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# Sharp lower bounds for the Widom factors on the real line

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ABSTRACT

We derive lower bounds for the $L^p(\mu)$ norms of monic extremal polynomials with respect to compactly supported probability measures $\mu$. We obtain a sharp universal lower bound for all $0 < p < \infty$ and all measures in the Szegő class and an improved lower bound on $L^2(\mu)$ norm for several classes of orthogonal polynomials including Jacobi polynomials, isospectral torus of a finite gap set, and orthogonal polynomials with respect to the equilibrium measure of an arbitrary non-polar compact subset of $\mathbb{R}$.



## 1. Introduction

Let $K$ be a non-polar compact subset of $\mathbb{C}$ and $\mu$ a probability Borel measure with $\operatorname{supp}(\mu) = K$. In this work we investigate lower bounds on the $L^p(\mu)$ norms of monic polynomials. A well known inequality that goes back to Szegő [41] (for a textbook presentation see [30, Theorem 5.5.4] or [34, Theorem 5.7.8]) provides such a lower bound for $L^\infty(K)$ norm,

$$\|P_n\|_\infty = \sup_{z\in K} |P_n(z)| \geq C(K)^n, \quad P_n \in \mathcal{P}_n,\ n \in \mathbb{N}, \tag{1.1}$$

where $\mathcal{P}_n$ is the set of monic polynomials of degree $n$ and $C(K)$ denotes the logarithmic capacity of $K$. The inequality (1.1) is sharp in the class of subsets of $\mathbb{C}$, however for compact sets $K \subset \mathbb{R}$, Schiefermayr [32] showed that the inequality can be improved to

$$\|P_n\|_\infty \geq 2C(K)^n, \quad P_n \in \mathcal{P}_n,\ n \in \mathbb{N}, \tag{1.2}$$

which is optimal in the class of subsets of $\mathbb{R}$.

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We are interested in finding sharp analogs of the above inequalities for $L^p(\mu)$ norms. To simplify the notation we introduce the Widom factors,

$$W_n^p(\mu) = \frac{\inf_{P_n \in \mathcal{P}_n} \|P_n\|_p}{C(K)^n}, \quad n \in \mathbb{N},\ 0 < p \le \infty, \tag{1.3}$$

where as usual $\|P_n\|_p = \left(\int |P_n(z)|^p d\mu(z)\right)^{1/p}$, $0 < p < \infty$. The infimum in (1.3) is attained for some polynomial in $\mathcal{P}_n$ and a minimizer is unique when $p \in (1, \infty]$ but not necessarily when $p \in (0, 1]$. We note that in the case $p = \infty$ the Widom factors are defined in terms of the (unweighted) supremum norm over $K$ and so do not depend on a particular choice of the measure $\mu$ and hence will be denoted by $W_n^\infty(K)$. Despite this difference in the definition of the Widom factors for $p = \infty$ and $p < \infty$ it follows from Hölder's inequality and continuity of $\|P_n\|_p$ with respect to $p \in (0, \infty]$ that for each fixed $n \in \mathbb{N}$ and probability measure $\mu$ with $\mathrm{supp}(\mu) = K$ the Widom factors are nondecreasing and continuous with respect to $p \in (0, \infty]$, in particular, $W_n^\infty(K) = \sup_{p>0} W_n^p(\mu)$.

From the application point of view there are two important cases $p = \infty$ and $p = 2$. The monic polynomials that have the smallest $L^\infty(K)$ norm are known as the Chebyshev polynomials and those that have the smallest $L^2(\mu)$ norm are the orthogonal polynomials with respect to $\mu$. We use the term *Widom factors* to commemorate the fundamental paper [48] where H. Widom studied asymptotics of the Chebyshev and orthogonal polynomials on sets $K$ consisting of a finite number of smooth Jordan curves and arcs. More recently, asymptotics and upper bounds on $W_n^\infty(K)$ have been studied in [5,6,11,14–16,18,20,42–47]. Due to monotonicity of the Widom factors, an upper bound on $W_n^\infty(K)$ is automatically an upper bound for all $W_n^p(\mu)$. The main contribution of the present work is complementary sharp lower bounds for the Widom factors $W_n^p(\mu)$.

For absolutely continuous measures $\mu$ on the unit circle a lower bound and asymptotics of $W_n^2(\mu)$ date back to the work of Szegő [38,39] (for a textbook presentation see [33, Sections 2.2 and 2.3]). Asymptotics of $W_n^2(\mu)$ for more general measures and on other sets have been actively studied ever since [3,4,7–10,13, 17,19,23,26,28,29,33–35,40,48]. In these works the central role is played by measures $\mu$ from the Szegő class which in the most general setting is defined as follows. Given a probability measure $\mu$ with $K = \mathrm{supp}(\mu)$ a non-polar compact subset of $\mathbb{C}$, denote by $\mu_K$ the equilibrium measure of $K$ (see [30] or [34, Section 5.5] for basic notions of logarithmic potential theory) and consider the Lebesgue decomposition of $\mu$ with respect to $\mu_K$, that is, $d\mu = f d\mu_K + d\mu_s$. The Szegő class consists of such measures $\mu$ that have finite relative entropy with respect to $\mu_K$, that is,

$$\int \log f(z) d\mu_K(z) > -\infty. \tag{1.4}$$

The relative entropy enters the asymptotics and lower bounds via the exponential relative entropy function

$$S(\mu) = \exp\left[\int \log f(z) d\mu_K(z)\right]. \tag{1.5}$$

As with the lower bound in the case $p = \infty$ (cf., (1.1) vs (1.2)) there is a difference in the asymptotics of $W_n^2(\mu)$ depending on whether the measure $\mu$ is supported on $\mathbb{R}$ or on $\mathbb{C}$, for example,

$$\lim_{n\to\infty} \left[W_n^2(\mu)\right]^2 = S(\mu) \tag{1.6}$$

for measures $\mu$ with $\mathrm{supp}(\mu) = \partial\mathbb{D}$ (see for example [33, Theorem 2.3.1]) and

$$\lim_{n\to\infty} \left[W_n^2(\mu)\right]^2 = 2S(\mu) \tag{1.7}$$

for measures $\mu$ with $\mathrm{supp}(\mu) = [-2, 2]$ (see for example [28], [33, Theorem 13.8.8]).

