi_\varepsilon^2) + (\rho_\varepsilon\chi_\varepsilon^2 - \rho\chi^2), \tag{3.37}$$

we deduce from (3.33), (3.35) (3.36) that

$$\int_0^T\int_\Omega \rho\chi_\varepsilon^2 \,dxdt \to \int_0^T\int_\Omega \rho\chi^2 \,dxdt \quad \text{as} \quad \varepsilon\to 0, \tag{3.38}$$

which implies

$$\chi_\varepsilon \to \chi \quad \text{strongly} \quad \text{in} \quad L^p(Q_T^+) \quad \text{for} \quad \text{all} \quad p\in[1,\infty), \tag{3.39}$$

$$\rho_\varepsilon \to \rho \quad \text{strongly} \quad \text{in} \quad L^p(Q_T^0) \quad \text{for} \quad \text{all} \quad p\in[1,\beta), \tag{3.40}$$

where $Q_T^+ = \{(x,t)\in(0,T)\times\Omega|\rho(x,t)>0\}$, $Q_T^0 = \{(x,t)\in(0,T)\times\Omega|\rho(x,t)=0\}$. It follows that

$$\iint_{Q_T^+} \overline{\rho\mu\chi}\,dxdt = \iint_{Q_T^+} \overline{\rho\mu}\chi\,dxdt, \tag{3.41}$$

and

$$\iint_{Q_T^0} \overline{\rho\mu\chi}\,dxdt = \iint_{Q_T^0} \rho\overline{\mu\chi}\,dxdt = 0, \tag{3.42}$$

$$\iint_{Q_T^0} \overline{\rho\mu}\chi\,dxdt = \iint_{Q_T^0} \rho\mu\chi\,dxdt = 0. \tag{3.43}$$

In view of (3.28), and (3.41)-(3.43), we prove (3.32) and the strong convergence of χ_ε. Consequently, we have

$$\chi_\varepsilon^3 - \chi_\varepsilon \to \chi^3 - \chi \quad \text{strongly} \quad \text{in} \quad L^p((0,T)\times\Omega) \quad \text{for} \quad \text{all} \quad p\in[1,\infty). \tag{3.44}$$

Letting $\varepsilon \to 0^+$, we conclude that the limit of $(\rho_\varepsilon, u_\varepsilon, \chi_\varepsilon, \mu_\varepsilon)$ satisfies the following system:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla \overline{P} = \nu \Delta u + (\nu + \lambda) \nabla \operatorname{div} u - \operatorname{div}(\nabla \chi \otimes \nabla \chi - \frac{|\nabla \chi|^2}{2} \mathbb{I}), \\ (\rho \chi)_t + \operatorname{div}(\rho \chi u) = -\mu, \\ \overline{\rho \mu} = -\Delta \chi + \rho(\chi^3 - \chi), \end{cases} \tag{3.45}$$

where $\overline{P} = \overline{\rho^\gamma + \delta(\rho^2 + \rho^\beta)}$.

3.3. Strong convergence of the density

Observing that the equation $(2.1)_1$ holds almost everywhere in $(0, T) \times \Omega$, we test it by $b'(\rho_\varepsilon)$ to obtain

$$\begin{aligned} \partial_t b(\rho_\varepsilon) + \operatorname{div}[b(\rho_\varepsilon) u_\varepsilon] + (b'(\rho_\epsilon)\rho_\varepsilon - b(\rho_\varepsilon)) \operatorname{div} u_\varepsilon \\ = \varepsilon \operatorname{div}(1_\Omega \nabla b(\rho_\varepsilon)) - \varepsilon 1_\Omega b''(\rho_\varepsilon) |\nabla \rho_\varepsilon|^2 \end{aligned} \tag{3.46}$$

in $\mathcal{D}'((0, T) \times \mathbb{R}^3)$, where b is a convex function belongs to $C^2[0, \infty)$ and 1_Ω is the characteristic function of Ω. Integrating over $(0, T) \times \Omega$, one has

$$\int_0^T \int_\Omega (b'(\rho_\varepsilon)\rho_\varepsilon - b(\rho_\varepsilon)) \operatorname{div} u_\varepsilon \, dxdt \leq \int_\Omega b(\rho_{0,\delta}) \, dx - \int_\Omega b(\rho_\varepsilon) \, dx. \tag{3.47}$$

Let $b_n(z) \in C_c^\infty(\mathbb{R})$ satisfying

$$b_n(z) = \begin{cases} z \log z, \ |z| \leq n, \\ (n+1) \log(n+1), \ |z| > n+1. \end{cases} \tag{3.48}$$

By Lebesgue convergence theorem, we are able to take $b(z) = z \log z$, which gives

$$\int_0^T \int_\Omega \rho_\varepsilon \operatorname{div} u_\varepsilon \, dxdt \leq \int_\Omega \rho_{0,\delta} \log \rho_\varepsilon \, dx - \int_\Omega \rho_\varepsilon \log \rho_\varepsilon \, dx. \tag{3.49}$$

Passing to the limit as $\varepsilon \to 0^+$, we arrive at

$$\int_0^T \int_\Omega \overline{\rho \operatorname{div} u} \, dxdt \leq \int_\Omega \rho_{0,\delta} \log \rho_{0,\delta} \, dx - \int_\Omega \overline{\rho \log \rho} \, dx. \tag{3.50}$$

Similarly, since (ρ, u) satisfies

$$\rho_t + \operatorname{div}(\rho u) = 0 \tag{3.51}$$

in the renormalized sense, we take $b(z) = z \log z$ and get

$$\int_0^T \int_\Omega \rho \operatorname{div} u \, dxdt = \int_\Omega \rho_{0,\delta} \log \rho_{0,\delta} \, dx - \int_\Omega \rho \log \rho \, dx. \tag{3.52}$$

From (3.50) and (3.52), we deduce that

$$\int_{\Omega} \overline{\rho \log \rho} - \rho \log \rho(\tau) \, dx \le \int_0^T \int_{\Omega} \rho \mathrm{div} u - \overline{\rho \mathrm{div} u} \, dx dt, \tag{3.53}$$

for any $\tau \in [0, T]$.

