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Existence and multiplicity results for some Schrödinger-Poisson system with critical growth

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ABSTRACT

In this paper we study the existence and multiplicity of positive solutions for the Schrödinger-Poisson system with critical growth:

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = f(u) + |u|^3 u\phi, & x \in \mathbb{R}^3, \\ -\varepsilon^2 \Delta \phi = |u|^5, & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u(x) > 0, & x \in \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$ is a parameter, $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function and $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function. Under a global condition for V we prove that the above problem has a ground state solution and relate the number of positive solutions with the topology of the set where V attains its minimum, by using variational methods.

1. Introduction

In this paper we are concerned with the following Schrödinger-Poisson system involving critical growth

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = f(u) + |u|^3 u\phi, & x \in \mathbb{R}^3, \\ -\varepsilon^2 \Delta \phi = |u|^5, & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u(x) > 0, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a parameter, V and f are satisfied some suitable conditions which will be stated below.

The investigation of equation (1.1) is motivated by recent studies of Schrödinger-Poisson system

$$\begin{cases} -\Delta u + bu + \lambda\phi g(u) = f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = 2G(u), & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

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where the functions $g(u)$ and $G(u)$ satisfy $|g(u)| \leq C(|u| + |u|^q)$ for some $q \in [1, 4), G(u) = \int_0^u g(t)dt$, and $f(u)$ satisfies $|f(u)| \leq C(|u| + |u|^p)$ for some $p \in (1, 5]$. Eq. (1.2) arises in many interesting mathematical physics contexts, such as in quantum electro-dynamics, to describe the interaction between a charge particle interacting with electromagnetic field, and also in semi-conductor theory, in nonlinear optics and in plasma physics. We refer to [7,4,12,34] for more details on physical aspects.

For subcritical nonlinearity f with $p \in (1, 5)$ and subcritical nonlocal term g with $q \in [1, 4)$, problem (1.2) was studied by several authors, see for instance, [8,28]. In [8], system (1.2) on bounded domain $\Omega \subset \mathbb{R}^3$ was considered for positive and negative value of λ. In [28] system (1.2) was studied and it was showed that there exists a positive solution for small $\lambda \geq 0$. When $g(u) = u^4$, Li, Li and Shi [26] proved the existence of positive solutions to (1.2) by using variational method which does not require usual compactness condition. Later, in [27] they studied the existence, nonexistence and multiplicity of positive solutions to (1.2) are influenced on the parameter ranges of λ.

Recently, Azzollini, d'Avenia and Vaira [10] considered the following Schrödinger-Newton type system which is equivalent to a nonlocal version of the well known Brezis Nirenberg problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-3} u\phi, & \text{in } \Omega, \\ -\Delta \phi = |u|^{2^*-1}, & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N, N \geq 3$ is a smooth bounded domain. They studied the existence and nonexistence results of positive solutions when $N = 3$ and existence of solutions in both resonance and the non-resonance case for higher dimensions. In [31], Liu studied the following asymptotically periodic Schrödinger-Poisson system with critical exponent

$$\begin{cases} -\Delta u + V(x)u - K(x)\phi|u|^3 u = f(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)|u|^5, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where V, K, f are asymptotically periodic functions of x. The author proved the existence of positive solutions to (1.4) by the mountain pass theorem and the concentration-compactness principle.

In the special case $g(u) = u$, system (1.2) reduces to the following well known Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi(x)u = f(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

which has been studied by many authors, see for example, [7,4,6,18,16,25,34,36–38,43,44,47,48] and the references therein. In [5,7,36], the existence and multiplicity of positive solutions were considered for various λ and p; when V depends on x and is not radial, and f is asymptotically linear at infinity, the existence of positive solution for small λ and the nonexistence of nontrivial solution for large λ were obtained in [43]; when V depends on x, the existence of a sign-changing solution was proved in [44]; when V depends on x and is sign-changing, the existence and multiplicity were investigated in [47]; existence of a nontrivial solution and concentration results were showed in [23,42,48]. Moreover, the ground state solutions for (1.5) were considered in [9]; the ground and bound state solutions for system (1.5) were studied in [24,38].

We notice that, in all the papers aforementioned, only few papers like [26,27,31] deal with problem (1.4) which is involved with the critical growth for the nonlocal term. The purpose of this paper is to prove that system (1.1) has a ground state solution and relate the number of positive solutions with the topology of the set where V attains its minimum. By using variational method and the Ljusternik-Schnirelmann category theory we shall establish the multiplicity of positive solutions to system (1.1) concentrating at the minimum points set of the potential V, when parameter ε is small enough. To the best of our knowledge, there is not any results for system (1.1) on the existence, multiplicity and concentration of positive solutions in the literature.

We remark that the lack of compactness caused by the unboundedness of the whole space $\mathbb{R}^3$ and the critical growth in the nonlocal term $\phi_u|u|^3u$ (see Section 2), makes the situation more complicated to handle with system (1.1). To overcome these obstacles, we shall transform system (1.1) into a nonlinear Schrödinger equation with a non-local term and apply the variational methods. The compactness involving Palais-Smale sequences are recovered by adopting some more delicate analysis and tricks.

In order to state the main result, we introduce some basics assumptions on the functions V and f. For the potential V, we assume that $V:\mathbb{R}^3\to\mathbb{R}$ is a continuous function satisfying

(V) $0<V_0=\inf_{x\in\mathbb{R}^3}V(x)<\liminf_{|x|\to\infty}V(x):=V_\infty$.

This kind of hypothesis was first introduced by Rabinowitz [35] in the study of a nonlinear Schrödinger equation, and in this paper we shall consider the case $V_\infty<\infty$ or $V_\infty=\infty$. Since we are only concerned with positive solutions of (1.1), we may assume that $f:\mathbb{R}\to\mathbb{R}$ is a function of C^1 class and satisfies the following conditions:

(f_1) $f(s)=0$ for all $s<0$;
(f_2) $\lim_{s\to 0^+}\frac{f(s)}{s}=0$;
(f_3) there exists $q\in(3,5)$ verifying $\lim_{s\to\infty}\frac{f(s)}{s^q}=0$;
(f_4) $\exists\theta>4$ such that $0<\theta F(s):=\theta\int_0^s f(\tau)d\tau\le sf(s)$ for all $s>0$;
(f_5) the function $s\to\frac{f(s)}{s^3}$ is increasing in $(0,\infty)$.

The assumptions on V and f are quite natural in this context. Assumption (V) was first employed in [35] to take into account potentials which are possibly not coercive. Hypothesis (f_1) is not restrictive since we are concerned with positive solutions, and $(f_2)-(f_5)$ are indispensable to use variational techniques which involve in the Palais-Smale condition, the Mountain Pass Theorem and the Nehari manifold. For this aim, we recall that $\{u_n\}$ is a Palais-Smale sequence for a C^1 functional I at level $c\in\mathbb{R}$, if $I(u_n)\to c$ and $I'(u_n)\to 0$. We shall abbreviate this by saying that $\{u_n\}$ is a $(PS)_c$ sequence. Furthermore, the functional I is said to satisfy the Palais-Smale condition at level c, if every $(PS)_c$ sequence has a strongly convergent subsequence.

In order to relate the number of solutions of (1.1) with the topology of the set of minima of the potential V, we introduce the set of global minima of V given by

$$M=\{x\in\mathbb{R}^3:\ V(x)=V_0=\inf_{x\in\mathbb{R}^3}V(x)\}.$$

In view of (V), the set M is compact. For any $\delta>0$, we denote by $M_\delta=\{x\in\mathbb{R}^3:\ \operatorname{dist}(x,M)\le\delta\}$ the closed δ-neighborhood of M.

Theorem 1.1. *Suppose that f satisfies $(f_1)-(f_5)$ and V verifies (V). Then, for any $\delta>0$, there exists $\varepsilon_\delta>0$ such that, for any $\varepsilon\in(0,\varepsilon_\delta)$, problem (1.1) has at least $cat_{M_\delta}(M)$ positive solutions, for any $\varepsilon\in(0,\varepsilon_\delta)$. Moreover, if u_ε denotes one of these positive solutions and $\eta_\varepsilon\in\mathbb{R}^3$ its global maximum point, then*

$$\lim_{\varepsilon\to 0}V(\eta_\varepsilon)=V_0.$$

We recall that if Y is a closed subset of a topological space X, the Ljusternik-Schnirelmann category $cat_X(Y)$ (if $X=Y$ we just write $cat(X)$) is the least number of closed and contractible sets in X which cover Y.

In order to obtain multiple solutions for (1.1), we use some techniques introduced by some papers of Benci, Cerami [11], and Cingolani, Lazzo [17]. The main idea is to make precisely comparisons between the category of some sublevel sets of the energy functional of (1.1) and the category of the set M. For more applications of the Ljusternik-Schnirelmann theory on the study of Schrödinger equations, p-Laplace equations, quasilinear equations, we refer the reader to [1,2,20,21] and references therein.

The paper is organized as follows. In Section 2 we present the abstract framework of the system as well as some preliminary results and present some compactness properties of the functional of the autonomous problem. In Section 3 we prove system (1.1) has a positive ground state solution. Section 4 is devoted to the proof of Theorem 1.1. A technical lemma is given in the Appendix.

As a matter of notation, we denote with $B_r(y)$, respectively B_r, the ball in $\mathbb{R}^N$ with radius $r > 0$ centered in y, respectively in 0. The L^p-norm in $\mathbb{R}^N$ is simply denoted with $|\cdot|_p$. If we need to specify the domain, let us say $A \subset \mathbb{R}^N$, we write $|\cdot|_{L^p(A)}$. From now on, the letter $C, C_1, i = 1, 2, \cdots$, will be repeatedly used to denote various positive constants whose exact values are irrelevant.

2. The variational framework and preliminary results

2.1. Variational framework and notations

Throughout the paper we suppose that the functions V and f satisfy conditions (V) and $(f_1) - (f_5)$, respectively. To fix some notations, we denote the standard norm of $H^1(\mathbb{R}^3)$ by

$$||u||^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx,$$

and the norm of $D^{1,2}(\mathbb{R}^3)$ by

$$||u||^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

For every $u \in H^1(\mathbb{R}^3)$, and any fixed $\varepsilon > 0$, the Lax-Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that (e.g. [31])

$$-\varepsilon^2 \Delta \phi_u = |u|^5. \tag{2.1}$$

Moreover,

$$\phi_u(x) = \frac{1}{4\pi\varepsilon^2} \int_{\mathbb{R}^3} \frac{|u(y)|^5}{|x - y|} dy. \tag{2.2}$$

We next summarize some properties about the solution ϕ_u of the Poisson equation in (1.1) which will be useful in the following.

Lemma 2.1. *For any $u \in H^1(\mathbb{R}^3) \backslash \{0\}$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ which is the solution of*

$$-\Delta \phi = |u|^5 \quad in \ \ \mathbb{R}^3, \tag{2.3}$$

and ϕ_u can be expressed as the form

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^5}{|x-y|} dy. \tag{2.4}$$

Moreover,

(i) $\phi_u(x) > 0$ *for* $x \in \mathbb{R}^3$*;*
(ii) $|\nabla \phi_u|_2^2 = \int_{\mathbb{R}^3} \phi_u |u|^5 dx$*;*
(iii) *for any* $t > 0, \phi_{tu} = t^5 \phi_u$*;*
(iv) $|\nabla \phi_u|_2 \le S^{-3} |\nabla u|_2^5$*, where* $S = \inf_{v \in H^1(\mathbb{R}^3)\setminus\{0\}} |\nabla v|_2^2 / |v|_6^2$*;*
(v) $|\nabla \phi_u|_2^2 \ge 2\delta |u|_6^6 - \delta^2 |\nabla u|_2^2$ *for any* $\delta > 0$*;*
(vi) *for any* $u, v \in D^{1,2}(\mathbb{R}^3)$*,*

$$\int_{\mathbb{R}^3} \phi_u |v|^5 dx = \int_{\mathbb{R}^3} \phi_v |u|^5 dx;$$

(vii) *for every* $u, u_1, u_2, \cdot, u_k \in H^1(\mathbb{R}^3)$*,*

$$\left| \phi_u - \sum_{i=1}^k \phi_{u_i} \right|_6 \le \frac{1}{S} \left| |u|^5 - \sum_{i=1}^k |u_i|^5 \right|_{\frac{6}{5}};$$

(viii) *if* $\{u_n\} \subset H^1(\mathbb{R}^3)$ *and* $u \in H^1(\mathbb{R}^3)$ *are such that* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$ *and* $u_n \to u$ *a.e. in* $\mathbb{R}^3$ *as* $n \to \infty$*, then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$*. Moreover,*

$$\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx - \int_{\mathbb{R}^3} \phi_{u_n - u} |u_n - u|^5 dx = \int_{\mathbb{R}^3} \phi_u |u|^5 dx + o_n(1).$$

Proof. The existence and uniqueness of ϕ_u follows from the Lax-Milgram theorem. The conclusions (i), (ii) and (iii) are clear from the definition of ϕ_u and (2.3)-(2.4).

(iv) Multiplying (2.3) by ϕ_u, integrating and using Hölder inequality, we have

$$|\nabla \phi_u|_2^2 = \int_{\mathbb{R}^3} \phi_u |u|^5 dx \le |\phi_u|_6 |u|_6^5 \le S^{-3} |\nabla u|_2^5 |\nabla \phi_u|_2$$

and then (iv) holds.

(v) Multiplying (2.3) by $|u|$ and integrating, we have

$$|u|_6^6 = \int_{\mathbb{R}^3} \nabla \phi_u \nabla |u| dx \le \frac{1}{2\delta} |\nabla \phi_u|_2^2 + \frac{\delta}{2} |\nabla u|_2^2 \quad \text{for any } \delta > 0$$

and so (v).

(vi) We observe that for any $u, v \in D^{1,2}(\mathbb{R}^3)$, one has

$$\int_{\mathbb{R}^3} \phi_v |u|^5 dx = \int_{\mathbb{R}^3} \nabla \phi_u \nabla \phi_v dx = \int_{\mathbb{R}^3} \phi_u |v|^5 dx,$$

and then (vi) follows.

(vii) By the definition of S, properties (ii) and (vi), and the Hölder inequality, we get

$$\begin{aligned}\left|\phi_u - \sum_{i=1}^{k} \phi_{u_i}\right|_6^2 &\leq \frac{1}{S}\left|\nabla(\phi_u - \sum_{i=1}^{k} \phi_{u_i})\right|_2^2 \\ &= \frac{1}{S}\int_{\mathbb{R}^3} (\phi_u - \sum_{i=1}^{k} \phi_{u_i})(|u|^5 - \sum_{i=1}^{k} |u_i|^5)dx \\ &\leq \frac{1}{S}\left|\phi_u - \sum_{i=1}^{k} \phi_{u_i}\right|_6 \left||u|^5 - \sum_{i=1}^{k} |u_i|^5\right|_{\frac{6}{5}}\end{aligned}$$

and (vii) follows.

