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Mathematical modeling of junctions in fluid mechanics via two-scale convergence

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ABSTRACT

We derive the effective models for describing the behavior of the fluid in 1D-1D junctions (pipes) and 2D-2D junctions. Starting from the Navier-Stokes system in thin domain and using the two-scale convergence, we justify the two-scale model describing the flow through a junction. Finally, separating the variables in the two-scale model, we obtain the effective junction condition.

1. Introduction

Flows in thin domains like pipes and fractures are important because they appear in various applications. We use the name thin domains for those whose extension in one or more (say ℓ) directions is small compared to the extension in other (say m) directions. Simple examples of such domains are thin pipes and bars ($\ell = 2$, $m = 1$) or thin plates and fractures ($\ell = 1$, $m = 2$). Numerical studies of PDEs in such domains are difficult due to their two-scale structure. The situation is particularly complicated when the domain consists of several thin domains (like the junction of pipes, bars, plates, fractures,...).

2. Two-scale convergence for thin domains

In this section we recall the definition and the basic properties of the two-scale convergence for thin domains introduced in [8].

Definition 1. Let $\omega \subset \mathbf{R}^m$ be a bounded domain and let $\varepsilon \ll 1$ be a small parameter. For each $x_1 \in \omega$ we denote by $S(x_1) \subset \mathbf{R}^\ell$ a bounded domain such that a family $\{S(x_1)\}_{x_1 \in \omega}$ forms a Lipschitz domain $\Omega \subset \mathbf{R}^{m+\ell}$ and that the measure $|S(x_1)| > 0$:

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$$\Omega = \{z = (x_1, y) \in \mathbf{R}^{m+\ell} \ ; \ x_1 \in \omega \ , \ y \in S(x_1)\}$$

and

$$\Gamma = \{z = (x_1, y) \in \mathbf{R}^{m+\ell} \ ; \ x_1 \in \omega \ , \ y \in \partial S(x_1)\}$$

For a small parameter $\varepsilon \ll 1$ we define a thin domain

$$\Omega_\varepsilon = \{x = (x^1, x^2) \in \mathbf{R}^{m+\ell} \ ; \ x^1 \in \omega \ , \ x^2 \in \varepsilon\, S(x_1)\}$$

and the surface

$$\Gamma_\varepsilon = \{x = (x^1, x^2) \in \mathbf{R}^{m+\ell} \ ; \ x^1 \in \omega \ , \ x^2 \in \varepsilon\, \partial S(x_1)\}$$

We say that a sequence $\{v_\varepsilon\}_{\varepsilon>0}$, such that $v_\varepsilon \in L^r(\Omega_\varepsilon)$, L^r-two-scale converges to a function $V \in L^r(\Omega)$ (we use the notation L^r-2s convergence in the sequel) if

$$\frac{1}{\varepsilon^\ell} \int\limits_{\Omega_\varepsilon} v_\varepsilon(x)\phi\left(x^1, \frac{x^2}{\varepsilon}\right) dx \to \int\limits_{\Omega} V(x^1, y)\phi(x^1, y) dx^1\, dy$$

for any $\phi \in L^{r'}(\Omega)$, where $1/r + 1/r' = 1$ if $1 < r < \infty$ and $r' = 1$ if $r = \infty$, $r' = \infty$ if $r = 1$.

The compactness theorem for such convergence can be found in [8]. We mention only briefly the main results:

- If

  $$\varepsilon^{-\ell/r} |v_\varepsilon|_{L^r(\Omega_\varepsilon)} \le C$$

  then there exists a subsequence (denoted by the same symbol) and $V^0 \in L^r(\Omega)$ such that

  $$v_\varepsilon \to V^0 \quad L^r - 2s \ \ .$$

- If, in addition

  $$\varepsilon^{1-\ell/r} |\nabla v_\varepsilon|_{L^r(\Omega_\varepsilon)} \le C$$

  then $V^0 \in Y^r = \{V \in L^r(\Omega) \ \ ; \ \ \nabla_y V \in L^r(\Omega)\}$ and

  $$\varepsilon \nabla v_\varepsilon \to \nabla_y V^0 \quad L^r - 2s \ \ .$$

  Furthermore, if $v_\varepsilon = 0$ on Γ_ε, then $V^0 = 0$ on Γ (we should notice that the trace on Γ makes sense for functions from Y^r).
- If $\mathbf{v}_\varepsilon$ are vector functions $L^r - 2s$ converging to $\mathbf{V}^0$, and, in addition

  $$\operatorname{div} \mathbf{v}_\varepsilon = 0$$

  then

  $$\operatorname{div}_y \mathbf{V}^0 = 0 \ \ , \quad \operatorname{div}_{x^1} \left(\int\limits_{S(x^1)} \mathbf{V}^0(x^1, y) dy \right) = 0 \ \ .$$

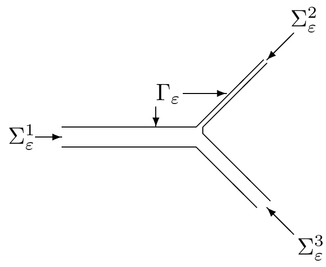


Fig. 1. Three thin pipes in junction.

3. Junction of pipes

In this section we look for the effective model to describe the flow through the junction of pipes. In real-life situations, two (or several) pipes are often interconnected (for instance watering systems and water-works are networks of thin pipes). Places where several pipes meet are called junctions. Multiple pipes systems may be as small as two pipes separating or rejoining or as complex as several hundreds of interconnected pipes forming a massive network. The basic principles of analysis are the same. Of course, the complexity of the computation depends on the complexity of the system. Therefore, in the present paper, we limit our study to the case of one junction point (Fig. 1).

The problem of junctions of elastic bars and other elastic structures has been extensively studied by several authors. For an exhaustive study of such problems we refer to the books of Ciarlet [2] and LeDret [5].

The case of junction of two intersected pipes with a flow governed by a body force was treated in [12]. The multiple junction problem with Dirichlet boundary condition was studied in [1]. An interesting algorithm for the domain decomposition as well as some numerical simulations were given. The results similar to the ones presented here were derived in [9] and [11] (see also [6] and [14] for generalizations) but using completely different, and much more complicated, approach of asymptotic expansions and boundary layers.

For some $k \in \mathbf{N}$ we define the set consisting of k pipes. After adimensionalization (taking the average length of the pipe for the characteristic length) we denote the m-th pipe by

$$\tilde{\mathcal{O}}_\varepsilon^m = \{(x^m, y^m)\ ;\ 0 < x^m < \ell_m\ ,\ y^m = (y_1^m, y_2^m) \in \omega_\varepsilon^m\},$$

where $0 < \varepsilon \ll 1$ is a small parameter (the ratio between the average pipe's thickness and length), $\ell_m > 0$ is the rescaled length of the m-th pipe and

$$\omega_\varepsilon^m = \varepsilon\, \omega^m \quad , \quad \omega^m \subset \mathbf{R}^2 \quad - \ \text{bounded set}$$

is the (rescaled) cross section of the m-th pipe. Each pipe is described using its own orthogonal coordinate system $(\mathbf{0}, \mathbf{i}_m, \mathbf{j}_m, \mathbf{k}_m)$. Those coordinate systems are (possibly) different but they all have the same origin $\mathbf{0}$.

Let

$$\mathcal{O}_\varepsilon^m = \varepsilon\, d_m\, \mathbf{i}_m + \tilde{\mathcal{O}}_\varepsilon^m$$

denote its translation for $\varepsilon\, d_m$ in direction of $\mathbf{i}_m$, the central axis of the m-th pipe. The number $d_m \geq 0$ is chosen so that each pipe might or might not contain the origin $\mathbf{0}$. However it is not too far from the origin. To join the pipes together we need the central set of the junction

$$\mathcal{O}_\varepsilon^0 = \varepsilon\, \mathcal{O}^0 \quad , \quad \mathcal{O}^0 \subset \mathbf{R}^3$$

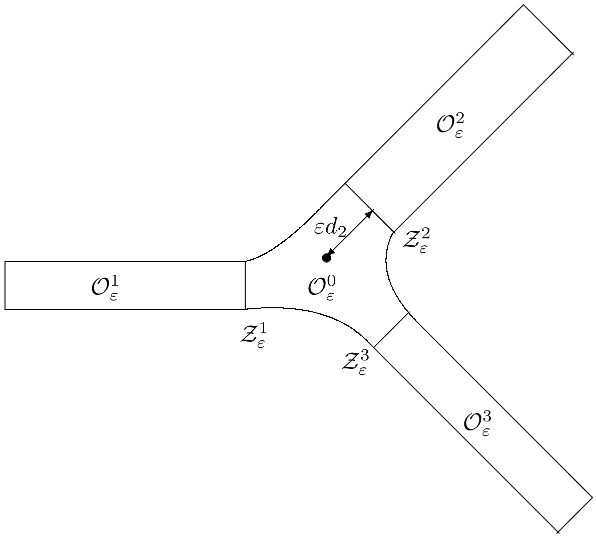


Fig. 2. Partition of the domain.

where $\mathcal{O}^0$ is a bounded set containing the origin $\mathbf{0}$ chosen to keep the pipes together in a bundle, i.e., such that

$$\Omega_\varepsilon = \bigcup_{m=0}^{k} \mathcal{O}_\varepsilon^m$$

is a bounded, connected set with Lipschitz boundary, as depicted in Fig. 2 for $k = 3$.

The boundary consists of the ends of the pipes

$$\Sigma_\varepsilon^m = \{(\ell_m^\varepsilon, y^m)\ ; \quad y^m = (y_1^m, y_2^m) \in \omega_\varepsilon^m\}\ , \quad \text{with} \quad \ell_m^\varepsilon = \ell_m + \varepsilon\, d_m \approx \ell_m\ ,$$

and the walls of the pipes

$$\Gamma_\varepsilon = \partial\Omega_\varepsilon \backslash \left(\bigcup_{m=1}^{k} \Sigma_\varepsilon^m \right) \quad .$$

The flow is governed by the stationary Navier-Stokes system. Denoting by p^ε the pressure and by $\mathbf{u}^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ the velocity, it reads

$$\begin{aligned} & -\Delta \mathbf{u}^\varepsilon + Re\,(\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon + \nabla p^\varepsilon = \frac{Re}{Fr^2}\,\mathbf{g}\ , \ \operatorname{div} \mathbf{u}^\varepsilon = 0 \ \text{ in } \Omega_\varepsilon \\ & \mathbf{u}^\varepsilon = 0\ , \quad \text{on } \Gamma_\varepsilon \\ & \mathbf{u}^\varepsilon \times \mathbf{i}_m = 0 \ \text{ and } \ p^\varepsilon = p_m \text{ on } \Sigma_\varepsilon^m\ . \end{aligned} \tag{1}$$

The prescribed values $p_m \in \mathbf{R}$ are assumed to be constants, for simplicity. The vector $\mathbf{g} = -g\ \mathbf{k}$ is the gravitational acceleration with $\mathbf{k}$ being the unit vector perpendicular to the surface of the Earth. Adimensional numbers Re (Reynolds) and Fr (Freude) could depend on ε.

Remark 1. It is not enough to impose the pressure on the end of the pipe to have a well-posed problem. We either impose the whole normal stress or we impose the condition that the components of the velocity perpendicular to the pipe are zero. We choose the second option. However, in case of thin pipe, imposing that the component of the velocity tangential to Σ_ε^m, $(\mathbf{I} - \mathbf{i}_m \otimes \mathbf{i}_m)\mathbf{u}^\varepsilon$, equals zero on Σ_ε^m is not a serious restriction because the only part of the velocity on the boundary that counts is the normal part, due to the St-Venant principle for thin domains. It is well-known that two flows with same normal velocities on Σ_ε^m and different tangential velocities, differ only in some small boundary layer in vicinity of Σ_ε^m (see [13] and [10]).

