



# Oscillating multipliers on rank one locally symmetric spaces

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## ABSTRACT

We prove  $L^p$ -boundedness of oscillating multipliers on certain wide classes of rank one locally symmetric spaces.

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## 1. Introduction and statement of the results

Oscillating multipliers on  $\mathbb{R}^n$  are bounded functions of the type

$$m_{\alpha,\beta}(\xi) = \|\xi\|^{-\beta} e^{i\|\xi\|^\alpha} \theta(\|\xi\|),$$

where  $\alpha, \beta > 0$  and  $\theta \in C_0^\infty(\mathbb{R})$  which vanishes near zero, and equals to 1 outside the ball  $B(0, 2)$ . Let  $T_{\alpha,\beta}$  be the operator which in the Fourier transform variables is given by

$$\widehat{(T_{\alpha,\beta}f)}(\xi) = m_{\alpha,\beta}(\xi)\hat{f}(\xi), \quad f \in C_0^\infty(\mathbb{R}^n),$$

i.e.  $T_{\alpha,\beta}$  is a convolution operator with kernel the inverse Fourier transform of  $m_{\alpha,\beta}$ . The  $L^p$ -boundedness of  $T_{\alpha,\beta}$  on  $\mathbb{R}^n$  is extensively studied. See for example [8,15,26,27,29] for  $\alpha \in (0, 1)$  and [24] for  $\alpha = 1$ . In the first case, for  $p \in (1, \infty)$ ,  $T_{\alpha,\beta}$  is bounded provided  $\operatorname{Re} \beta \geq \alpha n |1/p - 1/2|$ , and when  $\alpha = 1$ , when  $\operatorname{Re} \beta \geq (n-1)|1/p - 1/2|$  [24, p. 138–139].

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The  $L^p$ -boundedness of oscillating multipliers has been studied also in various geometric contexts as Riemannian manifolds, Lie groups and symmetric spaces. See for example [1,10,11,19,22] and the references therein.

In the present work we deal with oscillating multipliers on rank one locally symmetric spaces. To state our results, we need to introduce some notation (for details see Section 2). Let  $G$  be a semi-simple, non-compact, connected Lie group with finite center and let  $K$  be a maximal compact subgroup of  $G$ . We consider the symmetric space of non-compact type  $X = G/K$ . Let  $G = KAN$  be the Iwasawa decomposition of  $G$ . If  $A \cong \mathbb{R}$ , we say that  $X$  has rank one. Recall that rank one symmetric spaces are the real, complex, and quaternionic hyperbolic spaces, denoted  $\mathbb{H}^n(\mathbb{R})$ ,  $\mathbb{H}^n(\mathbb{C})$  and  $\mathbb{H}^n(\mathbb{H})$ ,  $n \geq 2$ , and the octonionic hyperbolic plane  $\mathbb{H}^2(\mathbb{O})$ . Throughout this paper we shall assume that  $\dim X = n \geq 2$  and  $\text{rank} X = 1$ .

In [11], Giulini and Meda consider the multiplier

$$m_{\alpha,\beta}(\lambda) = (\lambda^2 + \rho^2)^{-\beta/2} e^{i(\lambda^2 + \rho^2)^{\alpha/2}}, \alpha > 0, \text{Re } \beta \geq 0, \lambda > 0,$$

where  $\rho$  is the half sum of positive roots, counted with their multiplicities. As in the case of  $\mathbb{R}^n$ , we denote by  $T_{\alpha,\beta}$  the convolution operator with kernel  $\kappa_{\alpha,\beta}$ , the inverse spherical Fourier transform of  $m_{\alpha,\beta}$  in the sense of distributions. Then,

$$T_{\alpha,\beta}(f)(x) = \int_G \kappa_{\alpha,\beta}(xy^{-1})f(y)dy, \quad f \in C_0^\infty(X). \quad (1)$$

Note that

$$T_{\alpha,\beta} = \Delta_X^{-\beta/2} e^{i\Delta_X^{\alpha/2}}, \quad \alpha > 0, \text{Re } \beta \geq 0, \quad (2)$$

where  $\Delta_X$  is the Laplace-Beltrami operator on  $X$ . In [11], the  $L^p$ -boundedness of  $T_{\alpha,\beta}$  is investigated on rank one symmetric spaces and the following theorem is proved:

**Theorem 1** (Giulini, Meda). *If  $p \in (1, \infty)$ , then*

- (i) *If  $\alpha < 1$ , then  $T_{\alpha,\beta}$  is bounded on  $L^p(X)$ , provided that  $\text{Re } \beta > \alpha n |1/p - 1/2|$ ,*
- (ii) *if  $\alpha = 1$ , then  $T_{\alpha,\beta}$  is bounded on  $L^p(X)$ , provided that  $\text{Re } \beta > (n-1) |1/p - 1/2|$ ,*
- (iii) *if  $\alpha > 1$ , then  $T_{\alpha,\beta}$  is bounded on  $L^p(X)$  if and only if  $p = 2$ .*

As it is noticed in [11], the results above for the case  $\alpha \leq 1$  are less precise than in the Euclidean case, since it is not known what happens at the critical indices  $\text{Re } \beta = \alpha n |1/p - 1/2|$  for  $\alpha \in (0, 1)$  and  $\text{Re } \beta = (n-1) |1/p - 1/2|$  for  $\alpha = 1$ . Subsequently, Ionescu [16] sharpened (ii) above by proving that  $T_{\alpha,\beta}$  is bounded on  $L^p(X)$  also in the limiting case  $\text{Re } \beta = (n-1) |1/p - 1/2|$ .