(viii) For any $v \in H^1(\mathbb{R}^3) \hookrightarrow D^{1,2}(\mathbb{R}^3)$, using $u_n \rightharpoonup u$ in $L^6(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, we have $|u_n|^5 \rightharpoonup |u|^5$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$. Thus

$$(\phi_{u_n}, v)_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |u_n|^5 v dx \to \int_{\mathbb{R}^3} |u|^5 v dx = (\phi_u, v)_{D^{1,2}(\mathbb{R}^3)}.$$

Therefore, $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$. Furthermore, by applying (vi) we get

$$\begin{aligned}&\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 dx - \int_{\mathbb{R}^3} \phi_{u_n-u}|u_n - u|^5 dx \\ &= \int_{\mathbb{R}^3} (\phi_{u_n} - \phi_{u_n-u})(|u_n|^5 - |u_n - u|^5)dx \\ &\quad + 2\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_{u_n-u})|u_n - u|^5 dx\end{aligned} \tag{2.5}$$

An easy variant of the classical Brezis-Lieb Lemma (e.g. Lemma 2.5 [32]) yields that

$$|u_n|^5 - |u_n - u|^5 \to |u|^5 \quad \text{in} \;\; L^{\frac{6}{5}}(\mathbb{R}^3) \;\; \text{as} \;\; n \to \infty$$

and applying (vii) we get

$$\phi_{u_n} - \phi_{u_n-u} \to \phi_u \quad \text{in} \;\; L^6(\mathbb{R}^3) \;\; \text{as} \;\; n \to \infty. \tag{2.6}$$

Therefore,

$$\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_{u_n-u})(|u_n|^5 - |u_n - u|^5)dx \to \int_{\mathbb{R}^3} \phi_u|u|^5 dx \quad \text{as} \;\; n \to \infty. \tag{2.7}$$

Moreover, applying Proposition 5.4.7 [46], we have $|u_n - u| \rightharpoonup 0$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$. Hence, since $\phi_u \in L^6(\mathbb{R}^3)$ and using also (2.6),

$$\begin{aligned}&\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_{u_n-u})|u_n - u|^5 dx \\ &= \int_{\mathbb{R}^3} (\phi_{u_n} - \phi_{u_n-u} - \phi_u)|u_n - u|^5 dx + \int_{\mathbb{R}^3} \phi_u|u_n - u|^5 dx \to 0\end{aligned} \tag{2.8}$$

as $n \to \infty$. Combining (2.5)-(2.8) we get (viii). $\square$

Making the change of variable $\varepsilon z = x$, we can rewrite (1.1) as the following equivalent equation

$$\begin{cases} -\Delta u + V(\varepsilon x)u = f(u) + |u|^3 u\phi_u, & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u(x) > 0, & x \in \mathbb{R}^3. \end{cases} \tag{2.9}$$

For any $\varepsilon > 0$, let $H_\varepsilon = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)u^2 < \infty\}$ be the Sobolev space endowed with the norm

$$||u||_\varepsilon^2 = \int\limits_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2)dx.$$

At this step, we see that (2.9) is variational and its solutions are the critical points of the functional $I_\varepsilon : H^1(\mathbb{R}^3) \to \mathbb{R}$ given by

$$I_\varepsilon(u) = \frac{1}{2}\int\limits_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2)dx - \int\limits_{\mathbb{R}^3} F(u)dx - \frac{1}{10}\int\limits_{\mathbb{R}^3} \phi_{u^+}|u^+|^5 dx. \tag{2.10}$$

Moreover, I_ε belongs to $C^1(H_\varepsilon, \mathbb{R})$.

Next, we define the Nehari manifold [45] associated to I_ε by

$$\mathcal{N}_\varepsilon = \left\{ u \in H_\varepsilon \backslash \{0\} : \quad \int\limits_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2 dx = \int\limits_{\mathbb{R}^3} f(u)udx + \int\limits_{\mathbb{R}^3} \phi_{u^+}|u^+|^5 dx \right\},$$

and consider the following minimization problem

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u).$$

As we shall see in the sequel, it is important to compare the minimax value c_ε with the mountain pass level of the autonomous system

$$\begin{cases} -\Delta u + \mu u = f(u) + \phi|u|^3 u \ \text{ in } \mathbb{R}^3, \\ -\Delta\phi = |u|^5 \ \text{ in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \ u(x) > 0, \ \ \forall x \in \mathbb{R}^3, \end{cases} \tag{2.11}$$

where $\mu \in \mathbb{R}^+$. The solutions of (2.11) are precisely critical points of the functional defined by

$$E_\mu(u) = \frac{1}{2}\int\limits_{\mathbb{R}^3} (|\nabla u|^2 + \mu u^2)dx - \int\limits_{\mathbb{R}^3} F(u)dx - \frac{1}{10}\int\limits_{\mathbb{R}^3} \phi_{u^+}|u^+|^5 dx.$$

Let $\mathcal{M}_\mu$ be the Nehari manifold of E_μ given by

$$\mathcal{M}_\mu = \left\{ u \in H_\mu \backslash \{0\} : \quad \int\limits_{\mathbb{R}^3} (|\nabla u|^2 + \mu u^2)dx = \int\limits_{\mathbb{R}^3} f(u)udx + \int\limits_{\mathbb{R}^3} \phi_{u^+}|u^+|^5 dx \right\},$$

where $H_\mu = H^1(\mathbb{R}^3)$ is endowed with the norm $||u||_\mu^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + \mu u^2)dx$. We define m_μ by setting

$$m_\mu = \inf_{u\in\mathcal{M}_\mu} E_\mu(u).$$

The number m_μ and the manifold $\mathcal{M}_\mu$ have properties similar to those of c_ε and $\mathcal{N}_\varepsilon$.

2.2. Technical results

In this subsection we will show some lemmas concerned to the functional I_ε. Firstly, we have the following properties of the Nehari manifold $\mathcal{N}_\varepsilon$.

Lemma 2.2. *The following properties for the manifold $\mathcal{N}_\varepsilon$ hold true:*

(i) For any $u \in H_\varepsilon\backslash\{0\}$, there exists a unique $t_u > 0$ such that $I_\varepsilon(t_u u) = \max_{t\geq 0} I_\varepsilon(tu)$ and $t_u u \in \mathcal{N}_\varepsilon$.
(ii) There exists $r^* > 0$ such that $\|u\|_\varepsilon \geq r^*$ for $\forall u \in \mathcal{N}_\varepsilon$.

Proof. (i) Denote by the function $g(t) \triangleq I_\varepsilon(tu)$ for $t \geq 0$. It is easy to verify, using $(f_1)-(f_3)$ that $g(0) = 0$ and $g(t) < 0$ for $t > 0$ large. Therefore $\max_{t\geq 0} g(t)$ attains its maximum at some $t_u > 0$ such that $g'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. Suppose there exist $t_u^1 > t_u^2 > 0$ such that $t_u^i u \in \mathcal{N}_\varepsilon, i = 1, 2$. Then

$$\begin{aligned}&\left(\frac{1}{(t_u^1)^2} - \frac{1}{(t_u^2)^2}\right)\|u\|_\varepsilon^2\\ &= \int_{\mathbb{R}^3}\left[\frac{f(t_u^1 u)}{(t_u^1 u)^3} - \frac{f(t_u^2 u)}{(t_u^2 u)^3}\right]u^4 dx + [(t_u^1)^6 - (t_u^2)^6]\int_{\mathbb{R}^3}\phi_{u^+}|u^+|^5 dx,\end{aligned}$$

which is a contradiction by virtue of (f_6), and so $t_u^1 = t_u^2 > 0$. Moreover, the function $u \to t_u$ is continuous from $H_\varepsilon\backslash\{0\}$ to $(0, \infty)$ (e.g. [35]).

(ii) It follows from $(f_1)-(f_3)$ that, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|f(t)| \leq \epsilon|t| + C_\epsilon|t|^q, \quad \forall t \in \mathbb{R}. \tag{2.12}$$

Thus, for any $u \in \mathcal{N}_\varepsilon$, by the Hölder inequality, Sobolev inequality and (iv) of Lemma 2.1, one has

$$\begin{aligned}0 &= ||u||_\varepsilon^2 - \int_{\mathbb{R}^3} f(u)u dx - \int_{\mathbb{R}^3}\phi_{u^+}|u^+|^5 dx\\ &\geq ||u||_\varepsilon^2 - \epsilon\int_{\mathbb{R}^3} u^2 dx - C_\epsilon\int_{\mathbb{R}^3}|u|^{q+1}dx - \left(\int_{\mathbb{R}^3}|\phi_{u^+}|^6 dx\right)^{\frac{1}{6}}\left(\int_{\mathbb{R}^3}|u^+|^6 dx\right)^{\frac{5}{6}}\\ &\geq ||u||_\varepsilon^2 - \epsilon C||u||_\varepsilon^2 - CC_\epsilon||u||_\varepsilon^{q+1} - C\|u\|_\varepsilon^{10}\end{aligned}$$

from which

$$||u||_\varepsilon \geq r^* > 0 \quad \text{for } \forall u \in \mathcal{N}_\varepsilon. \quad \square \tag{2.13}$$

The functional I_ε satisfies the mountain pass geometry.

Lemma 2.3. *The functional I_ε satisfies the following properties.*

(i) There exist $\alpha, \rho > 0$ such that $I_\varepsilon(u) \geq \alpha$ with $||u||_\varepsilon = \rho$.
(ii) There exists $e \in B_\rho^c(0)$ with $I_\varepsilon(e) < 0$.

Proof. (i) For any $u \in H_\varepsilon \backslash \{0\}$ and $\epsilon > 0$ small, it follows from $(f_1) - (f_3)$ that there exists $C_\epsilon > 0$ such that

$$|F(t)| \le \frac{\epsilon}{2}t^2 + \frac{C_\epsilon}{q+1}|t|^{q+1}, \quad \forall t \in \mathbb{R}.$$

Now by the Sobolev embedding $H_\varepsilon \hookrightarrow L^p(\mathbb{R}^3)$ for $2 < p < 2^*$ and by the Hölder inequality and (iv) of Lemma 2.1, we have

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(\varepsilon x)u^2)dx - \int_{\mathbb{R}^3} F(u)dx - \frac{1}{10}\int_{\mathbb{R}^3} \phi_{u^+}|u^+|^5 dx \\ &\ge \frac{1}{2}\|u\|_\varepsilon^2 - \frac{\epsilon}{2}\int_{\mathbb{R}^3} u^2 dx - \frac{C_\epsilon}{q+1}\int_{\mathbb{R}^3}|u|^{q+1}dx - \frac{1}{10}|\phi_u|_6|u|_6^5 \\ &\ge \left(\frac{1}{2} - \frac{\epsilon}{2}\right)||u||_\varepsilon^2 - CC_\epsilon||u||_\varepsilon^{q+1} - C\|u\|_\varepsilon^{10}. \end{aligned}$$

Hence we can choose $\epsilon = \frac{1}{2}$ and some $\alpha, \rho > 0$ such that

$$I_\varepsilon(u) \ge \alpha \ \text{ with } ||u||_\varepsilon = \rho.$$

(ii) By $(f_1), (f_4)$, we have $F(t) \ge 0$ for all $t \in \mathbb{R}$. Take a $0 \le \varphi \in C_0^\infty(\mathbb{R}^3)$, then

$$\begin{aligned} I_\varepsilon(t\varphi) &= \frac{t^2}{2}\int_{\mathbb{R}^3}(|\nabla\varphi|^2 + V(\varepsilon x)\varphi^2)dx - \int_{\mathbb{R}^3} F(t\varphi)dx - \frac{t^{10}}{10}\int_{\mathbb{R}^3}\phi_\varphi|\varphi|^5 dx \\ &\le \frac{t^2}{2}\int_{\mathbb{R}^3}(|\nabla\varphi|^2 + V(\varepsilon x)\varphi^2)dx - \frac{t^{10}}{10}\int_{\mathbb{R}^3}\phi_\varphi|\varphi|^5 dx \\ &< 0 \end{aligned}$$

for $t > 0$ large enough. Hence, we can take $e = t^*\varphi$ with some $t^* > 0$ large and (ii) follows. □

It follows from Lemma 2.3 and the mountain pass theorem without (PS) condition [45], there exists a $(PS)_c$ sequence $\{u_n\} \subset H_\varepsilon$ such that $I_\varepsilon(u_n) \to c_\varepsilon$ and $I'_\varepsilon(u_n) \to 0$ in H_ε^{-1} with the minimax level

$$c_\varepsilon = \inf_{g\in\Gamma}\sup_{t\in[0,1]} I_\varepsilon(g(t)) > 0, \tag{2.14}$$

where $\Gamma = \{g \in C^1([0,1], H_\varepsilon) : \ g(0) = 0, I_\varepsilon(g(1)) < 0\}$. Moreover, we have the following assertion.

Lemma 2.4. *The sequence $\{u_n\}$ is bounded in H_ε.*

Proof. Let $\{u_n\}$ be a $(PS)_{c_\varepsilon}$ sequence for I_ε. From (f_5), it follows that:

$$\text{the function } \ t \in [0,\infty) \mapsto f(t)t - 4F(t) \in \mathbb{R} \ \text{ is strictly increasing.} \tag{2.15}$$

By $(f_1), (f_4)$ and (2.15) we get

$$f(t)t - 4F(t) \ge 0, \ \forall t \in \mathbb{R}. \tag{2.16}$$

Therefore,

$$\begin{aligned} o_n(1)\|u_n\|_\varepsilon + 4c_\varepsilon &= 4I_\varepsilon(u_n) - I'_\varepsilon(u_n)u_n \\ &= \|u_n\|_\varepsilon^2 + \int\limits_{\mathbb{R}^3} [f(u_n) - 4F(u_n)]dx + \frac{3}{5}\int\limits_{\mathbb{R}^3} \phi_{u_n^+}|u_n^+|^5 dx \\ &\geq \|u_n\|_\varepsilon^2. \end{aligned}$$

Hence, $\{u_n\}$ is bounded in H_ε. □

Remark 2.5. *If we denote $u_n^\pm = \max\{\pm u_n, 0\}$ as the positive (negative) part of u_n, then one has $I'_\varepsilon(u_n)(-u_n^-) = \|u_n^-\|_\varepsilon^2 = o_n(1)$. Note that $\|u_n^+\|_\varepsilon^2 \geq C > 0$ for n large. Otherwise, we would have $\|u_n\|_\varepsilon^2 = o_n(1)$ and $I_\varepsilon(u_n) \to 0$ as $n \to \infty$, which contradicts to (2.13). So, in the sequel, for any (PS) sequence $\{u_n\}$ of I_ε, we may assume that it is a nonnegative sequence.*

From Lemma 2.4, there exists a $u \in H_\varepsilon$ such that $u_n \rightharpoonup u$ in H_ε and $u_n \to u$ a.e. in $\mathbb{R}^3$. Adopting similar arguments as in Proposition 3.11 [35], we have the following equivalent characterization of c_ε, which is more adequate to our purpose.

$$c_\varepsilon = \inf_{u\in H_\varepsilon\setminus\{0\}} \sup_{t\geq 0} I_\varepsilon(tu) = \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u) > 0. \tag{2.17}$$

In the rest of this subsection, we shall show that m_μ can be compared with a suitable number which involves the best constant S.