3.1. A priori estimates

Sharp a priori estimates are essential for use of the two-scale convergence. To do so, we start from the weak formulation and use Poincaré and trace inequalities as well as the Sobolev embedding theorems. The constants in those inequalities depend on the domain geometry and thus on ε. To derive sharp a priori estimates we need to know exactly how those constants depend on ε. First of all, it is well known (see e.g. [7] and [8]) that the Poincaré constant behaves as ε and the embedding constant $H^1 \hookrightarrow L^4$ like $\varepsilon^{1/4}$ i.e.

$$|v|_{L^2(\mathcal{O}_\varepsilon^m)} \le C\,\varepsilon\,|\nabla v|_{L^2(\mathcal{O}_\varepsilon^m)} \text{ for all } v \text{ such that } v = 0 \text{ on } \Gamma_\varepsilon \tag{2}$$

$$|v|_{L^4(\mathcal{O}_\varepsilon^m)} \le C\,\varepsilon^{1/4}\,|\nabla v|_{L^2(\mathcal{O}_\varepsilon^m)} \text{ for all } v \text{ such that } v = 0 \text{ on } \Gamma_\varepsilon\ . \tag{3}$$

We will also need the Nečas inequality

$$|\varphi|_{L^2(\Omega_\varepsilon)} \le \frac{C}{\varepsilon}|\,\nabla\varphi\,|_{W^{-1,2}_{\Gamma_\varepsilon}(\Omega_\varepsilon)}\ , \tag{4}$$

for all $\varphi \in L^2_0(\Omega_\varepsilon)$ (see e.g. [10], [11] or [8] for the proof). Here

$$W^{1,2}_{\Gamma_\varepsilon}(\Omega_\varepsilon) = \{\mathbf{v} \in H^1(\Omega_\varepsilon)^3\ ;\ \mathbf{v} = 0 \text{ on } \Gamma_\varepsilon\ \}$$

and $W^{-1,2}_{\Gamma_\varepsilon}(\Omega_\varepsilon)$ denotes its dual space.

Finally we need the trace inequality. Lemma A2 from [11] claims that there exists a constant $C > 0$, independent on ε, such that, for every $\varphi \in W^{1,2}_{\Gamma_\varepsilon}(\Omega_\varepsilon)$ and $m = 1, \dots, k$ we have

$$|\varphi|_{L^2(\Sigma_\varepsilon^m)} \le C|\,\nabla\varphi\,|_{L^2(\Omega_\varepsilon)}\ . \tag{5}$$

Before we proceed, we define the renormalised pressure

$$q^\varepsilon = p^\varepsilon - \frac{Re}{Fr^2}\,\mathbf{g}\cdot\mathbf{x}\ .$$

Now our system reads

$$\begin{aligned} &-\Delta\mathbf{u}^\varepsilon + Re\,(\mathbf{u}^\varepsilon\cdot\nabla)\mathbf{u}^\varepsilon + \nabla q^\varepsilon = \mathbf{0}\ ,\ \operatorname{div}\mathbf{u}^\varepsilon = 0 \text{ in } \Omega_\varepsilon \\ &\mathbf{u}^\varepsilon = 0\ , \text{ on } \Gamma_\varepsilon\ ,\ \text{ finally for all } 1 \le j \le m \text{ we have} \\ &\mathbf{u}^\varepsilon \times \mathbf{i}^m = 0 \text{ and } q^\varepsilon = q^\varepsilon_m \equiv p_m - \frac{Re}{Fr^2}\,g\cdot\mathbf{x} \text{ on } \Sigma_\varepsilon^m\ . \end{aligned} \tag{6}$$

Note that, owing to the definition of ℓ^ε_m and the fact that the m-th pipe has small thickness, we have

$$q^\varepsilon_m = q_m + O(\varepsilon)\ ,$$

where

$$q_m = p_m - \ell_m\,g\,\frac{Re}{Fr^2}\cos\angle(\mathbf{k}\,,\,\mathbf{i}_m)\ .$$

Theorem 1. *Suppose that $\varepsilon^{5/2} Re \ll 1$ and $\frac{Re}{Fr^2} \le C$. Then the problem (6) has a solution that satisfies the following estimates*

$$\begin{aligned} |\nabla \mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)} &\le C\,\varepsilon^2 \\ |\mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)} &\le C\,\varepsilon^3 \\ |\nabla q^\varepsilon|_{W^{-1,2}_{\Gamma_\varepsilon}(\Omega_\varepsilon)} &\le C\,\varepsilon^2 \\ |q^\varepsilon|_{L^2(\Omega_\varepsilon)} &\le C\,\varepsilon \quad , \end{aligned} \tag{7}$$

where $C > 0$ is some constant independent on ε. In addition, all the above estimates hold if Ω_ε is replaced by $\mathcal{O}^m_\varepsilon$ for any $m \in \{1, \dots, k\}$.

Before we proceed we need to estimate the boundary integral appearing in the weak formulation of the problem.

Lemma 1. *Under the hypothesis $\frac{Re}{Fr^2} \le C$ there exists some $C_0 > 0$, such that the following estimate holds for any $\mathbf{w} \in \mathbf{V}$*

$$\left| \sum_{m=1}^{k} \int_{\Sigma^m_\varepsilon} q^\varepsilon_m \, \mathbf{w} \cdot \mathbf{i}_m \right| \le C_0 \varepsilon^2 |\nabla \mathbf{w}|_{L^2(\Omega_\varepsilon)} \ , \tag{8}$$

where

$$\mathbf{V} = \{ \mathbf{v} \in H^1(\Omega_\varepsilon)^3 \ ; \ \ \mathbf{v} = 0 \ \ on \ \ \Gamma_\varepsilon \ ; \ \ \mathbf{v} \times \mathbf{i}_m = 0 \ \ on \ \ \Sigma^m_\varepsilon \ , \ div\, \mathbf{v} = 0 \ \} \ .$$

Proof. Since the $\mathbf{w}$ is divergence free, integration leads to

$$0 = \int_{\mathcal{O}^m_\varepsilon \cap \{s < x^m_1 < t\}} \operatorname{div} \mathbf{w} = \int_{\omega^m_\varepsilon} \mathbf{w}(t, y^m) \cdot \mathbf{i}_m \, dy^m - \int_{\omega^m_\varepsilon} \mathbf{w}(s, y^m) \cdot \mathbf{i}_m \, dy^m$$

so that

$$t \mapsto \int_{\omega^m_\varepsilon} \mathbf{w}(s, y^m) \cdot \mathbf{i}_m \, dy^m$$

is constant. Thus

$$\int_{\omega^m_\varepsilon} \mathbf{w}(s, y^m) \cdot \mathbf{i}_m \, dy^m = \frac{1}{\ell^\varepsilon_m} \int_{\mathcal{O}^m_\varepsilon} \mathbf{w} \cdot \mathbf{i}_m$$

for any $d_m \varepsilon \le s \le \ell^\varepsilon_m$. Now, we easily obtain, using (2),

$$\left| \int_{\Sigma^m_\varepsilon} \mathbf{w} \cdot \mathbf{i}_m \right| = \left| \frac{1}{\ell^\varepsilon_m} \int_{\mathcal{O}^m_\varepsilon} \mathbf{w} \cdot \mathbf{i}_m \right| \le C \varepsilon^2 |\nabla \mathbf{w}|_{L^2(\Omega_\varepsilon)} \quad . \tag{9}$$

Now

$$\left| \int_{\Sigma_\varepsilon^m} q_m^\varepsilon \, \mathbf{w} \cdot \mathbf{i}_m \right| = \left| \int_{\Sigma_\varepsilon^m} (q_m + O(\varepsilon)\,)\, \mathbf{w} \cdot \mathbf{i}_m \right| \ .$$

It follows easily from (9)

$$\left| q_m \int_{\Sigma_\varepsilon^m} \mathbf{w} \cdot \mathbf{i}_m \right| \le C\varepsilon^2 |\nabla \mathbf{w}|_{L^2(\Omega_\varepsilon)}$$

and (5) implies

$$\left| \int_{\Sigma_\varepsilon^m} O(\varepsilon)\ \mathbf{w} \cdot \mathbf{i}_m \right| \le C\varepsilon |\Sigma_\varepsilon^m|^{1/2} |\mathbf{w}|_{L^2(\Sigma_\varepsilon^m)} \le C\varepsilon^2 |\nabla \mathbf{w}|_{L^2(\Omega_\varepsilon)} \ ,$$

finishing the proof of (8). □

Proof of Theorem 1. We follow the steps from [11] in standard fixed-point procedure. We define the ball

$$B_\varepsilon = \{ \mathbf{v} \in \mathbf{V}\ ;\ \ |\nabla \mathbf{v}|_{L^2(\Omega_\varepsilon)} \le 2C_0 \varepsilon^2 \} \ ,$$

where C_0 is the constant from the trace estimate (8). The operator $T : B_\varepsilon \to \mathbf{V}$ is defined by taking $T(\mathbf{v}) = \mathbf{u}$ where, for given $\mathbf{v} \in B_\varepsilon$, function $\mathbf{u} \in \mathbf{V}$ is the unique solution of the linear problem:

Find $\mathbf{u} \in \mathbf{V}$ such that for every $\mathbf{w} \in \mathbf{V}$

$$\int_{\Omega_\varepsilon} \nabla \mathbf{u} \cdot \nabla \mathbf{w} + Re \int_{\Omega_\varepsilon} (\mathbf{v} \cdot \nabla) \mathbf{u} \cdot \mathbf{w} = \sum_{m=1}^{k} \int_{\Sigma_\varepsilon^m} q_m^\varepsilon \, \mathbf{w} \cdot \mathbf{i}_m \ . \tag{10}$$

That problem has a unique solution, since the quadratic form

$$a(\mathbf{u}, \mathbf{w}) = \int_{\Omega_\varepsilon} \nabla \mathbf{u} \cdot \nabla \mathbf{w} + Re \int_{\Omega_\varepsilon} (\mathbf{v} \cdot \nabla) \mathbf{u} \cdot \mathbf{w}$$

is coercive on $\mathbf{V}$ for $\mathbf{v} \in B_\varepsilon$. Indeed, since $\mathbf{v} \in B_\varepsilon$, condition $\varepsilon^{5/2} Re \ll 1$ implies

$$a(\mathbf{u}, \mathbf{u}) \ge (1 - C Re \sqrt{\varepsilon} |\nabla \mathbf{v}|_{L^2(\Omega_\varepsilon)}) |\nabla \mathbf{u}|^2_{L^2(\Omega_\varepsilon)} \ge (1 - C Re\, \varepsilon^{5/2}) |\nabla \mathbf{u}|^2_{L^2(\Omega_\varepsilon)} \ ,$$

so that

$$a(\mathbf{u}, \mathbf{u}) \ge \frac{1}{2} |\nabla \mathbf{u}|^2_{L^2(\Omega_\varepsilon)} \ .$$

Then the Lax-Milgram theorem implies that T is well defined.

We intend to use the Banach (contraction) fixed-point theorem to prove the existence of the solution of the problem (6).

First we need to prove that $T(B_\varepsilon) \subset B_\varepsilon$. Since for $\mathbf{v} \in B_\varepsilon$ (Lemma 1)

$$\frac{1}{2} |\nabla T(\mathbf{v})|^2_{L^2(\Omega_\varepsilon)} \le a(T(\mathbf{v}), T(\mathbf{v})) = \sum_{m=1}^{k} \int_{\Sigma_\varepsilon^m} q_m^\varepsilon \, T(\mathbf{v}) \cdot \mathbf{i}_m \le C_0 \varepsilon^2 |\nabla T(\mathbf{v})|_{L^2(\Omega_\varepsilon)} \ ,$$

we have $T(\mathbf{v}) \in B_\varepsilon$.