Let us now present the case of locally symmetric spaces. Let  $\Gamma$  be a discrete and torsion free subgroup of  $G$  and let us consider the locally symmetric space  $M = \Gamma \backslash X = \Gamma \backslash G/K$ . Then  $M$ , equipped with the projection of the canonical Riemannian structure of  $X$ , becomes a Riemannian manifold.

To define oscillating multipliers on  $M$ , we first observe that if  $f \in C_0^\infty(M)$ , then, the function  $T_{\alpha,\beta}f$  defined by (1), is right  $K$ -invariant and left  $\Gamma$ -invariant. So  $T_{\alpha,\beta}$  can be considered as an operator acting on functions on  $M$ , which we shall denote by  $\hat{T}_{\alpha,\beta}$ . Note that the Laplace-Beltrami operator  $\Delta_M$  is the projection of  $\Delta_X$ . So from (2), it follows that

$$\hat{T}_{\alpha,\beta}f = \Delta_M^{-\beta/2} e^{i\Delta_M^{\alpha/2}} f, \quad f \in C_0^\infty(M). \quad (3)$$

In this paper we deal with the  $L^p$ -boundedness of  $\widehat{T}_{\alpha,\beta}$  and we prove the analogue of Theorem 1. We treat the cases  $\alpha \in (0, 1)$ ,  $\alpha = 1$  and  $\alpha > 1$ , separately.

**Case 1.**  $\alpha \in (0, 1)$ . The main ingredient for the proof of our results is the Kunze and Stein phenomenon on locally symmetric spaces, proved in [21], and which states that there exist  $\eta_\Gamma \in \mathfrak{a}^*$  and  $s(p) > 0$ ,  $p \in (1, \infty)$ , such that for every  $K$ -bi-invariant function  $\kappa$ , the convolution operator  $*|\kappa|$  with kernel  $|\kappa|$  satisfies the estimate

$$\|*|\kappa|\|_{L^p(M) \rightarrow L^p(M)} \leq \int_G |\kappa(g)| \varphi_{-i\eta_\Gamma}(g)^{s(p)} dg, \quad (4)$$

where  $\varphi_\lambda$  is the spherical function with index  $\lambda$ . For more details see Section 2.

We say that  $M = \Gamma \backslash X$  belongs in the class  $(KS)$  if the Kunze and Stein phenomenon is valid on it. The class  $(KS)$  is described in detail in [21, Section 1]. We note that  $M \in (KS)$  for all discrete groups  $\Gamma$ , if  $X = \mathbb{H}^n(\mathbb{H})$ ,  $\mathbb{H}^2(\mathbb{O})$ , while if  $X = \mathbb{H}^n(\mathbb{R})$ ,  $\mathbb{H}^n(\mathbb{C})$ , then  $M \in (KS)$ , provided that  $\Gamma$  is amenable [9,21].

We have the following theorem:

**Theorem 2.** *If  $M$  belongs in the class  $(KS)$ , then for  $\alpha \in (0, 1)$ ,  $\widehat{T}_{\alpha,\beta}$  is bounded on  $L^p(M)$ , provided that  $\operatorname{Re} \beta > \alpha n |1/p - 1/2|$ .*

**Case 2.**  $\alpha = 1$ . This case is of particular interest since by (3),  $\widehat{T}_{1,\beta} = \Delta_M^{-\beta/2} e^{i\Delta_M^{1/2}}$  and thus  $\widehat{T}_{1,\beta}$  is related to the wave operator.

The  $L^p$ -boundedness of the operator  $T_{1,\beta}$  is investigated in [24] for the case of  $\mathbb{R}^n$ , in [19] in a very general geometric context including Riemannian manifolds of bounded geometry and Lie groups of polynomial or exponential growth, and in [11] for rank one symmetric spaces.

Denote by  $\delta(\Gamma)$  the critical exponent of the group  $\Gamma$ :

$$\delta(\Gamma) = \inf \{s > 0 : P_s(x, y) < +\infty\}, \quad (5)$$

where

$$P_s(x, y) = \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma y)} \quad (6)$$

is the Poincaré series. Note that  $\delta(\Gamma) \in [0, 2\rho]$ .

We say that  $\Gamma$  is of convergence type according as the series  $P_{\delta(\Gamma)}(x, y)$  converges and of divergence type otherwise.

We say that  $\Gamma$  belongs in the class  $(CT)$  if  $\delta(\Gamma) = 2\rho$  and  $\Gamma$  is of convergence type. For example, if  $\Gamma \subset SO(n, 1)$ , then  $\Gamma \in (CT)$  if it is of the second kind, i.e. the limit set  $\Lambda(\Gamma)$  is not equal to the whole of  $\partial\mathbb{H}^n(\mathbb{R})$ , [23, Theorem 1.6.2]

In [25], Roblin gives a criterion of convergence for the case of rank one locally symmetric spaces. Recall that a limit point  $\xi$  belongs in the conical limit set  $\Lambda_c(\Gamma)$  if there is some geodesic ray  $\beta_t$  tending to  $\xi$  as  $t \rightarrow \infty$  and a constant  $c > 0$  such that the orbit  $\Gamma x_0$ ,  $x_0 \in X$ , accumulates to  $\xi$  within the closed  $c$ -neighborhood of  $\beta_t$ . In [25, Theorem 1.7] it is proved that  $\Gamma$  is of convergence type iff  $\mu_{y_0}(\Lambda_c(\Gamma)) = 0$ , where  $\mu_{y_0}$ ,  $y_0 \in X$  is a  $\Gamma$ -invariant Patterson-Sullivan density. For more details for the class  $(CT)$ , see Section 2.