In order to show the strong convergence of the density, we need additional regularity related to (ρ, u). It comes from the momentum equation and is now well known as the effective viscous flux, i.e.

$$H_\varepsilon := \rho_\varepsilon^\gamma + \delta(\rho_\varepsilon^2 + \rho_\varepsilon^\beta) - (2\nu + \lambda)\mathrm{div} u_\varepsilon, \tag{3.54}$$

and its limit function

$$\overline{H} = \overline{P} - (2\nu + \lambda)\mathrm{div} u, \tag{3.55}$$

where $\overline{P} = \overline{\rho^\gamma + \delta(\rho^2 + \rho^\beta)}$.

Lemma 3.2. *Let $(\rho_\varepsilon, u_\varepsilon, \chi_\varepsilon)$ be the sequences of the problem (2.1), (2.2), (2.3) and (ρ, u, χ) be their weak limit, then*

$$\lim_{\varepsilon \to 0+} \int_0^T \int_{\Omega} \psi(t)\phi(x) H_\varepsilon \rho_\varepsilon \, dx dt = \int_0^T \int_{\Omega} \psi(t)\phi(x)\overline{H}\rho \, dx dt, \tag{3.56}$$

where $\psi(t) \in \mathcal{D}(0, T)$, $\phi(x) \in \mathcal{D}(\Omega)$.

Proof. The trick of the proof is to introduce the operator

$$\mathcal{A}[v] = \Delta^{-1}\nabla[v], \tag{3.57}$$

where Δ^{-1} represents the inverse of the Laplace operator on $\mathbb{R}^3$. The operator $\mathcal{A}$ has some good properties as follows:

$$\begin{aligned}
&\mathcal{A}_j = \mathcal{F}^{-1}(-\frac{-i\xi_j}{|\xi|^2}), \ \mathrm{div}\mathcal{A}[v] = v; \\
&\|\mathcal{A}_i v\|_{W^{[1,s]}(\Omega)} \le c(s, \Omega)\|v\|_{L^s(R^3)} \quad \text{for} \quad \text{any} \quad 1 < s < \infty; \\
&\|\mathcal{A}_i v\|_{L^p(\Omega)} \le c(p, s, \Omega)\|v\|_{L^s(R^3)}, \quad p \quad \text{finite}, \quad \text{provided} \quad \frac{1}{p} \ge \frac{1}{s} - \frac{1}{3}; \\
&\|\mathcal{A}_i v\|_{L^\infty(\Omega)} \le c(s, \Omega)\|v\|_{L^s(R^3)} \quad \text{for} \quad s > 3.
\end{aligned} \tag{3.58}$$

Choosing

$$\Psi(x, t) = \psi(t)\phi(x)\mathcal{A}[1_\Omega \rho_\varepsilon], \quad \text{where} \quad \psi \in \mathcal{D}(0, T), \quad \phi \in \mathcal{D}(\Omega),$$

as a test function in the momentum equation (2.1), we have

$$
\begin{aligned}
&\int_0^T \psi \int_\Omega \phi H_\varepsilon \rho_\varepsilon dxdt \\
&= \int_0^T \partial_t \psi \int_\Omega \phi \rho_\varepsilon u_\varepsilon \mathcal{A}[1_\Omega \rho_\varepsilon]\ dxdt + \int_0^T \psi \int_\Omega \phi \rho_\varepsilon u_\varepsilon \mathcal{A}[\operatorname{div}(\rho_\varepsilon u_\varepsilon)]\ dxdt \\
&\quad - \int_0^T \psi \int_\Omega (\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) : (\nabla \phi \otimes \mathcal{A}[1_\Omega \rho_\varepsilon])\ dxdt - \int_0^T \psi \int_\Omega \phi \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla \mathcal{A}[1_\Omega \rho_\varepsilon]\ dxdt \\
&\quad - \int_0^T \psi \int_\Omega (\rho_\varepsilon^\gamma + \delta(\rho_\varepsilon^2 + \rho_\varepsilon^\beta))(\nabla \phi \cdot \mathcal{A}[1_\Omega \rho_\varepsilon])\ dxdt + \int_0^T \psi \int_\Omega \nabla u_\varepsilon \nabla \phi \otimes \mathcal{A}[1_\Omega \rho_\varepsilon]\ dxdt \\
&\quad + \int_0^T \psi \int_\Omega \phi \nabla u_\varepsilon \cdot \mathcal{A}[1_\Omega \rho_\varepsilon]\ dxdt + \int_0^T \psi \int_\Omega \operatorname{div} u_\varepsilon \nabla \phi \cdot \mathcal{A}[1_\Omega \rho_\varepsilon]\ dxdt \\
&\quad - \int_0^T \psi \int_\Omega \left[(\nabla \chi_\varepsilon \otimes \nabla \chi_\varepsilon - \frac{|\nabla \chi_\varepsilon|^2}{2} \mathbb{I}) \nabla \phi \right] \cdot \mathcal{A}[1_\Omega \rho_\varepsilon]\ dxdt \\
&\quad - \int_0^T \psi \int_\Omega \phi (\nabla \chi_\varepsilon \otimes \nabla \chi_\varepsilon - \frac{|\nabla \chi_\varepsilon|^2}{2} \mathbb{I}) : \nabla \mathcal{A}[1_\Omega \rho_\varepsilon]\ dxdt \\
&\quad - \varepsilon \int_0^T \psi \int_\Omega \nabla \rho_\varepsilon \cdot \nabla u_\varepsilon \cdot \mathcal{A}[1_\Omega \rho_\varepsilon]\ dxdt - \varepsilon \int_0^T \psi \int_\Omega \rho_\varepsilon u_\varepsilon \cdot \mathcal{A}[\operatorname{div}(1_\Omega \nabla \rho_\varepsilon)]\ dxdt = \sum_{j=1}^{12} I_j. \qquad (3.59)
\end{aligned}
$$