Next we prove that T is a contraction. We take $\mathbf{v}, \mathbf{w} \in B_\varepsilon$. Then

$$|\nabla T(\mathbf{v}) - \nabla T(\mathbf{w})|^2_{L^2(\Omega_\varepsilon)} = Re \int_{\Omega_\varepsilon} (\mathbf{v} \cdot \nabla)(T(\mathbf{v}) - T(\mathbf{w})) \cdot (T(\mathbf{v}) - T(\mathbf{w})) +$$

$$+ Re \int_{\Omega_\varepsilon} ((\mathbf{v} - \mathbf{w}) \cdot \nabla) T(\mathbf{w}) \cdot (T(\mathbf{v}) - T(\mathbf{w})) \leq$$

$$\leq Re |\mathbf{v}|_{L^4(\Omega_\varepsilon)} \, |\nabla(T(\mathbf{v}) - T(\mathbf{w}))|_{L^2(\Omega_\varepsilon)} \, |T(\mathbf{v}) - T(\mathbf{w})|_{L^4(\Omega_\varepsilon)} +$$

$$+ Re |\mathbf{v} - \mathbf{w}|_{L^4(\Omega_\varepsilon)} \, |\nabla(T(\mathbf{w})|_{L^2(\Omega_\varepsilon)} \, |T(\mathbf{v}) - T(\mathbf{w})|_{L^4(\Omega_\varepsilon)} \leq$$

$$\leq C Re \, \varepsilon^{5/2} (\, |\nabla(T(\mathbf{v}) - T(\mathbf{w}))|^2_{L^2(\Omega_\varepsilon)} +$$

$$+ |\nabla(T(\mathbf{v}) - T(\mathbf{w}))|_{L^2(\Omega_\varepsilon)} |\nabla(\mathbf{v} - \mathbf{w})|_{L^2(\Omega_\varepsilon)})$$

Since, by assumption, $Re \, \varepsilon^{5/2} \ll 1$ we have

$$|\nabla T(\mathbf{v}) - \nabla T(\mathbf{w})|_{L^2(\Omega_\varepsilon)} \leq C Re \, \varepsilon^{5/2} |\nabla(\mathbf{v} - \mathbf{w})|_{L^2(\Omega_\varepsilon)}$$

proving that

$$|T(\mathbf{v}) - T(\mathbf{w})|_{\mathbf{V}} \leq \lambda |\mathbf{v} - \mathbf{w}|_{\mathbf{V}}$$

with $\lambda < 1$. Thus $T : B_\varepsilon \to B_\varepsilon$ is a contraction and, due to the Banach fixed-point theorem it has a unique fixed point, proving the existence of a solution $\mathbf{u}^\varepsilon \in B_\varepsilon$. At the same time, since that solution is in the ball B_ε, we have proved the first estimate from (7). The second one follows from the Poincaré inequality (2).

The existence of the pressure can be proved in the same way as in case of the Dirichlet boundary condition. Indeed, we know that

$$\mathcal{P} \equiv -\Delta \mathbf{u}^\varepsilon + Re \, (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{u}^\varepsilon \; \in \; \mathbf{H}^{-1}(\Omega_\varepsilon)$$

where $\mathbf{H}^{-1}(\Omega_\varepsilon)$ stands for the dual space of $H^1_0(\Omega_\varepsilon)^3$. Furthermore, if $\mathbf{z} \in \mathcal{W}$, with

$$\mathcal{W} = \{\mathbf{z} \in H^1_0(\Omega_\varepsilon)^3 \; ; \; \operatorname{div} \mathbf{z} = 0\} \subset \mathbf{V} \quad ,$$

then

$$\langle \mathcal{P} \mid \mathbf{z} \rangle = 0 \quad .$$

Due to the DeRham theorem we know that

$$\mathcal{W}^0 = \{\mathcal{Q} \in \mathbf{H}^{-1}(\Omega_\varepsilon) \; ; \; \mathcal{W} \subset Ker \; \mathcal{Q}\} = \{\nabla q \;\; ; \; q \in L^2_0(\Omega_\varepsilon)\}$$

Thus, there exists some $q^\varepsilon \in L^2_0(\Omega_\varepsilon)$ such that

$$\Delta \mathbf{u}^\varepsilon - Re \, (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{u}^\varepsilon = \nabla q^\varepsilon \quad .$$

The boundary condition

$$q^\varepsilon = q^\varepsilon_m \;\; \text{on} \quad \Sigma^m_\varepsilon$$

holds in the sense of $H_n^{-1/2}(\Sigma_\varepsilon^m) = H_n^{1/2}(\Sigma_\varepsilon^m)'$, and

$$H_n^{1/2}(\Sigma_\varepsilon^m) = \{\mathbf{w} \in H^{1/2}(\Sigma_\varepsilon^m)^3 \ ; \ \mathbf{w} \times \mathbf{i}_m = 0 \ \}$$

(see e.g. [8]). To estimate the pressure we take a test function $\mathbf{v}$ with zero trace on Γ_ε. We have

$$\int_{\Omega_\varepsilon} \nabla \mathbf{u}^\varepsilon \cdot \nabla \mathbf{v} + Re \int_{\Omega_\varepsilon} (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon \, \mathbf{v} + \langle \nabla p^\varepsilon \mid \mathbf{v}\rangle = \sum_{m=1}^k \int_{\Sigma_\varepsilon^m} q_m^\varepsilon \, \mathbf{v} \cdot \mathbf{i}_m \ ,$$

so that

$$| \ \langle \nabla q^\varepsilon \mid \mathbf{v}\rangle \ | \le C\varepsilon^2 |\nabla \mathbf{v}|_{L^2(\Omega_\varepsilon)} \ .$$

Finally, the Nečas inequality (4) finishes the proof of the theorem. □

Remark 2. It is worth noticing that we have not proved the uniqueness of the solution. The contraction theorem does guarantee the uniqueness of the fixed point, meaning that our problem has only one solution inside the ball B_ε. However we can not exclude the possibility that there exists some solution with larger norm, since we have lost the energy equality due to the pressure boundary condition. Even for small data. We have proved that there exists a solution satisfying the a priori estimate (7) but we have not proved that all possible solutions satisfy such estimate. Obviously, for given velocity, the corresponding pressure is unique due to the linearity.

4. Convergence

Before we proceed we define the rescaled pipes

$$\tilde{\mathcal{O}}^m = \{(x^m, z^m) \ ; \ 0 < x^m < \ell_m \ , \ z^m = (z_1^m, z_2^m) \in \omega^m\}$$

and their lateral boundaries

$$\tilde{\Gamma}^m = \{(x^m, z^m) \ ; \ 0 < x^m < \ell_m \ , \ z^m = (z_1^m, z_2^m) \in \partial\omega^m\} \ .$$

Using the compactness theorem ([8], Theorem 1, Proposition 4 and Proposition 7) and denoting by $\mathbf{u}^{\varepsilon,m}$ and $q^{\varepsilon,m}$ the restrictions of $\mathbf{u}^\varepsilon$ and q^ε on m-th pipe $\mathcal{O}_\varepsilon^m$, we have the following convergences (after possible extraction of subsequences). There exist functions

$$\begin{aligned} &\mathbf{U}^m \in \mathbf{Y}_m^2 = \{\mathbf{V} \in L^2(\tilde{\mathcal{O}}^m)^3 \ , \ \nabla_{z^m}\mathbf{V} \in L^2(\tilde{\mathcal{O}}^m)^6\} \ , \ |\mathbf{V}|_{\mathbf{Y}_m^2} = |\nabla_{z^m}\mathbf{V}|_{L^2(\tilde{\mathcal{O}}^m)} \\ &Q^m \in L^2(0, \ell_m) \quad , \end{aligned}$$

such that

$$\begin{aligned} &\varepsilon^{-2}\mathbf{u}^{\varepsilon,m} \ \rightarrow \ \mathbf{U}^m \quad L^2(\tilde{\mathcal{O}}^m) - \text{two scale} \\ &\varepsilon^{-1}\nabla\mathbf{u}^{\varepsilon,m} \ \rightarrow \ \nabla_{y^m}\mathbf{U}^m \quad L^2(\tilde{\mathcal{O}}^m) - \text{two scale} \\ &q^{\varepsilon,m} \ \rightarrow \ Q^m(x^m) \quad L^2(\tilde{\mathcal{O}}^m) - \text{two scale} \quad . \end{aligned} \tag{11}$$

Furthermore

$$\mathbf{U}^m = 0 \quad \text{on} \quad \tilde{\Gamma}^m \tag{12}$$

$$\mathrm{div}_{z^m} \mathbf{U}^m(x^m, z^m) = 0 \quad \text{in} \quad \tilde{\mathcal{O}}^m \tag{13}$$

$$\frac{\partial}{\partial x^m}\left(\int_{\omega^m} U_1^m(x^m, z^m) dz^m\right) = 0 \quad , \quad U_1^m = \mathbf{U}^m \cdot \mathbf{i}_m \tag{14}$$

$$\sum_{m=1}^{k}\left(\int_{\omega^m} U_1^m(x^m, z^m) dz^m\right) = 0 \tag{15}$$

Our next step is to choose the appropriate test functions. We need the functions of the form

$$\Psi_\varepsilon^m = \Psi^m\left(\varepsilon\, d_m + x^m, \frac{y^m}{\varepsilon}\right) \quad .$$

Each function $\Psi^m(x^m, z^m)$ is defined on m-th rescaled pipe $\tilde{\mathcal{O}}^m$ and they should be equal to zero on the wall of the pipe $\tilde{\Gamma}^m$. Furthermore we need them to be divergence-free

$$\mathrm{div}_{z^m} \Psi^m(x^m, z^m) = 0 \quad .$$

In addition, we impose an important condition

$$\sum_{m=1}^{k} \int_{\omega^m} \Psi^m(0, z^m) \cdot \mathbf{i}_m \quad dz^m = 0 \tag{16}$$

Now, we need to extend those functions to $\mathcal{O}_\varepsilon^0$ in order to construct a continuous test function on the entire Ω_ε. The easy way to do it is to impose that Ψ^m and Ψ^0 have the same value on the interface between $\mathcal{O}_\varepsilon^m$ and $\mathcal{O}_\varepsilon^0$, denoted by $\mathcal{Z}_\varepsilon^m$ (see Fig. 2). We denote by $x^0 = (x_1^0, y_1^0, y_2^0)$ the standard Cartesian coordinates and by $z^0 = x^0/\varepsilon$. Now

$$\Psi^0 = \Psi^0(z^0) \quad .$$

$$\mathrm{div}_{z^0} \Psi^0\left(z^0\right) = 0 \quad , \ \Psi^0 = \Psi^m \ \text{ on } \ \mathcal{Z}_\varepsilon^m \ , \ \Psi^0 = 0 \quad \text{on } \partial\mathcal{O}_\varepsilon^0 \setminus \bigcup_{m=1}^{k} \mathcal{Z}_\varepsilon^m \tag{17}$$

Such function exists due to the condition (16). Finally

$$\Psi_\varepsilon^0 = \Psi^0(x^0/\varepsilon) \quad .$$

With such test-function

$$\Psi_\varepsilon = \Psi_\varepsilon^m \quad \text{on} \quad \mathcal{O}_\varepsilon^m \quad , \quad m = 0, 1, \ldots, k \quad ,$$

we are ready to pass to the limit. The variational formulation reads

$$\varepsilon^{-2}\int_{\Omega_\varepsilon} \nabla\mathbf{u}^\varepsilon \cdot \nabla\Psi_\varepsilon + Re\, \varepsilon^{-2}\int_{\Omega_\varepsilon} (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon \cdot \Psi_\varepsilon -$$

$$- \varepsilon^{-2}\int_{\Omega_\varepsilon} q^\varepsilon \ \mathrm{div}\ \Psi_\varepsilon = \varepsilon^{-2}\sum_{m=1}^{k} q_m^\varepsilon \int_{\Sigma_\varepsilon^m} \Psi_\varepsilon \cdot \mathbf{i}_m \ .$$