**Theorem 3.** *If either  $\delta(\Gamma) < 2\rho$  or  $\Gamma \in (CT)$ , then  $\widehat{T}_{1,\beta}$  is bounded on  $L^p(M)$ ,  $p \in (1, \infty)$ , provided that  $\operatorname{Re} \beta > (n-1)|1/p - 1/2|$ .*

**Case 3.**  $\alpha > 1$ . As it is shown in Section 3,

$$\widehat{T}_{\alpha,\beta} = \frac{1}{\Gamma(\beta/\alpha)} \int_0^\infty \sigma^{(\beta/\alpha)-1} e^{(i-\sigma)\Delta_M^{\alpha/2}} d\sigma.$$

But, by the spectral theorem

$$\left\| e^{(i-\sigma)\Delta_M^{\alpha/2}} \right\|_{L^2(M) \rightarrow L^2(M)} \leq e^{-\sigma\rho^\alpha}.$$

So,

$$\left\| \widehat{T}_{\alpha,\beta} \right\|_{L^2(M) \rightarrow L^2(M)} \leq c,$$

and this is the only we can say for the  $L^p$ -boundedness of  $\widehat{T}_{\alpha,\beta}$  if  $\alpha > 1$ . On the contrary, in [11] Giulini and Meda observe that the multiplier  $m_{\alpha,\beta}$  is not bounded on any strip  $S_\varepsilon = \{|\operatorname{Im} \lambda| < \varepsilon\rho\}$ ,  $\varepsilon \leq 1$ , and consequently by the necessary part of the multiplier theorem of Clerc and Stein [3], they conclude that  $T_{\alpha,\beta}$  is bounded on  $L^p(X)$  iff  $p = 2$ .

The paper is organized as follows. In Section 2 we present the necessary tools we need for our proofs, and in Section 3 we give the proofs of our results.

## 2. Preliminaries

In this section we recall some basic facts about symmetric spaces and locally symmetric spaces we will use for the proof of our results. For details see [2,9,13,17,21].

Let  $G$  be a semisimple Lie group, connected, noncompact, with finite center,  $K$  be a maximal compact subgroup of  $G$  and consider the symmetric space  $X = G/K$ . In the sequel we assume that  $\dim X = n$ . Denote by  $\mathfrak{g}$  and  $\mathfrak{k}$  the Lie algebras of  $G$  and  $K$ . Let also  $\mathfrak{p}$  be the subspace of  $\mathfrak{g}$  which is orthogonal to  $\mathfrak{k}$  with respect to the Killing form. The Killing form induces a  $K$ -invariant scalar product on  $\mathfrak{p}$  and hence a  $G$ -invariant metric on  $G/K$ . Denote by  $\Delta_X$  the Laplace-Beltrami operator on  $X$ , by  $d(\cdot, \cdot)$  the Riemannian distance and by  $dx$  the associated measure on  $X$ . Let  $\Gamma$  be a discrete torsion free subgroup of  $G$ . Then, the locally symmetric space  $M = \Gamma \backslash X$ , equipped with the projection of the canonical Riemannian structure of  $X$ , becomes a Riemannian manifold. We denote by  $\Delta_M$  the Laplacian on  $M$ .

Fix  $\mathfrak{a}$  a maximal abelian subspace of  $\mathfrak{p}$  and denote by  $\mathfrak{a}^*$  the real dual of  $\mathfrak{a}$ . Let  $\mathfrak{a}_+$  be a positive Weyl chamber. Recall that we assumed that  $X$  has rank one, that is,  $\dim \mathfrak{a} = 1$  and consequently  $\mathfrak{a}_+ = \mathbb{R}_+$ . Let  $A$  be the analytic subgroup of  $G$  with Lie algebra  $\mathfrak{a}$ . Put  $A^+ = \exp \mathfrak{a}_+$ . Its closure in  $G$  is  $\overline{A^+} = \exp \overline{\mathfrak{a}_+}$ . We have the Cartan decomposition

$$G = K(\overline{A^+})K = K(\exp \overline{\mathfrak{a}_+})K.$$

Denote by  $x_0 = eK$  the origin of  $X$ . If  $x, y \in X$ , then there are isometries  $g, h \in G$  such that  $x = gx_0$  and  $y = hx_0$ . Then,

$$d(x, y) = d(gx_0, hx_0) = d(x_0, g^{-1}hx_0). \quad (7)$$

But, by the Cartan decomposition

$$g^{-1}h = k(\exp H(g^{-1}h))k', \quad k, k' \in K, \quad H(g^{-1}h) \in \overline{\mathfrak{a}_+}. \quad (8)$$

In the rank one case, we have that

$$A^+ = \{\exp H : H \in \overline{\mathfrak{a}_+}\} = \{\exp tH_0 : t > 0\},$$

where  $H_0 \in \overline{\mathfrak{a}_+}$  with  $\|H_0\| = 1$ . From (7) and (8) it follows that

$$d(x, y) = \|H\| = t\|H_0\| = t. \quad (9)$$

Normalize the Haar measure  $dk$  of  $K$  such that  $\int_K dk = 1$ . Then from the Cartan decomposition it follows that

$$\int_G f(g)dg = \int_K dk_1 \int_{\mathfrak{a}_+} \delta(H)dH \int_K f(k_1 \exp(H)k_2)dk_2, \quad (10)$$

and  $\delta(H) \leq ce^{2\rho(H)}$ .

### 2.1. The spherical Fourier transform and the Kunze and Stein phenomenon

Denote by  $S(K \backslash G/K)$  the Schwartz space of  $K$ -bi-invariant functions on  $G$ . The spherical Fourier transform  $\mathcal{H}$  is defined by

$$\mathcal{H}f(\lambda) = \int_G f(x)\varphi_\lambda(x) dx, \quad \lambda \in \mathfrak{a}^*, \quad f \in S(K \backslash G/K),$$

where  $\varphi_\lambda$  are the elementary spherical functions on  $G$ .