Lemma 2.6. *For any $\mu > 0$, there exists $u_\epsilon \in H_\mu\setminus\{0\}$ such that*

$$\max_{t\geq 0} E_\mu(tu_\epsilon) < \frac{2}{5}S^{\frac{3}{2}}.$$

In particular $m_\mu < \frac{2}{5}S^{\frac{3}{2}}$.

Proof. For each $\epsilon > 0$, consider the function

$$U_\epsilon(x) = \frac{(3\epsilon^2)^{\frac{1}{4}}}{(\epsilon^2 + |x|^2)^{\frac{1}{2}}}.$$

We recall that U_ϵ solves

$$-\Delta u = u^5, \quad \text{in } \mathbb{R}^3.$$

By a result due to Talenti [39] we have

$$\int\limits_{\mathbb{R}^3} |\nabla U_\epsilon|^2 dx = \int\limits_{\mathbb{R}^3} |U_\epsilon|^6 dx = S^{\frac{3}{2}}.$$

Let $\eta \in C_0^\infty(\mathbb{R}^3, [0,1])$ be such that $0 \leq \eta \leq 1, \eta(x) = 1$ if $|x| < 1$ and $\eta(x) = 0$ if $|x| \geq 2$. Setting $u_\epsilon = \eta U_\epsilon |\eta U_\epsilon|_6^{-1}$, and computing as in [23], [15], we have

$$|\nabla u_\epsilon|_2^2 = S + O(\epsilon) \tag{2.18}$$

and

$$|u_\epsilon|_r^r = \begin{cases} O(\epsilon^{\frac{r}{2}}), & \text{if } r \in [2,3); \\ O(\epsilon^{\frac{3}{2}}|\log \epsilon|), & \text{if } r = 3; \\ O(\epsilon^{\frac{6-r}{2}}), & \text{if } r \in (3,6). \end{cases} \tag{2.19}$$

By Lemma 2.2-(i), there exists $t_{u_\epsilon} > 0$ such that $t_{u_\epsilon} u_\epsilon \in \mathcal{M}_\mu$ and $E_\mu(t_{u_\epsilon} u_\epsilon) = \max_{t\geq 0} E_\mu(tu_\epsilon)$. We claim that there exist constants $k_1, k_2 > 0$ such that

$$0 < k_1 < t_{u_\epsilon} < k_2 < \infty. \tag{2.20}$$

In fact, using $t_{u_\epsilon} u_\epsilon \in \mathcal{M}_\mu$ and Lemma 2.1-(ii), (v) with $\delta = |\nabla u_\epsilon|_2^{-2}$, we have

$$\begin{aligned} t_{u_\epsilon}^2 \|u_\epsilon\|_\mu^2 &= \int_{\mathbb{R}^3} f(t_{u_\epsilon} u_\epsilon) t_{u_\epsilon} u_\epsilon dx + t_{u_\epsilon}^{10} \int_{\mathbb{R}^3} \phi_{u_\epsilon} |u_\epsilon|^5 dx \\ &\geq t_{u_\epsilon}^{10} \int_{\mathbb{R}^3} \phi_{u_\epsilon} |u_\epsilon|^5 dx \\ &\geq t_{u_\epsilon}^{10} \left[2\delta \int_{\mathbb{R}^3} |u_\epsilon|^6 dx - \delta^2 \int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 dx \right] \\ &= t_{u_\epsilon}^{10} |\nabla u_\epsilon|_2^{-2} \end{aligned} \tag{2.21}$$

which implies that t_{u_ϵ} is bounded from above by some constant $k_2 > 0$ by virtue of (2.18). On the other hand, by the first equality of (2.21) and (f_2) we see that t_{u_ϵ} is bounded from below by some constant $k_1 > 0$. Thus (2.20) holds true.

Again, by using Lemma 2.1-(ii) and Lemma 2.1-(v) with $\delta = 1$, we infer that

$$\begin{aligned} E_\mu(u) &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + \mu u^2) dx - \int_{\mathbb{R}^3} F(u) dx - \frac{1}{10} \int_{\mathbb{R}^3} \phi_u |u|^5 dx \\ &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + \mu u^2) dx - \int_{\mathbb{R}^3} F(u) dx - \frac{1}{10} \int_{\mathbb{R}^3} |\nabla \phi_u|^2 dx \\ &\leq \frac{3}{5} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\mu}{2} \int_{\mathbb{R}^3} u^2 dx - \frac{1}{5} \int_{\mathbb{R}^3} u^6 dx - \int_{\mathbb{R}^3} F(u) dx. \end{aligned}$$

Therefore, we have

$$\begin{aligned} E_\mu(tu_\epsilon) &\leq \frac{3t^2}{5} \int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 dx + \frac{\mu t^2}{2} \int_{\mathbb{R}^3} u_\epsilon^2 dx - \frac{t^6}{5} - \int_{\mathbb{R}^3} F(tu_\epsilon) dx \\ &\triangleq g(t). \end{aligned} \tag{2.22}$$

By $(f_1) - (f_3)$ we see that $\lim_{t\to\infty} g(t) = -\infty$ and $g(t) > 0$ as t is closed to 0. So, $\sup_{t\geq 0} g(t)$ is attained at some $t_\epsilon > 0$.

From

$$0 = g'(t_\epsilon) = t_\epsilon \left(\frac{6}{5} \int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 dx + \mu \int_{\mathbb{R}^3} u_\epsilon^2 dx - \frac{6t_\epsilon^4}{5} - \int_{\mathbb{R}^3} f(t_\epsilon u_\epsilon) u_\epsilon t_\epsilon^{-1} dx \right),$$

we have

$$\frac{6}{5}\int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 dx + \mu \int_{\mathbb{R}^3} u_\epsilon^2 dx = \frac{6t_\epsilon^4}{5} + \int_{\mathbb{R}^3} f(t_\epsilon u_\epsilon) u_\epsilon t_\epsilon^{-1} dx$$
$$\geq \frac{6t_\epsilon^4}{5},$$

which implies that t_ϵ is bounded from above by some $t_2 > 0$. On the other hand, by $(f_1)-(f_3)$, for any fixed $\tau > 0$, there exist $C_\tau > 0$ such that $f(t) \leq \tau(t + t^q) + C_\tau t^2, \quad \forall t \geq 0$, and so

$$\frac{6}{5}\int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 dx \leq \frac{6t_\epsilon^4}{5} + \int_{\mathbb{R}^3} f(t_\epsilon u_\epsilon) u_\epsilon t_\epsilon^{-1} dx$$
$$\leq \frac{6t_\epsilon^4}{5} + \int_{\mathbb{R}^3} [\tau u_\epsilon^2 + \tau t_2^{q-1} u_\epsilon^{q+1} + C_\tau t_2 u_\epsilon^3] dx.$$

Choosing ϵ small enough, by (2.18), (2.19), we obtain

$$(t_\epsilon)^4 > \frac{S}{2}.$$

That is, we get a lower bound $t_1 > 0$ for t_ϵ independent of ϵ. Thus, $0 < t_1 < t_\epsilon < t_2$.

Now we estimate $g(t)$. Set $\bar{g}(t) = \frac{3t^2}{5}\int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 dx - \frac{t^6}{5}$. Then $\bar{g}(t)$ attains its maximum at

$$t_{max} = \left(\int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 dx\right)^{\frac{1}{4}}$$

and

$$\bar{g}(t_{max}) = \frac{2}{5}\left(\int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 dx\right)^{\frac{3}{2}} = \frac{2}{5}(S + O(\epsilon))^{\frac{3}{2}} = \frac{2}{5}S^{\frac{3}{2}} + O(\epsilon). \tag{2.23}$$

Consequently, by (2.18), (2.19) and (2.23) we get

$$\begin{aligned} g(t_\epsilon) &= \bar{g}(t_\epsilon) + \frac{\mu t_\epsilon^2}{2}\int_{\mathbb{R}^3} u_\epsilon^2 dx - \int_{\mathbb{R}^3} F(t_\epsilon u_\epsilon) dx \\ &\leq \bar{g}(t_{max}) + C_1 \int_{\mathbb{R}^3} |u_\epsilon|^2 dx - \int_{\mathbb{R}^3} F(t_\epsilon u_\epsilon) dx \\ &\leq \frac{2}{5}S^{\frac{3}{2}} + O(\epsilon) - \int_{\mathbb{R}^3} F(t_\epsilon u_\epsilon) dx. \end{aligned} \tag{2.24}$$

By the definition of u_ϵ, one has

$$t_\epsilon u_\epsilon \geq C t_1 \epsilon^{-\frac{1}{2}} \quad \text{if} \quad |x| \leq \epsilon \leq 1.$$

By virtue of (f_4), we see that $F(s) \geq C_1 s^\theta - C_2, \forall s \in \mathbb{R}^+$, for some $C_1, C_2 > 0$. Then for any $K > 0$, we have $F(t_\epsilon u_\epsilon) \geq K(t_\epsilon u_\epsilon)^4 \geq K(Ct_1\epsilon^{-\frac{1}{2}})$ if $|x| \leq \epsilon << 1$. Therefore,

$$\int_{\mathbb{R}^3} F(t_\epsilon u_\epsilon) dx \geq \int_{B_\epsilon(0)} F(t_\epsilon u_\epsilon) dx \geq K(Ct_1\epsilon^{-\frac{1}{2}})^4 \int_{B_\epsilon(0)} dx = KC_1\epsilon. \tag{2.25}$$

Combining (2.24) and (2.25) we obtain

$$O(\epsilon) - \int_{\mathbb{R}^3} F(t_\epsilon u_\epsilon)dx < 0$$

for sufficiently small ϵ and sufficiently large K. Therefore, $\max_{t\geq 0} E_\mu(tu_\epsilon) < \frac{2}{5}S^{\frac{3}{2}}$, as desired. □

Remark 2.7. *Note that by lemma above, in case $V_\infty < \infty$, we have $m_{V_\infty} < \frac{2}{5}S^{\frac{3}{2}}$.*

The following result presents an interesting property of the Palais-Smale sequences of E_μ.

Lemma 2.8. *Let $\{u_n\} \subset H_\varepsilon$ be a $(PS)_c$ sequence for I_ε with $c < \frac{2}{5}S^{\frac{3}{2}}$ and $u_n \rightharpoonup 0$ in H_ε. Then one of the following conclusions holds.*

(a) *$u_n \to 0$ in H_ε, or*
(b) *there exist a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} u_n^2 dx \geq \beta > 0.$$

Proof. Suppose that (b) does not occur. Then we have

$$\sup_{y\in\mathbb{R}^3} \int_{B_R(y)} u_n(x)^2 dx \to 0 \quad \text{as} \quad n \to \infty.$$

By Lemma 1.1 in [30], we get

$$u_n \to 0 \ \text{ in } L^t(\mathbb{R}^3) \quad \text{for } t \in (2,6).$$

Given $\epsilon > 0$, from $(f_1) - (f_3)$, one has

$$0 \leq \int_{\mathbb{R}^3} f(u_n)u_n dx \leq \epsilon \int_{\mathbb{R}^3} u_n^2 dx + C_\epsilon \int_{\mathbb{R}^3} |u_n|^{q+1} dx. \tag{2.26}$$

Using the fact that $\{u_n\}$ is bounded in $H_\varepsilon, u_n \to 0$ in $L^{q+1}(\mathbb{R}^3)$ and that ϵ can be small arbitrarily, we can conclude that

$$\int_{\mathbb{R}^3} f(u_n)u_n dx \to 0.$$

Recalling that $I'_\varepsilon(u_n)u_n \to 0$, we get

$$\|u_n\|_\varepsilon^2 = \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 dx + o_n(1).$$

Since $\{u_n\} \subset H_\varepsilon$ is bounded, up to a subsequence, we have

$$\|u_n\|_\varepsilon^2 \to l \geq 0 \ \text{ and } \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 dx \to l \geq 0.$$

Suppose, by contradiction, that $l > 0$. Since

$$\begin{aligned} I_\varepsilon(u_n) &= \frac{1}{2}\|u_n\|_\varepsilon^2 - \frac{1}{10}\int\limits_{\mathbb{R}^3}\phi_{u_n}|u_n|^5dx + o_n(1) \\ &= \frac{1}{2}l - \frac{1}{10}l + o_n(1) \\ &= c + o_n(1), \end{aligned}$$

it follows that $\frac{2}{5}l = c$. On the other hand, from (ii), (iv) of Lemma 2.1, we get

$$\int\limits_{\mathbb{R}^3}\phi_{u_n}|u_n|^5dx = \int\limits_{\mathbb{R}^3}|\nabla\phi_{u_n}|^2 \le S^{-6}\|u_n\|_\varepsilon^{10},$$

which implies that $l \ge S^{\frac{3}{2}}$ or $c \ge \frac{2}{5}S^{\frac{3}{2}}$, which is a contradiction with our assumption. Thus $l = 0$, and so $u_n \to 0$ in H_ε. □

Lemma 2.9. *Assume that $V_\infty < \infty$ and let $\{u_n\}$ be a $(PS)_d$ sequence for the functional I_ε with $d < \frac{2}{5}S^{\frac{3}{2}}$ and $u_n \rightharpoonup 0$ in H_ε. If $u_n \nrightarrow 0$ in H_ε, then $d \ge m_{V_\infty}$.*

Proof. Let $t_n > 0, \forall n \in \mathbb{N}$ such that $\{t_nu_n\} \subset \mathcal{M}_{V_\infty}$. We claim that $\sup_{n\to\infty} t_n \le 1$.