The first term is a sum of integrals I_m such that

$$I_m = \varepsilon^{-2} \int\limits_{\mathcal{O}_\varepsilon^m} \nabla \mathbf{u}^\varepsilon \cdot \nabla \Psi_\varepsilon^m = \varepsilon^{-2} \int\limits_{\mathcal{O}_\varepsilon^m} \frac{1}{\varepsilon} \nabla_{y^m} \mathbf{u}^\varepsilon \cdot \nabla_{z^m} \Psi^m \left(x^m, \frac{y^m}{\varepsilon} \right) +$$

$$+ \varepsilon^{-2} \int\limits_{\mathcal{O}_\varepsilon^m} \frac{\partial \mathbf{u}^\varepsilon}{\partial x^m} \frac{\partial \Psi^m}{\partial x^m} \left(x^m, \frac{y^m}{\varepsilon} \right) + O(\varepsilon) \to$$

$$\to \int\limits_0^{\ell_m} \int\limits_{\omega^m} \nabla_{z^m} \mathbf{U}^m \cdot \nabla_{z^m} \Psi^m \left(x^m, z^m \right) dx^m \; dz^m \quad ,$$

for $m > 0$, while for $m = 0$

$$I_0 = \varepsilon^{-2} \int\limits_{\mathcal{O}_\varepsilon^0} \nabla \mathbf{u}^\varepsilon \cdot \nabla \Psi_\varepsilon^0 = \varepsilon^{-3} \int\limits_{\mathcal{O}_\varepsilon^0} \nabla \mathbf{u}^\varepsilon \cdot \nabla_{z^0} \Psi^0 \leq$$

$$\leq \varepsilon^{-3} |\mathcal{O}_\varepsilon^0|^{1/2} \; |\nabla \mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)} |\nabla_{z^0} \Psi^0|_{L^\infty} \leq C\sqrt{\varepsilon} \to 0 \quad .$$

The second integral tends to zero, since

$$Re\, \varepsilon^{-2} \int\limits_{\Omega_\varepsilon} (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{u}^\varepsilon \cdot \Psi_\varepsilon \leq Re\, \varepsilon^{-2} |\nabla \mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)} |\mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)} |\Psi_\varepsilon|_{L^\infty} \leq C Re\, \varepsilon^3 \quad .$$

The third integral is the sum of integrals

$$J_m = \varepsilon^{-2} \int\limits_{\mathcal{O}_\varepsilon^m} q^{\varepsilon,m} \operatorname{div} \Psi_\varepsilon^m = \varepsilon^{-2} \int\limits_{\mathcal{O}_\varepsilon^m} q^{\varepsilon,m} \frac{\partial \Psi_1^m}{\partial x^m} \to$$

$$\to \int\limits_0^{\ell_m} \int\limits_{\omega^m} Q^m \frac{\partial \Psi_1^m}{\partial x^m} \left(x^m, z^m \right) dx^m \; dz^m \quad ,$$

for $m > 0$. Here

$$\Psi_1^m = \Psi^m \cdot \mathbf{i}_m \quad .$$

It should be noticed that, due to (17), we have $J_0 = 0$. For the last integral on the right-hand side we have

$$\varepsilon^{-2} \sum_{m=1}^{k} \int\limits_{\Sigma_\varepsilon^m} q_m^\varepsilon \, \Psi_\varepsilon \cdot \mathbf{i}_m = \sum_{m=1}^{k} q_m \int\limits_{\Sigma^m} \Psi_1^m(\ell_m, z^m) + O(\varepsilon) \quad .$$

We now have the two-scale problem for the limit functions $(\mathbf{U}^m, Q^m)$

$$\sum_{m=1}^{k} \Bigg(\int\limits_{\tilde{\mathcal{O}}^m} \nabla_{z^m} \mathbf{U}^m \cdot \nabla_{z^m} \Psi^m \left(x^m, z^m \right) dx^m \, dz^m - \tag{18}$$

$$- \int\limits_{\tilde{\mathcal{O}}^m} Q^m \frac{\partial \Psi_1^m}{\partial x^m} \left(x^m, z^m \right) dx^m \, dz^m \Bigg) = \sum_{m=1}^{k} q_m \int\limits_{\Sigma^m} \Psi_1^m(\ell_m, z^m) \quad .$$

To define the two-scale problem properly, we need an appropriate functional space:

$$\mathbf{W}^m = \Big\{ \mathbf{v} \in \mathbf{Y}_m^2 \, ; \, \mathrm{div}_{z^m} \mathbf{v} = 0 \ , \ \ \mathbf{v} = 0 \ \text{ on } \tilde{\Gamma}^m \ ,$$

$$\int_{\omega^m} \mathbf{v}(x^m, z^m) \cdot \mathbf{i}^m \quad dz^m = const. \Big\} \ , \ \ m = 1, \ldots, k \ ,$$

$$\mathbf{W} \ = \Big\{ \mathbf{V} = (\mathbf{V}^1, \ldots, \mathbf{V}^k) \, ; \, \mathbf{V}^m \in \mathbf{W}^m \ , \ \sum_{m=1}^{k} \int_{\omega^m} \mathbf{V}^m(0, z^m) \cdot \mathbf{i}_m \, dz^m = 0 \Big\}$$

$$|\mathbf{V}^m|_{\mathbf{W}^m} = |\mathbf{V}^m|_{\mathbf{Y}_m^2} \ , \ \ |\mathbf{V}|_{\mathbf{W}} = \sum_{m=1}^{k} |\mathbf{V}^m|_{\mathbf{W^m}} \ .$$

We notice that for $\Psi^m \in \mathbf{W}^m$

$$\sum_{m=1}^{k} \int_{\tilde{\mathcal{O}}^m} Q^m \frac{\partial \Psi_1^m}{\partial x^m} = \sum_{m=1}^{k} \int_0^{\ell_m} Q^m(x^m) \frac{\partial}{\partial x^m} \int_{\omega^m} \Psi^m \cdot \mathbf{i}_m \ (x^m, z^m) \, dz^m \ dx^m = 0$$

The problem now reads: Find $\mathbf{U} = (U^1, \ldots, U^k) \in \mathbf{W}$ such that for all $\Psi = (\Psi^1, \ldots, \Psi^k) \in \mathbf{W}$

$$\sum_{m=1}^{k} \int_{\tilde{\mathcal{O}}^m} \nabla_{z^m} \mathbf{U}^m \cdot \nabla_{z^m} \Psi^m \, (x^m, z^m) \, dx^m \, dz^m = \sum_{m=1}^{k} q_m \int_{\Sigma^m} \Psi^m(\ell_m, z^m) \cdot \mathbf{i}_m \, dz^m \ . \tag{19}$$

Here

$$\Sigma^m = \{\ell_m\} \times \omega^m$$

denotes the end of the pipe $\tilde{\mathcal{O}}^m$.

The quadratic form

$$\mathbf{A}(\mathbf{U}, \mathbf{W}) = \sum_{m=1}^{k} \int_{\tilde{\mathcal{O}}^m} \nabla_{z^m} \mathbf{U}^m \cdot \nabla_{z^m} \mathbf{V}^m \ dx^m \ dz^m$$

defined on $\mathbf{W}$ is obviously coercive since

$$\mathbf{A}(\mathbf{U}, \mathbf{U}) = \sum_{m=1}^{k} \int_{\tilde{\mathcal{O}}^m} |\nabla_{z^m} \mathbf{U}^m|^2 \ dx^m \ dz^m = |\mathbf{U}|^2_{\mathbf{W}} \ .$$

Thus the above two-scale problem has a unique solution, due to the Lax-Milgram theorem. Uniqueness implies that in (11) the whole sequence converges and not the subsequence (i.e. it has only one accumulation point). Obviously $U_2^m = \mathbf{U}^m \cdot \mathbf{j}_m = 0$ and $U_3^m = \mathbf{U}^m \cdot \mathbf{k}_m = 0$ for all $m = 1, \ldots, k$. Since our unknown function is scalar in each pipe, the two-scale problem (19) reduces to:

Find $U = (U_1, \ldots, U_k) \in W$ such that for all $\psi = (\psi_1, \ldots, \psi_k) \in W$

$$\sum_{m=1}^{k} \int_{\tilde{\mathcal{O}}^m} \nabla_{z^m} U^m \cdot \nabla_{z^m} \psi^m \, (x^m, z^m) \, dx^m \ dz^m = \sum_{m=1}^{k} q_m \int_{\Sigma^m} \psi^m(\ell_m, z^m) \ dz^m \ \ , \tag{20}$$

with

$$W^m = \left\{ v \in Y_m^2 \, ; \; , \; v = 0 \text{ on } \Gamma^m \, , \int_{\omega^m} v(x^m, z^m) \quad dz^m = const. \right\} ,$$

$$m = 1, \dots, k \; ,$$

$$W = \left\{ V = (V^1, \dots, V^m) \, ; \, V^m \in W^m \, , \; \sum_{m=1}^{k} \int_{\omega^m} V^m(0, z^m) \; dz^m = 0 \right\}$$

$$Y_m^2 = \{ V \in L^2(\tilde{\mathcal{O}}^m) \, , \, \nabla_{z^m} V \in L^2(\tilde{\mathcal{O}}^m) \} \; , \; |V|_{Y_m^2} = |\nabla_{z^m} V|_{L^2(\tilde{\mathcal{O}}^m)} \; .$$

Furthermore (18) implies that

$$\sum_{m=1}^{k} \Big(\int_{\tilde{\mathcal{O}}^m} \nabla_{z^m} U^m \cdot \nabla_{z^m} \psi^m \, (x^m, z^m) \, dx^m \; dz^m - \\ - \int_{\tilde{\mathcal{O}}^m} Q^m \frac{\partial \psi^m}{\partial x^m} \, (x^m, z^m) \, dx^m \; dz^m \Big) = \sum_{m=1}^{k} q_m \int_{\Sigma^m} \psi^m(\ell_m, z^m) \; dz^m \; , \tag{21}$$

for any $(\psi^1, \dots, \psi^k)$ such that $\psi^m \in L^2(0, \ell_m; H_0^1(\omega^m))$. Since U^m are unique, so are the Q^m.