Let  $S(\mathfrak{a}^*)$  be the usual Schwartz space on  $\mathfrak{a}^*$ , and let us denote by  $S(\mathfrak{a}^*)^W$  the subspace of  $W$ -invariants in  $S(\mathfrak{a}^*)$ , where  $W$  is the Weyl group associated to the root system of  $(\mathfrak{g}, \mathfrak{a})$ . Then, by a celebrated theorem of Harish-Chandra,  $\mathcal{H}$  is an isomorphism between  $S(K \backslash G/K)$  and  $S(\mathfrak{a}^*)$  and its inverse is given by

$$(\mathcal{H}^{-1}f)(x) = c \int_{\mathfrak{a}^*} f(\lambda)\varphi_{-\lambda}(x) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}, \quad x \in G, \quad f \in S(\mathfrak{a}^*)^W,$$

where  $\mathbf{c}(\lambda)$  is the Harish-Chandra function.

In [21] it is proved the following analogue [6] of Kunze and Stein phenomenon, for convolution operators on a class of locally symmetric spaces. Let  $C_\rho$  be the convex body in  $\mathfrak{a}^*$  generated by the vectors  $\{w\rho; w \in W\}$ . Let also  $\lambda_0$  be the bottom of the  $L^2$ -spectrum of  $\Delta_M$ . Then there exists a vector  $\eta_\Gamma \in C_\rho \cap S(0, (\rho^2 - \lambda_0)^{1/2})$ , where  $S(0, r)$  is the Euclidean sphere of  $\mathfrak{a}^*$ , such that for all  $p \in (1, \infty)$  and for every  $K$ -bi-invariant function  $\kappa$ , the convolution operator  $*|\kappa|$  with kernel  $|\kappa|$  satisfies the estimate

$$\|*|\kappa|\|_{L^p(M) \rightarrow L^p(M)} \leq \int_G |\kappa(g)|\varphi_{-i\eta_\Gamma}(g)^{s(p)} dg, \quad (11)$$

where

$$s(p) = 2 \min((1/p), (1/p')),$$

and  $p'$  is the conjugate of  $p$ . For more details, see [9,21].

## 2.2. The class (CT)

In [25], Roblin investigates the question of convergence or divergence type of a discrete group  $\Gamma$  of isometries of  $CAT(-1)$  spaces. Note that  $CAT(-1)$  spaces contain all rank one symmetric spaces. To state the results of Roblin, we need to introduce some notation.

Denote by  $\mu_x$ ,  $x \in X$ , a  $\Gamma$ -invariant Patterson-Sullivan density: that is a family of finite and mutually absolutely continuous measures on  $\partial X$  satisfying the following conditions:

- (i) for any  $x, y \in X$ ,

$$\frac{d\mu_y}{d\mu_x}(\xi) = e^{-\delta(\Gamma)\beta_\xi(x,y)}, \quad \xi \in \partial X.$$

Here  $\beta_\xi(x, y)$  is the Busemann function:

$$\beta_\xi(x, y) = \lim_{t \rightarrow \infty} (d(\xi_t, x) - d(\xi_t, y)),$$

where  $\xi_t$  is a geodesic ray tending to  $\xi$  as  $t \rightarrow \infty$ ;

- (ii) for any  $\gamma \in \Gamma$  and  $x \in X$ ,  $\gamma^*\mu_x = \mu_{\gamma x}$ .

Note that for any  $x \in X$ ,  $\mu_x$  is supported on the limit set  $\Lambda(\Gamma)$ . Note also that the celebrated Patterson-Sullivan construction insures the existence of such conformal densities in various geometric contexts as Hadamard manifolds and  $CAT(-1)$  spaces, [4,30].

If  $\mu_{x_0}$  is normalized to be a probability measure, then in [25, Theorem 1.7], Roblin proves that  $\Gamma$  is of convergence type iff  $\mu_{x_0}(\Lambda_c(\Gamma)) = 0$ .

As far as it concerns divergence type groups, the question is investigated in [28,5] for rank one symmetric spaces and in [7,18] for rank greater than 2. More precisely in [28, Proposition 2] Sullivan treats the case of the real hyperbolic space and shows that a discrete group  $\Gamma$  of  $SO(n, 1)$  is of divergence type if it is geometrically finite and convex co-compact, i.e.  $\Gamma \backslash C(\Lambda(\Gamma))$  is compact, where  $C(\Lambda(\Gamma))$  is the convex hull of the limit set of  $\Gamma$ . In [5, Proposition 3.7] Corlette and Iozzi treat the case of all rank one symmetric spaces and improve the result of [28] in proving that if  $\Gamma$  is geometrically finite subgroup of isometries of a rank one symmetric space, then  $\Gamma$  is of divergence type.

In [18] Leuzinger investigates the case when  $G$  possesses Kazhdan's property (T), i.e. when  $G$  has no simple factors locally isomorphic to  $SO(n, 1)$  or  $SU(n, 1)$ , [12, Ch. 2]. For example, the rank one symmetric spaces which possess property (T) are  $H^n(\mathbb{H}) = Sp(n, 1)/Sp(n)$ , and  $H^2(\mathbb{O}) = F_4^{-20}/Spin(9)$ . In [18, Main Theorem] it is proved that if  $G$  is as above, then the following are equivalent:

- (i)  $M$  is a lattice, i.e.  $\text{Vol}(M) < \infty$ ,
- (ii)  $\delta(\Gamma) = 2\rho$ ,
- (iii)  $\Gamma$  is of divergence type.