Assume by contradiction, there exist $\delta > 0$ and a subsequence still denoted by $\{t_n\}$ such that

$$t_n > 1 + \delta \quad \text{for all} \quad n \in \mathbb{N}.$$

By Lemma 2.4, the sequence $\{u_n\}$ is bounded and from $E'_\varepsilon(u_n)u_n = o_n(1)$, we get

$$\int\limits_{\mathbb{R}^N}(|\nabla u_n|^2 + V(\varepsilon x)u_n^2)dx = \int\limits_{\mathbb{R}^N} f(u_n)u_ndx + \int\limits_{\mathbb{R}^3}\phi_{u_n}|u_n|^5dx + o_n(1). \tag{2.27}$$

Recalling that $t_nu_n \in \mathcal{M}_{V_\infty}$, we have

$$t_n^2\int\limits_{\mathbb{R}^3}(|\nabla u_n|^2 + V_\infty u_n^2)dx = \int\limits_{\mathbb{R}^N} f(t_nu_n)t_nu_ndx + t_n^{10}\int\limits_{\mathbb{R}^3}\phi_{u_n}|u_n|^5dx. \tag{2.28}$$

Combining (2.27), (2.28) and $t_n > 1 + \delta$, we get

$$\begin{aligned} &\left(\frac{1}{t_n^2} - 1\right)\int\limits_{\mathbb{R}^3}|\nabla u_n|^2dx + \int\limits_{\mathbb{R}^3}\left(\frac{V_\infty}{t_n^2} - V(\varepsilon x)\right)u_n^2dx \\ &= \int\limits_{\mathbb{R}^3}\left(\frac{f(t_nu_n)}{t_n^3u_n^3} - \frac{f(u_n)}{u_n^3}\right)u_n^4dx + \int\limits_{\mathbb{R}^3}(t_n^6 - 1)\phi_{u_n}|u_n|^5dx + o_n(1) \\ &\ge \int\limits_{\mathbb{R}^3}\left(\frac{f(t_nu_n)}{t_n^3u_n^3} - \frac{f(u_n)}{u_n^3}\right)u_n^4dx + o_n(1). \end{aligned} \tag{2.29}$$

From (V) and $t_n > 1$ there exists $R = R(\epsilon) > 0$ such that

$$V(\varepsilon x) \ge V_\infty - \epsilon > \frac{V_\infty}{t_n^2} - \epsilon \ \text{ for all } \ |x| \ge R. \tag{2.30}$$

Since $\|u_n\|_\varepsilon \leq C$ and $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^3)$, we deduce from Lemma 2.8 that there exist $\{y_n\} \subset \mathbb{R}^3$ and $R_1, \beta > 0$ such that

$$\int_{B_{R_1}(y_n)} u_n^2 dx \geq \beta. \tag{2.31}$$

If we set $\widetilde{u}_n(x) = u_n(x+y_n)$, then there exists a nonnegative function $\widetilde{u}$ small that, up to a subsequence, $\widetilde{u}_n \rightharpoonup \widetilde{u}$ in H_ε, $\widetilde{u}_n \to \widetilde{u}$ in $L^2_{B_{R_1}(0)}$ and $\widetilde{u}_n \to \widetilde{u}$ a.e. in $\mathbb{R}^3$. Moreover, by (2.31), there exists a subset $\Lambda \subset B_{R_1(0)}$ with positive measure such that $\widetilde{u} > 0$ a.e. in Λ. It follows from (f_5), (2.29)-(2.31) and $t_n \geq 1+\delta$ that

$$0 < \int_\Lambda \left(\frac{f((1+\delta)\widetilde{u}_n)}{((1+\delta)\widetilde{u}_n)^3} - \frac{f(\widetilde{u}_n)}{\widetilde{u}_n^3} \right) \widetilde{u}_n^4 \leq \epsilon C + o_n(1),$$

for any $\epsilon > 0$. Taking limit in the above inequality as $n \to \infty$ and applying Fatou's lemma, we get

$$0 < \int_\Lambda \left(\frac{f((1+\delta)\widetilde{u})}{((1+\delta)\widetilde{u})^3} - \frac{f(\widetilde{u})}{\widetilde{u}^3} \right) \widetilde{u}^4 \leq \epsilon C$$

for any $\epsilon > 0$, which yields a contradiction.

We next distinguish the following two cases:

Case 1. $\limsup_{n\to\infty} t_n = 1$. In this case, there exists a subsequence, still denoted by $\{t_n\}$ such that $t_n \to 1$ as $n \to \infty$. Hence

$$d + o_n(1) = I_\varepsilon(u_n) \geq I_\varepsilon(u_n) + m_{V_\infty} - E_{V_\infty}(t_n u_n). \tag{2.32}$$

Note that

$$\begin{aligned} I_\varepsilon(u_n) - E_{V_\infty}(t_n u_n) = {} & \frac{1}{2}\int_{\mathbb{R}^3} (1-t_n^2)|\nabla u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x) u_n^2 dx - \frac{t_n^2}{2}\int_{\mathbb{R}^3} V_\infty u_n^2 dx \\ & + \frac{1}{10}\int_{\mathbb{R}^3} (t_n^{10}-1)\phi_{u_n}|u_n|^5 + \int_{\mathbb{R}^3} (F(t_n u_n) - F(u_n)) dx. \end{aligned} \tag{2.33}$$

Now, from condition (V), given $\xi > 0$, there exists $R = R(\xi) > 0$ such that $V(\varepsilon x) \geq V_\infty - \xi$ for any $|x| \geq R$. Let $C > 0$ such that $\|u_n\|^2_\varepsilon \leq C$, for any $n \in \mathbb{N}$. By (2.33) we have

$$I_\varepsilon(u_n) - E_{V_\infty}(t_n u_n) \geq o_n(1) - \xi C + \int_{\mathbb{R}^3} (F(t_n u_n) - F(u_n)) dx.$$

Moreover, by virtue of the Mean Value Theorem,

$$\int_{\mathbb{R}^3} (F(t_n u_n) - F(u_n)) dx = o_n(1),$$

therefore,

$$d + o_n(1) \geq m_{V_\infty} - \xi C + o_n(1),$$

and taking limits, we obtain $d \geq m_{V_\infty}$.

Case 2. $\limsup_{n\to\infty} t_n = t_0 < 1$. In this case, we may suppose that there exists a subsequence, still denoted by $\{t_n\}$, satisfying

$$t_n \to t_0 \quad \text{and} \quad t_n < 1 \quad \forall n \in \mathbb{N}.$$

From (2.30), $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^3)$ and $\|u_n\|_\varepsilon \le C$, we see that

$$\int\limits_{\mathbb{R}^3} (V_\infty - V(\varepsilon x)u_n^2 dx \le \epsilon C + o_n(1)$$

for any given $\epsilon > 0$. Since $\frac{1}{4}f(s)s - F(s)$ is increasing, we deduce that

$$\begin{aligned} m_{V_\infty} &\le E_{V_\infty}(t_n u_n) - \frac{1}{4}\langle E'_{V_\infty}(t_n u_n), t_n u_n\rangle \\ &= \frac{t_n^2}{4}\int\limits_{\mathbb{R}^3}(|\nabla(u_n)|^2 + V_\infty u_n^2)dx + \int\limits_{\mathbb{R}^3}\left(\frac{1}{4}f(t_n u_n)t_n u_n - F(t_n u_n)\right)dx \\ &\quad + \frac{3}{20}t_n^{10}\int\limits_{\mathbb{R}^3}\phi_{u_n}|u_n|^5 dx \\ &\le \frac{1}{4}\int\limits_{\mathbb{R}^3}(|\nabla u_n|^2 + V(\varepsilon x)u_n^2)dx + \int\limits_{\mathbb{R}^3}\left(\frac{1}{4}f(u_n)u_n - F(u_n)\right) \\ &\quad + \frac{3}{20}\int\limits_{\mathbb{R}^3}\phi_{u_n}|u_n|^5 dx + \epsilon C + o(1) \\ &= I_\varepsilon(u_n) - \frac{1}{4}I'_\varepsilon(u_n)u_n + \epsilon C + o_n(1). \end{aligned} \tag{2.34}$$

Hence, taking the limit as $n \to \infty, \epsilon \to 0$ at last inequality we have $d \ge m_{V_\infty}$. □

2.3. Compactness properties for I_ε and E_μ

In order to apply the Ljusternik-Schnirelmann category theory, we need to check that I_ε satisfies the Palais-Smale condition on $\mathcal{N}_\varepsilon$. As the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3), 2 \le s < 2^*$, is continuous but is not compact, it is well known that, in general, such a condition is not fulfilled. Nevertheless, we shall prove that Palais-Smale condition holds in a suitable sublevel, related to the ground energy "at infinity".

Proposition 2.10. *The functional I_ε satisfies the $(PS)_c$ condition at any level $c < m_{V_\infty}$ if $V_\infty < \infty$ and at any level $c < \frac{2}{5}S^{\frac{3}{2}}$ if $V_\infty = \infty$.*

Proof. Let $\{u_n\} \subset H_\varepsilon$ be such that $I_\varepsilon(u_n) \to c$ and $I'_\varepsilon(u_n) \to 0$ in $(H_\varepsilon)^{-1}$. Since $\{u_n\}$ is bounded, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H_\varepsilon, u_n \to u$ in $L^r_{loc}(\mathbb{R}^3)$ for $1 \le r < 2^*$ and $u_n \to u$ a.e. in $\mathbb{R}^3$. Moreover, u is a critical point of I_ε. To see this, one only needs to prove

$$I'_\varepsilon(u)\varphi = 0 \quad \text{for any} \quad \varphi \in C_0^\infty(\mathbb{R}^3).$$

It follows from $(f_1) - (f_3)$ that, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|f(t)| \le C_\epsilon + \epsilon|t|^5 \quad \text{for all} \quad t \in \mathbb{R}.$$

Denote the support set of φ by Ω_φ, then for any subset $\mathcal{O} \subset \Omega_\varphi$ with $|\mathcal{O}| < \delta = \delta(\epsilon) = \frac{\epsilon^2}{C_\epsilon^2 |\varphi|_2^2} > 0$, we have

$$\begin{aligned}\left|\int_{\mathcal{O}} f(u_n)\varphi dx\right| &\leq C_\epsilon \int_{\mathcal{O}} |\varphi| dx + \epsilon \int_{\mathcal{O}} |u_n|^5 |\varphi| dx \\ &\leq C_\epsilon |\mathcal{O}|^{\frac{1}{2}} |\varphi|_2 + \epsilon |u_n|_6^5 |\varphi|_6 \\ &< \epsilon + C_1\epsilon = (1+C_1)\epsilon.\end{aligned}$$

By Vitali convergence theorem, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} f(u_n)\varphi dx = \int_{\mathbb{R}^3} f(u)\varphi dx. \tag{2.35}$$

By Lemma 2.1-(viii) we have $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ and so, $\phi_{u_n} \rightharpoonup \phi_u$ in $L^6(\mathbb{R}^3)$. Then

$$\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u)|u|^3 u\varphi dx \to 0 \quad \text{as} \quad n \to \infty. \tag{2.36}$$

Using $u_n \to u$ a.e. in $\mathbb{R}^3$ and

$$\begin{aligned}\int_{\mathbb{R}^3} |\phi_{u_n}(|u_n|^3 u_n - |u|^3 u)|^{\frac{6}{5}} dx &\leq 2^{\frac{6}{5}} \int_{\mathbb{R}^3} |\phi_{u_n}|^{\frac{6}{5}} [|u_n|^{\frac{24}{5}} + |u|^{\frac{24}{5}}] dx \\ &\leq C|\phi_{u_n}|_6^{\frac{6}{5}} (|u_n|_6^{\frac{24}{5}} + |u|_6^{\frac{24}{5}}) \\ &\leq C_1,\end{aligned}$$

we have $\phi_{u_n}(|u_n|^3 u_n - |u|^3 u) \rightharpoonup 0$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$ and thus

$$\int_{\mathbb{R}^3} \phi_{u_n}(|u_n|^3 u_n - |u|^3 u)\varphi \to 0 \quad \text{as} \quad n \to \infty, \tag{2.37}$$

which together with (2.36) implies that

$$\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^3 u_n \varphi dx \to \int_{\mathbb{R}^3} \phi_u |u|^3 u\varphi dx \quad \text{as} \quad n \to \infty. \tag{2.38}$$

Combining (2.37) with the weak convergence $u_n \rightharpoonup u$ in H_ε, we have

$$I'_\varepsilon(u)\varphi = \lim_{n\to\infty} I'_\varepsilon(u_n)\varphi = 0, \quad \forall\, \varphi \in C^\infty(\mathbb{R}^3),$$

which means that $I'_\varepsilon(u) = 0$. Thus u is a critical point of I_ε because $C_0^\infty(\mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3)$. Furthermore,

$$I_\varepsilon(u) = I_\varepsilon(u) - \frac{1}{4} I'_\varepsilon(u)u = \frac{1}{4}\|u\|_\varepsilon^2 + \frac{1}{4}\int_{\mathbb{R}^3} [f(u)u - 4F(u)] dx + \frac{3}{20}\int_{\mathbb{R}^3} \phi_u |u|^5 dx \geq 0. \tag{2.39}$$

Setting $v_n = u_n - u$, from a result due to Brezis-Lieb (e.g. [45]), we get

$$\|v_n\|_\varepsilon^2 = \|u_n\|_\varepsilon^2 - \|u\|_\varepsilon^2 + o_n(1),$$

and

$$\int_{\mathbb{R}^3} F(v_n)dx = \int_{\mathbb{R}^3} F(u_n)dx - \int_{\mathbb{R}^3} F(u)dx + o_n(1).$$

Thus, using Lemma 2.1-(viii) we infer that

$$I_\varepsilon(v_n) = I_\varepsilon(u_n) - I_\varepsilon(u) + o_n(1). \tag{2.40}$$

Now, we are going to show that

$$\|I'_\varepsilon(v_n) - I'_\varepsilon(u_n) + I'_\varepsilon(u)\|_{H_\varepsilon^{-1}} = o_n(1). \tag{2.41}$$