On the other hand, we use $\psi(t)\phi(x)\mathcal{A}[1_\Omega \rho]$ as a test function for $(3.45)_2$ to obtain:

$$
\begin{aligned}
&\int_0^T \psi \int_\Omega \phi \overline{H} \rho\ dxdt \\
&= \int_0^T \partial_t \psi \int_\Omega \phi \rho u \mathcal{A}[1_\Omega \rho]\ dxdt + \int_0^T \psi \int_\Omega \phi \rho u \mathcal{A}[\operatorname{div}(\rho u)]\ dxdt \\
&\quad - \int_0^T \psi \int_\Omega (\rho u \otimes u) : (\nabla \phi \otimes \mathcal{A}[1_\Omega \rho])\ dxdt - \int_0^T \psi \int_\Omega \phi \rho u \otimes u : \nabla \mathcal{A}[1_\Omega \rho]\ dxdt \\
&\quad - \int_0^T \psi \int_\Omega (\overline{\rho^\gamma + \delta(\rho^2 + \rho^\beta)})(\nabla \phi \cdot \mathcal{A}[1_\Omega \rho])\ dxdt + \int_0^T \psi \int_\Omega \nabla u \nabla \phi \otimes \mathcal{A}[1_\Omega \rho]\ dxdt \\
&\quad + \int_0^T \psi \int_\Omega \phi \nabla u \cdot \mathcal{A}[1_\Omega \rho]\ dxdt + \int_0^T \psi \int_\Omega \operatorname{div} u \nabla \phi \cdot \mathcal{A}[1_\Omega \rho]\ dxdt \\
&\quad - \int_0^T \psi \int_\Omega \left[(\nabla \chi \otimes \nabla \chi - \frac{|\nabla \chi|^2}{2} \mathbb{I}) \nabla \phi \right] \cdot \mathcal{A}[1_\Omega \rho]\ dxdt
\end{aligned}
$$

$$-\int_0^T \psi \int_\Omega \phi(\nabla\chi \otimes \nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I}) : \nabla\mathcal{A}[1_\Omega \rho]\, dxdt = \sum_{j=1}^{10} I'_j. \tag{3.60}$$

It is easy to deduce from (3.8), (3.59) that

$$\mathcal{A}_i[\rho_\varepsilon] \to \mathcal{A}_i[\rho] \ in\ C(\overline{(0,T)\times\Omega}) \quad \text{and} \quad \partial_j\mathcal{A}_i[\rho_\varepsilon] \to \partial_j\mathcal{A}_i[\rho] \quad \text{in} \quad C([0,T]; L^\beta_{weak}(\Omega)). \tag{3.61}$$

Following the same analysis of [7], we observe that the I_j converge to I'_j for $j \le 8$ and I_{11}, I_{12} tend to zero. It remains to show I_9, I_{10} converge to I'_9, I'_{10} respectively. Thanks to (3.9), (3.26), (3.31), and the imbedding $W^{2,\frac{2\beta}{2+\beta}} \hookrightarrow W^{1,\frac{6\beta}{6+\beta}}$, we have

$$\nabla\chi_\varepsilon \otimes \nabla\chi_\varepsilon \to \nabla\chi \otimes \nabla\chi \quad \text{strongly} \quad \text{in} \quad L^2(0,T; L^{\frac{3\beta}{3+2\beta}}(\Omega)), \tag{3.62}$$

$$|\nabla\chi_\varepsilon|^2 \to |\nabla\chi|^2 \quad \text{strongly} \quad \text{in} \quad L^2(0,T; L^{\frac{3\beta}{3+2\beta}}(\Omega)). \tag{3.63}$$

Combining with (3.61), one obtains

$$\begin{aligned} &\int_0^T \psi \int_\Omega \left[(\nabla\chi_\varepsilon \otimes \nabla\chi_\varepsilon - \frac{|\nabla\chi_\varepsilon|^2}{2}\mathbb{I})\nabla\phi\right] \cdot \mathcal{A}[1_\Omega\rho_\varepsilon]\, dxdt \\ &\to \int_0^T \psi \int_\Omega \left[(\nabla\chi \otimes \nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I})\nabla\phi\right] \cdot \mathcal{A}[1_\Omega\rho]\, dxdt \quad \text{as} \quad \varepsilon \to 0, \end{aligned} \tag{3.64}$$

$$\begin{aligned} &\int_0^T \psi \int_\Omega \phi(\nabla\chi_\varepsilon \otimes \nabla\chi_\varepsilon - \frac{|\nabla\chi_\varepsilon|^2}{2}\mathbb{I}) : \nabla\mathcal{A}[1_\Omega\rho_\varepsilon]\, dxdt \\ &\to \int_0^T \psi \int_\Omega \phi(\nabla\chi \otimes \nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I}) : \nabla\mathcal{A}[1_\Omega\rho]\, dxdt \quad \text{as} \quad \varepsilon \to 0. \end{aligned} \tag{3.65}$$

Taking limit in (3.59), (3.60), we complete the proof. □

Making use of Lemma 3.2 and the monotonicity and convexity of $\rho^\gamma + \delta(\rho^2 + \rho^\beta)$, one has

$$\int_0^T \int_\Omega \rho \mathrm{div} u - \overline{\rho \mathrm{div} u}\, dxdt \le \frac{1}{2\nu + \lambda} \int_0^T \int_\Omega (\overline{P}\rho - \overline{\rho^{\gamma+1} + \delta(\rho^3 + \rho^{\beta+1})})dxdt \le 0. \tag{3.66}$$

With the help of (3.53), we obtain

$$\int_\Omega \overline{\rho\log\rho} - \rho\log\rho\, dx \le 0. \tag{3.67}$$

On the other hand, since

$$\overline{\rho\log\rho} \ge \rho\log\rho, \tag{3.68}$$

due to the convexity of $z \mapsto z\log z$, we deduce that

$$\overline{\rho\log\rho} = \rho\log\rho \quad \text{a.e. in } (0,T)\times\Omega, \tag{3.69}$$

which implies

$$\rho_\varepsilon \to \rho \quad \text{strongly in } L^1((0,T)\times\Omega). \tag{3.70}$$

With the help of (3.8), (3.11) and (3.70), one obtains

$$\rho_\varepsilon\mu_\varepsilon \to \rho\mu \quad \text{weakly} \quad \text{in} \quad L^2(0,T;L^{\frac{2\beta}{2+\beta}}(\Omega)). \tag{3.71}$$

Letting $\varepsilon \to 0$, we have the following result.