Recently a lower bound for the Widom factors $W_n^2(\mu)$ was obtained in [1] for the equilibrium measure $\mu = \mu_K$ of a general non-polar compact set $K \subset \mathbb{R}$ and in [2] for a general Szegő class measure $\mu$ on $\mathbb{C}$,

$$\left[W_n^2(\mu)\right]^2 \geq S(\mu). \tag{1.8}$$

The goal of the present work is to extend (1.8) to all Widom factors $W_n^p(\mu)$ and investigate to what extent such a lower bound is sharp and whether it can be improved for measures supported on $\mathbb{R}$ with a special emphasis on the case $p = 2$. It turns out that the lower bound (1.8) as well as its extension to $0 < p < \infty$ (2.1) are sharp in the Szegő class even for measures with $\mathrm{supp}(\mu) = [-2, 2]$. Nevertheless, we will show that for several special classes of measures on $\mathbb{R}$ the lower bound (1.8) can be improved. In particular, for the equilibrium measures $\mu = \mu_K$ of compact non-polar sets $K \subset \mathbb{R}$ the lower bound (1.8) improves by a factor of 2,

$$\left[W_n^2(\mu)\right]^2 \geq 2S(\mu). \tag{1.9}$$

In light of the asymptotics (1.7), the lower bound (1.9) is the best possible. We also obtain similar improvements on the lower bounds of Widom factors $W_n^p(\mu_K)$ for $p > 1$. Besides the equilibrium measure we prove the optimal lower bound (1.9) for measures from the finite gap isospectral torus of half-line Jacobi matrices and for Jacobi weights $d\mu_{\alpha,\beta}(x) = c_{\alpha,\beta}(1-x)^\alpha(1+x)^\beta\chi_{[-1,1]}(x)dx$ on $[-1, 1]$ for a certain range of parameters $\alpha, \beta$.

We want to emphasize that much less is known on the asymptotics of $L^p(\mu)$ extremal polynomials for a general $p$ ($0 < p < \infty$ with $p \neq 2$). We refer the reader to [22,24,27] for some of the previous attempts on this problem. Peherstorfer and Steinbauer suggested this as an open problem in Problem 3, p. 314, [27]. Theorems 2.1 and 2.2 below can be seen as a partial answer to this open problem for a large class of measures. Besides, we hope that our conjecture (6.2) will generate further research in this direction.

The plan of the paper is as follows. In Section 2, we extend the lower bound of (1.8) to the case of general Widom factors $W_n^p(\mu)$ and show that the bound is optimal not only in the class of measures on the complex plane but also on the real line. In particular, for measures on $\mathbb{R}$ the Szegő condition alone is insufficient for (1.9). In Section 3 we obtain increased lower bounds on $W_n^p(\mu_K)$ for the equilibrium measures $\mu_K$ on compact non-polar subsets of $\mathbb{R}$. In Section 4 we consider lower bounds on $W_n^2(\mu_{\alpha,\beta})$ for the Jacobi weights over the full range of parameters. In Section 5, we prove (1.9) for measures $\mu$ associated with half-line Jacobi matrices from finite gap isospectral tori. Finally, in Section 6, we discuss some open problems.

## 2. A sharp lower bound for the Widom factors

In this section we extend the lower bound (1.8) to the general Widom factors and show that our lower bound is optimal in the class of Szegő measures even if the support of the measure is an interval on the real line.

**Theorem 2.1.** *Let $0 < p < \infty$ and $\mu$ be a Borel probability measure with $K = \mathrm{supp}(\mu)$ a non-polar compact subset of $\mathbb{C}$. Then*

$$W_n^p(\mu) \geq S(\mu)^{1/p}, \quad n \in \mathbb{N}. \tag{2.1}$$

**Proof.** If $S(\mu) = 0$ there is nothing to prove. Let us assume that $S(\mu) > 0$. We modify the argument used in the proof of Theorem 1.2 in [2]. Let $d\mu = f\mu_K + d\mu_s$ and write $P_n \in \mathcal{P}_n$ as $P_n(z) = \prod_{j=1}^n (z - z_j)$. Then

$$\|P_n\|_p^p = \left(\int |P_n|^p f\, d\mu_K + \int |P_n|^p d\mu_s\right) \geq \int |P_n|^p f\, d\mu_K \tag{2.2}$$

$$= \exp\left[\log\left(\int |P_n|^p f \, d\mu_K\right)\right] \tag{2.3}$$

$$\geq \exp\left[\int \log\left(|P_n|^p f\right) d\mu_K\right] \tag{2.4}$$

$$= \exp\left[\int \log f \, d\mu_K\right] \exp\left[p \int \sum_{j=1}^{n} \log|z - z_j| d\mu_K(z)\right] \tag{2.5}$$

$$\geq S(\mu) C(K)^{np}. \tag{2.6}$$

Note that, (2.4) follows from Jensen's inequality and (2.6) follows from Frostman's theorem, see Theorem 3.3.4 (a) in [30]. The inequality (2.1) follows by taking $p$-th root and dividing by $C(K)^n$. $\square$

It is easy to see that (2.1) is sharp in the class of probability measures on the complex plane since for the equilibrium measure on the unit circle $\mu_{\partial\mathbb{D}}$ we have $1 = S(\mu_{\partial\mathbb{D}})^{1/p} \leq W_n^p(\mu) \leq \|z^n\|_p = 1$ for all $n \in \mathbb{N}$ and $0 < p < \infty$.