In fact, for all $\psi \in H_\varepsilon$ with $\|\psi\|_\varepsilon \le 1$, we have

$$\begin{aligned}
&|[I'_\varepsilon(v_n) - I'_\varepsilon(u_n) + I'_\varepsilon(u)]\psi| \\
&= \Big| \int_{\mathbb{R}^3} [\nabla v_n \nabla\psi + V(\varepsilon x)v_n\psi - f(v_n)\psi - \phi_{v_n^+}|v_n^+|^4\psi]dx \\
&\quad - \int_{\mathbb{R}^3} [\nabla u_n \nabla\psi + V(\varepsilon x)u_n\psi - f(u_n)\psi - \phi_{u_n^+}|u_n^+|^4\psi]dx \\
&\quad + \int_{\mathbb{R}^3} [\nabla u \nabla\psi + V(\varepsilon x)u\psi - f(u)\psi - \phi_{u^+}|u^+|^4\psi]dx \Big| \\
&= \Big| \int_{\mathbb{R}^3} (\nabla v_n - \nabla u_n + \nabla u)\nabla\psi dx + \int_{\mathbb{R}^3} V(\varepsilon x)[v_n - u_n + u]\psi dx \\
&\quad - \int_{\mathbb{R}^3} [f(v_n) - f(u_n) + f(u)]\psi dx - \int_{\mathbb{R}^3} [\phi_{v_n^+}|v_n^+|^4 - \phi_{u_n^+}|u_n^+|^4 + \phi_{u^+}|u^+|^4]\psi dx \Big| \\
&\le \int_{\mathbb{R}^3} |\nabla v_n - \nabla u_n + \nabla u||\nabla\psi|dx + \int_{\mathbb{R}^3} V(\varepsilon x)|v_n - u_n + u||\psi|dx \\
&\quad + \int_{\mathbb{R}^3} |f(v_n) - f(u_n) + f(u)||\psi|dx + \int_{\mathbb{R}^3} \left|\phi_{v_n^+}|v_n^+|^4 - \phi_{u_n^+}|u_n^+|^4 + \phi_{u^+}|u^+|^4 dx\right| |\psi|dx.
\end{aligned} \tag{2.42}$$

Moreover, using Lemma 3.1 [3], it is possible to check that

$$\left(\int_{\mathbb{R}^3} |\nabla v_n - \nabla u_n + \nabla u|^2 dx\right)^{\frac{1}{2}} = o_n(1), \qquad \left(\int_{\mathbb{R}^3} V(\varepsilon x)|v_n - u_n + u|^2 dx\right)^{\frac{1}{2}} = o_n(1), \tag{2.43}$$

and

$$\left(\int_{\mathbb{R}^3} |f(v_n) - f(u_n) + f(u)|^r dx\right)^{\frac{1}{r}} = o_n(1) \tag{2.44}$$

with $1 \le r \le \frac{6}{q}$. From Lemma A, we get

$$\left(\int_{\mathbb{R}^3} \left|\phi_{v_n^+}|v_n^+|^4 - \phi_{u_n^+}|u_n^+|^4 + \phi_{u^+}|u^+|^4\right|^{\frac{6}{5}} dx\right)^{\frac{5}{6}} = o_n(1). \tag{2.45}$$

Combining (2.42)-(2.45), we derive (2.41), and so,

$$I'_\varepsilon(v_n) \to 0 \quad \text{in} \quad H_\varepsilon^{-1}.$$

While, from (2.40) we obtain

$$I_\varepsilon(v_n) = c - I_\varepsilon(u) + o_n(1) := d + o_n(1)$$

and consequently, if $V_\infty < \infty$, we have by (2.39), that

$$d \le c < m_{V_\infty}.$$

It follows from Lemma 2.9 that $v_n \to 0$ in H_ε, and so $u_n \to u$ in H_ε.

If $V_\infty = \infty$, then V is coercive and by [19], the continuous embedding $H_\varepsilon \hookrightarrow L^r(\mathbb{R}^3)$ is compact for $2 < r < 2^*$. Hence, up to a subsequence, $v_n \to 0$ in $L^r(\mathbb{R}^3)$ and by $(f_1) - (f_3)$,

$$\|v_n\|_\varepsilon^2 = \int_{\mathbb{R}^3} \phi_{v_n^+}|v_n^+|^5 dx + o_n(1).$$

Since $\{v_n\} \subset H_\varepsilon$ is bounded, we may assume that

$$\|v_n\|_\varepsilon^2 \to L \ge 0 \quad \text{and} \quad \int_{\mathbb{R}^3} \phi_{v_n^+}|v_n^+|^5 dx \to L \ge 0,$$

perhaps for a subsequence. Suppose, by contradiction, that $L > 0$. Since $I_\varepsilon(v_n) = d + o_n(1)$, it follows that $\frac{2}{5}L = d$. But from (ii), (iv) of Lemma 2.1, we get

$$\int_{\mathbb{R}^3} \phi_{v_n^+}|v_n^+|^5 dx \le \int_{\mathbb{R}^3} \phi_{v_n}|v_n|^5 dx = \int_{\mathbb{R}^3} |\nabla \phi_{v_n}|^2 \le S^{-6}\|v_n\|_\varepsilon^{10},$$

which implies that $L \ge S^{\frac{3}{2}}$ or $d \ge \frac{2}{5}S^{\frac{3}{2}}$, which is a contradiction with our assumption. Thus $L = 0$, and so $v_n \to 0$ in H_ε, and so $u_n \to u$ in H_ε. □

Proposition 2.11. *The functional I_ε restricted to $\mathcal{N}_\varepsilon$ satisfies the $(PS)_c$ condition at any level $c < m_{V_\infty}$ if $V_\infty < \infty$ and at any level $c < \frac{2}{5}S^{\frac{3}{2}}$ if $V_\infty = \infty$.*

Proof. Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ be such that $I_\varepsilon(u_n) \to c$ and $I'_\varepsilon(u_n) \to 0$ in H_ε^{-1}. Then there exists $\{\lambda_n\} \subset \mathbb{R}$ such that

$$I'_\varepsilon(u_n) = \lambda_n J'_\varepsilon(u_n) + o_n(1), \tag{2.46}$$

where $J_\varepsilon : H_\varepsilon \to \mathbb{R}$ is given by

$$J_\varepsilon(u) = \int_{\mathbb{R}^3} [|\nabla u|^2 + V(\varepsilon x)u^2] dx - \int_{\mathbb{R}^3} f(u)u dx - \int_{\mathbb{R}^3} \phi_{u^+}|u^+|^5 dx.$$

Notice that

$$J'_\varepsilon(u_n)u_n = 2\int_{\mathbb{R}^3}[|\nabla u_n|^2 + V(\varepsilon x)u_n^2]dx - \int_{\mathbb{R}^3} f(u_n)u_n dx - \int_{\mathbb{R}^3} f'(u_n)u_n^2 dx$$
$$- 10\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 dx,$$

and by $u_n \in \mathcal{N}_\varepsilon$ and (f_5) we get

$$J'_\varepsilon(u_n)u_n = \int_{\mathbb{R}^3} f(u_n)u_n dx - \int_{\mathbb{R}^3} f'(u_n)u_n^2 dx - 8\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 dx$$
$$\leq -8\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 dx \leq 0.$$

We may assume that $J'_\varepsilon(u_n)u_n \to \gamma \leq 0$. If $\gamma = 0$, then by Lemma 2.1-(ii), we get

$$\int_{\mathbb{R}^3} |\nabla\phi_{u_n}|^2 dx = \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 dx \to 0. \tag{2.47}$$

Choosing $\delta = \delta_n = (\int_{\mathbb{R}^3} |\nabla\phi_{u_n}|^2 dx)^{1/2}$, using Lemma 2.1-(v) and the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^3)$, we have,

$$\begin{aligned}\int_{\mathbb{R}^3} |u_n|^6 dx &\leq \frac{\delta_n}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{2\delta_n}\int_{\mathbb{R}^3} |\nabla\phi_{u_n}|^2 dx \\ &\leq \frac{\delta_n}{2}C + \frac{1}{2}\delta_n \to 0\end{aligned} \tag{2.48}$$

as $n \to \infty$. From $(f_1) - (f_3)$, we have by interpolation

$$0 \leq \int_{\mathbb{R}^3} f(u_n)u_n dx \leq \epsilon\int_{\mathbb{R}^3}[u_n^2 + u_n^6]dx + C_\epsilon\int_{\mathbb{R}^3} |u_n|^{q+1} dx \to 0 \tag{2.49}$$

as $n \to \infty, \epsilon \to 0$. Consequently, from (2.47)-(2.49) we have

$$\|u_n\|_\varepsilon^2 = \int_{\mathbb{R}^3} f(u_n)u_n dx + \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 dx \to 0 \quad \text{as} \quad n \to \infty.$$

But this contradicts (2.13), therefore, $\gamma \neq 0$. Using $\langle I'_\varepsilon(u_n), u_n\rangle = 0$, we get

$$\lambda_n\langle J'_\varepsilon(u_n), u_n\rangle = o_n(1),$$

consequently, $\lambda_n = o_n(1)$, which yields that

$$I_\varepsilon(u_n) \to c \ \text{ and } \ I'_\varepsilon(u_n) \to 0.$$

Thus, $\{u_n\}$ is a $(PS)_c$ sequence for I_ε in H_ε and the result follows from Proposition 2.10. □

Corollary 2.12. *The critical points of functional I_ε on $\mathcal{N}_\varepsilon$ are critical points of functional I_ε in H_ε.*

Proof. The proof follows by using similar arguments employed in the last proposition. □

Now we pass to the functional related to the autonomous problem (2.11).

Lemma 2.13. *(Ground state for the autonomous problem) Let $\{u_n\} \subset \mathcal{M}_\mu$ be a sequence satisfying $E_\mu(u_n) \to m_\mu$. Then, up to subsequences the following alternative holds:*

(a) $\{u_n\}$ strongly converges in $H^1(\mathbb{R}^3)$;
(b) there exists a sequence $\{\widetilde{y}_n\} \subset \mathbb{R}^3$ such that $u_n(\cdot + \widetilde{y}_n)$ strongly converges in $H^1(\mathbb{R}^3)$.

In particular, there exists a minimizer $w_\mu \geq 0$ for m_μ.

Proof. By using the Ekeland Variational Principle as in the proof of Proposition 2.11, we may suppose that $\{u_n\}$ is a $(PS)_{m_\mu}$ sequence for E_μ and $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Thus going to a subsequence if necessary, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$. Moreover, u is a critical point of E_μ. Hence, $u \in H_\mu$ is a weak solution of (2.11). If $u \neq 0$, it remains to show that $E_\mu(u) = m_\mu$. By the fact $E'_\mu(u)u = 0$ and Fatou's lemma, we get

$$\begin{aligned}
m_\mu &\leq E_\mu(u) = E_\mu(u) - \frac{1}{4}\langle I'_\mu(u), u\rangle \\
&= \frac{1}{4}\int_{\mathbb{R}^3} (|\nabla u|^2 + \mu u^2)dx + \int_{\mathbb{R}^3} \left(\frac{1}{4}f(u)u - F(u)\right)dx + \frac{3}{20}\int_{\mathbb{R}^3} \phi_{u^+}|u^+|^5 dx \\
&\leq \liminf_{n\to\infty}\left\{\frac{1}{4}\int_{\mathbb{R}^3} (|\nabla u_n|^2 + \mu u_n^2)dx + \int_{\mathbb{R}^3} \left(\frac{1}{4}f(u_n)u_n - F(u_n)\right)dx + \frac{3}{20}\int_{\mathbb{R}^3} \phi_{u_n^+}|u_n^+|^5 dx\right\} \\
&= \liminf_{n\to\infty}\left(E_\mu(u_n) - \frac{1}{4}E'_\mu(u_n)u_n\right) \\
&\leq m_\mu.
\end{aligned}$$

Therefore, we have that $u_n \to u$ in $H^1(\mathbb{R}^3)$ and $E_\mu(u) = m_\mu$.

Now, we consider the case $u \equiv 0$. In this case, since $\{u_n\} \subset \mathcal{M}_\mu$, and $E_\mu(u_n) \to m_\mu \in (0, \frac{2}{5}S^{\frac{3}{2}})$, we conclude that $\|u_n\|_\mu \nrightarrow 0$. Therefore, arguing as in the proof of by Lemma 2.8, we deduce that there exist $R, \eta > 0$ and $\widetilde{y}_n \in \mathbb{R}^3$ such that

$$\liminf_{n\to\infty} \int_{B_R(\widetilde{y}_n)} u_n^2 dx \geq \eta.$$

Let $v_n(x) = u_n(x+\widetilde{y}_n)$, then we can use the invariance of $\mathbb{R}^3$ by translations to conclude that $E_\mu(v_n) \to m_\mu$ and $I'_\mu(v_n) \to 0$. Moreover, up to a subsequence, $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, and $v_n \to v$ in $L^2(B_R(0))$, with v being a critical point of E_μ. Since

$$\int_{B_R(0)} |v|^2 dx = \liminf_{n\to\infty} \int_{B_R(0)} |v_n|^2 dx = \liminf_{n\to\infty} \int_{B_R(\widetilde{y}_n)} |u_n|^2 dx \geq \eta > 0,$$

we conclude that $v \neq 0$, and the conclusion follows as in the first case of the proof.

Denote by $u^{\pm} = \max\{\pm u, 0\}$ the positive (negative) part of u, we get

$$0 = E'_\mu(u)u^- = -\|u^-\|_\mu^2 - \int\limits_{\mathbb{R}^3} f(u)u^- dx - \int\limits_{\mathbb{R}^3} \phi_{u^+}|u^+|^4 u^- dx = -\|u^-\|_\mu^2$$

and therefore $u \geq 0$ in $\mathbb{R}^3$. $\square$

3. Existence of a ground state solution

In this section, we show there exists a ground state solution to (2.9), that is, a positive solution u_ε of (2.9) with $I_\varepsilon(u_\varepsilon) = c_\varepsilon$. To study the regularization of the ground state solution, we recall the following two propositions in our case $N = 3$. The first one is an adequate version, for our aim, from a result due to Brezis and Kato [14].

Proposition 3.1. *Let $u \in H^1(\mathbb{R}^3)$ satisfying*

$$-\Delta u + (b(x) - q(x))u = f(x, u) \quad \textit{in } \mathbb{R}^3,$$

where $q \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $b : \mathbb{R}^3 \to \mathbb{R}^+$ is a $L^\infty_{loc}(\mathbb{R}^3)$ function; $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^+$ is a Caratheodory function such that

$$0 \leq f(x, s) \leq C_f(s^r + s) \quad \textit{for all } s > 0, x \in \mathbb{R}^3$$

and $r \in (1, 5)$. Then $u \in L^t(\mathbb{R}^3)$ for all $t \in [2, \infty)$. Moreover, there is a positive constant C_t depending on t, C_f and q such that

$$|u|_t \leq C_t\|u\|.$$

The dependence on q of the constant C_t can be given uniformly on a Cauchy sequence $q_k(x)$ in $L^{\frac{3}{2}}(\mathbb{R}^3)$.