Proposition 3.1. *Let $\beta > max\{\frac{9}{2}, \gamma, \frac{6\gamma}{2\gamma-3}\}$. Then for any fixed $\delta > 0$, there exists a weak solution (ρ, u, χ, μ) of the problem*

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u\otimes u) + \nabla\rho^\gamma + \delta\nabla(\rho^2+\rho^\beta) = \nu\Delta u + (\nu+\lambda)\nabla\operatorname{div}u - \operatorname{div}(\nabla\chi\otimes\nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I}), \\ (\rho\chi)_t + \operatorname{div}(\rho\chi u) = -\mu, \\ \rho\mu = -\Delta\chi + \rho(\chi^3-\chi), \end{cases} \tag{3.72}$$

satisfying the initial and boundary conditions

$$(\rho, \rho u, \chi)|_{t=0} = (\rho_{0,\delta}, m_{0,\delta}, \chi_{0,\delta}) \quad \text{on} \quad \bar{\Omega}, \tag{3.73}$$

$$u|_{\partial\Omega} = 0, \quad \nabla\chi\cdot n|_{\partial\Omega} = 0. \tag{3.74}$$

Moreover, the energy inequality

$$\frac{d}{dt}\int\limits_\Omega \left[\frac{1}{2}\rho|u|^2 + \frac{1}{\gamma-1}\rho^\gamma + \delta\left(\rho^2 + \frac{1}{\beta-1}\rho^\beta\right) + \frac{1}{2}|\nabla\chi|^2 + \frac{1}{2}\rho(\chi^2-1)^2\right] dx$$
$$+\int\limits_\Omega \mu^2\, dx + \int\limits_\Omega \left[\nu|\nabla u|^2 + (\nu+\lambda)|\operatorname{div}u|^2\right] dx \le 0, \tag{3.75}$$

holds in $\mathcal{D}'(0,T)$.

Finally, the following estimates

$$\sup_{t\in[0,T]} \|\sqrt{\rho(t)}u(t)\|^2_{L^2(\Omega)} \le CE_\delta[\rho_0, m_0, \chi_0], \tag{3.76}$$

$$\sup_{t\in[0,T]} \|\rho(t)\|^\gamma_{L^\gamma(\Omega)} \le CE_\delta[\rho_0, m_0, \chi_0], \tag{3.77}$$

$$\delta \sup_{t\in[0,T]} \|\rho(t)\|^\beta_{L^\beta(\Omega)} \le CE_\delta[\rho_0, m_0, \chi_0], \tag{3.78}$$

$$\sup_{t\in[0,T]} \|\nabla\chi(t)\|^2_{L^2(\Omega)} \le CE_\delta[\rho_0, m_0, \chi_0], \tag{3.79}$$

$$\|u\|_{L^2(0,T;H^1_0(\Omega))} \le CE_\delta[\rho_0, m_0, \chi_0], \tag{3.80}$$

$$\|\mu\|_{L^2((0,T)\times\Omega)} \le CE_\delta[\rho_0, m_0, \chi_0], \tag{3.81}$$

hold, where the constant C is independent of δ.

4. The vanishing artificial pressure limit

In this section, we will pass the limit as $\delta \to 0^+$ and complete the proof of Theorem 1.1. To relax the constraint of the adiabatic constant γ as much as possible, it still needs to improve the integrability of the density. Note that the case for $\gamma > 2$ is not difficulty for compressible Navier-Stokes equations for isentropic flow nowadays. But for the compressible Navier-Stokes/Allen-Cahn system, the stronger nonlinearity and the poor regularity of the function χ require us to use the cut-off function. It is mainly due to the convergence in effective viscous flux, see Remark 4.1.

4.1. Higher integrability of the density

Similar to Section 3.1, we shall choose a suitable function to test the pressure term and get better estimate for the density. The difference comes from the continuity equation $(3.72)_1$, which only holds in $\mathcal{D}'((0,T)\times\Omega)$. We have to regularize its renormalized equation (for better γ) to obtain a new equation, which holds almost everywhere, i.e.

$$\partial_t S_m[b(\rho_\delta)] + \operatorname{div}(S_m[b(\rho_\delta)]u_\delta) + S_m\left[(b(\rho_\delta) - b'(\rho_\delta)\rho_\delta)\operatorname{div}u_\delta\right] = r_m, \quad \text{in } (0,T)\times\mathbb{R}^3, \tag{4.1}$$

$$r_m \to 0 \quad \text{in} \quad L^\alpha((0,T)\times\mathbb{R}^3) \quad \text{for} \quad \text{all} \quad \alpha\in[1,2), \tag{4.2}$$

where $S_m[f] = \eta_m * f$ is the standard regularized operator and η_m is the modified kernel (see details in [7]). Therefore, we construct the identity

$$\Phi(t,x) = \psi(t)\mathcal{B}\left[S_m[b(\rho_\delta)] - \fint_\Omega S_m[b(\rho_\delta)]\,dx\right], \quad \psi(t)\in C_0^\infty(0,T), \quad 0\le\psi(t)\le 1,$$

$$\fint_\Omega S_m[b(\rho_\delta)]\,dx = \frac{1}{|\Omega|}\int_\Omega S_m[b(\rho_\delta)]\,dx,$$

as a test function for the momentum equation, which leads to the following result.

Lemma 4.1. *Let $\gamma > 2$, $(\rho_\delta, u_\delta, \chi_\delta)$ be the sequences of the problem (3.72)-(3.74), then*

$$\int_0^T\int_\Omega \left\{\rho_\delta^{\gamma+\theta} + \delta(\rho_\delta^{2+\theta} + \rho_\delta^{\beta+\theta})\right\} dxdt \le C, \tag{4.3}$$

where $\theta = \min\{1, \frac{\gamma}{2} - 1\}$.

Proof. Taking $\Phi(t,x)$ as a test function for $(3.72)_2$, integrating over $(0,T)\times\Omega$ and integrating by parts, one arrives at

$$\begin{aligned}&\int_0^T \psi\int_\Omega (\rho_\delta^\gamma + \delta(\rho_\delta^2 + \rho_\delta^\beta))S_m[b(\rho_\delta)]\,dxdt\\ =&(2\nu+\lambda)\int_0^T\psi\int_\Omega S_m[b(\rho_\delta)]\operatorname{div}u_\delta\,dxdt\end{aligned}$$