It follows then that if  $\Gamma \in Sp(n, 1)$  (resp.  $\Gamma \in F_4^{-20}$ ) and  $\Gamma \backslash \mathbb{H}^n(\mathbb{H})$  (resp.  $\Gamma \backslash \mathbb{H}^2(\mathbb{O})$ ) is a lattice, then  $\Gamma$  is of divergence type. So, only discrete subgroups  $\Gamma$  of  $SO(n, 1)$  or  $SU(n, 1)$  with  $\delta(\Gamma) = 2\rho$  can possibly be in the class (CT).

### 3. Proof of the results

#### 3.1. Proof of Theorem 2

We start by performing a decomposition of the kernel  $\kappa_{\alpha,\beta} = \mathcal{H}^{-1}(m_{\alpha,\beta})$ . We write

$$\kappa_{\alpha,\beta} = \kappa_{\alpha,\beta}^0 + \kappa_{\alpha,\beta}^\infty,$$

with  $\kappa_{\alpha,\beta}^0, \kappa_{\alpha,\beta}^\infty$ ,  $K$ -bi-invariant functions that satisfy  $\text{supp}(\kappa_{\alpha,\beta}^0) \subset B(0,2)$  and  $\text{supp}(\kappa_{\alpha,\beta}^\infty) \subset B(0,1)^c$ . Denote by  $\hat{T}_{\alpha,\beta}^0$  and  $\hat{T}_{\alpha,\beta}^\infty$  the corresponding convolution operators with kernels  $\kappa_{\alpha,\beta}^0$  and  $\kappa_{\alpha,\beta}^\infty$ .

The operator  $\hat{T}_{\alpha,\beta}^0$  is the “local” part of  $\hat{T}_{\alpha,\beta}$  and has the same behavior as its analogue in the Euclidean context.

**Proposition 4.** For  $\alpha \in (0,1)$ ,  $\hat{T}_{\alpha,\beta}^0$  is bounded on  $L^p(M)$ , provided that  $\text{Re } \beta > \alpha n |1/p - 1/2|$ .

**Proof.** The continuity of  $\hat{T}_{\alpha,\beta}^0$  on  $L^p(M)$  follows from [20, Proposition 13]. Indeed, observe first that  $\hat{T}_{\alpha,\beta}^0$  can be defined as an operator on the group  $G$ , and then, apply the local result of [11], to conclude its boundedness on  $L^p(G)$ , for all  $p \in (1, \infty)$  such that  $\text{Re } \beta > \alpha n |1/p - 1/2|$ . The continuity of  $\hat{T}_{\alpha,\beta}^0$  on  $L^p(M)$ , follows by applying Herz’s Theorem A, [14].  $\square$

Note that Proposition 4 is valid for all discrete and torsion free subgroups of  $G$ .

To finish the proof of Theorem 2 it remains to prove the  $L^p$  boundedness of  $\hat{T}_{\alpha,\beta}^\infty$ . Here we shall need the assumption that the Kunze and Stein phenomenon is valid on  $M$ .

**Proposition 5.** If  $M$  belongs in the class  $(KS)$ , and  $\alpha \in (0,1)$ , then  $\hat{T}_{\alpha,\beta}^\infty$  is bounded on  $L^p(M)$  for all  $p \in (1, \infty)$ .

For the proof of the proposition above we shall make use of the Kunze and Stein phenomenon. For that we need to introduce some notation.

For  $p \in (1, \infty)$ , set

$$v_\Gamma(p) = 2 \min\{(1/p), (1/p')\} \frac{|\eta_\Gamma|}{\rho} + |(2/p) - 1|, \quad (12)$$

where  $p'$  is the conjugate of  $p$  and  $\eta_\Gamma \in \mathfrak{a}^*$  is the vector appearing in (11).

For  $N \in \mathbb{N}$ ,  $v \in \mathbb{R}$  and  $\theta \in (0,1)$ , we say that the multiplier  $m$  belongs in the class  $\mathcal{M}(v, N, \theta)$ , if

- $m$  is analytic inside the strip  $\mathcal{T}^v = \mathfrak{a}^* + ivC_\rho$  and
- for all  $k \in \mathbb{N}$  with  $k \leq N$ ,  $\partial^k m(\lambda)$  extends continuously to the whole of  $\mathcal{T}^v$  with

$$|\partial^k m(\lambda)| \leq c(1 + |\lambda|^2)^{-k\theta/2} := \langle \lambda \rangle^{-k\theta}. \quad (13)$$

We have the following:

**Proposition 6.** Fix  $p \in (1, \infty)$  and consider an even function  $m \in \mathcal{M}(v, N, \theta)$ , with  $v > v_\Gamma(p)$  and  $N = \left\lceil \frac{n+1}{2\theta} \right\rceil + 1$ . If  $\kappa$  is the inverse spherical transform of  $m$  in the sense of distributions, and  $\kappa^\infty$  its part away from the origin, then, the convolution operator  $\hat{T}_\kappa^\infty$  with kernel  $\kappa^\infty$ , is bounded on  $L^p(M)$ .