The next result shows that for $0 < p < \infty$, $S(\mu)$ is the best possible lower bound for $W_n^p(\mu)$ in the Szegő class of probability measures on the real line, in fact, even on an interval.

**Theorem 2.2.** *For each $0 < p < \infty$ and $n \in \mathbb{N}$ fixed,*

$$\inf_{\mu} \left[W_n^p(\mu)\right]^p / S(\mu) = 1, \tag{2.7}$$

*where the infimum is taken over probability measures on $K = [-2, 2]$ with $S(\mu) > 0$.*

**Proof.** First, assume that $np \geq 1$. Let $N$ be the integer satisfying $np - 1 < N \leq np$ and consider the measures

$$d\mu_\varepsilon(x) = c_\varepsilon |x|^{N-np} \prod_{j=1}^{N} |x^2 - j^2\varepsilon^2|^{-1/2} d\mu_K(x), \quad \varepsilon > 0,$$

where $c_\varepsilon > 0$ is the normalization constant chosen such that $\mu_\varepsilon(K) = 1$. The equilibrium measure is given by $d\mu_K(x) = \frac{1}{\pi} \frac{\chi_K(x)}{\sqrt{4-x^2}} dx$. Since $K$ is a regular set for potential theory, by Frostman's theorem, the logarithmic potential $U_K(z) = \int \log|x - z| d\mu_K(x)$ equals $\log C(K) = 0$ for all $z \in K$, so for each $0 < \varepsilon < 1/N$ we get

$$S(\mu_\varepsilon) = \exp\left[\int \log\left(c_\varepsilon |x|^{N-np} \prod_{j=1}^{N} |x^2 - j^2\varepsilon^2|^{-1/2}\right) d\mu_K(x)\right]$$

$$= c_\varepsilon \exp\left[(N - np) U_K(0) - \frac{1}{2} \sum_{j=1}^{N} \left(U_K(j\varepsilon) + U_K(-j\varepsilon)\right)\right] = c_\varepsilon. \tag{2.8}$$

On the other hand, we have

$$W_n^p(\mu_\varepsilon)^p \leq \int |x^n|^p d\mu_\varepsilon(x) = c_\varepsilon \int \frac{|x|^N}{\prod_{j=1}^N |x^2 - j^2\varepsilon^2|^{1/2}} d\mu_K(x) \to c_\varepsilon \tag{2.9}$$

as $\varepsilon \to 0$ since

$$\frac{|x|^N}{\prod_{j=1}^N |x^2 - j^2\varepsilon^2|^{1/2}} = \frac{1}{\prod_{j=1}^N |1 - (j\varepsilon/x)^2|^{1/2}} \leq \frac{1}{|1 - \frac{N^2}{(N+1)^2}|^{N/2}} \text{ for } |x| \geq (N+1)\varepsilon$$

so by the dominated convergence theorem,

$$\int\limits_{|x|\geq(N+1)\varepsilon} \frac{|x|^N}{\prod_{j=1}^N |x^2 - j^2\varepsilon^2|^{1/2}} d\mu_K(x) \to \int d\mu_K = 1 \text{ as } \varepsilon \to 0$$

and

$$\int\limits_{|x|<(N+1)\varepsilon} \frac{|x|^N}{\prod_{j=1}^N |x^2 - j^2\varepsilon^2|^{1/2}} d\mu_K(x) = \int\limits_{-1}^{1} \frac{|t|^N}{\prod_{j=1}^N |t^2 - \frac{j^2}{(N+1)^2}|^{1/2}} d\mu_K((N+1)\varepsilon t)$$
$$\leq \frac{1}{\pi} \int\limits_{-1}^{1} \frac{|t|^N}{\prod_{j=1}^N |t^2 - \frac{j^2}{(N+1)^2}|^{1/2}} \frac{(N+1)\varepsilon dt}{\sqrt{4-(N+1)^2\varepsilon^2 t^2}} \to 0 \text{ as } \varepsilon \to 0.$$

Thus, by (2.8) and (2.9), $\limsup_{\varepsilon\to 0} W_n^p(\mu_\varepsilon)^p/S(\mu_\varepsilon) \leq 1$. This combined with (2.1) yields (2.7).

Next, assume that $np < 1$. Consider the measure $d\nu(x) = c|x|^{-np} d\mu_K(x)$ where $c > 0$ is chosen so that $\nu(K) = 1$. Then

$$S(\nu) = \exp\left[\int \log\left(c|x|^{-np}\right) d\mu_K(x)\right]$$
$$= c \exp[(-np)U_K(0)] = c.$$

We also have

$$W_n^p(\nu)^p \leq \int |x^n|^p d\nu(x) = c \int d\mu_K(x) = c$$

Thus, $W_n^p(\nu)^p/S(\nu) \leq 1$. This combined with (2.1) yields (2.7) in the case $np < 1$. $\square$

## 3. Lower bounds for the equilibrium measures on subsets of $\mathbb{R}$

In this section we improve the lower bound (2.1) for equilibrium measures on general compact non-polar subsets of $\mathbb{R}$.