$$+\int_0^T \psi\left(\int_\Omega \rho_\delta^\gamma+\delta(\rho_\delta^2+\rho_\delta^\beta)\ dx\right)\left(\fint_\Omega S_m\left[b(\rho_\delta)\right]\ dx\right)dt$$
$$-\int_0^T \psi_t\int_\Omega \rho_\delta u_\delta\cdot\mathcal{B}\left(S_m[b(\rho_\delta)]-\fint_\Omega S_m[b(\rho_\delta)]\ dx\right)dxdt$$
$$+\int_0^T \psi\int_\Omega (\nu\nabla u_\delta-\rho_\delta u_\delta\otimes u_\delta):\nabla\mathcal{B}\left(S_m[b(\rho_\delta)]-\fint_\Omega S_m[b(\rho_\delta)]\ dx\right)dxdt$$
$$+\int_0^T \psi\int_\Omega \rho_\delta u_\delta\cdot\mathcal{B}\left(S_m\left[(b(\rho_\delta)-b'(\rho_\delta)\rho_\delta)\mathrm{div}u_\delta\right]-\fint_\Omega S_m\left[(b(\rho_\delta)-b'(\rho_\delta)\rho_\delta)\mathrm{div}u_\delta\right]\ dx\right)dxdt$$
$$+\int_0^T \psi\int_\Omega \rho_\delta u_\delta\cdot\mathcal{B}\left(r_m-\fint_\Omega r_m\ dx\right)dxdt-\int_0^T \psi\int_\Omega \rho_\delta u_\delta\cdot\mathcal{B}\left(\mathrm{div}\left(S_m[b(\rho_\delta)]u_\delta\right)\right)\ dxdt$$
$$-\int_0^T \psi\int_\Omega \left(\nabla\chi_\delta\otimes\nabla\chi_\delta-\frac{1}{2}|\nabla\chi_\delta|^2\mathbb{I}\right):\nabla\mathcal{B}\left(S_m[b(\rho_\delta)]-\fint_\Omega S_m[b(\rho_\delta)]\ dx\right)dxdt. \tag{4.4}$$

With the help of (4.2), we are able to pass the limit $m\to\infty$ and take $b(\rho_\delta)=\rho_\delta^\theta$ to obtain

$$\int_0^T \psi\int_\Omega \rho_\delta^{\gamma+\theta}+\delta(\rho_\delta^{2+\theta}+\rho_\delta^{\beta+\theta})\ dxdt$$
$$=\int_0^T \psi\left(\int_\Omega \rho_\delta^\gamma+\delta(\rho_\delta^2+\rho_\delta^\beta)\ dx\right)\left(\fint_\Omega \rho_\delta^\theta\ dx\right)\ dt-\int_0^T \psi_t\int_\Omega \rho_\delta u_\delta\cdot\mathcal{B}\left(\rho_\delta^\theta-\fint_\Omega \rho_\delta^\theta\ dx\right)\ dxdt$$
$$+\int_0^T \psi\int_\Omega \nu\nabla u_\delta:\nabla\mathcal{B}\left(\rho_\delta^\theta-\fint_\Omega \rho_\delta^\theta\ dx\right)\ dxdt-\int_0^T \psi\int_\Omega \rho_\delta u_\delta\otimes u_\delta:\nabla\mathcal{B}\left(\rho_\delta^\theta-\fint_\Omega \rho_\delta^\theta\ dx\right)\ dxdt$$
$$+(1-\theta)\int_0^T \psi\int_\Omega \rho_\delta u_\delta\cdot\mathcal{B}\left(\rho_\delta^\theta\mathrm{div}u-\fint_\Omega \rho_\delta^\theta\mathrm{div}u\ dx\right)\ dxdt-\int_0^T \psi\int_\Omega \rho_\delta u_\delta\cdot\mathcal{B}\left(\mathrm{div}(\rho_\delta^\theta u_\delta)\right)\ dxdt$$
$$-\int_0^T \psi\int_\Omega \left(\nabla\chi_\delta\otimes\nabla\chi_\delta-\frac{1}{2}|\nabla\chi_\delta|^2\mathbb{I}\right):\nabla\mathcal{B}\left(\rho_\delta^\theta-\fint_\Omega \rho_\delta^\theta\ dx\right)\ dxdt=\sum_{k=1}^7 J_k. \tag{4.5}$$

The estimates of J_1-J_7 are treated as follows:

(1)

$$\begin{aligned}|J_1|=&\left|\int_0^T \psi\left(\int_\Omega \rho_\delta^\gamma+\delta(\rho_\delta^2+\rho_\delta^\beta)\ dx\right)\left(\fint_\Omega \rho_\delta^\theta\ dx\right)\ dt\right|\\ \le& C\|\rho_\delta\|_{L^\infty(0,T;L^\gamma(\Omega))}\int_0^T\int_\Omega \rho_\delta^\gamma+\delta(\rho_\delta^2+\rho_\delta^\beta)dxdt\\ \le& C(\rho_0,m_0,\chi_0,T),\end{aligned}$$

provided $\theta \le \gamma$.

(2)

$$
\begin{aligned}
|J_2| =& \Big| \int_0^T \psi_t \int_\Omega \rho_\delta u_\delta \cdot \mathcal{B}\left(\rho_\delta^\theta - \fint_\Omega \rho_\delta^\theta \, dx\right) \, dxdt \Big| \\
\le& C \int_0^T |\psi_t| \|\sqrt{\rho_\delta}\|_{L^2(\Omega)} \|\sqrt{\rho_\delta} u_\delta\|_{L^2(\Omega)} \|\mathcal{B}\left(\rho_\delta^\theta - \fint_\Omega \rho_\delta^\theta \, dx\right)\|_{L^\infty(\Omega)} dt \\
\le& C \int_0^T |\psi_t| \|\rho_\delta^\theta - \fint_\Omega \rho_\delta^\theta \, dx\|_{L^{\frac{\gamma}{\theta}}(\Omega)} dt \\
\le& C \int_0^T |\psi_t| dt
\end{aligned}
$$

provided $\theta \le \frac{\gamma}{3}$.

(3)

$$
\begin{aligned}
|J_3| =& \Big| \int_0^T \psi \int_\Omega \nu \nabla u_\delta : \nabla \mathcal{B}\left(\rho_\delta^\theta - \fint_\Omega \rho_\delta^\theta \, dx\right) dxdt \Big| \\
\le& C \int_0^T \|\nabla u_\delta\|_{L^2(\Omega)} \|\nabla \mathcal{B}\left(\rho_\delta^\theta - \fint_\Omega \rho_\delta^\theta \, dx\right)\|_{L^2(\Omega)} \, dt \\
\le& C \int_0^T \|\nabla u_\delta\|_{L^2} \|\rho_\delta^\theta - \fint_\Omega \rho_\delta^\theta \, dx\|_{L^2(\Omega)} dt
\end{aligned}
$$

provided $\theta \le \frac{\gamma}{2}$.