**Proof.** Since  $M$  belongs in the class  $(KS)$ , then, according to Kunze and Stein phenomenon, we have that

$$\begin{aligned} \|\widehat{T}_\kappa^\infty\|_{L^p(M) \rightarrow L^p(M)} &\leq \int_G |\kappa^\infty(g)| \varphi_{-i\eta_\Gamma}(g)^{s(p)} dg \\ &\leq \int_{|g| \geq 1} |\kappa(g)| \varphi_{-i\eta_\Gamma}(g)^{s(p)} dg := I. \end{aligned} \quad (14)$$

So, to finish the proof of Proposition 6, we have to show that if  $m$  satisfies (13), then the integral  $I$  in (14) is finite. To prove that, we will slightly modify the proof of the Theorem 1 in [21] which is based on Proposition 5 of [2]. To estimate the integral  $I$ , we shall estimate  $\kappa^\infty$  over concentric shells. Set

$$V_r = \{H \in \mathfrak{a} : \|H\| \leq r\},$$

and

$$V_r^+ = V_r \cap \overline{\mathfrak{a}_+}.$$

Note that in the rank one case,  $V_r = [-r, r]$ , and  $V_r^+ = [0, r]$ . Set also

$$U_r = \{x = k_1(\exp H)k_2 \in G : k_1, k_2 \in K, H \in V_r^+\} = K(\exp V_r^+)K.$$

Using the Cartan decomposition of  $G$ , the integral  $I$  is written as

$$I = \sum_{j \geq 1} \int_{U_{j+1} \setminus U_j} |\kappa(g)| \varphi_{-i\eta_\Gamma}(g)^{s(p)} dg := \sum_{j \geq 1} I_j. \quad (15)$$

Set  $b = n - 1$  and let  $b'$  be the smallest integer  $\geq b/2$ . Set also  $N = \lceil \frac{n+1}{2\theta} \rceil + 1$ ,  $\theta \in (0, 1)$ . In [21, p. 645], using [2, p. 608], it is proved that if  $m \in \mathcal{M}(v, N, \theta)$ , then for  $v > v_\Gamma(p)$  and  $j \geq 1$ ,

$$I_j \leq c j^{-N} \sum_{0 \leq k \leq N} \left( \int_{\mathfrak{a}^*} \langle \lambda \rangle^{b'-N+k} |\partial_\lambda^k m(\lambda + i\rho)|^2 d\lambda \right)^{1/2}. \quad (16)$$

Then, using the estimates of the derivatives of  $m(\lambda)$

$$|\partial^k m(\lambda)| \leq c \langle \lambda \rangle^{-k\theta},$$

we obtain that

$$\begin{aligned} I_j &\leq c j^{-N} \sum_{0 \leq k \leq N} \left( \int_{\mathfrak{a}^*} \langle \lambda \rangle^{b'-N+k} \langle \lambda \rangle^{-k\theta} d\lambda \right)^{1/2} \\ &\leq c j^{-N} \sum_{0 \leq k \leq N} \left( \int_{\mathfrak{a}^*} \langle \lambda \rangle^{b'-N+k(1-\theta)} d\lambda \right)^{1/2} \\ &\leq c j^{-N} \left( \int_{\mathfrak{a}^*} \langle \lambda \rangle^{2(b'-\theta N)} d\lambda \right)^{1/2} \end{aligned}$$



$$\leq c j^{-N} \left( \int_0^\infty (1 + \lambda^2)^{b' - \theta(\lceil \frac{n+1}{2\theta} \rceil + 1)} d\lambda \right)^{1/2}.$$

Using the fact that  $b' \leq n/2$  and that  $\lceil \frac{n+1}{2\theta} \rceil = \frac{n+1}{2\theta} - q$ ,  $q \in [0, 1)$ , it follows that

$$\begin{aligned} I_j &\leq c j^{-N} \left( \int_0^\infty (1 + \lambda^2)^{\frac{n}{2} - \theta(\frac{n+1}{2\theta} - q) - \theta} d\lambda \right)^{1/2} \\ &\leq c j^{-N} \left( \int_0^\infty (1 + \lambda^2)^{-\frac{1}{2} - \theta(1-q)} d\lambda \right)^{1/2} \\ &\leq c j^{-N}, \end{aligned} \quad (17)$$

since  $2(\frac{1}{2} + \theta(1-q)) > 1$ .

From (17) it follows that

$$\int_{|g| \geq 1} |\kappa^\infty(g)| \varphi_{-i\eta_\Gamma}(g)^{s(p)} dg = \sum_{j \geq 1} I_j \leq c \sum_{j \geq 1} j^{-N} < \infty,$$

since  $N > 1$ .  $\square$

**Proof of Proposition 5.** Note first that  $m_{\alpha, \beta}(\lambda)$  has poles only at  $\lambda = \pm i\rho$ . So, the function  $\lambda \rightarrow m_{\alpha, \beta}(\lambda)$  is analytic in the strip  $S_\rho = \{z \in \mathbb{C} : |\operatorname{Im} z| < \rho\}$ . Secondly, for every  $k \in \mathbb{N}$ , it holds that

$$|\partial^k m_{\alpha, \beta}(\lambda)| \leq c(1 + \lambda)^{-Re\beta - k(1-a)} \leq c(1 + \lambda^2)^{-k(1-\alpha)/2}, \quad \lambda \in S_\rho. \quad (18)$$

Note also that for any  $p \in (1, \infty)$ , from (12) it follows that  $v_\Gamma(p) < 1$ . So, for every  $p \in (1, \infty)$ , there exists  $v'(p) < 1$ , such that  $v'(p) > v_\Gamma(p)$ . It follows that  $\mathfrak{a}^* + iv'(p)C_\rho \subset S_\rho$ , which combined with (18), implies that for any  $p \in (1, \infty)$ , and  $N \in \mathbb{N}$ ,  $m_{\alpha, \beta} \in \mathcal{M}(v'(p), N, 1 - \alpha)$ . Thus, Proposition 6 applies and Proposition 5 follows.  $\square$