The next proposition is a very particular version of Theorem 8.17 in [22], due to Trudinger.

Proposition 3.2. *Suppose that $t > 3, g \in L^{\frac{t}{2}}(\Omega)$ and $u \in H^(\Omega)$ satisfies in the weak sense*

$$-\Delta u \leq g(x) \quad \textit{in} \quad \Omega,$$

where Ω is an open subset of $\mathbb{R}^3$. Then for any $R > 0$ and any ball $B_{2R}(y) \subset \Omega$,

$$\sup_{x \in B_R(y)} u(x) \leq C(|u^+|_{L^2(B_{2R}(y))} + |g|_{L^{\frac{t}{2}}(B_{2R}(y))})$$

where C depends on t and R.

In the end of this section we show that the existence of a ground state solution to (2.9), that is, a positive solution u_ε of (2.9) satisfying $I_\varepsilon(u_\varepsilon) = c_\varepsilon$.

Theorem 3.3. *Suppose that f satisfies $(f_1) - (f_5)$ and V verifies (V). Then there exists some $\varepsilon^* > 0$, such that for every $\varepsilon \in (0, \varepsilon^*)$, system (2.9) possesses a positive ground state solution $u_\varepsilon \in C^{1,\alpha}_{loc}(\mathbb{R}^3)$.*

Proof. By Lemma 2.3, the functional I_ε satisfies the geometry of the Mountain Pass Theorem in H_ε. Then, by a version of Mountain Pass Theorem due to Ambrosetti and Rabinowitz without (PS) condition (e.g. [5]), there exists a sequence $\{u_n\} \subset H_\varepsilon$ satisfying $I_\varepsilon(u_n) \to c_\varepsilon$ and $I'_\varepsilon(u_n) \to 0$ in $(H_\varepsilon)^{-1}$.

If $V_\infty < \infty$, we may assume, without loss of generality, that

$$V(0) = V_0 = \inf_{x\in\mathbb{R}^3} V(x).$$

Let $\mu > 0$ such that $V_0 < \mu < V_\infty$. Then

$$m_{V_0} < m_\mu < m_{V_\infty}. \tag{3.1}$$

By Lemma 2.13, there exists a nonnegative function $w_\mu \in H^1(\mathbb{R}^3)$ such that

$$E_\mu(w_\mu) = \max_{t\geq 0} E_\mu(tw_\mu) = m_\mu.$$

For $r > 0$, let η_r a smooth cut-off function in $\mathbb{R}^3$ which equals to 1 on $B_r(0)$ and with support in $B_{2r}(0)$. Let $\omega_r := \eta_r w_\mu$ and $t_r > 0$ such that $t_r\omega_r \in \mathcal{M}_\mu$. If it were, $E_\mu(t_r\omega_r) \geq m_{V_\infty}, \forall r > 0$, in view of $\omega_r \to w_\mu$ in $H^1(\mathbb{R}^3)$ as $r \to \infty$, we would have $t_r \to 1$ and then

$$m_{V_\infty} \leq \liminf_{r\to\infty} E_\mu(t_r\omega_r) = E_\mu(w_\mu) = m_\mu$$

which contradicts (3.1). Therefore, there exists some $r^* > 0$ such that $\varphi := t_{r^*}\omega_{r^*}$ satisfies $E_\mu(\varphi) < m_{V_\infty}$. Consequently, the condition (V) implies that for some $\varepsilon^* > 0$

$$V(\varepsilon x) \leq \mu, \quad \text{for all } \ x \in \operatorname{supp}\varphi \ \text{ and } \ \varepsilon \in (0, \varepsilon^*), \tag{3.2}$$

and so

$$\int_{\mathbb{R}^3} V(\varepsilon x)\varphi^2 dx \leq \int_{\mathbb{R}^3} \mu\varphi^2 dx \quad \text{for all } \ \varepsilon \in (0, \varepsilon^*). \tag{3.3}$$

Consequently

$$I_\varepsilon(t\varphi) \leq E_\mu(t\varphi) \leq E_\mu(\varphi) \ \text{ for all } \ \varepsilon \in (0, \varepsilon^*), \ t \geq 0.$$

Hence

$$\max_{t\geq 0} I_\varepsilon(t\varphi) \leq E_\mu(\varphi) < m_{V_\infty}, \quad \text{for any } \ \varepsilon \in (0, \varepsilon^*)$$

and so $c_\varepsilon < m_{V_\infty}$.

If $V_\infty = \infty$, for any $\mu > V_0$, from Lemma 2.6, we can choose some $t_\epsilon > 0$ such that

$$\max_{t\geq 0} E_\mu(tu_\epsilon) = E_\mu(t_\epsilon u_\epsilon) < \frac{2}{5} S^{\frac{3}{2}},$$

where u_ϵ is given in Lemma 2.6 with support in $B_2(0)$. By a similar argument as in (3.2), (3.3), we have

$$I_\varepsilon(tu_\epsilon) \leq E_\mu(tu_\epsilon) \leq \max_{t\geq 0} E_\mu(tu_\epsilon) < \frac{2}{5} S^{\frac{3}{2}} \ \text{ for all } \ t \geq 0,$$

which implies $c_\varepsilon < \frac{2}{5} S^{\frac{3}{2}}$.

In virtue of Proposition 2.10, we see that there exists some $u_\varepsilon \in H_\varepsilon$ such that $u_n \to u_\varepsilon$ in H_ε with $I_\varepsilon(u_\varepsilon) = c_\varepsilon$. Denote by $u_\varepsilon^\pm = \max\{\pm u_\varepsilon, 0\}$ the positive (negative) part of u_ε, we get

$$0 = E'_\mu(u_\varepsilon)u_\varepsilon^- = -\|u_\varepsilon^-\|_\mu^2 - \int_{\mathbb{R}^3} f(u_\varepsilon)u_\varepsilon^- dx - \int_{\mathbb{R}^3} \phi_{u_\varepsilon^+}|u_\varepsilon^+|^4 u_\varepsilon^- dx = -\|u_\varepsilon^-\|_\mu^2$$

and therefore $u_\varepsilon \geq 0$ in $\mathbb{R}^3$. Since $u_\varepsilon \in H^1(\mathbb{R}^3)$, we see that

$$\lim_{R\to\infty} \int_{|x|\geq R} (u_\varepsilon^2 + u_\varepsilon^5)dx \to 0.$$

Using Proposition 3.1 with $q(x) = \phi_{u_\varepsilon} u_\varepsilon^3 \in L^{\frac{3}{2}}(R^3), b(x) = \mu$ and $h(x, u_\varepsilon) = f(u_\varepsilon) \leq C_1 u_\varepsilon + C_2 u_\varepsilon^r$ for some $r \in (3, 5)$, we can infer that

$$|u_\varepsilon|_t \leq C_t\|u_\varepsilon\|.$$

Applying Proposition 3.2 in the following inequality

$$-\Delta u_\varepsilon \leq -\Delta u_\varepsilon + \mu u_\varepsilon = \phi_{u_\varepsilon}|u_\varepsilon|^3 u + f(u_\varepsilon) := g(x),$$

there exists a $t > 3$ such that $|g|_{\frac{3}{2}} \leq C$, and all $y \in \mathbb{R}^3$

$$\sup_{x\in B_1(y)} u_\varepsilon \leq C(|u_\varepsilon|_{L^2(B_2(y))} + |g|_{L^{\frac{t}{2}}(B_2(y))})$$

which implies that $|u|_\infty \leq C$. Moreover, combining with the last limit we reach

$$\lim_{|x|\to\infty} u_\varepsilon(x) = 0.$$

Then, by the regularity theory [29,40,13,33], there exists $\alpha \in (0, 1)$ such that $u_\varepsilon \in C^{1,\alpha}_{loc}(\mathbb{R}^3)$. Now applying Harnack's inequality [41] we have that $u_\varepsilon(x) > 0$ in $\mathbb{R}^3$. $\square$

4. Multiplicity of solutions to (2.9)

In this section we are going to prove the multiplicity of solutions and study the behavior of their maximum points in relation to the set M. The main result in this section has the following statement.

Theorem 4.1. *Suppose that f satisfies $(f_1)-(f_5)$ and V verifies (V). Then, for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, problem (2.9) has at least $cat_{M_\delta}(M)$ positive solutions, for any $\varepsilon \in (0, \varepsilon_\delta)$. Moreover, if u_ε denotes one of these positive solutions and $z_\varepsilon \in \mathbb{R}^3$ its global maximum point, then*

$$\lim_{\varepsilon\to 0} V(\varepsilon z_\varepsilon) = V_0.$$

In order to prove the above theorem, in the next subsection we fix some notation and show some preliminary lemmas.

4.1. Preliminary results

Let w be a ground state solution of problem (2.11) with $\mu = V_0$ and ψ be a smooth nonincreasing function defined in $[0, \infty)$ such that $\psi(s) = 1$ if $0 \leq s \leq \delta/2$ and $\psi(s) = 0$ if $s \geq \delta$.

For any $y \in M$, we define

$$\pi_{\varepsilon,y}(x) = \psi(|\varepsilon x - y|)w\left(\frac{\varepsilon x - y}{\varepsilon}\right)$$

and $t_\varepsilon > 0$ satisfying

$$\max_{t\geq 0} I_\varepsilon(t\pi_{\varepsilon,y}) = I_\varepsilon(t_\varepsilon\pi_{\varepsilon,y})$$

and define $\Phi_\varepsilon : \ M \to \mathcal{N}_\varepsilon$ by

$$\Phi_\varepsilon(y) = t_\varepsilon\pi_{\varepsilon,y}.$$

By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$.

Lemma 4.2. *The function Φ_ε has the following property:*

$$\lim_{\varepsilon\to 0} I_\varepsilon(\Phi_\varepsilon(y)) = m_{V_0} \quad \textit{uniformly in} \;\; y \in M.$$

Proof. Suppose by contradiction that, there exist some $\delta_0 > 0, \{y_n\} \subset M$ and $\varepsilon_n \to 0$ such that

$$|I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - m_{V_0}| \geq \delta_0. \tag{4.1}$$

Now we claim that $\lim_{n\to\infty} t_{\varepsilon_n} = 1$. In fact, by the definition of t_{ε_n} and (2.13) we have

$$\begin{aligned} r^* &\leq \int\limits_{\mathbb{R}^3} \left[|\nabla(t_{\varepsilon_n}\pi_{\varepsilon_n,y_n})|^2 + V(\varepsilon_n x)(t_{\varepsilon_n}\pi_{\varepsilon_n,y_n})^2\right] dx \\ &= \int\limits_{\mathbb{R}^3} f(t_{\varepsilon_n}\pi_{\varepsilon_n,y_n})t_{\varepsilon_n}\pi_{\varepsilon_n,y_n} dx + |t_{\varepsilon_n}|^{10} \int\limits_{\mathbb{R}^3} \phi_{\pi_{\varepsilon_n,y_n}} |\pi_{\varepsilon_n,y_n}|^5 dx. \end{aligned} \tag{4.2}$$

Clearly, t_{ε_n} can not go zero, therefore, $t_{\varepsilon_n} \geq t_0 > 0$ for some $t_0 > 0$. Note that

$$\begin{aligned} &\int\limits_{\mathbb{R}^3} (|\nabla\pi_{\varepsilon_n,y_n}|^2 + V(\varepsilon_n x)|\pi_{\varepsilon_n,y_n}|^2) dx \\ &= \int\limits_{\mathbb{R}^3} \frac{f(t_{\varepsilon_n}\pi_{\varepsilon_n,y_n})}{t_{\varepsilon_n}\pi_{\varepsilon_n,y_n}} \pi^2_{\varepsilon_n,y_n} + |t_{\varepsilon_n}|^8 \int\limits_{\mathbb{R}^3} \phi_{\pi_{\varepsilon_n,y_n}} |\pi_{\varepsilon_n,y_n}|^5 dx \\ &\geq |t_{\varepsilon_n}|^8 \int\limits_{\mathbb{R}^3} \phi_{\pi_{\varepsilon_n,y_n}} |\pi_{\varepsilon_n,y_n}|^5 dx. \end{aligned} \tag{4.3}$$

By using the Lebesgue's theorem, we can verify that

$$\lim_{n\to\infty} ||\pi_{\varepsilon_n,y_n}||^2_{\varepsilon_n} = ||w||^2_{V_0}, \qquad \lim_{n\to\infty} \int\limits_{\mathbb{R}^3} \phi_{\pi_{\varepsilon_n,y_n}} |\pi_{\varepsilon_n,y_n}|^5 dx = \int\limits_{\mathbb{R}^3} \phi_w |w|^5 dx,$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} f(\pi_{\varepsilon_n,y_n})\pi_{\varepsilon_n,y_n} = \int_{\mathbb{R}^3} f(w)wdx.$$

If $t_{\varepsilon_n} \to \infty$, then the right side of (4.3) tends to ∞, which yields a contradiction. Hence, $0 < t_0 \le t_{\varepsilon_n} \le t_1$. Assuming that $t_{\varepsilon_n} \to T > 0$, then we get

$$\frac{1}{T^2}\int_{\mathbb{R}^3}(|\nabla w|^2 + V_0 w^2)dx = \int_{\mathbb{R}^3}\frac{f(Tw)}{T^3w^3}w^4dx + T^6\int_{\mathbb{R}^3}\phi_w|w|^5. \tag{4.4}$$

Since w is a ground state solution of (2.11) with $\mu = V_0$, one has

$$\int_{\mathbb{R}^3}(|\nabla w|^2 + V_0 w^2)dx = \int_{\mathbb{R}^3} f(w)wdx + \int_{\mathbb{R}^3}\phi_w|w|^5. \tag{4.5}$$

Combining (4.4), (4.5), we have

$$\left(\frac{1}{T^2} - 1\right)\int_{\mathbb{R}^3}(|\nabla w|^2 + V_0 w^2) = \int_{\mathbb{R}^3}\left(\frac{f(Tw)}{(Tw)^3}dx - \frac{f(w)}{w^3}\right)w^4dx + (T^6 - 1)\int_{\mathbb{R}^3}\phi_w|w|^5. \tag{4.6}$$

By (f_5), we conclude that $T = 1$.