**Theorem 3.1.** *Let $K \subset \mathbb{R}$ be a compact non-polar set. Then for each $p > 1$,*

$$W_n^p(\mu_K) \geq 2\left(\frac{(m!)^2}{(2m)!}\right)^{\frac{1}{2m}} > S(\mu_K)^{1/p} = 1, \quad n \in \mathbb{N}, \tag{3.1}$$

*where $m = \lceil \frac{p}{2(p-1)} \rceil$. In particular, for $p \geq 2$,*

$$W_n^p(\mu_K) \geq \sqrt{2}, \quad n \in \mathbb{N}, \tag{3.2}$$

*and the case $p = 2$ is the improved lower bound (1.9) which is sharp in the class of equilibrium measures of non-polar compact subsets of $\mathbb{R}$.*

**Proof.** First, note that (3.2) is a special case of (3.1) and $m \geq p/[2(p-1)]$ is equivalent to $p \geq 2m/(2m-1)$. Since $W_n^p(\mu_K)$ is nondecreasing with respect to $p$ it suffices to prove (3.1) for $p = 2m/(2m-1)$, $m \in \mathbb{N}$.

Next, we prove (3.1) in the special case of a finite gap compact set $K \subset \mathbb{R}$. In this setting we recall the uniformization map for finite gap sets as discussed in [12] or Sections 9.5–9.7 in [34]. The uniformization map is a unique conformal map $\mathbf{x} : \mathbb{D} \to \overline{\mathbb{C}} \backslash K$ normalized by $\mathbf{x}(0) = \infty$ and $\lim_{z\to 0} z\mathbf{x}(z) > 0$. It is known that $\mathbf{x}$ is symmetric under complex conjugation, $\mathbf{x}(\bar{z}) = \overline{\mathbf{x}(z)}$, has an analytic extension to $\overline{\mathbb{C}}\backslash\Lambda$, where $\Lambda \subset \partial\mathbb{D}$ is a certain null set, and $\mathbf{x} : \partial\mathbb{D}\backslash\Lambda \to K$ preserves the equilibrium measure (cf., Corollary 4.6 in [12] or Theorem 9.7.6 in [34]),

$$\int_K f(x)d\mu_K(x) = \int_0^{2\pi} f(\mathbf{x}(e^{i\theta}))\frac{d\theta}{2\pi}, \quad f \in L^1(d\mu_K). \tag{3.3}$$

In the following we will also need the associated Blaschke product $B(z)$ which is the unique bounded analytic function on $\mathbb{D}$ with $|B(e^{i\theta})| = 1$ a.e. on $\partial\mathbb{D}$, zeros at $\mathbf{x}^{-1}(\infty)$, and normalized by $\lim_{z\to 0} z^{-1}B(z) > 0$. By Theorem 4.4 in [12] or Theorem 9.7.5 in [34] the Blaschke product $B(z)$ has a connection to the Green function $G_K(z)$ of the domain $\overline{\mathbb{C}}\backslash K$ via $|B(z)| = \exp[-G_K(\mathbf{x}(z))]$ and it satisfies (cf. (9.7.35) and (9.7.37) in [34])

$$\lim_{z\to 0} \mathbf{x}(z)B(z) = C(K). \tag{3.4}$$

Now consider an arbitrary monic polynomial $P_n(x)$ of degree $n$, and let $Q_n(x) = \mathrm{Re}(P_n(x))$, $x \in \mathbb{R}$. Then $Q_n(x)$ is a monic polynomial with coefficients given by the real parts of the coefficients of $P_n(x)$, $Q_n(x)$ is real-valued on $K$, and satisfies $\|P_n\|_p \geq \|\mathrm{Re}(P_n)\|_p = \|Q_n\|_p$. In addition, $B(z)^n Q_n(\mathbf{x}(z))$ has only removable singularities on $\mathbb{D}$ and hence can be identified with a bounded analytic function with $\lim_{z\to 0} B(z)^n Q_n(\mathbf{x}(z)) = C(K)^n$ by (3.4). Thus,

$$C(K)^n = \int_0^{2\pi} Q_n(\mathbf{x}(e^{i\theta}))B(e^{i\theta})^n \frac{d\theta}{2\pi}.$$

Since the complex conjugation does not change the LHS and $Q_n(\mathbf{x}(e^{i\theta}))$ we have

$$2C(K)^n = \int_0^{2\pi} Q_n(\mathbf{x}(e^{i\theta}))\big(B(e^{i\theta})^n + \overline{B(e^{i\theta})^n}\big)\frac{d\theta}{2\pi}.$$

Applying Hölder's inequality, (3.3), and noting that $B(e^{i\theta})^{-1} = \overline{B(e^{i\theta})}$ we obtain

$$\begin{aligned}
2C(K)^n &\leq \left[\int_0^{2\pi} |Q_n(\mathbf{x}(e^{i\theta}))|^p \frac{d\theta}{2\pi}\right]^{\frac{1}{p}} \left[\int_0^{2\pi} \big(B(e^{i\theta})^n + B(e^{i\theta})^{-n}\big)^{2m} \frac{d\theta}{2\pi}\right]^{\frac{1}{2m}} \\
&= \left[\int_K |Q_n(x)|^p d\mu_K(x)\right]^{\frac{1}{p}} \left[\int_0^{2\pi} \sum_{j=0}^{2m} \binom{2m}{j} B(e^{i\theta})^{2n(m-j)} \frac{d\theta}{2\pi}\right]^{\frac{1}{2m}} \\
&= \|Q_n\|_p \left[\binom{2m}{m} + 2\mathrm{Re}\sum_{j=0}^{m-1} \binom{2m}{j} \int_0^{2\pi} B(e^{i\theta})^{2n(m-j)} \frac{d\theta}{2\pi}\right]^{\frac{1}{2m}}.
\end{aligned} \tag{3.5}$$

Since $\int_0^{2\pi} B(e^{i\theta})^k \frac{d\theta}{2\pi} = B^k(0) = 0$ for all $k \in \mathbb{N}$ and $\|Q_n\|_p \leq \|P_n\|_p$ we get

$$2C(K)^n \leq \|P_n\|_p \binom{2m}{m}^{\frac{1}{2m}} = \|P_n\|_p \left(\frac{(2m)!}{(m!)^2}\right)^{\frac{1}{2m}} \tag{3.6}$$

which after rearranging yields (3.1) for finite gap sets $K \subset \mathbb{R}$.