One solution to that problem is easy to construct by taking

$$U^m = G^m(z^m) \; \frac{\partial Q^m}{\partial x^m}(x^m) \tag{22}$$

with

$$-\Delta G^m = 1 \text{ in } \omega^m \; , \; G^m = 0 \text{ on } \partial\omega^m \; . \tag{23}$$

Furthermore $\frac{\partial Q^m}{\partial x^m}(x^m) = const. = A_m$ so that

$$Q^m(x^m) = A_m x^m + B \; . \tag{24}$$

The constant A_m is to be determined from the condition

$$\sum_{m=1}^{k} \int_{\omega^m} U^m(0, z^m) \; dz^m = 0 \; ,$$

since $U = (U^1, \dots, U^k) \in W$ and B is independent on m (continuity of the pressure). That leads to

$$\sum_{m=1}^{k} A_m \int_{\omega^m} G^m(z^m) \; dz^m = 0 \; .$$

Furthermore, we must impose

$$Q^m(\ell_m) = q_m$$

leading to

$$A_m \ell^m + B = q_m \quad \Rightarrow \quad A_m = \frac{q_m - B}{\ell_m} \quad .$$

Combining, we obtain

$$\sum_{m=1}^{k} \frac{q_m - B}{\ell_m} \; \overline{G}^m = 0$$

where

$$\overline{G}^m = \int_{\omega^m} G^m(z^m) \; dz^m \quad .$$

It implies the Kirchoff law for B, the pressure at the origin

$$B = \left(\sum_{m=1}^{k} \frac{\overline{G}^m}{\ell_m} \right)^{-1} \; \sum_{m=1}^{k} \frac{q_m \; \overline{G}^m}{\ell_m} \tag{25}$$

Now (22)-(25) defines one solution of the two-scale problem (21). As the solution is unique, that is the only solution. We have proved the following theorem

Theorem 2. *Let* $(\mathbf{u}^\varepsilon, q^\varepsilon)$ *be the solution of the Navier-Stokes system (6) and let* $\mathbf{u}^{\varepsilon,m}$ *and let* $q^{\varepsilon,m}$ *be their restrictions on* $\mathcal{O}_\varepsilon^m$*. Then*

$$\varepsilon^{-2}\mathbf{u}^{\varepsilon,m} \to U^m \, \mathbf{i}_m \quad L^2 - \textit{two scale on } \mathcal{O}_\varepsilon^m$$
$$q^{\varepsilon,m} \to Q^m(x^m) \quad L^2 - \textit{two scale on } \mathcal{O}_\varepsilon^m \quad ,$$

where $U = (U^1, \dots, U^k) \in W$ *and* $Q = (Q^1, ..., Q^k) \in L^2(0, \ell_1) \times \dots \times L^2(0, \ell_k)$ *is the unique solution of the two-scale problem (21). Furthermore, (21) can be decoupled and its solution is given by (22)-(25).*

It is important to notice that the pressure B in the junction node is given by the Kirchoff formula (25). Denoting by $\mathbf{r}$ the vector

$$\mathbf{r} = \frac{g \; Re \; \sum_{j=1}^k \; \overline{G}^j \, \mathbf{i}_j}{Fr^2 \; \sum_{j=1}^k \; \frac{\overline{G}^j}{\ell_j}}$$

and by

$$\overline{P} = \left(\sum_{j=1}^{k} \frac{\overline{G}^j}{\ell_j} \right)^{-1} \; \sum_{j=1}^{k} \frac{p_j \; \overline{G}^j}{\ell_j}$$

the weighted mean of the exterior pressures p_j, it reads

$$B = \overline{P} - \mathbf{r} \cdot \mathbf{k} \tag{26}$$

5. Junction of 2D thin domains

In this section we want to find a model for describing the flow through two thin domains with extensions in two dimensions much larger than in the third (fractures, gaps,...). The results can be easily generalized to the case of more than two domains. It can also be generalized on situations when those thin domains are not intersecting on their boundaries but somewhere in the middle. We chose to treat the case of a layer of liquid lubricant with slider that has one edge.

5.1. The geometry

Let $(\mathbf{O}, \mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1)$ and $(\mathbf{O}, \mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2)$ be two orthonormal basis with joint origin $\mathbf{O}$. Let Π_1 and Π_2 be two plains such that Π_m is spanned by $(\mathbf{i}_m, \mathbf{j}_m)$ and $\mathbf{k}_m$ is its normal, with $m = 1, 2$. We assume that they are not parallel and that their intersection is a line $\mathcal{L}$ spanned by the vector $\mathbf{e}$. Both those plains Π_1 and Π_2 separate the space $\mathbf{R}^3$ in two half-spaces

$$\begin{aligned}
M_1^+ &= \{\mathbf{x} \in \mathbf{R}^3 \ ; \ \mathbf{x} \cdot \mathbf{k}_1 > 0\} \\
M_1^- &= \{\mathbf{x} \in \mathbf{R}^3 \ ; \ \mathbf{x} \cdot \mathbf{k}_1 < 0\} \\
M_2^+ &= \{\mathbf{x} \in \mathbf{R}^3 \ ; \ \mathbf{x} \cdot \mathbf{k}_2 > 0\} \\
M_2^- &= \{\mathbf{x} \in \mathbf{R}^3 \ ; \ \mathbf{x} \cdot \mathbf{k}_2 < 0\} \ .
\end{aligned}$$

Without loosing generality we can choose that vector to be the first vector of the both canonical bases, i.e.

$$\mathbf{i}_1 = \mathbf{i}_2 = \mathbf{e} \ .$$

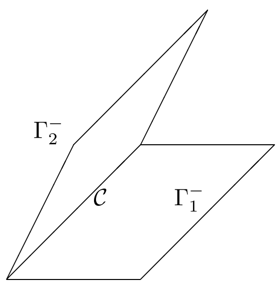
Now, let $\omega^m \subset \mathbf{R}^2$, $m = 1, 2$ be bounded domains with Lipschitz boundaries. We identify the domains ω^m with sets

$$\Gamma_m^- = \{\mathbf{x} = x\,\mathbf{e} + y^m \mathbf{j}_m \in \Pi_m \ ; \ (x, y^m) \in \omega^m\} \subset \Pi_m \ , \ m = 1, 2 \ .$$

Those are the lower boundaries of the fluid domain.

We assume that the boundaries $\partial\omega^1$ and $\partial\omega^2$ intersect and that their intersection is a segment $\mathcal{J} = [0, \ell] \times \{0\}$. Thus the intersection of Γ_1^- and Γ_2^+ is the line $\mathcal{C}$, starting from the origin $\mathbf{O}$, with length ℓ

$$\mathcal{C} = \{\mathbf{O} + \lambda \mathbf{e} \ ; \ 0 < \lambda < \ell\} \ .$$



Let $h_m : \overline{\omega^m} \to]0, +\infty[$ be smooth functions such that $h_m \geq \alpha_0 > 0$. For $m = 1, 2$, we denote by

$$\Omega_\varepsilon^m = \{\mathbf{x} = x\,\mathbf{e} + y^m \mathbf{j}_m + z^m \mathbf{k}_m \in \mathbf{R}^3 \ ; \ (x, y^m) \in \omega^m \ ; \ 0 < z^m < \varepsilon\, h_m(x, y^m)\} \ ,$$

and for $\varepsilon = 1$ we get Ω_1^m. If the angle between the two domains Ω_ε^1 and Ω_ε^2 is obtuse or right, then $\Omega_\varepsilon^2 \subset M_1^+$ and $\Omega_\varepsilon^1 \subset M_2^+$. Thus the fluid domain is

$$\Omega_\varepsilon = \Omega_\varepsilon^1 \cup \Omega_\varepsilon^2 \quad .$$

If the angle is acute then we need to cut-off the edges

$$E_\varepsilon^1 = \Omega_\varepsilon^1 \cap M_2^- \quad , \quad E_\varepsilon^2 = \Omega_\varepsilon^2 \cap M_1^-$$

so that

$$\Omega_\varepsilon = (\Omega_\varepsilon^1 \cup \Omega_\varepsilon^2) \backslash (E_\varepsilon^1 \cup E_\varepsilon^2) \quad .$$

The intersection

$$\mathcal{C}_\varepsilon = \Omega_\varepsilon^1 \cap \Omega_\varepsilon^2$$

is nonempty, by construction.

It is a thin set with one edge equal to $\mathcal{C}$. It can be described as

$$\mathcal{C}_\varepsilon = \{\mathbf{x} \in \Omega_\varepsilon \ ; \ 0 < \mathbf{k}_m \cdot \mathbf{x} < \varepsilon\, h_m(x, y^m) \ , \ m = 1, 2 \ ; \ 0 < \mathbf{e} \cdot \mathbf{x} < \ell\} \ ,$$

with $x = \mathbf{e} \cdot \mathbf{x}$, $y^m = \mathbf{j}_m \cdot \mathbf{x}$.

We could add a domain $\mathcal{O}_\varepsilon^0$ to join Ω_ε^1 and Ω_ε^2 together, as we did with pipes in the previous section, but we choose not to, for diversity.

We denote the parts of the boundary of Ω_ε as follows (again $m = 1, 2$):

Upper boundary:

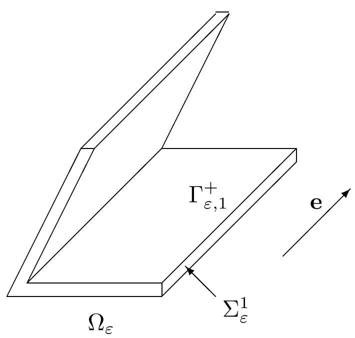
$$\Gamma_{\varepsilon,m}^+ = \{\mathbf{x} = x\,\mathbf{e} + y^m \mathbf{j}_m + \varepsilon\, h_m(x, y^m)\mathbf{k}_m \in \overline{\Omega_\varepsilon} \ ; \ (x, y^m) \in \omega^m \ \} \ .$$

The lateral boundary:

$$\Sigma_\varepsilon^m = \{ \ \mathbf{x} = x\,\mathbf{e} + y^m \mathbf{j}_m + z^m \mathbf{k}_m \in \overline{\Omega_\varepsilon}\,; \, (x, y^m) \in \partial\omega^m \setminus \mathcal{J}\,; \\ 0 < z^m < \varepsilon\, h_m(x, y^m)\} \, .$$

For $\varepsilon = 1$ we get $\Gamma_{1,m}^+$ and Σ_1^m. Finally

$$\Gamma_\varepsilon^+ = \Gamma_{\varepsilon,1}^+ \cup \Gamma_{\varepsilon,2}^+ \ , \ \Gamma^- = \Gamma_1^- \cup \Gamma_2^- \ , \ \Sigma_\varepsilon = \Sigma_\varepsilon^1 \cup \Sigma_\varepsilon^2 \ .$$



The flow is governed by the Navier-Stokes system

$$\begin{aligned}
& -\Delta \mathbf{u}^\varepsilon + Re\,(\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon + \nabla p^\varepsilon = \frac{Re}{Fr^2}\,\mathbf{g}\ ,\ \operatorname{div}\mathbf{u}^\varepsilon = 0 \ \text{ in } \Omega_\varepsilon \\
& \mathbf{u}^\varepsilon = 0\ ,\ \text{ on } \Gamma_\varepsilon^+\ ,\ \mathbf{u}^\varepsilon = \varepsilon^2 \mathbf{w}\ ,\ \text{ on } \Gamma^- \\
& \mathbf{u}^\varepsilon \times \mathbf{n} = 0 \ \text{ and } \ p^\varepsilon = q \text{ on } \Sigma_\varepsilon\ ,
\end{aligned} \tag{27}$$

where

$$\mathbf{w} = w\,\mathbf{e}$$

is a constant vector, representing the velocity of relative motion for two lubricated surfaces. For simplicity, we have chosen the usual scaling ε^2 for that velocity, in order to get the standard Reynolds equation (see e.g. [8] or [3]), but other scalings are also possible and can be treated likewise.

As the domain is thin, it is reasonable to assume that, on each of the boundaries Σ_ε^1 and Σ_ε^2 the boundary pressure q does not depend on the third variable z^m, $m = 1, 2$. Before we proceed we subtract the hydrostatic pressure from p^ε. We define

$$r^\varepsilon = p^\varepsilon - \frac{Re}{Fr^2}\,\mathbf{x}\cdot\mathbf{g}\ . \tag{28}$$

Now

$$-\Delta \mathbf{u}^\varepsilon + Re\,(\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon + \nabla r^\varepsilon = \mathbf{0} \ \text{ in } \Omega_\varepsilon\ ,\ r^\varepsilon = r \equiv q - \frac{Re}{Fr^2}\,\mathbf{x}\cdot\mathbf{g} \text{ on } \Sigma_\varepsilon\ . \tag{29}$$

5.2. A priori estimates

We start by recalling that we again have

$$|v|_{L^2(\Omega_\varepsilon)} \le C_p\,\varepsilon\,|\nabla v|_{L^2(\Omega_\varepsilon)} \ \text{ for all } \ v \text{ such that } v = 0 \ \text{ on } \ \Gamma_\varepsilon^+ \tag{30}$$

$$|v|_{L^4(\Omega_\varepsilon)} \le C_4\,\varepsilon^{1/4}\,|\nabla v|_{L^2(\Omega_\varepsilon)} \ \text{ for all } \ v \text{ such that } v = 0 \ \text{ on } \ \Gamma_\varepsilon^+\ . \tag{31}$$

Theorem 3. *Suppose that $\varepsilon^2 Re \ll 1$ and $\frac{Re}{Fr^2} \le C$. Assume, in addition, that $q \in C^1(\mathbf{R}^3)$. Then the problem (27) has a solution such that the following estimates hold*

$$\begin{aligned}
|\nabla \mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)} &\le C\,\varepsilon\sqrt{\varepsilon} \\
|\mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)} &\le C\,\varepsilon^2\sqrt{\varepsilon} \\
|r^\varepsilon|_{L^2(\Omega_\varepsilon)} &\le C\,\sqrt{\varepsilon}\ ,
\end{aligned} \tag{32}$$

for some constant $C > 0$ independent on ε.

