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Long time asymptotics for the focusing nonlinear Schrödinger equation in the solitonic region with the presence of high-order discrete spectrum

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ABSTRACT

In this paper, we study the initial value problem for focusing nonlinear Schrödinger (fNLS) equation with non-generic weighted Sobolev initial data that allows for the presence of high-order discrete spectrum. More precisely, we show how to characterize the properties of the eigenfunctions and scattering coefficients in the presence of high-order poles; Further the initial value problem is formulated into an appropriate enlarged RH problem, which is transformed into a solvable model after a series of deformations. Finally, we obtain the asymptotic expansion of the solution of the fNLS equation in any fixed space-time cone:

$$\mathcal{S}(x_1, x_2, v_1, v_2) := \left\{(x,t) \in \mathbb{R}^2 : x = x_0 + vt,\ x_0 \in [x_1, x_2],\ v \in [v_1, v_2]\right\}.$$

Our result is a verification of the soliton resolution conjecture for the fNLS equation in the solitonic region with the presence of high-order discrete spectrum. The leading order term of this solution includes a high-order pole-soliton whose parameters are affected by soliton-soliton interactions through the cone and soliton-radiation interactions on continuous spectrum. The error term is up to $\mathcal{O}(t^{-3/4})$ which comes from the corresponding $\bar{\partial}$ equation.

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1. Introduction

In this paper, we study the long time asymptotic behavior of the Cauchy problem for the focusing nonlinear Schrödinger (fNLS) equation in the solitonic region with high-order discrete spectrum

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q = 0, \tag{1.1}$$

$$q(x,0) = q_0(x) \in H^{1,1}(\mathbb{R}), \tag{1.2}$$

where $H^{1,1}(\mathbb{R})$ is a weighted Sobolev space

$$H^{1,1}(\mathbb{R}) = \left\{ f(x) \in L^2(\mathbb{R}) : f'(x), xf(x) \in L^2(\mathbb{R}) \right\}. \tag{1.3}$$

The NLS equation is an important model in applied mathematics and theoretical physics due to both its surprisingly rich mathematical structure and its physical significance and broad applicability to a number of different areas [2,16,26]. The NLS equation is also a completely integrable system. Its Lax pair was first derived by Zakharov and Shabat in 1972 [34]. For sufficient smoothness of the initial data, Zakharov and Shabat developed the inverse scattering transform (IST) for the initial value problem of the NLS on the line for initial conditions with sufficiently rapid decay at infinity [34]. Later, the IST for the defocusing NLS equation on the line with nonzero boundary conditions (NZBC) at infinity was developed [35]. The periodic problem for NLS was studied by Its and Kotlyarov in 1976 [18]. Biondini and Kovacic established asymptotic expressions for the multiple pole solutions of the fNLS equation via the IST method [27]. The N-soliton solutions for the fNLS equation with NZBC at infinity and double zeros of the analytic scattering coefficients has been studied by Pichler and Biondini [25]. Recently, Weng and Yan found a kind of new tri-pole solutions of the focusing NLS hierarchy with NZBCs via the RH approach [30]. In addition, the well-posedness of the NLS equation on the line with initial data in L^2 and in Sobolev spaces H^s, $s > 0$ was proved by Tsutsumi and Bourgain respectively [8,29].

The long time asymptotic behavior of the defocusing NLS equation with Schwartz initial data was first studied by Zakharov and Manakov by the IST method [36]. The focusing NLS equation with nonzero boundary conditions by the IST method were presented by Kawata, Inoue and Ma in [21,22]. Using monodromy theory, Its was able to reduce the RH problem formulation for the NLS equation to a model case, which can then be solved explicitly, giving the desired asymptotics [17]. A perturbation theory for the NLS equation with non-vanishing boundary conditions was put forward in [15], where particular attention was paid to the stability of the Ma soliton. Whitham theory results for the focusing NLS with step-like data can be found by Bikbaev in [4]. In particular, a nonlinear steepest descent method for oscillatory RH problem was developed by Deift and Zhou in 1993 [38], which is a new great achievement in the further development of the IST method. After that, a numerous new significant results on long-time asymptotics for NLS equation also other integrable nonlinear equations have been obtained in a rigorous and transparent form with this new method [14,31,32]. Kamvissis obtained the long time behavior for the focusing NLS equation with real spectral singularities [20]. Boutet de Monvel et al. studied long time asymptotic behavior of the fNLS equation with time-periodic boundary condition on the half-line [11], with step-like initial data [12], and more general step-like initial data recently [13]. By using a variant of IST and by employing Deift-Zhou nonlinear steepest descent method, Biondini studied the long time asymptotic behavior of the focusing NLS equation on the line with symmetric, nonzero boundary conditions at infinity [5], and recently with nonzero boundary conditions in the presence of a discrete spectrum [6]. Chen and Yan obtained long-time asymptotic behavior of the third-order NLS equation with NZBCs by employing Deift-Zhou nonlinear steepest descent method [9].