Finally, we extend (3.1) to general non-polar compact sets $K \subset \mathbb{R}$ via an approximation argument of [1]. By Theorem 5.8.4 in [34] there exist finite gap sets $\{K_j\}_{j=1}^{\infty}$ such that $K \subset \cdots \subset K_{j+1} \subset K_j \subset \cdots \subset K_1 \subset \mathbb{R}$, $K = \cap_{j=1}^{\infty} K_j$, $C(K_j) \to C(K)$, and $d\mu_{K_j} \to d\mu_K$ in the weak star sense as $j \to \infty$. Then for every monic polynomial $P_n(x)$ of degree $n$ we have by the finite gap lower bound that

$$\begin{aligned}\|P_n\|_{p,\mu_K} &= \liminf_{j\to\infty} \|P_n\|_{p,\mu_{K_j}} \geq \liminf_{j\to\infty} W_n^p(\mu_{K_j}) C(K_j)^n \\ &\geq 2\left(\frac{(m!)^2}{(2m)!}\right)^{\frac{1}{2m}} \liminf_{j\to\infty} C(K_j)^n = 2\left(\frac{(m!)^2}{(2m)!}\right)^{\frac{1}{2m}} C(K)^n. \end{aligned} \tag{3.7}$$

Dividing by $C(K)^n$ yields (3.1) for arbitrary non-polar compact set $K \subset \mathbb{R}$.

In the case $K = [-2, 2]$ the orthogonal polynomials with respect to the equilibrium measure $\mu_K$ are the Chebyshev polynomials of the first kind and a straightforward computation shows that equality in (3.2) is attained for all $n \in \mathbb{N}$ proving that the lower bound (3.2) is sharp. □

## 4. Lower bounds for the Jacobi weights

In this section we obtain sharp lower bounds for the norms of monic Jacobi polynomials. Let $K = [-1, 1]$ and consider the normalized Jacobi weights,

$$d\mu_{\alpha,\beta}(x) = c_{\alpha,\beta}(1-x)^{\alpha}(1+x)^{\beta}\chi_K(x)dx, \tag{4.1}$$

where $\alpha, \beta > -1$ are parameters and $c_{\alpha,\beta}$ is a normalization constant such that $\mu_{\alpha,\beta}(K) = 1$. We denote the corresponding monic orthogonal polynomials by $P_n^{\alpha,\beta}$. By [37, Section VII.1, Equation (25)],

$$\|P_n^{\alpha,\beta}\|_2^2 = c_{\alpha,\beta}\frac{2^{\alpha+\beta+2n+1}n!}{\alpha+\beta+2n+1}\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)^2}. \tag{4.2}$$

The equilibrium measure on $K = [-1, 1]$ is given by $d\mu_K(x) = \frac{1}{\pi}\frac{\chi_K(x)}{\sqrt{1-x^2}}dx$ hence we have $d\mu_{\alpha,\beta}(x) = c_{\alpha,\beta}\pi(1-x)^{\alpha+\frac{1}{2}}(1+x)^{\beta+\frac{1}{2}}d\mu_K(x)$. Using Frostman's theorem and noting that $C(K) = \frac{1}{2}$ we get

$$S(\mu_{\alpha,\beta}) = c_{\alpha,\beta}\pi C(K)^{\alpha+\beta+1} = \frac{c_{\alpha,\beta}\pi}{2^{\alpha+\beta+1}}. \tag{4.3}$$

Now, consider the ratios $R_n = \left[W_n^2(\mu_{\alpha,\beta})\right]^2 / S(\mu_{\alpha,\beta})$. By (4.2) and (4.3) we have

$$R_n = \frac{2^{2\alpha+2\beta+4n+2}n!}{\pi(\alpha+\beta+2n+1)}\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)^2}. \tag{4.4}$$

The optimal constant in the lower bound for $\left[W_n^2(\mu_{\alpha,\beta})\right]^2$ is given by $\inf_n R_n$. We are interested in finding the parameters for which $\inf_n R_n$ is maximal. While estimating $\inf_n R_n$ directly is difficult, we can find values of the parameters $\alpha, \beta$ so that the sequence $R_n$ is decreasing. In this case the improved lower bound (1.9) follows from the Szegő asymptotics (1.7). In the other extreme, if $R_n$ is strictly increasing then, by (1.7), the optimal constant in the lower bound is strictly less than 2 and is given by $R_1$.

Define the quantities

$$D_n = \frac{R_{n+1}}{R_n} - 1, \quad n \in \mathbb{N}, \tag{4.5}$$

so that the sign of $D_n$ determines whether $R_n$ is increasing, constant, or decreasing. Using the identity $\Gamma(x+1) = x\Gamma(x)$ and introducing $s_n = \alpha + \beta + 2(n+1)$ we obtain

$$D_n = \frac{16(n+1)(\alpha+n+1)(\beta+n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)^2(\alpha+\beta+2n+3)} - 1$$
$$= \frac{[s_n^2-(\alpha+\beta)^2][s_n^2-(\alpha-\beta)^2]}{s_n^2(s_n^2-1)} - 1 \tag{4.6}$$
$$= \frac{(\alpha^2-\beta^2)^2+s_n^2[1-2(\alpha^2+\beta^2)]}{s_n^2(s_n^2-1)}. \tag{4.7}$$