Again we first need to estimate the boundary integral appearing in the weak formulation of the problem and coming from the pressure boundary condition.

Lemma 2. *Under the hypothesis of Theorem 3 there exists some $C_\ell > 0$, such that the following estimate holds for any $\mathbf{w} \in \mathbf{V}$*

$$\left| \int_{\Sigma_\varepsilon} \left(q - \frac{Re}{Fr^2}\,\mathbf{x}\cdot\mathbf{g} \right) \mathbf{w}\cdot\mathbf{n} \right| \le C_\ell \varepsilon^{3/2} |\nabla \mathbf{w}|_{L^2(\Omega_\varepsilon)}\ , \tag{33}$$

where

$$\mathbf{V} = \{\mathbf{v} \in H^1(\Omega_\varepsilon)^3 \; ; \;\; \mathbf{v} = 0 \;\; on \;\; \Gamma_\varepsilon^+ \; ; \;\; \mathbf{v} \times \mathbf{n} = 0 \;\; on \;\; \Sigma_\varepsilon \; , \; div\, \mathbf{v} = 0 \, \} \;\; .$$

Proof. Let $\mathbf{w} \in \mathbf{V}$. Then

$$\int_{\Sigma_\varepsilon} \left(q - \frac{Re}{Fr^2}\, \mathbf{x} \cdot \mathbf{g} \right) (\, \mathbf{w} \cdot \mathbf{n}) = \int_{\Omega_\varepsilon} \operatorname{div} \left\{ \left(q - \frac{Re}{Fr^2}\, \mathbf{x} \cdot \mathbf{g} \right) \mathbf{w} \right\} =$$

$$= \int_{\Omega_\varepsilon} \left(\nabla q - \frac{Re}{Fr^2}\, \mathbf{g} \right) \cdot \mathbf{w} \le C \left(|\nabla q|_{L^\infty(\Omega_\varepsilon)} + \frac{Re}{Fr^2}\, g \right) |\mathbf{w}|_{L^1(\Omega_\varepsilon)} \le$$

$$\le C |\Omega_\varepsilon|^{1/2} |\mathbf{w}|_{L^2(\Omega_\varepsilon)} \le (\text{ due to } (30)\,) \le C\varepsilon^{3/2} |\nabla \mathbf{w}|_{L^2(\Omega_\varepsilon)} \;\; . \quad \square$$

Next step is to lift the non-homogeneous boundary condition on Γ^-.

Lemma 3. *There exists a function $\mathbf{b}_\varepsilon \in H^1(\Omega_\varepsilon)^3$ such that*

$$div\, \mathbf{b}_\varepsilon = 0 \;\; in\; \Omega_\varepsilon \;,\;\; \mathbf{b}_\varepsilon = 0 \;\; on\; \Gamma_\varepsilon^+ \;,\;\; \mathbf{b}_\varepsilon = \varepsilon^2 \mathbf{w} \;\; on\; \Gamma^- \;,\;\; \mathbf{b}_\varepsilon \times \mathbf{n} = 0 \;\; on\; \Sigma_\varepsilon \; .$$

Furthermore, the function $\mathbf{b}_\varepsilon$ can be chosen such that there exists a constant $C_b > 0$ satisfying

$$|\nabla \mathbf{b}_\varepsilon|_{L^2(\Omega_\varepsilon)} \;\le\; C_b\, \varepsilon \sqrt{\varepsilon} \quad . \tag{34}$$

Proof. Outside of the thin layer near the junction, function $\mathbf{b}_\varepsilon = b_\varepsilon \cdot \mathbf{e}$ can be chosen such that b_ε equals

$$\varepsilon^2 \left(1 - \frac{z^m}{\varepsilon\, h_m} \right) w$$

on each Ω_ε^m (except in a small boundary layer) and is smoothly connected between. We need to construct the appropriate connection in the boundary layer near the junction. Before we proceed we have to precise the geometry of the layer. Let

$$h_m^\infty = |h_m|_{L^\infty(\omega^m)}$$

and let

$$\begin{aligned} S_\varepsilon^1 &= \{\mathbf{x} \in \mathbf{R}^3 \, ; \, 0 < x < \ell \, , \, 0 < y^1 < \varepsilon\, h_2^\infty \; , \;\; 0 < z^1 < \varepsilon\, h_1(x, y^1) \} \\ S_\varepsilon^2 &= \{\mathbf{x} \in \mathbf{R}^3 \, ; \, 0 < x < \ell \, , \, 0 < y^2 < \varepsilon\, h_1^\infty \; , \;\; 0 < z^2 < \varepsilon\, h_2(x, y^2) \} \, , \end{aligned} \tag{35}$$

with

$$\mathbf{x} = x\, \mathbf{e} + y^m\, \mathbf{j}_m + z^m\, \mathbf{k}_m \;\; \text{in} \;\; S_\varepsilon^m \;, \;\; m = 1, 2 \; .$$

Then the layer around the junction is

$$\mathcal{S}_\varepsilon = S_\varepsilon^1 \cup S_\varepsilon^2 \tag{36}$$

and the interfaces between $\mathcal{S}_\varepsilon$ and $\Omega_\varepsilon^m \backslash \mathcal{S}_\varepsilon$ are denoted

$$\begin{aligned}\gamma_\varepsilon^1 &= \{\mathbf{x}\in\mathbf{R}^3\,;\, 0<x<\ell\,,\, y^1=\varepsilon\, h_2^\infty\,,\ 0<z^1<\varepsilon\, h_1(x,\varepsilon\, h_2^\infty)\}\\ \gamma_\varepsilon^2 &= \{\mathbf{x}\in\mathbf{R}^3\,;\, 0<x<\ell\,,\, y^2=\varepsilon\, h_1^\infty\,,\ 0<z^2<\varepsilon\, h_2(x,\varepsilon\, h_1^\infty)\}\,.\end{aligned} \tag{37}$$

Let $\mathbf{x}^\perp = y^1\,\mathbf{j}_1 + z^1\,\mathbf{k}_1 = y^2\,\mathbf{j}_2 + z^2\,\mathbf{k}_2$ be such that $\mathbf{x} = x\,\mathbf{e} + \mathbf{x}^\perp$ and let $\zeta^\perp = \mathbf{x}^\perp/\varepsilon$. Depending on the need it can be written as

$$\zeta^\perp = \eta^1\,\mathbf{j}_1 + \xi^1\,\mathbf{k}_1 = \eta^2\,\mathbf{j}_2 + \xi^2\,\mathbf{k}_2$$

with

$$\eta^m = y^m/\varepsilon\,,\ \ \xi^m = z^m/\varepsilon\ \ .$$

We now construct the rest of the function $\mathbf{b}_\varepsilon$ in the layer $\mathcal{S}_\varepsilon$ by taking

$$\widetilde{\mathbf{b}}_\varepsilon(x,\mathbf{x}^\perp) = B_\varepsilon^1(x,\zeta^\perp)\,\mathbf{e} + \varepsilon\, B_\varepsilon^2(x,\zeta^\perp)\,\mathbf{j}_1 + \varepsilon\, B_\varepsilon^3(x,\zeta^\perp)\,\mathbf{k}_1$$

We now define the rescaled layer

$$\mathcal{S} = \{(x,\zeta^\perp)\in\mathbf{R}^3\,;\ (x,\varepsilon\,\zeta^\perp)\in\mathcal{S}_\varepsilon\ \}\ \ , \tag{38}$$

with boundary parts

$$\begin{aligned}\gamma^1 &= \{(x,\zeta^\perp)\in\mathbf{R}^3\,;\, 0<x<\ell\,,\, \eta^1 =\ h_2^\infty\,,\ 0<\xi^1<\ h_1(x,\varepsilon\, h_2^\infty)\}\\ \gamma^2 &= \{(x,\zeta^\perp)\in\mathbf{R}^3\,;\, 0<x<\ell\,,\, \eta^2 =\ h_1^\infty\,,\ 0<\xi^2<\ h_2(x,\varepsilon\, h_1^\infty)\}\\ \Gamma^- &= \{(x,\zeta^\perp)\in\overline{\mathcal{S}}\,;\ \ \xi^1=0\ \text{ or }\ \xi^2=0\ \}\\ \Gamma^+ &= \{(x,\zeta^\perp)\in\overline{\mathcal{S}}\,;\ \ \xi^1=h_1(x,\varepsilon\,\eta^1)\ \text{ or }\ \xi^2=h_2(x,\varepsilon\,\eta^2)\ \}\ .\end{aligned} \tag{39}$$

In order to meet the divergence-free condition as well as the boundary conditions we choose $\mathbf{B}_\varepsilon = (B_\varepsilon^1, B_\varepsilon^2, B_\varepsilon^3)$ such that

$$\frac{\partial B_\varepsilon^1}{\partial x} + \frac{\partial B_\varepsilon^2}{\partial \eta^1} + \frac{\partial B_\varepsilon^3}{\partial \xi^1} = 0\ \text{ in }\ \mathcal{S}$$

with boundary conditions

$$\mathbf{B}_\varepsilon = 0\ \text{ on }\ \Gamma^-\ \ ,\ \ \mathbf{B}_\varepsilon = \varepsilon^2\mathbf{w}\ \text{ on }\ \Gamma^+$$

and

$$\mathbf{B}_\varepsilon = \varepsilon^2\left(1-\frac{\xi^m}{h_m}\right)\mathbf{w}\ \text{ on }\ \gamma^m\ \ ,\ \ m=1,2\ \ .$$

Since $\mathbf{w}$ is perpendicular to the normal vector on Γ^- as well as on γ^m, the necessary condition for the existence of such B_ε is satisfied. Furthermore, as the domain $\mathcal{S}$ does not depend on ε, there exists such $\mathbf{B}_\varepsilon$ satisfying the estimate

$$|\mathbf{B}_\varepsilon|_{H^1(\mathcal{S})} \le C\varepsilon^2$$

(see e.g. [4]). A direct computation now yields

$$|\widetilde{\mathbf{b}}_\varepsilon|_{H^1(\mathcal{S}_\varepsilon)} \le C\varepsilon^2 \ .$$

Now the function

$$\mathbf{b}_\varepsilon(\mathbf{x}) = \begin{cases} \varepsilon^2 \left(1 - \frac{z^1}{\varepsilon\ h_1(x,y^1)}\right) \mathbf{w} \ \text{ in } \ \Omega^1_\varepsilon \backslash \mathcal{S}_\varepsilon \\ \varepsilon^2 \left(1 - \frac{z^2}{\varepsilon\ h_2(x,y^2)}\right) \mathbf{w} \ \text{ in } \ \Omega^2_\varepsilon \backslash \mathcal{S}_\varepsilon \\ \widetilde{\mathbf{b}}_\varepsilon\,(\mathbf{x}) \ \text{ in } \ \mathcal{S}_\varepsilon \end{cases}$$

has all the required properties. □

Proof of Theorem 3. The discussion on the existence of the solution is very similar as in Theorem 1 from the previous section, and the key role has the a priori estimate. We only scratch the proof, pointing out the differences. Let

$$\mathbf{W} = \{\mathbf{u} \in H^1(\Omega_\varepsilon)^3 \ ; \ \operatorname{div}\mathbf{u} = 0 \ \ , \ \mathbf{u} = 0 \ \text{ on } \Gamma^+_\varepsilon \cup \Gamma^- \ \ \mathbf{u} \times \mathbf{n} = 0 \ \text{ on } \Sigma_\varepsilon \ \} \ .$$