(4)

$$
\begin{aligned}
|J_4| =& \Big| \int_0^T \psi \int_\Omega \rho_\delta u_\delta \otimes u_\delta : \nabla \mathcal{B}\left(\rho_\delta^\theta - \fint_\Omega \rho_\delta^\theta \, dx\right) \, dxdt \Big| \\
\le& C \int_0^T \|\rho_\delta\|_{L^\gamma(\Omega)} \|u_\delta\|_{L^6(\Omega)}^2 \|\nabla \mathcal{B}\left(\rho_\delta^\theta - \fint_\Omega \rho_\delta^\theta \, dx\right)\|_{L^{\frac{3\gamma}{2\gamma-3}}(\Omega)} dt \\
\le& C \int_0^T \|u_\delta\|_{L^6(\Omega)}^2 \|\rho_\delta^\theta - \fint_\Omega \rho_\delta^\theta \, dx\|_{L^{\frac{3\gamma}{2\gamma-3}}(\Omega)} dt
\end{aligned}
$$

provided $\theta \le \frac{2\gamma}{3} - 1$.

(5)

$$
|J_5| = \Big| (1-\theta) \int_0^T \psi \int_\Omega \rho_\delta u_\delta \cdot \mathcal{B}\left(\rho_\delta^\theta \mathrm{div} u - \fint \rho_\delta^\theta \mathrm{div} u \, dx\right) \, dxdt \Big|
$$

$$
\begin{aligned}
&\leq C\int_0^T \|\rho_\delta\|_{L^\gamma(\Omega)}\|u_\delta\|_{L^6(\Omega)}\|\mathcal{B}\left(\rho_\delta^\theta \mathrm{div}u - \oint \rho_\delta^\theta \mathrm{div}u\ dx\right)\|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega)}dt\\
&\leq C\int_0^T \|u_\delta\|_{H^1}\|\mathcal{B}\left(\rho_\delta^\theta \mathrm{div}u - \oint \rho_\delta^\theta \mathrm{div}u\ dx\right)\|_{W^{1,s}(\Omega)}dt,
\end{aligned}
$$

where

$$
s = \begin{cases} \dfrac{6\gamma}{7\gamma-6}, & \text{if } \gamma < 6;\\ \dfrac{3}{2}, & \text{if } \gamma \geq 6, \end{cases}
$$

and provided $\theta \leq \frac{2\gamma}{3} - 1$, $\theta \leq 1$.

(6)

$$
\begin{aligned}
|J_6| =& \left|\int_0^T \psi \int_\Omega \rho_\delta u_\delta \cdot \mathcal{B}\left(\mathrm{div}(\rho_\delta^\theta u_\delta)\right)\ dxdt\right|\\
\leq& C\int_0^T \|\rho_\delta\|_{L^\gamma(\Omega)}\|u_\delta\|_{L^6(\Omega)}\|\mathcal{B}\left(\mathrm{div}(\rho_\delta^\theta u_\delta)\right)\|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega)}dt\\
\leq& C\int_0^T \|u_\delta\|^2_{L^6(\Omega)}\|\rho_\delta^\theta\|_{L^{\frac{3\gamma}{2\gamma-3}}(\Omega)}dt
\end{aligned}
\tag{4.6}
$$

provided $\theta \leq \frac{2\gamma}{3} - 1$.

(7)

$$
\begin{aligned}
|J_7| =& \left|\int_0^T \psi \int_\Omega \left(\nabla\chi_\delta \otimes \nabla\chi_\delta - \frac{1}{2}|\nabla\chi_\delta|^2\mathbb{I}\right) : \nabla\mathcal{B}\left(\rho_\delta^\theta - \oint_\Omega \rho_\delta^\theta\ dx\right) dxdt\right|\\
\leq& C\int_0^T \||\nabla\chi_\delta|^2\|_{L^{\frac{\gamma}{\gamma-\theta}}(\Omega)}\left\|\nabla\mathcal{B}\left(\rho_\delta^\theta - \oint_\Omega \rho_\delta^\theta\ dx\right)\right\|_{L^{\frac{\gamma}{\theta}}(\Omega)} dt\\
\leq& C\int_0^T \||\nabla\chi_\delta|^2\|_{L^{\frac{\gamma}{\gamma-\theta}}(\Omega)}\|\rho_\delta^\theta\|_{L^{\frac{\gamma}{\theta}}(\Omega)}\ dt\\
\leq& C\int_0^T \|\nabla\chi_\delta\|^2_{L^{\frac{2\gamma}{\gamma-\theta}}(\Omega)}dt\\
\leq& C\int_0^T \left(\|\chi_\delta\|_{W^{2,\frac{2\gamma}{2+\gamma}}(\Omega)}\|\chi_\delta\|_{L^\infty(\Omega)} + C\|\chi_\delta\|_{L^\infty(\Omega)}\right)dt\\
\leq& C\int_0^T \|\rho_\delta\|_{L^\gamma(\Omega)}(1 + \|\mu_\delta\|_{L^2(\Omega)})dt.
\end{aligned}
\tag{4.7}
$$

Here we have used the Gagliardo-Nirenberg inequality and the condition

$$\frac{2\gamma}{\gamma-\theta} \leq \frac{4\gamma}{2+\gamma},$$

holds if $\theta \leq \frac{\gamma}{2} - 1$.