Then the sequence $R_n$ is decreasing if and only if $D_n \le 0$ for all $n \in \mathbb{N}$ which by (4.6) holds if $|\alpha|+|\beta| \ge 1$. The sequence $R_n$ is constant if and only if $D_n = 0$ for all $n \in \mathbb{N}$ which by (4.7) holds if and only if $|\alpha| = |\beta| = \frac{1}{2}$. Similarly, the sequence $R_n$ is strictly increasing if and only if $D_n > 0$ for all $n \in \mathbb{N}$ which by (4.7) holds if $\alpha^2+\beta^2 \le \frac{1}{2}$ except $|\alpha| = |\beta| = \frac{1}{2}$. When $\alpha^2+\beta^2 > \frac{1}{2}$ we can use $s_n > 2$ to estimate $D_n$ by

$$D_n \le \frac{(\alpha^2+\beta^2)^2+4[1-2(\alpha^2+\beta^2)]}{s_n^2(s_n^2-1)} \tag{4.8}$$

which implies that $D_n < 0$ for all $n \in \mathbb{N}$ if $4-2\sqrt{3} < \alpha^2+\beta^2 < 4+2\sqrt{3}$, so the sequence $R_n$ is decreasing in this case. More generally, in the case $\alpha^2+\beta^2 > \frac{1}{2}$ it follows from $s_n \uparrow \infty$ and (4.7) that there exists $n_0$ such that $D_n < 0$ for all $n \ge n_0$ and $D_n \ge 0$ for $n < n_0$. Thus, for any $\alpha, \beta > -1$ the sequence $R_n$ is either increasing or decreasing or increases for $n < n_0$ and decreases for $n \ge n_0$. Since $\lim_{n\to\infty} R_n = 2$ by (1.7), it follows that in any case the infimum of $R_n$ is equal to $\min\{2, R_1\}$. Combining these special cases we get the following result:

**Theorem 4.1.** *Let $\alpha, \beta > -1$. Then the Widom factors for the normalized Jacobi weight $\mu_{\alpha,\beta}$ satisfy*

$$\left[W_n^2(\mu_{\alpha,\beta})\right]^2 \ge L_{\alpha,\beta}\, S(\mu_{\alpha,\beta}), \quad n \in \mathbb{N}, \tag{4.9}$$

*with the optimal constant $L_{\alpha,\beta}$ given by*

$$L_{\alpha,\beta} = \min\left\{2, \frac{2^{2\alpha+2\beta+6}}{\pi(\alpha+\beta+2)} B(\alpha+2, \beta+2)\right\}, \tag{4.10}$$

*where $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ denotes the beta function.*

*In addition, we have: (a) $W_n^2(\mu_{\alpha,\beta}) = 2$ for all $n \in \mathbb{N}$ if and only if $|\alpha| = |\beta| = \frac{1}{2}$; (b) $L_{\alpha,\beta} = 2$, that is, (1.9) holds if either $|\alpha|+|\beta| \ge 1$ or $\alpha^2+\beta^2 > 4-2\sqrt{3} \approx 0.536$; and (c) $L_{\alpha,\beta} < 2$ if $\alpha^2+\beta^2 \le \frac{1}{2}$ except $|\alpha| = |\beta| = \frac{1}{2}$. In particular, in the symmetric case $|\alpha| = |\beta|$ the lower bound (1.9) holds if and only if $|\alpha| \ge \frac{1}{2}$.*

## 5. Lower bounds for measures from the isospectral tori

In this section we discuss spectral measures of one-sided Jacobi matrices from finite gap isospectral tori and give another improvement (5.8) of the lower bound (1.8). While the material of this section is known to experts [13,34,36], the explicit form of the main lower bound (5.8) has not previously appeared in the literature.

For a finite gap set

$$K = [\alpha_1, \beta_1] \cup \cdots \cup [\alpha_{\ell+1}, \beta_{\ell+1}] \tag{5.1}$$

with $\alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_{\ell+1} < \beta_{\ell+1}$, the isospectral torus $\mathcal{T}_K$ consists of two sided Jacobi matrices $J = \{a_k, b_k\}_{k=-\infty}^{\infty}$ with the spectrum $\sigma(J) = K$ which are reflectionless on $K$, that is, the diagonal Green functions $G_{n,n}(z) = \langle \delta_n, (J-z)^{-1}\delta_n \rangle$, $n \in \mathbb{Z}$, of $J$ have purely imaginary boundary values a.e. on $K$, see for example Sections 5.13 and 7.5 in [34]. By Craig's formula (cf., Theorem 5.4.19 in [34]), the diagonal Green functions $G_{n,n}(z)$ of reflectionless Jacobi matrices are of the form

$$G_{n,n}(z) = -\prod_{j=1}^{\ell}(z-\gamma_{n,j}) \left[\prod_{j=1}^{\ell+1}(z-\alpha_j)(z-\beta_j)\right]^{-1/2}, \quad n \in \mathbb{Z}, \tag{5.2}$$

where $\gamma_{n,j} \in [\beta_j, \alpha_{j+1}]$, $j = 1, \ldots, \ell$, $n \in \mathbb{Z}$.

In this section we investigate the Widom factors for the spectral measure $\mu_n$ of the one-sided truncation $J_n = \{a_{n+k}, b_{n+k}\}_{k=1}^{\infty}$ of $J \in \mathcal{T}_K$. Alternatively, such one-sided Jacobi matrices $J_n$ are characterized by the property of the associated $m$-function $m_n(z) = \langle \delta_1, (J_n - z)^{-1}, \delta_1 \rangle$ being a minimal Herglotz function on the two sheeted Riemann surface with branch cuts along $K$ (cf., Theorems 5.13.10, 5.13.12, and 7.5.1 in [34]). The minimal Herglotz functions are characterized by Theorem 5.13.2 in [34] which implies that the spectral measures $\mu_n$ consist of an absolutely continuous component on $K$ and a finite number of mass points at the discrete eigenvalues of $J_n$, $\sigma_d(J_n) \subset \mathbb{R}\backslash K$ (cf., (5.13.19), (5.13.24), (5.13.25) in [34]),

$$d\mu_n(x) = \frac{1}{\pi}\mathrm{Im}[m_n(x+i0)]\chi_K(x)dx + \sum_{\lambda \in \sigma_d(J)} \mathrm{res}_{z=\lambda}[m_n(z)]d\delta_\lambda(x). \tag{5.3}$$