On the other hand,

$$\begin{aligned} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = \frac{t_{\varepsilon_n}^2}{2}\int_{\mathbb{R}^3}&[|\nabla(\psi(|\varepsilon_n x|)w)|^2 + V(\varepsilon_n x + y_n)|\psi(|\varepsilon_n x|)w|^2]dx \\ &- \int_{\mathbb{R}^3} F(t_{\varepsilon_n}\psi(|\varepsilon_n x|)w)dx - \frac{t_{\varepsilon_n}^{10}}{10}\int_{\mathbb{R}^3}\phi_{\psi(|\varepsilon_n x|)w}|\psi(|\varepsilon_n x|)w|^5dx. \end{aligned}$$

Let $n \to \infty$, we get $\lim_{n\to\infty} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = I_{V_0}(w) = m_{V_0}$, which contradicts to (4.1). This completes the proof. $\square$

For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. Define $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ as $\chi(x) = x$ for $|x| \le \rho$ and $\chi(x) = \rho x/|x|$ for $|x| \ge \rho$. Finally, let us consider the map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^3$ given by

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^3}\chi(\varepsilon x)u^2dx}{\int_{\mathbb{R}^3}u^2dx}.$$

Since $M \subset B_\rho(0)$, by the definition of χ and the Lebesgue's theorem, we conclude that

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(\Phi_\varepsilon(y)) = y \ \text{ uniformly in } y \in M.$$

To continue our argument, we need the following compactness result.

Proposition 4.3. *Let $\varepsilon_n \to 0$ and $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$ be such that $I_{\varepsilon_n}(u_n) \to m_{V_0}$. Then there exists a sequence $\{\widetilde{y}_n\} \subset \mathbb{R}^3$ such that $v_n(x) = u_n(x + \widetilde{y}_n)$ has a convergent subsequence in $H^1(\mathbb{R}^3)$. Moreover, up to a subsequence, $y_n := \varepsilon_n\widetilde{y}_n \to y \in M$.*

Proof. By Lemma 2.8, we can obtain a sequence $\{\widetilde{y}_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(\widetilde{y}_n)} u_n^2 dx \geq \beta > 0.$$

If we define $v_n(x) = u_n(x + \widetilde{y}_n)$, along a subsequence, we have $v_n \rightharpoonup v \neq 0$ in $H^1(\mathbb{R}^3)$. Let $t_n > 0$ be such that $\widetilde{v}_n := t_n v_n \in \mathcal{M}_{V_0}$. Set $y_n = \varepsilon_n \widetilde{y}_n$. Using $u_n \in \mathcal{N}_{\varepsilon_n}$, we have

$$\begin{aligned}
m_{V_0} &\leq E_{V_0}(\widetilde{v}_n) \\
&\leq \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla \widetilde{v}_n|^2 + V(\varepsilon_n(x + \widetilde{y}_n))\widetilde{v}_n^2\right) dx - \frac{1}{10} \int_{\mathbb{R}^3} \phi_{\widetilde{v}_n^+} |\widetilde{v}_n^+|^5 dx - \int_{\mathbb{R}^3} F(\widetilde{v}_n) dx \\
&= \frac{t_n^2}{2} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(\varepsilon_n x) u_n^2 dx - \frac{t_n^{10}}{10} \int_{\mathbb{R}^3} \phi_{u_n^+} |u_n^+|^5 dx - \int_{\mathbb{R}^3} F(t_n u_n) dx \\
&= I_{\varepsilon_n}(t_n u_n) \\
&\leq I_{\varepsilon_n}(u_n) = m_{V_0} + o_n(1).
\end{aligned}$$

Hence $\lim_{n\to\infty} I_{V_0}(\widetilde{v}_n) = m_{V_0}$.

We claim that, up to subsequence, $t_n \to t^* > 0$. Indeed, since $v_n \nrightarrow 0$ in $H^1(\mathbb{R}^3)$, there exists $\gamma > 0$ such that $0 < \gamma \leq ||v_n||$. Hence, $0 < \gamma^* \leq ||v_n||_{V_0}$ with $\gamma^* = \gamma \min\{1, V_0\}$. It follows that,

$$0 \leq t_n \gamma^* \leq ||t_n v_n||_{V_0} = ||\widetilde{v}_n||_{V_0} \leq C$$

for some $C > 0$. Thus $\{t_n\}$ is bounded and we can suppose that $t_n \to t^* \geq 0$. If $t^* = 0$, then, since $\{v_n\}$ is bounded, we infer that $\widetilde{v}_n = t_n v_n \to 0$. Hence $I_{V_0}(\widetilde{v}_n) \to 0$, which contradicts $m_{V_0} > 0$. So, $t^* > 0$ and the weak limit of $\{\widetilde{v}_n\}$ is different from zero. Let $\widetilde{v}$ be the weak limit of $\{\widetilde{v}_n\}$ in $H^1(\mathbb{R}^3)$. Since $t_n \to t^* > 0$ and $v_n \rightharpoonup v \not\equiv 0$, we have from the uniqueness of the weak limit that $\widetilde{v} = t^* v \not\equiv 0$. From Lemma 2.13, $\widetilde{v}_n \to \widetilde{v}$ in $H^1(\mathbb{R}^3)$, and so, $v_n \to v$ in $H^1(\mathbb{R}^3)$. This proves the first part of the lemma.

We next show that $\{y_n\}$ has a bounded subsequence. Suppose by contradiction that $|y_n| \to \infty$. Considering first the case $V_\infty = \infty$, the following inequality

$$\begin{aligned}
\int_{\mathbb{R}^3} V(\varepsilon_n x + y_n) v_n^2 dx &\leq \int_{\mathbb{R}^3} (|\nabla v_n|^2 + V(\varepsilon_n x + y_n) v_n^2) dx \\
&= \int_{\mathbb{R}^3} f(v_n) v_n dx + \int_{\mathbb{R}^3} \phi_{v_n^+} |v_n^+|^5 dx
\end{aligned}$$

together with Fatou's lemma imply

$$\infty = \liminf_{n\to\infty} \left[\int_{\mathbb{R}^3} f(v_n) v_n dx + \int_{\mathbb{R}^3} \phi_{v_n^+} |v_n^+|^5 dx \right]$$

which leads to a contradiction, since the sequence $\{f(v_n) v_n + \phi_{v_n^+} |v_n^+|^5\}$ is bounded in $L^1(\mathbb{R}^3)$.

Now, we consider that case $V_\infty < \infty$. By virtue of $\widetilde{v}_n \to \widetilde{v}$ in $H^1(\mathbb{R}^3)$ and $V_0 < V_\infty$, we have

$$\begin{aligned}
m_{V_0} &= I_{V_0}(\widetilde{v}) < I_{V_\infty}(\widetilde{v}) \\
&\le \liminf_{n\to\infty}\left\{\frac{1}{2}\int\limits_{\mathbb{R}^3}(|\nabla\widetilde{v}_n|^2 + V(\varepsilon_n x + y_n)\widetilde{v}_n^2)dx - \frac{1}{10}\int\limits_{\mathbb{R}^3}\phi_{\widetilde{v}_n^+}|\widetilde{v}_n^+|^5dx - \int\limits_{\mathbb{R}^3}F(\widetilde{v}_n)dx\right\} \\
&= \liminf_{n\to\infty}\left\{\frac{t_n^2}{2}\int\limits_{\mathbb{R}^3}\left(|\nabla u_n|^2 + V(\varepsilon_n x)u_n^2\right)dx - \frac{t_n^{10}}{10}\int\limits_{\mathbb{R}^3}\phi_{u_n^+}|u_n^+|^5dx - \int\limits_{\mathbb{R}^3}F(t_n u_n)dx\right\} \\
&= \liminf_{n\to\infty} I_{\varepsilon_n}(t_n u_n) \\
&\le \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) = m_{V_0},
\end{aligned}$$

which does not make sense. Therefore, $\{y_n\}$ is bounded and up to a subsequence, $y_n \to y$ in $\mathbb{R}^3$. If $y \notin M$, then $V(y) > V_0$ and we obtain a contradiction arguing as above. Thus, $y \in M$ and the lemma is proved. □

Let $h: \mathbb{R}^+ \to \mathbb{R}^+$ be any positive function satisfying $h(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$. Define the set

$$\widetilde{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : \ I_\varepsilon(u) \le m_{V_0} + h(\varepsilon)\}.$$

Given $y \in M$, by Lemma 4.2 we see that $h(\varepsilon) = |I_\varepsilon(\Phi_\varepsilon(y)) - m_{V_0}|$ satisfies $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Thus, $\Phi_\varepsilon(y) \in \widetilde{\mathcal{N}}_\varepsilon$ and $\widetilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$.

Lemma 4.4. *For any $\delta > 0$, there holds that*

$$\lim_{\varepsilon\to 0}\sup_{u\in\widetilde{\mathcal{N}}_\varepsilon} dist(\beta_\varepsilon(u), M_\delta) = 0.$$

Proof. Let $\{\varepsilon_n\} \subset \mathbb{R}^+$ be such that $\varepsilon_n \to 0$. By definition, there exists $\{u_n\} \subset \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\mathrm{dist}(\beta_{\varepsilon_n}(u_n), M_\delta) = \sup_{u\in\widetilde{\mathcal{N}}_{\varepsilon_n}} \mathrm{dist}(\beta_{\varepsilon_n}(u), M_\delta) + o_n(1).$$

Thus, it suffices to find a sequence $\{y_n\} \subset M_\delta$ such that

$$|\beta_{\varepsilon_n}(u_n) - y_n| = o_n(1). \tag{4.7}$$

By virtue of $I_{V_0}(tu_n) \le I_\varepsilon(tu_n)$ for $t \ge 0$ and $\{u_n\} \subset \widetilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we obtain

$$m_{V_0} \le c_{\varepsilon_n} \le I_{\varepsilon_n}(u_n) \le m_{V_0} + h(\varepsilon_n).$$

This leads to $I_\varepsilon(u_n) \to m_{V_0}$. Thus we can invoke Proposition 4.3 to obtain a sequence $\{\widetilde{y}_n\} \subset \mathbb{R}^3$ such that $y_n = \varepsilon_n\widetilde{y}_n \in M_\delta$ for n sufficiently large. Hence

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^3}(\chi(\varepsilon_n x + y_n) - y_n)u_n^2(x + \widetilde{y}_n)dx}{\int_{\mathbb{R}^3} u_n^2(x + \widetilde{y}_n)dx}.$$

Since $\varepsilon_n x + y_n \to y \in M$, we have that $\beta_{\varepsilon_n}(u_n) = y_n + o_n(1)$ and the sequence $\{y_n\}$ verifies (4.7). □

The next two lemmas play a fundamental role in the study of the behavior of the maximum points of the solutions.

Lemma 4.5. *Let v_n be s solution of the following problem*

$$\begin{cases} -\Delta v_n + V_n(x)v_n = f(v_n) + \phi_{v_n}|v_n|^3 v_n, & x \in \mathbb{R}^3, \\ v_n \in H^1(\mathbb{R}^3), \quad v_n(x) > 0, & x \in \mathbb{R}^3, \end{cases} \tag{4.8}$$

where $V_n(x) = V(\varepsilon_n x + \varepsilon_n \widetilde{y}_n)$. Assume that the conditions (V) and $(f_1) - (f_5)$ hold and that $v_n \to v$ in $H^1(\mathbb{R}^3)$ with $v \neq 0$, then $v_n \in L^\infty(\mathbb{R}^3)$ and there exists $C > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^3)} \leq C$ for all $n \in \mathbb{N}$. Furthermore

$$\lim_{|x|\to\infty} v_n(x) = 0 \quad \textit{uniformly in} \quad n.$$

Proof. From Proposition 4.3 we have $\varepsilon_n \widetilde{y}_n \to y \in M$. Since $v_n \to v$ in $H^1(\mathbb{R}^3)$ with $v \neq 0$, we infer that v satisfies the equation

$$-\Delta v + V_0 v = f(v) + |v|^3 v \phi_v, \quad x \in \mathbb{R}^3. \tag{4.9}$$

Moreover,

$$\lim_{R\to\infty} \int\limits_{|x|\geq R} (v_n^2 + v_n^6)dx = 0 \text{ uniformly for } n \in \mathbb{N}. \tag{4.10}$$

Using Proposition 3.1 with $q(x) = \phi_{v_n} v_n^3 \in L^{\frac{3}{2}}(R^3), b(x) = V_n(x)$ and $h(x, v_n) = f(v_n) \leq C_1 v_n + C_2 v_n^r$ for some $r \in (3, 5)$, we can infer that

$$|v_n|_t \leq C_t \|v_n\|$$

where C_t is independent of n. Applying Proposition 3.2 in the following inequality

$$-\Delta v_n \leq -\Delta v_n + V_n(x)v_n = \phi_{v_n}|v_n|^3 v_n + f(v_n) := g_n(x),$$

there exists a $t > 3$ such that $|g_n|_{\frac{3}{2}} \leq C$, and all $y \in \mathbb{R}^3$

$$\sup_{x\in B_1(y)} u_\varepsilon \leq C(|v_n|_{L^2(B_2(y))} + |g_n|_{L^{\frac{t}{2}}(B_2(y))})$$

which implies that $\|v_n\|_{L^\infty(\mathbb{R}^3)} \leq C$ uniformly for $n \in \mathbb{N}$. Moreover, combining with the last limit we reach

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly for } n \in \mathbb{N}. \quad \square$$

Lemma 4.6. *There exists $\delta > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^3)} \geq \delta$.*

Proof. Suppose by contradiction that, $\|v_n\|_{L^\infty(\mathbb{R}^3)} \to 0$. Taking $\varepsilon_0 = \frac{V_0}{2}$, it follows from (f_2) that there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\frac{f(\|v_n\|_{L^\infty(\mathbb{R}^3)})}{\|v_n\|_{L^\infty(\mathbb{R}^3)}} < \varepsilon_0.$$

Therefore, by Lemma 2.1-(iv), Hölder inequality, Sobolev inequality and boundedness of $\{v_n\}$ in $H^1(\mathbb{R}^3)$, we get

$$\begin{aligned}
\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \int_{\mathbb{R}^3} V_0 v_n^2 dx &\leq \int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \int_{\mathbb{R}^3} V_n(x) v_n^2 dx \\
&= \int_{\mathbb{R}^3} f(v_n) v_n dx + \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 dx \\
&\leq \int_{\mathbb{R}^3} \frac{f(\|v_n\|_{L^\infty(\mathbb{R}^3)})}{\|v_n\|_{L^\infty(\mathbb{R}^3)}} v_n^2 dx + \left(\int_{\mathbb{R}^3} |\phi_{v_n}|^6 dx \right)^{\frac{1}{6}} \left(\int_{\mathbb{R}^3} |v_n|^6 dx \right)^{\frac{5}{6}} \\
&\leq \varepsilon_0 \int_{\mathbb{R}^3} v_n^2 dx + S^{-\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nabla \phi_{v_n}|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |v_n|^6 dx \right)^{\frac{5}{6}} \\
&\leq \varepsilon_0 \int_{\mathbb{R}^3} v_n^2 dx + S^{-\frac{1}{2}} S^{-3} \left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \right)^{\frac{5}{2}} \left(\int_{\mathbb{R}^3} |v_n|^6 dx \right)^{\frac{5}{6}} \\
&\leq \varepsilon_0 \int_{\mathbb{R}^3} v_n^2 dx + C \|v_n\|_{L^\infty(\mathbb{R}^3)}^{\frac{10}{3}} \left(\int_{\mathbb{R}^3} |v_n|^2 dx \right)^{\frac{5}{6}},
\end{aligned}$$

which leads to $\|v_n\| \to 0$ as $n \to \infty$, contradicting to $v_n \to v$ in $H^1(\mathbb{R}^3)$ with $v \not\equiv 0$. Then there exists $\delta > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^3)} \geq \delta, \forall n \in \mathbb{N}$. $\square$

4.2. Proof of Theorem 4.1

We can finish the proof of Theorem 4.1 in two parts.