As in the proof of Theorem 1, we define the ball

$$B_\varepsilon = \{\mathbf{v} \in \mathbf{V} \ ; \ \ |\nabla \mathbf{v}|_{L^2(\Omega_\varepsilon)} \le M\varepsilon^{3/2}\} \ \ ,$$

where the constant $M > 0$ is defined as

$$M = 3\sqrt{2}(C_b + C_\ell)$$

and C_b, C_ℓ are the constants from Lemma 3 and Lemma 2, respectively. The operator $T : B_\varepsilon \to \mathbf{V}$ is defined by taking $T(\mathbf{v}) = \mathbf{u}$ where, for given $\mathbf{v} \in B_\varepsilon$, function $\mathbf{u} \in \mathbf{V}$ is the unique solution of the linear problem:

Find $\mathbf{u} \in \ \mathbf{V}$ such that $\mathbf{u} = \varepsilon^2\mathbf{w}$ on Γ^- and, for every $\mathbf{w} \in \mathbf{W}$

$$\int\limits_{\Omega_\varepsilon} \nabla\mathbf{u} \ \cdot \ \nabla\mathbf{w} + Re \int\limits_{\Omega_\varepsilon} (\mathbf{v} \ \cdot \ \nabla)\mathbf{u} \ \cdot \ \mathbf{w} = \int\limits_{\Sigma_\varepsilon} q\,\mathbf{w} \ \cdot \ \mathbf{n} \ . \tag{40}$$

Due to the non-homogeneous boundary condition on Γ^-, we look for the solution of (40) in the form $\mathbf{u} = \mathbf{z} + \mathbf{b}_\varepsilon$, with $\mathbf{z} \in \mathbf{W}$ (function $\mathbf{b}_\varepsilon$ lifts the boundary value and was constructed in Lemma 2). Then the bilinear form

$$H(\mathbf{z}, \mathbf{w}) = \int\limits_{\Omega_\varepsilon} \nabla\mathbf{z} \ \cdot \ \nabla\mathbf{w} + Re \int\limits_{\Omega_\varepsilon} (\mathbf{v} \ \cdot \ \nabla)\mathbf{z} \ \cdot \ \mathbf{w}$$

is coercive on $\mathbf{W} \times \mathbf{W}$. Now the Lax-Milgram theorem guarantees the existence and uniqueness of the solution of the problem

$$H(\mathbf{z}, \mathbf{w}) = \int\limits_{\Sigma_\varepsilon} q\,\mathbf{w} \ \cdot \ \mathbf{n} - \int\limits_{\Omega_\varepsilon} \nabla\mathbf{b}_\varepsilon \cdot \nabla\mathbf{w} + Re \int\limits_{\Omega_\varepsilon} (\mathbf{v} \cdot \nabla)\mathbf{b}_\varepsilon \cdot \mathbf{w} \ , \ \ \mathbf{w} \in \mathbf{W} \ .$$

We proceed by proving that $T(B_\varepsilon) \subset B_\varepsilon$. To construct the test function for the estimate, we need to homogenize the boundary condition for the velocity on Γ^-. Let $\mathbf{b}_\varepsilon$ be the function constructed in Lemma 2. We use the function

$$\mathbf{w} = \mathbf{u} - \ \mathbf{b}_\varepsilon \ \in \mathbf{V}$$

as the test function in (40). It gives (using Lemma 2 and Lemma 3)

$$\int_{\Omega_\varepsilon} |\nabla \mathbf{u}|^2 + Re \int_{\Omega_\varepsilon} (\mathbf{v} \cdot \nabla)\mathbf{u} \cdot \mathbf{u} = \int_{\Omega_\varepsilon} \nabla \mathbf{u} \cdot \nabla \mathbf{b}_\varepsilon +$$
$$+ Re \int_{\Omega_\varepsilon} (\mathbf{v} \cdot \nabla)\mathbf{u} \cdot \mathbf{b}_\varepsilon + \int_{\Sigma_\varepsilon} \left(q - \frac{Re}{Fr^2}\, \mathbf{x} \cdot \mathbf{g} \right) (\mathbf{u} - \mathbf{b}_\varepsilon) \cdot \mathbf{n} \leq$$
$$\leq \left[(C_b + C_\ell)\varepsilon^{3/2} + C_4^2 C_b Re\, \varepsilon^2\, |\nabla \mathbf{v}|_{L^2(\Omega_\varepsilon)} \right] |\nabla \mathbf{u}|_{L^2(\Omega_\varepsilon)} + C_\ell C_b \varepsilon^3 \quad .$$

Since, by assumption, $Re\, \varepsilon^2 \ll 1$, we have $M\, C_4^2 C_b Re\, \varepsilon^2 \leq (C_b + C_\ell)$, for ε sufficiently small. On the other hand, again for ε small enough

$$Re \int_{\Omega_\varepsilon} (\mathbf{v} \cdot \nabla)\mathbf{u} \cdot \mathbf{u} \leq Re\, \varepsilon^2\, C_4^2 M |\nabla \mathbf{u}|^2_{L^2(\Omega_\varepsilon)} \leq \frac{1}{2} |\nabla \mathbf{u}|^2_{L^2(\Omega_\varepsilon)}$$

so that

$$|\nabla \mathbf{u}|^2_{L^2(\Omega_\varepsilon)} \leq 4(C_b + C_\ell)\varepsilon^{3/2} |\nabla \mathbf{u}|_{L^2(\Omega_\varepsilon)} + 2 C_\ell C_b \varepsilon^3 \leq$$
$$\leq \frac{1}{2} |\nabla \mathbf{u}|^2_{L^2(\Omega_\varepsilon)} + \left[8(C_b + C_\ell)^2 + 2 C_\ell C_b \right] \varepsilon^3$$
$$\leq \frac{1}{2} |\nabla \mathbf{u}|^2_{L^2(\Omega_\varepsilon)} + \left[3(C_b + C_\ell)\, \varepsilon^{3/2} \right]^2 \quad ,$$

implying that

$$|\nabla T(\mathbf{v})|_{L^2(\Omega_\varepsilon)} \leq M \varepsilon^{3/2} \quad .$$

The rest of the proof follows exactly the same arguments as in the proof of Theorem 1. □

5.3. Two-scale limit

Theorem 4. *Let $(\mathbf{u}^\varepsilon, r^\varepsilon)$ be the solution to the problem (29) and let $m \in \{1, 2\}$. Then*

$$\varepsilon^{-2} \mathbf{u}^{\varepsilon,m} \to U_x^m\, \mathbf{e} + U_y^m\, \mathbf{j}_m \quad L^2(\Omega^m) - \textit{two scale} \tag{41}$$
$$\varepsilon^{-1} \nabla \mathbf{u}^{\varepsilon,m} \to \nabla_{xy^m} \mathbf{U}^m \quad L^2(\Omega^m) - \textit{two scale}$$
$$r^{\varepsilon,m} \to R^m(x, y^m) \quad L^2(\Omega^m) - \textit{two scale} \quad , \tag{42}$$

where

$$\mathbf{U}^m = (U_x^m, U_y^m) \in \mathbf{Y}_m^2 = \left\{ \mathbf{V} \in L^2(\Omega^m)^2\ ,\ \frac{\partial \mathbf{V}}{\partial \xi^m} \in L^2(\Omega^m)^2 \right\} \tag{43}$$
$$R^m \in L^2(\omega^m) \tag{44}$$
$$\mathbf{U}^m = 0 \quad on \quad \Gamma_m^+ \tag{45}$$
$$\mathbf{U}^m = w\, \mathbf{e} \quad on \quad \Gamma_m^- \tag{46}$$
$$div_{xy^m} \left(\int_0^{h_m(x,y^m)} \mathbf{U}^m(x, y^m, \xi^m) d\xi^m \right) = 0 \tag{47}$$

$$\nabla_{xy^m}\phi = \frac{\partial \phi}{\partial x}\,\mathbf{e} + \frac{\partial \phi}{\partial y^m}\,\mathbf{j}_m \tag{48}$$

$$div_{xy^m}\mathbf{U}^m = \frac{\partial U_x^m}{\partial x} + \frac{\partial U_y^m}{\partial y^m}\ , \tag{49}$$

and satisfies the two-scale problem

$$\sum_{m=1}^{2}\int_{\Omega^m}\left(\frac{\partial \mathbf{U}^m}{\partial \xi^m}\cdot\frac{\partial \Psi^m}{\partial \xi^m} + R^m\ div_{xy^m}\,\Psi^m\right) =$$

$$= \sum_{m=1}^{2}\int_{\Sigma_1^m}\left(q - \frac{Re}{Fr^2}\,\overline{\mathbf{x}}\cdot\mathbf{g}\right)(\Psi^m\cdot\mathbf{n}) \tag{50}$$

$$\textit{for any } \Psi^m = (\Psi_x^m, \Psi_y^m)\in \mathbf{Y}_m^2 \textit{ such that } \Psi^m = 0 \textit{ on } \Gamma_m^{\pm}\ , \tag{51}$$

with $\overline{\mathbf{x}} = x\,\mathbf{e} + y^m\,\mathbf{j}_m$. *Furthermore, functions* $\mathbf{U}^m$ *and* R^m *satisfy the coupled Reynolds equations*

$$\mathbf{U}^m = \frac{\xi^m}{2}(h_m - \xi^m)\nabla_{xy^m}R^m + w\left(1 - \frac{\xi^m}{h_m}\right)\mathbf{e} \tag{52}$$

$$div_{xy^m}\left(h_m^3\nabla_{xy^m}R^m\right) = 6\,w\,\frac{\partial h_m}{\partial x}\quad in\ \omega^m \tag{53}$$

$$R^m = q - \frac{Re}{Fr^2}\,\overline{\mathbf{x}}\cdot\mathbf{g}\ \ on\ \partial\omega^m\backslash\mathcal{C}$$

$$\int_0^{h_1}\mathbf{U}^1\cdot\mathbf{j}_1\ d\xi^1 = \int_0^{h_2}\mathbf{U}^2\cdot\mathbf{j}_2\ d\xi^2\ \ on\ \mathcal{C}\ \ \textit{(the junction condition)}. \tag{54}$$

Proof. Using the a priori estimates (32), we can now pass to the two-scale limit in each Ω_ε^m, using the compactness theorem ([8], Theorem 1, Proposition 4 and Proposition 7). Indeed, denoting $\mathbf{u}^{\varepsilon,m}$ and $r^{\varepsilon,m}$ the restrictions on Ω_ε^m, those a priori estimates imply

$$\begin{aligned}
&\varepsilon^{-2}\mathbf{u}^{\varepsilon,m}\ \rightarrow\ U_x^m(x,y^m,\xi^m)\,\mathbf{e} + U_y^m(x,y^m,\xi^m)\,\mathbf{j}_m\ \ L^2-\text{two scale on }\Omega_\varepsilon^m\\
&\varepsilon^{-1}\nabla\mathbf{u}^{\varepsilon,m}\ \rightarrow\ \frac{\partial \mathbf{U}^m}{\partial \xi^m}\quad L^2-\text{two scale on }\ \Omega_\varepsilon^m\\
&r^{\varepsilon,m}\ \rightarrow\ R^m(x,y^m)\quad L^2-\text{two scale on }\ \Omega_\varepsilon^m\ \ ,
\end{aligned} \tag{55}$$

where (43)-(49) holds. As in a previous case, we need to construct a good test-function, write down a variational formulation and pass to the two-scale limit. We start by taking two functions

$$\Psi^m\in\mathbf{Y}_m^2\cap C^1(\overline{\Omega_1^m})^2\ \ ,\ \ m = 1,2,$$

where Ω_1^m stands for Ω_ε^m with $\varepsilon = 1$. We need to correct the test function in a boundary layer near the junction. Before we proceed we have to precise the geometry of the layer. Let S_ε^1 and S_ε^2 be defined by (35). Then the layer around the junction is defined as in (36) by

$$\mathcal{S}_\varepsilon = S_\varepsilon^1\cup S_\varepsilon^2$$

and the interfaces between $\mathcal{S}_\varepsilon$ and $\Omega_\varepsilon^m\backslash\mathcal{S}_\varepsilon$ are denoted γ_ε^1 and γ_ε^2 like in (37). We impose the condition

$$\int_0^{h_1(x,0)} \Psi^1(x,0,\xi^1)\cdot \mathbf{j}_1\, d\xi^1 = \int_0^{h_2(x,0)} \Psi^2(x,0,\xi^2)\cdot \mathbf{j}_2\, d\xi^2$$