There is a connection between $G_{n,n}$ and $m_n$ obtained in the proof of Theorem 5.13.12 in [34],

$$\mathrm{Im}[a_n^2 m_n(x+i0)] = \frac{1}{2}\mathrm{Im}[-G_{n,n}(x+i0)^{-1}] \text{ for a.e. } x \in K, \tag{5.4}$$

and the zeros of $G_{n,n}(z)$ correspond to the poles of $m_n$ on either the first or the second sheet of the Riemann surface, hence $\sigma_d(J_n)$ is a subset of $\{\gamma_{n,j}\}_{j=1}^{\ell}$, the zero set of $G_{n,n}$. Thus, using (5.2), (5.4), and (5.13.24), (5.13.25) in [34] we get an explicit form of $\mu_n$,

$$\begin{aligned} d\mu_n(x) =& \frac{1}{2a_n^2\pi}\frac{\sqrt{\prod_{j=1}^{\ell+1}|x-\alpha_j||x-\beta_j|}}{\prod_{j=1}^{\ell}|x-\gamma_{n,j}|}\chi_K(x)dx \\ &+ \sum_{k:\gamma_{n,k}\in\sigma_d(J_n)} \frac{1}{a_n^2}\frac{\sqrt{\prod_{j=1}^{\ell+1}|x-\alpha_j||x-\beta_j|}}{\prod_{j=1,j\neq k}^{\ell}|\gamma_{n,k}-\gamma_{n,j}|}d\delta_{\gamma_{n,k}}(x). \end{aligned} \tag{5.5}$$

By Theorem 5.5.22 and (5.4.96) in [34]), the equilibrium measure $\mu_K$ of a finite gap set $K$ is given by

$$d\mu_K(x) = \frac{1}{\pi}\frac{\prod_{j=1}^{\ell}|x-c_j|}{\sqrt{\prod_{j=1}^{\ell+1}|x-\alpha_j||x-\beta_j|}}\chi_K(x)dx, \tag{5.6}$$

where $c_j \in (\beta_j, \alpha_{j+1})$, $j = 1, \ldots, \ell$, are the critical points of the Green function $G_K(z)$ for the domain $\overline{\mathbb{C}}\backslash K$ with a logarithmic pole at infinity. Combining (5.5) and (5.6) then gives the Lebesgue decomposition of $\mu_n$ with respect to $\mu_K$,

$$\begin{aligned} d\mu_n(x) =& \frac{1}{2a_n^2}\frac{\prod_{j=1}^{\ell+1}|x-\alpha_j||x-\beta_j|}{\prod_{j=1}^{\ell}|x-c_j||x-\gamma_{n,j}|}d\mu_K(x) \\ &+ \sum_{k=1}^{\ell} \frac{s_{n,k}}{a_n^2}\frac{\sqrt{\prod_{j=1}^{\ell+1}|x-\alpha_j||x-\beta_j|}}{\prod_{j=1,j\neq k}^{\ell}|\gamma_{n,k}-\gamma_{n,j}|}d\delta_{\gamma_{n,k}}(x), \end{aligned} \tag{5.7}$$

where $s_{n,k} = 1$ if $\gamma_{n,k} \in \sigma_d(J_n)$ and $s_{n,k} = 0$ otherwise. The factor $a_n^{-2}$ plays a role of the normalization constant and hence is uniquely determined by $\{\gamma_{n,j}, s_{n,k}\}_{j=1}^{\ell}$. By Theorems 5.13.5 and 7.5.1 in [34], the class of such measures $\mu_0$ as $J$ runs through $\mathcal{T}_K$ consists of all possible choices of $\gamma_{0,j} \in [\beta_j, \alpha_{j+1}]$ and $s_{0,j} \in \{0, 1\}$ with $s_{0,k} = 0$ if $\gamma_{0,j}$ is at an edge $\beta_j$ or $\alpha_{j+1}$, $j = 1, \ldots, \ell$.

**Theorem 5.1.** *Let $K \subset \mathbb{R}$ be a finite gap set and $\mu_0$ be the spectral measure of a half-line truncation $J_0$ of $J \in \mathcal{T}_K$, that is, $\mu_0$ is of the form (5.7). Then*

$$\left[W_n^2(\mu_0)\right]^2 \geq 2E(\mu_0)^2 S(\mu_0), \quad n \in \mathbb{N}, \tag{5.8}$$

*where $E(\mu_0)$ is the eigenvalue function given by*

$$E(\mu_0) = \exp\left[\sum_{x \in \operatorname{supp}(\mu_0) \setminus K} G_K(x)\right]. \tag{5.9}$$

*In particular, since $E(\mu_0) \geq 1$, the improved lower bound (1.9) holds.*

**Proof.** The proof will be based on the step-by-step sum rule of [13]. Let $\mu_n$ denote the spectral measure of the one-sided truncation $J_n$ of $J$, $n \geq 1$. Then, by Theorem 4.2 in [13] or Proposition 9.10.5 in [34], we have

$$W_n^2(\mu_0) = \frac{a_1 \cdots a_n}{C(K)^n} = \frac{E(\mu_0) S(\mu_0)^{1/2}}{E(\mu_n) S(\mu_n)^{1/2}}. \tag{5.10}$$