### 3.2. Proof of Theorem 3

To begin with, let us recall that in [26], the operator  $T_{\alpha, \beta}$ ,  $Re\beta \geq 0$ ,  $\alpha > 0$  is also expressed by the integral:

$$T_{\alpha, \beta}(f)(x) = \frac{1}{\Gamma(\beta/\alpha)} \int_0^\infty \sigma^{(\beta/\alpha)-1} (f * q_{\sigma, \alpha})(x) d\sigma, \quad f \in C_0^\infty(X), \quad (19)$$

where  $q_{\sigma, \alpha}$ ,  $\sigma > 0$ , is the inverse spherical Fourier transform of the function  $\lambda \rightarrow e^{(i-\sigma)(\lambda^2 + \rho^2)^{\alpha/2}}$ . Note that  $q_{\sigma, \alpha}$  is  $K$ -bi-invariant as the inverse spherical Fourier transform of an even function, [2]. Then, observe that the convolution operator  $T_{q_{\sigma, \alpha}} = *q_{\sigma, \alpha}$  is equal to  $e^{(i-\sigma)\Delta_X^{\alpha/2}}$ . So, by the spectral theorem,  $T_{q_{\sigma, \alpha}}$  is bounded on  $L^2(X)$ , with

$$\|T_{q_{\sigma, \alpha}}\|_{L^2(X) \rightarrow L^2(X)} \leq \sup_{\lambda > 0} \left| e^{(i-\sigma)(\lambda^2 + \rho^2)^{\alpha/2}} \right| \leq e^{-\sigma\rho^\alpha}. \quad (20)$$

For simplicity, for  $\alpha = 1$ , set  $q_{\sigma,1} = q_{\sigma}$ . Let us now define the operator  $\widehat{T}_{1,\beta}$  on the quotient  $M = \Gamma \backslash X$ . For that, recall that  $\widehat{T}_{q_{\sigma}}$ , as the case of  $\widehat{T}_{1,\beta}$ , is initially defined as a convolution operator

$$\widehat{T}_{q_{\sigma}} f(x) = \int_G q_{\sigma}(y^{-1}x) f(y) dy, \quad f \in C_0^{\infty}(M). \quad (21)$$

Set  $q_{\sigma}(x, y) = q_{\sigma}(y^{-1}x)$  and

$$\widehat{q}_{\sigma}(x, y) = \sum_{\gamma \in \Gamma} q_{\sigma}(x, \gamma y) = \sum_{\gamma \in \Gamma} q_{\sigma}((\gamma y)^{-1}x). \quad (22)$$

**Proposition 7.** *If either  $\delta(\Gamma) < 2\rho$  or  $\Gamma \in (CT)$ , then the series (22) is convergent and the operator  $\widehat{T}_{q_{\sigma}}$  on  $M$  is given by*

$$\widehat{T}_{q_{\sigma}} f(x) = \int_M \widehat{q}_{\sigma}(x, y) f(y) dy, \quad f \in C_0^{\infty}(M). \quad (23)$$

**Proof.** By the Cartan decomposition of  $G$ , we may write

$$(\gamma y)^{-1}x = k_{\gamma} \exp(t_{\gamma} H_0) k'_{\gamma},$$

where  $t_{\gamma} > 0$ ,  $k_{\gamma}, k'_{\gamma} \in K$ , and  $H_0 \in \mathfrak{a}_+$  with  $\|H_0\| = 1$ . Then, since  $q_{\sigma}$  is  $K$ -bi-invariant, we get that  $q_{\sigma}((\gamma y)^{-1}x) = q_{\sigma}(\exp t_{\gamma} H_0)$ .

Recall that in [11, p. 103] it is proved that

$$|q_{\sigma}(\exp t H_0)| \leq \begin{cases} c\sigma(t+1)^{-3/2}e^{-2\rho t}, & \sigma > 1, t > 0, \\ c(t+1)^{-3/2}e^{-2\rho t}, & 0 < \sigma \leq 1, t > 2. \end{cases} \quad (24)$$

Then,

$$\begin{aligned} |\widehat{q}_{\sigma}(\exp t H_0)| &\leq \sum_{\gamma \in \Gamma} |q_{\sigma}(\exp t_{\gamma} H_0)| \\ &\leq \sum_{\{\gamma \in \Gamma: t_{\gamma} \leq 2\}} |q_{\sigma}(\exp t_{\gamma} H_0)| + \sum_{\{\gamma \in \Gamma: t_{\gamma} > 2\}} |q_{\sigma}(\exp t_{\gamma} H_0)|. \end{aligned}$$

Note that the first sum above is finite, since  $\Gamma$  is discrete. Thus, we shall deal only with the second sum. Using (9) and the estimates (24) of  $q_{\sigma}$ , we have

$$\begin{aligned} \sum_{\{\gamma \in \Gamma: t_{\gamma} > 2\}} |q_{\sigma}(\exp t_{\gamma} H_0)| &\leq \sum_{\{\gamma \in \Gamma: t_{\gamma} > 2\}} c_{\sigma}(t_{\gamma} + 1)^{-3/2}e^{-2\rho t_{\gamma}} \\ &\leq c_{\sigma} \sum_{\{\gamma \in \Gamma: d(x, \gamma y) > 2\}} e^{-2\rho d(x, \gamma y)} \\ &\leq c_{\sigma} P_{2\rho}(x, y) \end{aligned}$$

which is convergent, since by our assumption, either  $\delta(\Gamma) < 2\rho$  or  $\Gamma \in (CT)$ .