We chose Ψ^3 on $\mathcal{S}$ (here $\mathcal{S}$ denotes $\mathcal{S}_\varepsilon$ with $\varepsilon = 1$, as in (38) and γ^m denotes γ_ε^m with $\varepsilon = 1$ as in (39)) such that

$$\begin{aligned}
&\mathrm{div}_{\eta^1\xi^1}\, \Psi^3 = \frac{\partial \Psi_y}{\partial \eta^1} + \frac{\partial \Psi_z}{\partial \xi^1} = 0 \ \text{ in } \ \mathcal{S}_1 \\
&\Psi^3(x, h_2^\infty, \xi^1) = \Psi^1(x, \varepsilon h_2^\infty, \xi^1) \ \text{ for } \ 0 < x < \ell\,,\ 0 < \xi^1 < h_1(x,0) \\
&\Psi^3(x, h_1^\infty, \xi^2) = \Psi^2(x, \varepsilon h_1^\infty, \xi^2) \ \text{ for } \ 0 < x < \ell\,,\ 0 < \xi^2 < h_1(x,0) \\
&\Psi^3 = 0 \ \text{ on } \ \partial\mathcal{S}_1 \backslash (\gamma^2 \cup \gamma^1)\ .
\end{aligned}$$

Then we pose

$$\Psi_\varepsilon(\mathbf{x}) = \begin{cases} \Psi^1\left(x, y^1, \frac{z^1}{\varepsilon}\right) & \text{in } \ \Omega_\varepsilon^1 \backslash \mathcal{S}_\varepsilon \\ \Psi^2\left(x, y^2, \frac{z^2}{\varepsilon}\right) & \text{in } \ \Omega_\varepsilon^2 \backslash \mathcal{S}_\varepsilon \\ \Psi^3\left(x, \frac{y^1}{\varepsilon}, \frac{z^1}{\varepsilon}\right) & \text{in } \ \mathcal{S}_\varepsilon \end{cases}$$

Using Ψ_ε as a test function leads to

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla \mathbf{u}^\varepsilon \cdot \nabla \Psi_\varepsilon + Re \int_{\Omega_\varepsilon} (\mathbf{u}^\varepsilon\cdot\nabla)\mathbf{u}^\varepsilon \cdot \Psi_\varepsilon - \int_{\Omega_\varepsilon} r^\varepsilon\, \mathrm{div}\, \Psi_\varepsilon = \\
&= \int_{\Sigma_\varepsilon} \left(q - \frac{Re}{Fr^2}\, \mathbf{x}\cdot\mathbf{g}\right)(\Psi_\varepsilon\cdot\mathbf{n})
\end{aligned} \tag{56}$$

We multiply the equation (56) by $\frac{1}{\varepsilon}$ and we get for the right hand side

$$\frac{1}{\varepsilon}\int_{\Sigma_\varepsilon} \left(q - \frac{Re}{Fr^2}\, \mathbf{x}\cdot\mathbf{g}\right)(\Psi_\varepsilon\cdot\mathbf{n}) = \int_{\Sigma_1} \left(q - \frac{Re}{Fr^2}\, \overline{\mathbf{x}}\cdot\mathbf{g}\right)(\Psi_1\cdot\mathbf{n}) + O(\varepsilon)\ ,$$

with Ψ_1 denoting Ψ_ε for $\varepsilon = 1$. The third integral on the left hand-side

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} r^\varepsilon\, \mathrm{div}\, \Psi_\varepsilon = \frac{1}{\varepsilon}\sum_{m=1}^2 \int_{\Omega_\varepsilon^m\backslash\mathcal{S}_\varepsilon} r^\varepsilon\, \mathrm{div}_{xy^m}\, \Psi_\varepsilon + \frac{1}{\varepsilon}\int_{\mathcal{S}_\varepsilon} r^\varepsilon\, \frac{\partial(\Psi_\varepsilon)_1}{\partial x} = \\
&= \frac{1}{\varepsilon}\sum_{m=1}^2 \int_{\Omega_\varepsilon^m} r^\varepsilon\, \mathrm{div}_{xy^m}\, \Psi_\varepsilon + O(\varepsilon) \to \sum_{m=1}^2 \int_{\Omega^m} R^m\, \mathrm{div}_{xy^m}\, \Psi^m
\end{aligned}$$

The second integral on the left-hand side

$$\frac{Re}{\varepsilon}\int_{\Omega_\varepsilon} (\mathbf{u}^\varepsilon\cdot\nabla)\mathbf{u}^\varepsilon\cdot\ \Psi_\varepsilon \le C\ Re\, |\nabla\mathbf{u}^\varepsilon|^2_{L^2(\Omega_\varepsilon)} |\Psi_\varepsilon|_{L^\infty(\Omega_\varepsilon)} \le C\varepsilon^3\ Re \to 0\ .$$

Finally, for the first integral on the left-hand side

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\nabla\mathbf{u}^\varepsilon\cdot\nabla\Psi_\varepsilon=\frac{1}{\varepsilon}\sum_{m=1}^{2}\int_{\Omega_\varepsilon^m\setminus\mathcal{S}_\varepsilon}\nabla\mathbf{u}^\varepsilon\cdot\nabla\Psi_\varepsilon+\frac{1}{\varepsilon}\int_{\mathcal{S}_\varepsilon}\nabla\mathbf{u}^\varepsilon\cdot\nabla\Psi_\varepsilon= \\ =I_1+I_2\quad .$$

Now

$$I_2\le\frac{1}{\varepsilon^2}|\nabla\mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)}|\nabla\Psi_\varepsilon|_{L^\infty(\mathcal{S}_\varepsilon)}|\mathcal{S}_\varepsilon|^{1/2}\le C\varepsilon^{1/2}\to 0$$

and

$$I_1=\frac{1}{\varepsilon}\int_{\Omega_\varepsilon^1}\nabla\mathbf{u}^\varepsilon\cdot\nabla\left[\Psi^1\left(x,y^1,\frac{z^1}{\varepsilon}\right)\right]+\frac{1}{\varepsilon}\int_{\Omega_\varepsilon^2}\nabla\mathbf{u}^\varepsilon\cdot\nabla\left[\Psi^2\left(x,y^2,\frac{z^2}{\varepsilon}\right)\right]- \\ -\frac{1}{\varepsilon}\left\{\int_{\mathcal{S}_\varepsilon\cap\Omega_\varepsilon^1}\nabla\mathbf{u}^\varepsilon\cdot\nabla\left[\Psi^1\left(x,y^1,\frac{z^1}{\varepsilon}\right)\right]+\int_{\mathcal{S}_\varepsilon\cap\Omega_\varepsilon^2}\nabla\mathbf{u}^\varepsilon\cdot\nabla\left[\Psi^2\left(x,y^2,\frac{z^2}{\varepsilon}\right)\right]\right\}\ .$$

The last two integrals can be estimated in the same manner as I_2 and they tend to zero as $\varepsilon\to 0$. For smooth Ψ^m we obviously have

$$\nabla\Psi^1\left(x,y^1,\frac{z^1}{\varepsilon}\right)=\nabla_{xy^1}\Psi^1\left(x,y^1,\frac{z^1}{\varepsilon}\right)+\varepsilon^{-1}\,\frac{\partial\Psi^1}{\partial\xi_1}\left(x,y^1,\frac{z^1}{\varepsilon}\right)$$

and

$$\nabla\Psi^2\left(x,y^2,\frac{z^2}{\varepsilon}\right)=\nabla_{xy^2}\Psi^2\left(x,y^2,\frac{z^2}{\varepsilon}\right)+\varepsilon^{-1}\,\frac{\partial\Psi^2}{\partial\xi_2}\left(x,y^2,\frac{z^2}{\varepsilon}\right)\quad .$$

Since

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon^1}\nabla_{xy^1}\mathbf{u}^\varepsilon\cdot\nabla_{xy^1}\left[\Psi^1\left(x,y^1,\frac{z^1}{\varepsilon}\right)\right]\le\frac{\sqrt{|\Omega_\varepsilon^1|}}{\varepsilon}|\nabla\mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)}|\nabla\Psi^1|_{L^\infty(\Omega^1)}\le C\varepsilon\to 0$$

and

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon^2}\nabla_{xy^2}\mathbf{u}^\varepsilon\cdot\nabla_{xy^2}\left[\Psi^2\left(x,y^2,\frac{z^2}{\varepsilon}\right)\right]\le\frac{\sqrt{|\Omega_\varepsilon^2|}}{\varepsilon}|\nabla\mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)}|\nabla\Psi^2|_{L^\infty(\Omega^2)}\le C\varepsilon\to 0\quad .$$

It only remains to pass to the limit in the integrals

$$\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon^m}\frac{\partial\mathbf{u}^\varepsilon}{\partial\xi^m}\cdot\left(\frac{\partial\Psi^m}{\partial\xi^m}\right)\left(x,y^m,\frac{z^m}{\varepsilon}\right)\quad .$$

Using the two-scale compactness (55), we get for $m=1,2$

$$\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon^m}\frac{\partial\mathbf{u}^\varepsilon}{\partial\xi^m}\cdot\left(\frac{\partial\Psi^m}{\partial\xi^m}\right)\left(x,y^m,\frac{z^m}{\varepsilon}\right)\to\int_{\Omega^m}\frac{\partial\mathbf{U}^0}{\partial\xi^m}\cdot\frac{\partial\Psi^m}{\partial\xi^m}\quad .$$

Finally, we have arrived at the two-scale problem (50). As in the previous section, we can prove that the two-scale problem (45), (46), (47) and (50) has a unique solution. That solution can be constructed by separating the scales in (50) and it satisfies the coupled Reynolds equations (53). □

Remark 3. The junction condition (54) combined with (52) gives the continuity of the fluxes across the junction line $\mathcal{C}$

$$h_1^3 \frac{\partial R^1}{\partial y^1} = h_2^3 \frac{\partial R^2}{\partial y^2} \quad \text{on} \quad \mathcal{C} \ .$$

Furthermore, we need to remember that we have redefined the pressure by taking (28). If we go back to the original pressure

$$p^\varepsilon = r^\varepsilon + \frac{Re}{Fr^2}\mathbf{x}\cdot\mathbf{g}$$

in each Ω^m, $m = 1, 2$, converges to

$$P^m = R^m + \frac{Re}{Fr^2}(x\,\mathbf{e}\cdot\mathbf{g} + y_m\,\mathbf{j}_m\cdot\mathbf{g}) = R^m + \frac{Re}{Fr^2}\,\overline{\mathbf{x}}\cdot\mathbf{g} \ .$$

Now the junction condition for physical pressure P reads

$$h_1^3 \frac{\partial P^1}{\partial y^1} + \frac{Re}{Fr^2}\,\mathbf{j}_1\cdot\mathbf{g} = h_2^3 \frac{\partial P^2}{\partial y^2} + \frac{Re}{Fr^2}\,\mathbf{j}_2\cdot\mathbf{g} \quad \text{on} \quad \mathcal{C} \ .$$

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