The proof is completed. $\square$

4.2. The limit passage

Due to the energy inequality, Lemma 4.1 and (2.11), letting $\delta \to 0^+$(take the subsequence if necessary), we have

$$\delta(\rho_\delta^2 + \rho_\delta^\beta) \to 0 \quad \text{in } L^1((0,T)\times\Omega), \tag{4.8}$$

$$\rho_\delta \to \rho \quad \text{in} \quad C([0,T]; L^\gamma_{weak}(\Omega)) \quad \text{and} \quad \text{weakly–}(^*) \quad \text{in } L^\infty(0,T;L^\gamma(\Omega)), \tag{4.9}$$

$$\chi_\delta \to \chi \quad \text{weakly–}(^*) \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)) \cap L^\infty((0,T)\times\Omega), \tag{4.10}$$

$$u_\delta \to u \quad \text{weakly} \quad \text{in} \quad L^2(0,T;H_0^1(\Omega)), \tag{4.11}$$

$$\mu_\delta \to \mu \quad \text{weakly} \quad \text{in} \quad L^2((0,T)\times\Omega), \tag{4.12}$$

$$\rho_\delta^\gamma \to \overline{\rho^\gamma} \quad \text{weakly in } L^{\frac{\gamma+\theta}{\gamma}}((0,T)\times\Omega). \tag{4.13}$$

Moreover, by $(3.3)_2$, (3.76), (3.77) and (3.80), we deduce that

$$\rho_\delta u_\delta \to \rho u \quad \text{in} \quad C([0,T]; L^{\frac{2\gamma}{2+\gamma}}_{weak}(\Omega)) \quad \text{and} \quad \text{weakly–}(^*) \quad \text{in} \quad L^\infty(0,T;L^\gamma(\Omega)), \tag{4.14}$$

which combines (4.11) and the compactness of $L^{\frac{2\gamma}{2+\gamma}}(\Omega) \hookrightarrow H^{-1}(\Omega)$, $\gamma > \frac{3}{2}$ and yields

$$\rho_\delta(u_\delta \otimes u_\delta) \to \rho(u\otimes u) \quad \text{in} \quad \mathcal{D}'((0,T)\times\Omega). \tag{4.15}$$

On the other hand, similar to the arguments in Subsection 3.2, one obtains

$$\rho_\delta\chi_\delta \to \rho\chi \quad \text{in} \quad C([0,T]; L^\gamma_{weak}(\Omega)), \tag{4.16}$$

$$\rho_\delta\chi_\delta u_\delta \to \rho\chi u \quad \text{weakly} \quad \text{in} \quad L^2(0,T;L^{\frac{6\gamma}{6+\gamma}}(\Omega)), \tag{4.17}$$

$$\Delta\chi_\delta \to \Delta\chi \quad \text{weakly} \quad \text{in} \quad L^2(0,T;L^{\frac{2\gamma}{2+\gamma}}(\Omega)), \tag{4.18}$$

$$\chi_\delta \to \chi \quad \text{strongly} \quad \text{in} \quad L^2(0,T;H^1(\Omega)), \tag{4.19}$$

$$\chi_\delta^3 - \chi_\delta \to \chi^3 - \chi \quad \text{strongly} \quad \text{in} \quad L^p((0,T)\times\Omega) \quad \text{for} \quad \text{all} \quad p\in[1,\infty), \tag{4.20}$$

$$\rho_\delta\mu_\delta \to \overline{\rho\mu} \quad \text{weakly} \quad \text{in} \quad L^2(0,T;L^{\frac{2\gamma}{2+\gamma}}(\Omega)). \tag{4.21}$$

Consequently, letting $\delta \to 0^+$, the limit of $(\rho_\delta, u_\delta, \chi_\delta, \mu_\delta)$ satisfies the system

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u\otimes u) + \nabla\overline{\rho^\gamma} = \nu\Delta u + (\nu+\lambda)\nabla \operatorname{div} u - \operatorname{div}(\nabla\chi\otimes\nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I}), \\ (\rho\chi)_t + \operatorname{div}(\rho\chi u) = -\mu, \\ \overline{\rho\mu} = -\Delta\chi + \rho(\chi^3-\chi), \end{cases} \tag{4.22}$$

in $\mathcal{D}'((0,T)\times\Omega)$.

4.3. Strong convergence of the density

It remains to show the strong convergence of ρ_δ in $L^1((0,T)\times\Omega)$ and then we are able to prove $\overline{\rho^\gamma}=\rho^\gamma$ and $\overline{\rho\mu}=\rho\mu$. Firstly, we denote the cut-off functions $T_k(z)=kT(\frac{z}{k})$ for $z\in\mathbb{R}$, $k=1,2,3,\cdots$ and $T\in C^\infty(\mathbb{R})$ is a concave function satisfying

$$T(z)=\begin{cases} z, & z\le 1,\\ 2, & z\ge 3.\end{cases}$$

It follows that

$$\overline{T_k(\rho)}\to\rho \quad\text{in}\quad L^p((0,T)\times\Omega)\quad\text{for}\quad\text{any}\quad 1\le p<\gamma+\theta,\quad\text{as}\quad k\to\infty, \tag{4.23}$$

since

$$\|\overline{T_k(\rho)}-\rho\|_{L^p((0,T)\times\Omega)}\le\liminf_{\delta\to 0}\|T_k(\rho_\delta)-\rho_\delta\|_{L^p((0,T)\times\Omega)},$$

and

$$\|T_k(\rho_\delta)-\rho_\delta\|^p_{L^p((0,T)\times\Omega)}\le 2^pk^{p-\gamma-\theta}\|\rho_\delta\|^{\gamma+\theta}_{L^{\gamma+\theta}((0,T)\times\Omega)}\le C2^pk^{p-\gamma-\theta}.$$

As ρ_δ,u_δ is a renormalized solution of the continuity equation in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$, we have

$$\partial_t T_k(\rho_\delta)+\operatorname{div}(T_k(\rho_\delta)u_\delta)+(T_k'(\rho_\delta)\rho_\delta-T_k(\rho_\delta))\operatorname{div}u_\delta=0\quad\text{in}\quad\mathcal{D}'((0,T)\times\mathbb{R}^3).$$

With the help of (4.9), (4.11), (4.14), one can pass to the limit $\delta\to 0$ to deduce that

$$\partial_t\overline{T_k(\rho)}+\operatorname{div}(\overline{T_k(\rho)}u)+\overline{(T_k'(\rho)\rho-T_k(\rho))\operatorname{div}u}=0\quad\text{in}\quad\mathcal{D}'((0,T)\times\mathbb{R}^3),$$

where

$$\begin{aligned} &T_k(\rho_\delta)\to\overline{T_k(\rho)}\quad\text{in}\quad C([0,T];L^p_{weak}(\Omega))\quad\text{for}\quad\text{in}\quad 1\le p<\infty,\\ &(T_k'(\rho_\delta)\rho_\delta-T_k(\rho_\delta))\operatorname{div}u_\delta\to\overline{(T_k'(\rho)\rho-T_k(\rho))\operatorname{div}u}\quad\text{weakly}\quad\text{in}\quad L^2((0,T)\times\Omega).\end{aligned}$$

Then we reach a situation to discuss the effective viscous flux in this level of approximation. The proof is similar to the previous sections and this time we use the function

$$\Psi(x,t)=\psi(t)\phi(x)\mathcal{A}[T_k(\rho_\delta)],\quad\text{where}\quad\psi\in\mathcal{D}(0,T),\quad\phi\in\mathcal{D}(\Omega),$$

as a test function for the momentum equation. Similar analysis leads the following auxiliary result.