To prove (23), we follow [9, Prop. 4]. Since  $q_{\sigma}$  and  $f$  are right  $K$ -invariant, from (21) we get that

$$\widehat{T}_{q_{\sigma}}(f)(x) = \int_G q_{\sigma}(xy^{-1}) f(y) dy = \int_X q_{\sigma}(x, y) f(y) dy.$$

Further, since  $f$  is left  $\Gamma$ -invariant, by Weyl's formula we obtain that

$$\begin{aligned}\widehat{T}_{q_\sigma}(f)(x) &= \int_X q_\sigma(x, y) f(y) dy = \int_{\Gamma \backslash X} \left( \sum_{\gamma \in \Gamma} q_\sigma(x, \gamma y) f(\gamma y) \right) dy \\ &= \int_M \widehat{q}_\sigma(x, y) f(y) dy. \quad \square\end{aligned}$$

**Proposition 8.** *If either  $\delta(\Gamma) < 2\rho$  or  $\Gamma \in (CT)$ , then for all  $p \in (1, \infty)$ , there are constants  $c_p, k_p > 0$  such that*

$$\|\widehat{T}_{q_\sigma}\|_{L^p(M) \rightarrow L^p(M)} \leq c_p \begin{cases} e^{-k_p \sigma}, & \sigma \geq 1, \\ \sigma^{(1-n)(1/2-1/p)}, & \sigma < 1. \end{cases} \quad (25)$$

**Proof.** Recall that  $\widehat{T}_{q_\sigma} = e^{(i-\sigma)\Delta_M^{1/2}}$ . Thus, by the spectral theorem,

$$\|\widehat{T}_{q_\sigma}\|_{L^2(M) \rightarrow L^2(M)} \leq \sup_{\lambda > 0} \left| e^{-(i-\sigma)(\lambda^2 + \rho^2)^{1/2}} \right| \leq e^{-\sigma \rho}. \quad (26)$$

Next, note that if  $f \in C_0^\infty(M)$ , then  $\|f\|_{L^\infty(M)} = \|f\|_{L^\infty(X)}$ , since  $f$  is left  $\Gamma$ -invariant. Then, again by Weyl's formula it follows that

$$\begin{aligned}|\widehat{T}_{q_\sigma} f(x)| &= \left| \int_M \widehat{q}_\sigma(x, y) f(y) dy \right| = \left| \int_{\Gamma \backslash X} \sum_{\gamma \in \Gamma} q_\sigma(x, \gamma y) f(y) dy \right| \\ &= \left| \int_X q_\sigma(x, y) f(y) dy \right| \leq \|f\|_{L^\infty(X)} \|q_\sigma\|_{L^1(X)} \\ &= \|f\|_{L^\infty(M)} \|q_\sigma\|_{L^1(X)}.\end{aligned} \quad (27)$$

But, in [11, p. 101] it is proved that

$$\|q_\sigma\|_{L^1(X)} \leq c \begin{cases} \sigma, & \sigma \geq 1, \\ \sigma^{(1-n)/2}, & \sigma < 1. \end{cases} \quad (28)$$

From (27) and (28), it follows that  $\widehat{T}_{q_\sigma}$  is bounded on  $L^\infty(M)$ , with

$$\|\widehat{T}_{q_\sigma}\|_{L^\infty(M) \rightarrow L^\infty(M)} \leq \|q_\sigma\|_{L^1(X)}. \quad (29)$$

By interpolation between (26) and (29) and duality, we deduce the boundedness of  $\widehat{T}_{q_\sigma}$  on  $L^p(M)$ ,  $p > 1$ . Further, by the Riesz-Thorin interpolation theorem we have

$$\|\widehat{T}_{q_\sigma}\|_{L^{p_\theta}(M) \rightarrow L^{p_\theta}(M)} \leq c \|\widehat{T}_{q_\sigma}\|_{L^2(M) \rightarrow L^2(M)}^{1-\theta} \|\widehat{T}_{q_\sigma}\|_{L^\infty(M) \rightarrow L^\infty(M)}^\theta,$$

with  $\frac{1}{p_\theta} = \frac{1-\theta}{2}$ . Choosing  $\theta = 1 - (2/p)$ ,  $p > 2$ , we have that

$$\|\widehat{T}_{q_\sigma}\|_{L^p(M) \rightarrow L^p(M)} \leq c \|\widehat{T}_{q_\sigma}\|_{L^2(M) \rightarrow L^2(M)}^{2/p} \|\widehat{T}_{q_\sigma}\|_{L^\infty(M) \rightarrow L^\infty(M)}^{1-(2/p)},$$

and from (26), (28) and (29), it is straightforward to obtain (25) for  $p > 2$ . The claim for  $p \in (1, 2)$  follows by duality.  $\square$

**End of proof of Theorem 3.** Using (19), we have that

$$\begin{aligned} \|\widehat{T}_{1,\beta}\|_{L^p(M) \rightarrow L^p(M)} &\leq \frac{1}{\Gamma(\beta)} \int_0^\infty \sigma^{\beta-1} \|\widehat{T}_{q_\sigma}\|_{L^p(M) \rightarrow L^p(M)} d\sigma \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^1 \sigma^{\beta-1} \|\widehat{T}_{q_\sigma}\|_{L^p(M) \rightarrow L^p(M)} d\sigma \\ &\quad + \frac{1}{\Gamma(\beta)} \int_1^\infty \sigma^{\beta-1} \|\widehat{T}_{q_\sigma}\|_{L^p(M) \rightarrow L^p(M)} d\sigma. \end{aligned}$$

Applying the estimates (25), we get:

$$\begin{aligned} \|\widehat{T}_{1,\beta}\|_{L^p(M) \rightarrow L^p(M)} &\leq \frac{1}{\Gamma(\beta)} \int_0^1 \sigma^{\beta-1} \sigma^{(1-n)(1/2-1/p)} d\sigma \\ &\quad + \frac{1}{\Gamma(\beta)} \int_1^\infty \sigma^{\beta-1} e^{-k_p \sigma} d\sigma. \end{aligned}$$

The second integral above is finite, while the first is convergent provided that

$$\operatorname{Re} \beta - 1 - (n-1)(1/2 - 1/p) > -1, \text{ or } \operatorname{Re} \beta / (n-1) > 1/2 - 1/p,$$

and the claim follows for  $p > 2$ . The case  $p < 2$  follows by duality.  $\square$

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