Part 1: Multiplicity of solutions. We fix a small $\varepsilon > 0$. Then, by Lemmas 4.2, 4.4 we concluded that $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the inclusion map $Id: \ M \to M_\delta$. This fact and Lemma 4.3 [11] imply that

$$\operatorname{cat}_{\widetilde{\mathcal{N}_\varepsilon}}(\widetilde{\mathcal{N}_\varepsilon}) \geq \operatorname{cat}_{M_\delta}(M).$$

Since I_ε satisfies the $(PS)_c$ condition for all $c \in (m_{V_0}, m_{V_0} + h(\varepsilon))$ by the Ljusternil-Schnirelmann theory of critical points [45], I_ε restricted to $\mathcal{N}_\varepsilon$ possesses at least $\operatorname{cat}_{M_\delta}(M)$ critical points. Consequently by Corollary 2.12, we see that I_ε has at least $\operatorname{cat}_{M_\delta}(M)$ critical points in H_ε.

Part 2: The behavior of maximum points. If u_{ε_n} is a solution of problem

$$\begin{cases} -\Delta v_n + V_n(x) v_n = f(v_n) + |v_n|^3 v_n \phi_{v_n}, & x \in \mathbb{R}^3, \\ v_n \in H^1(\mathbb{R}^3), \quad v_n(x) > 0, & x \in \mathbb{R}^3, \end{cases}$$

where $V_n(x) = V(\varepsilon_n x + \varepsilon_n \widetilde{y}_n)$ and $\{\widetilde{y}_n\}$ is given in Proposition 4.3. Moreover, up to a subsequence, $v_n \to v$ in $H^1(\mathbb{R}^3)$ and $y_n = \varepsilon_n \widetilde{y}_n \to y$ in M. Denoting p_n the global maximum point of v_n, by Lemmas 4.5, 4.6, we have that $p_n \in B_R(0)$ for some $R > 0$. Thus, the global maximum point of u_{ε_n} is $z_{\varepsilon_n} = p_n + \widetilde{y}_n$ and therefore

$$\varepsilon_n z_{\varepsilon_n} = \varepsilon_n p_n + \varepsilon_n \widetilde{y}_n = \varepsilon_n p_n + y_n.$$

Since $\{p_n\}$ is bounded, we have

$$\lim_{n\to\infty} V(\varepsilon_n z_{\varepsilon_n}) = V_0. \quad \square$$

4.3. Final comments

If u_ε is a positive solution of (2.9), the function $w_\varepsilon = u_\varepsilon(x/\varepsilon)$ is a positive solution of (1.1). Thus, the maximum points η_ε and z_ε of w_ε and u_ε, respectively, satisfy the identity

$$\eta_\varepsilon = \varepsilon z_\varepsilon,$$

consequently,

$$\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = V_0.$$

Theorem 1.1 follows from Theorem 4.1 and the last limit.

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Appendix A

As we point out in the introduction, the nonlocal term is involved with critical growth, and we shall encounter the problem of the convergence of integral with nonlocal term. To this end, we present a technical lemma which is useful in proving that the functional I_ε satisfies the $(PS)_c$ condition, see Proposition 2.10. For notational convenience, we denote by $I_\alpha(x) = \frac{1}{4\pi}\frac{1}{|x|^\alpha}$, $\alpha \in (0, 3)$ and then

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{|u(y)|^5}{|x-y|} dy = (I_1 * |u|^5)(x).$$

Lemma A. *Let (ξ_n) be a bounded sequence in $D^{1,2}(\mathbb{R}^3)$ such that $\xi_n \to 0$ a.e. in $\mathbb{R}^3$. Denote by $A(u) = \phi_u |u|^3 u$. Then for each $w \in D^{1,2}(\mathbb{R}^3)$, we have the following estimates:*

$$\int_{\mathbb{R}^3} |A(\xi_n + w) - A(\xi_n) - A(w)|^{\frac{6}{5}} dx = o_n(1).$$

Proof. By assumption, we can rewrite $A(u) = \phi_u |u|^3 u$ as $(I_1 * |u|^5)|u|^3 u$. Then by the mean value theorem and Young inequality, we derive that

$$\begin{aligned}
&|A(\xi_n+w)-A(\xi_n)| \\
&=\left|\int_0^1 \frac{d}{dt}\left\{(I_1*|\xi_n+tw|^5)|\xi_n+tw|^3[\xi_n+tw]\right\}dt\right| \\
&=\Big|\int_0^1 (I_\alpha*(5|\xi_n+tw|^3[\xi_n+tw]w))|\xi_n+tw|^3[\xi_n+tw]dt \\
&\quad +4\int_0^1 (I_1*|\xi_n+tw|^5)|\xi_n+tw|^3 w dt\Big| \qquad\qquad (A.1)\\
&\le 5\int_0^1 (I_1*(|\xi_n+tw|^4|w|))|\xi_n+tw|^4 dt+4\int_0^1 (I_1*|\xi_n+tw|^5)|\xi_n+tw|^3|w|dt \\
&\le C_1(I_1*((|\xi_n|^4+|w|^4)|w|))[|\xi_n|^4+|w|^4]+C_2(I_1*(|\xi_n|^5+|w|^5)(|\xi_n|^3+|w|^3)|w| \\
&\le C_1(I_1*(\varepsilon|\xi_n|^5+C_\varepsilon|w|^5))[|\xi_n|^4+|w|^4]+C_2(I_1*(|\xi_n|^5+|w|^5)(\varepsilon|\xi_n|^4+C_\varepsilon|w|^4) \\
&\le \varepsilon C_3\left[(I_1*|\xi_n|^5)|\xi_n|^4+(I_1*|w|^5)|\xi_n|^4+(I_1*|\xi_n|^5)|w|^4\right] \\
&\qquad +C_\varepsilon C_4\left[(I_1*|\xi_n|^5)|w|^4+(I_1*|w|^5)|\xi_n|^4\right]+C_\varepsilon C_5(I_1*|w|^5)|w|^4 \\
&\qquad \triangleq \mathcal{Q}_{\varepsilon,n}(x)+C_\varepsilon C_5(I_1*|w|^5)|w|^4.
\end{aligned}$$

Recall the Hardy-Littlewood-Sobolev inequality Theorem 4.3 [32]: if $\theta\in\left(1,\frac{N}{N-\alpha}\right)$ then for every $v\in L^\theta(\mathbb{R}^N)$, $I_1*v\in L^{\frac{N\theta}{N-(N-\alpha)\theta}}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N}|I_1*v|^{\frac{N\theta}{N-(N-\alpha)\theta}}\le C\left(\int_{\mathbb{R}^N}|v|^\theta dx\right)^{\frac{N}{N-(N-\alpha)\theta}}. \tag{A.2}$$

For each $\varepsilon>0$, let us consider the function $G_{\varepsilon,n}$ given by

$$G_{\varepsilon,n}(x)=\max\{|A(\xi_n+w)-A(\xi_n)-A(w)|-\mathcal{Q}_{\varepsilon,n}(x),0\},$$

which satisfies

$$G_{\varepsilon,n}(x)\to 0 \quad \text{a.e. in } \ \mathbb{R}^3$$

and using Hölder inequality and $(A.2)$ we see that

$$0\le G_{\varepsilon,n}(x)\le C_6(I_1*|w|^5)|w|^4\in L^{\frac{6}{5}}(\mathbb{R}^3).$$

Therefore, by the Lebesgue Dominated Convergence Theorem we have

$$\int_{\mathbb{R}^3}|G_{\varepsilon,n}(x)|^{\frac{6}{5}}dx\to 0 \quad \text{as} \quad n\to\infty. \tag{A.3}$$

From the definition of $G_{\varepsilon,n}(x)$, we get

$$|A(\xi_n+w)-A(\xi_n)-A(w)|\le \mathcal{Q}_{\varepsilon,n}(x)+C_7G_{\varepsilon,n}(x),$$

which yields that

$$|A(\xi_n + w) - A(\xi_n) - A(w)|^{\frac{6}{5}} \leq |\mathcal{Q}_{\varepsilon,n}(x)|^{\frac{6}{5}} + C_8|G_{\varepsilon,n}(x)|^{\frac{6}{5}}.$$

By $(A.3)$, we obtain the following estimates

$$\begin{aligned} &\int_{\mathbb{R}^3} |A(\xi_n + w) - A(\xi_n) - A(w)|^{\frac{6}{5}} dx \\ &\leq \int_{\mathbb{R}^3} |\mathcal{Q}_{\varepsilon,n}(x)|^{\frac{6}{5}} dx + C_8 \int_{\mathbb{R}^3} |G_{\varepsilon,n}(x)|^{\frac{6}{5}} dx \\ &= \int_{\mathbb{R}^3} |\mathcal{Q}_{\varepsilon,n}(x)|^{\frac{6}{5}} dx + o_n(1). \end{aligned} \tag{A.4}$$

Now we estimate the last integral of $(A.4)$. Since (ξ_n) is bounded in $D^{1,2}(\mathbb{R}^3)$, $\xi_n \to 0$ a.e. in $\mathbb{R}^3$, then $\xi_n \rightharpoonup 0$ in $L^6(\mathbb{R}^3)$. By the definition of $\mathcal{Q}_{\varepsilon,n}(x)$, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} |\mathcal{Q}_{\varepsilon,n}(x)|^{\frac{6}{5}} dx \\ &\leq \varepsilon C \int_{\mathbb{R}^3} \left[(I_1 * |\xi_n|^5)|\xi_n|^4\right]^{\frac{6}{5}} dx \\ &\quad + \varepsilon C \int_{\mathbb{R}^3} \left[(I_1 * |w|^5)|\xi_n|^4\right]^{\frac{6}{5}} dx + \varepsilon C \int_{\mathbb{R}^3} \left[(I_1 * |\xi_n|^5)|w|^4\right]^{\frac{6}{5}} dx \\ &\quad + C_\varepsilon C \int_{\mathbb{R}^3} \left[(I_1 * |w|^5)|\xi_n|^4\right]^{\frac{6}{5}} dx + C_\varepsilon C \int_{\mathbb{R}^3} \left[(I_1 * |\xi_n|^5)|w|^4\right]^{\frac{6}{5}} dx \\ &= \varepsilon C \int_{\mathbb{R}^3} \left[(I_1 * |\xi_n|^5)|\xi_n|^4\right]^{\frac{6}{5}} dx \\ &\quad + D_\varepsilon \int_{\mathbb{R}^3} \left[(I_1 * |w|^5)|\xi_n|^4\right]^{\frac{6}{5}} dx + D_\varepsilon \int_{\mathbb{R}^3} \left[(I_1 * |\xi_n|^5)|w|^4\right]^{\frac{6}{5}} dx \\ &= \Gamma_1 + \Gamma_2 + \Gamma_3, \end{aligned} \tag{A.5}$$

where $D_\varepsilon = C(\varepsilon + C_\varepsilon)$. Next we estimate the three integrals in the right-side of $(A.5)$.

For Γ_1, by Hölder inequality, $(A.2)$ and (ξ_n) is bounded in $D^{1,2}(\mathbb{R}^3)$, we have

$$\begin{aligned} \Gamma_1 &= \varepsilon C \int_{\mathbb{R}^3} (I_1 * |\xi_n|^5)^{\frac{6}{5}} |\xi_n|^{\frac{24}{5}} dx \\ &\leq \varepsilon C \left[\int_{\mathbb{R}^3} (I_1 * |\xi_n|^5)^6 dx\right]^{\frac{1}{5}} \left[\int_{\mathbb{R}^3} |\xi_n|^6 dx\right]^{\frac{4}{5}} \\ &\leq \varepsilon C_1 \int_{\mathbb{R}^3} |\xi_n|^6 dx \left[\int_{\mathbb{R}^3} |\xi_n|^6 dx\right]^{\frac{4}{5}} \end{aligned} \tag{A.6}$$

$$\leq \varepsilon C_2 \left(\int_{\mathbb{R}^3} |\nabla \xi_n|^2 dx \right)^{\frac{27}{5}}$$
$$\leq \varepsilon C_3.$$

For Γ_2, we have

$$\begin{aligned} \Gamma_2 &= D_\varepsilon \int_{\mathbb{R}^3} (I_1 * |w|^5)^{\frac{6}{5}} |\xi_n|^{\frac{24}{5}} dx \\ &= o_n(1), \end{aligned} \tag{A.7}$$

by virtue of $(I_1 * |w|^5)^{\frac{6}{5}} \in L^5(\mathbb{R}^3)$ and $|\xi_n|^{\frac{24}{5}} \rightharpoonup 0$ in $L^{\frac{5}{4}}(\mathbb{R}^3)$.

For Γ_3, we have

$$\begin{aligned} \Gamma_3 &= D_\varepsilon \int_{\mathbb{R}^3} (I_1 * |\xi_n|^5)^{\frac{6}{5}} |w|^{\frac{24}{5}} dx \\ &= o_n(1), \end{aligned} \tag{A.8}$$

by virtue of $(I_1 * |\xi_n|^5)^{\frac{6}{5}} \rightharpoonup 0$ in $L^5(\mathbb{R}^3)$ and $|w|^{\frac{24}{5}} \in L^{\frac{5}{4}}(\mathbb{R}^3)$. From $(A.4) - (A.8)$, we obtain

$$\int_{\mathbb{R}^3} |A(\xi_n + w) - A(\xi_n) - A(w)|^{\frac{6}{5}} dx \to 0 \quad \text{as} \quad n \to \infty,$$

which completes the proof. $\square$

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