Lemma 4.2. *Let (ρ_δ,u_δ) be the sequence of the approximate solution obtained in Proposition 3.1. Then*

$$\begin{aligned}&\lim_{\delta\to 0}\int_0^T\psi\int_\Omega\phi(\rho_\delta^\gamma-(2\nu+\lambda)\operatorname{div}u_\delta)T_k(\rho_\delta)\,dxdt\\ &=\int_0^T\psi\int_\Omega\phi(\overline{\rho^\gamma}-(2\nu+\lambda)\operatorname{div}u)\overline{T_k(\rho)}\,dxdt,\end{aligned}\tag{4.24}$$

for any $\psi\in\mathcal{D}(0,T)$, $\phi\in\mathcal{D}(\Omega)$.

Remark 4.1. Although $\gamma > 2$, we still need the cut-off function for instance to ensure the following convergence, i.e.,

$$\int_0^T \psi \int_\Omega \phi(\nabla\chi_\delta \otimes \nabla\chi_\delta - \frac{|\nabla\chi_\delta|^2}{2}\mathbb{I}) : \nabla\mathcal{A}[T_k(\rho_\delta)]\ dxdt$$
$$\to \int_0^T \psi \int_\Omega \phi(\nabla\chi \otimes \nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I}) : \nabla\mathcal{A}[\overline{T_k(\rho)}]\ dxdt \quad \text{as} \quad \delta \to 0,$$

since we only have

$$\nabla\chi_\delta \to \nabla\chi \quad \text{strongly in } L^2(0,T; L^{\frac{4\gamma}{2+\gamma}}(\Omega)),$$
$$\rho_\delta \to \rho \quad \text{weakly–}(^*) \quad \text{in } L^\infty(0,T; L^\gamma(\Omega)),$$

and

$$T_k(\rho_\delta) \to \overline{T_k(\rho)} \quad \text{weakly–}(^*) \quad \text{in } L^\infty((0,T)\times\Omega).$$

As in [7], we introduce a sequence of functions $L_k \in C^1(\mathbb{R})$:

$$L_k(z) = \begin{cases} z\log z, & 0 \le z < k; \\ z\log(k) + z\int_k^z \frac{T_k(s)}{s^2}ds, & z \ge k. \end{cases}$$

Let

$$\beta_k = \log k + \int_k^{3k} \frac{T_k(s)}{s^2}ds + \frac{2}{3},$$

we denote

$$b_k(z) = L_k(z) - \beta_k(z).$$

Then b_k belongs to $C^1(0,\infty) \cap C[0,\infty)$, $b'(z) \equiv 0$ for all $z \in \mathbb{R}$ large enough with

$$b'(z)z - b(z) = T_k(z). \tag{4.25}$$

Therefore, we are able to deduce that

$$\partial_t L_k(\rho_\delta) + \mathrm{div}(L_k(\rho_\delta)u_\delta) + T_k(\rho_\delta)\mathrm{div}u_\delta = 0. \tag{4.26}$$

Since the limit functions ρ, u are already renormalized solutions of the continuity equation for $\gamma > 2$, we have

$$\partial_t L_k(\rho) + \mathrm{div}(L_k(\rho)u) + T_k(\rho)\mathrm{div}u = 0 \tag{4.27}$$

in $\mathcal{D}'((0,T)\times\Omega)$. By (4.9), one can assume

$$L_k(\rho_\delta) \to \overline{L_k(\rho)} \quad \text{in} \quad C([0,T]; L^\gamma_{weak}(\Omega)). \tag{4.28}$$

Taking the difference of (4.26) and (4.27), integrating with respect to time t, one obtains

$$\begin{aligned}&\int_{\Omega}(L_k(\rho_\delta)-L_k(\rho))\phi\, dx\\&=\int_0^T\int_{\Omega}(L_k(\rho_\delta)u_\delta-L_k(\rho)u)\cdot\nabla\phi+(T_k(\rho)\mathrm{div}u-T_k(\rho_\delta)\mathrm{div}u_\delta)\phi\, dxdt\end{aligned}\tag{4.29}$$

for any $\phi\in\mathcal{D}(\Omega)$. Following the argument in [7] and thanks to $u\in L^2(0,T;H_0^1(\Omega))$, we have

$$\begin{aligned}&\int_{\Omega}(\overline{L_k(\rho)}-L_k(\rho))(t)dx\\&=\int_0^T\int_{\Omega}T_k(\rho)\mathrm{div}u\, dxdt-\lim_{\delta\to 0}\int_0^T\int_{\Omega}T_k(\rho_\delta)\mathrm{div}u_\delta\, dxdt.\end{aligned}\tag{4.30}$$

Finally, making use of Lemma 4.2 and the monotonicity of the pressure, we are able to estimate the right-hand side of (4.30) as below

$$\begin{aligned}&\int_0^T\int_{\Omega}T_k(\rho)\mathrm{div}u\, dxdt-\lim_{\delta\to 0}\int_0^T\int_{\Omega}T_k(\rho_\delta)\mathrm{div}u_\delta\, dxdt\\&\le\int_0^T(T_k(\rho)-\overline{T_k(\rho)})\mathrm{div}u\, dxdt.\end{aligned}\tag{4.31}$$

With the help of (4.3), (4.23), and $\gamma>2$, the right-hand side of (4.31) tends to zero as $k\to\infty$. Then we conclude that

$$\overline{\rho\log(\rho)(t)}=\rho\log(\rho)(t)\quad\text{for all } t\in[0,T],$$

as $k\to\infty$. The above identity implies the strong convergence of ρ_δ in $L^1((0,T)\times\Omega)$ and thus $\overline{\rho^\gamma}=\rho^\gamma$, $\overline{\rho\mu}=\rho\mu$.

Therefore we complete the proof of Theorem 1.1.

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