Since in each gap of $K$ the Green function $G_K(x)$ is positive and attains its maximal value at the critical points we have the estimate

$$1 \leq E(\mu_n) \leq \exp\left[\sum_{j=1}^{\ell} G_K(\gamma_{n,j})\right]. \tag{5.11}$$

Recalling that $G_K(z) = -\log C(K) + \int \log|z - x| d\mu_K(x)$, we get from (5.7),

$$\begin{aligned} S(\mu_n) &= \frac{1}{2a_n^2} \exp\left[\int \log\left(\frac{\prod_{j=1}^{\ell+1} |x - \alpha_j||x - \beta_j|}{\prod_{j=1}^{\ell} |x - c_j||x - \gamma_{n,j}|}\right) d\mu_K(x)\right] \\ &= \frac{C(K)^2}{2a_n^2} \exp\left[-\sum_{j=1}^{\ell} G_K(c_j) - \sum_{j=1}^{\ell} G_K(\gamma_{n,j})\right], \end{aligned} \tag{5.12}$$

and hence,

$$E(\mu_n)^2 S(\mu_n) \leq \frac{C(K)^2}{2a_n^2} \exp\left[\sum_{j=1}^{\ell} \left[G_K(\gamma_{n,j}) - G_K(c_j)\right]\right] \leq \frac{C(K)^2}{2a_n^2}. \tag{5.13}$$

Squaring (5.10) and using (5.13) give

$$\frac{a_1^2 \cdots a_n^2}{C(K)^{2n}} = W_n^2(\mu_0)^2 = \frac{E(\mu_0)^2 S(\mu_0)}{E(\mu_n)^2 S(\mu_n)} \geq \frac{2a_n^2}{C(K)^2} E(\mu_0)^2 S(\mu_0). \tag{5.14}$$

Cancelling $a_n^2/C(K)^2$ term and utilizing (5.11) we obtain

$$\frac{a_1^2 \cdots a_{n-1}^2}{C(K)^{2(n-1)}} = W_{n-1}^2(\mu_0)^2 \geq 2E(\mu_0)^2 S(\mu_0), \quad n \in \mathbb{N}. \quad \square$$

## 6. Open problems

*Problem 1.* In Theorem 3.1, the sharp lower bound for $W_n^p(\mu_K)^p$ is obtained for $p = 2$. The sharp lower bound for $W_n^p(\mu_K)^p$ when $p \neq 2$ and $K \subset \mathbb{R}$ is an open problem. At least, we have a natural candidate for this lower bound: It is known that on an interval $K = [-1, 1]$ the monic Chebyshev polynomials of the first kind minimize $L^p(\mu_K)$ norms for all $1 \leq p \leq \infty$ (see for example p. 96 in [31]), hence the corresponding Widom factors can be evaluated explicitly in this case,

$$\left[W_n^p(\mu_K)\right]^p = \frac{2^p}{\pi} \int_0^\pi |\cos\theta|^p \, d\theta = \frac{2^p}{\sqrt{\pi}} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2}+1)}, \quad n \in \mathbb{N}. \tag{6.1}$$

Note that the right hand side of (6.1) is independent of $n$. When $p = 2$ (6.1) gives the sharp lower bound (1.9) and the limit as $p \to \infty$ of the $p$-th root of (6.1) gives the sharp lower bound (1.2). We conjecture that

$$\left[W_n^p(\mu_K)\right]^p \geq \frac{2^p}{\sqrt{\pi}} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2}+1)}, \quad n \in \mathbb{N}, \tag{6.2}$$

when $K$ is a non-polar compact subset of $\mathbb{R}$ and $1 \leq p < \infty$.

*Problem 2.* Let $K$ be a finite gap set. Besides the equilibrium measure $\mu_K$ and measures from the isospectral torus of $K$, an important class of measures is the class of reflectionless measures. These are the measures appearing in the Herglotz representation of $G_{n,n}$ from (5.2), that is, given by $d\mu_{n,n}(x) = \frac{1}{\pi}\mathrm{Im}[G_{n,n}(x+i0)]\chi_K(x)dx$. The equilibrium measure $\mu_K$ is a member of this class. We conjecture that (1.9) holds for all reflectionless measures on a finite gap set.

*Problem 3.* Is there a simple characterization of Szegő class measures on a finite gap set or even an interval for which (1.9) holds?

*Problem 4.* If $K$ is a finite gap set and $\mu$ is a Borel probability measure which is purely singular continuous with respect to $\mu_K$ and $\mathrm{supp}(\mu) = K$, then $W_n^2(\mu) \to 0$ since $S(\mu) = 0$ by Theorem 4.5 in [13].

If $K_1 = \overline{\mathbb{D}}$ and $\mu_1$ is the normalized area measure on $K_1$, then $P_n(z) = z^n$ is the $n$-th monic orthogonal polynomial with respect to $\mu_1$ and a straightforward calculation shows that $[W_n^2(\mu_1)]^2 = \frac{1}{n+1}$. Since $\mu_{K_1}$ is the normalized arc-measure on the unit circle, $\mu_1$ is purely singular continuous with respect to $\mu_{K_1}$ and we have $W_n^2(\mu_1) \to 0$. It is also true that Widom factors for the normalized area measure on Jordan domains with analytic boundary go to 0, see Theorem 4.1 in [21].

If $K_2$ is the Cantor ternary set and $\mu_2$ is the Cantor measure, then $\mu_2$ is purely singular continuous with respect to $\mu_{K_2}$ by [25]. However, in this case it was conjectured in [23, Conjecture 3.2] that $\liminf W_n^2(\mu_2) > 0$ based on numerical evidence.

It would be interesting to develop the theory of Widom factors for purely singular continuous measures (w.r.t. the equilibrium measure of the support). Proving or disproving existence of such a measure $\mu$ satisfying the condition $\liminf W_n^2(\mu) > 0$ would be a good start.

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