Most recently, for weighted Sobolev initial data $q_0(x) \in H^{1,1}(\mathbb{R})$, Borghese et al. applied the $\bar{\partial}$ steepest descent method to obtain asymptotic expansion in any fixed space-time cone for the focusing NLS equation in solitonic region [7]; the $\bar{\partial}$ steepest descent method was first applied to analyze the asymptotics of orthogonal polynomials on the unit circle and on real line by McLaughlin and Miller in 2006 [23,24]. Later, this method was further generalized to widely study the long time asymptotics of integrable systems. For example, Cuccagna and Jenkins studied the large-time leading order approximation and the asymptotic stability of N-soliton solutions of the defocusing NLS equation in 2016 [10]. Jenkins et al. obtained the soliton resolution property of the derivative NLS equation. As t approaches infinity, the solutions can be described by a finite sum of localized solitons and a dispersive component [19]. We recently obtained long time asymptotics of short pulse equation in solitonic region [33]. The advantages of this method are not only avoiding delicate estimates of Cauchy projection operators but also improving error estimates without additional restrictions on the initial data.

For the defocusing NLS equation, its ZS-AKNS operator is self adjoint, no soliton solutions appear due to empty discrete spectrum for finite mass initial data $q_0(x) \in H^{1,1}(\mathbb{R})$. Soliton solutions have no effect on the long-time asymptotic behavior. However, for the focusing case, the ZS-AKNS operator is non-self adjoint that allow for presence of solitons anywhere in $\mathbb{C} \setminus \mathbb{R}$. It is necessary to consider effects of soliton solutions when we study long time asymptotic behavior. Therefore, the long-time behavior of solutions of fNLS are necessarily more detailed than in the defocusing case due to the presence of solitons which correspond to discrete spectrum of the non self-adjoint ZS-AKNS scattering operator. The corresponding reflection coefficient $r(z)$ is a mapping defined on the real axis $r(z) : \mathbb{R} \to \mathbb{C}$. It is possible for $r(z)$ to possess singularities along the real line and we call these points spectral singularities. The initial data q_0, which has no spectral singularities and produces only simple discrete spectrum, is generic. If spectral singularities or high-order discrete spectrum exist, the initial data q_0 is called the non-generic.

For the focusing NLS equation with zero boundary conditions (ZBC), it has been known that more general solutions corresponding to double order poles exist since the original work of Zakharov and Shabat [34]. More general high-order pole solutions of the focusing NLS equation with ZBC were also studied by Aktosun et al. [3,28]. Such solutions also exist in the focusing case with NZBC and describe the interaction of two solitons with same amplitude and velocity parameters, which diverge from each other logarithmically as in the case of zero boundary conditions [25]. Indeed for focusing NLS equation, it can be shown that high order discrete spectrum may appear see in the following Lemma 1. As is common, long time asymptotic expressions of the focusing NLS equation in the solitonic region are limited to the case in which all discrete spectrum are simple [5–7,13]. In this work, we apply the $\bar{\partial}$ steepest descent techniques to obtain the long-time asymptotic behavior of solutions for the Cauchy problem (1.1)-(1.2) of the fNLS equation with non-generic initial data which allows for the presence of high-order discrete spectrum. More precisely, we show how to characterize the properties of the eigenfunctions and scattering coefficients in the presence of high-order poles; Further the initial value problem is formulated into an appropriate enlarged RH problem, which is transformed into a solvable model after a series of deformations. We then obtain the long time asymptotic expression of the focusing NLS equation in solitonic region with the presence of high-order discrete spectrum.

The structure of this work is the following: In Section 2, we simply recall the basic scattering theory about the fNLS equation, such as the Lax pair, the analyticity and the symmetry of the corresponding eigenfunctions, for the details, see [7]. In Section 3, we consider the high-order discrete spectrum and compute the residue condition and the coefficients of negative power terms. In Section 4, we formulate an RH problem $m(z)$ to characterize the Cauchy problem (1.1)-(1.2) with high-order poles. In Section 5, in order to regularize the RH problem $m(z)$, we first study the property of the jump matrices and introduce a transformation $T(z)$ to get $m^{(1)}(z)$. We then make continuous extension of these jump matrices to obtain a mixed $\bar{\partial}$-RH problem $m^{(2)}(z)$. In Section 6, we decompose $m^{(2)}(z)$ into a pure RH problem $m^{(2)}_{RHP}(z)$ and a pure $\bar{\partial}$-problem $m^{(3)}(z)$, while the RH problems about $m^{(2)}_{RHP}(z)$ and $m^{(3)}(z)$ can be shown solved respectively. Finally, we give an explicit formula for the solution of the fNLS equation in Section 7. Moreover, the property of soliton resolution can be obtained after analyzing the form of solution.

2. The Lax pair and spectral analysis

The fNLS equation (1.1) admits the Lax pair [7]

$$\Phi_x + iz\sigma_3\Phi = Q_1\Phi, \quad \Phi_t + iz^2\sigma_3\Phi = Q_2\Phi, \tag{2.1}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad Q_2 = \frac{1}{2}\begin{pmatrix} i\left|q\right|^2 & iq_x + 2zq \\ i\bar{q}_x - 2z\bar{q} & -i\left|q\right|^2 \end{pmatrix}.$$

Given the initial condition (1.2), the Lax pair (2.1) has a solution of the following asymptotic form

$$\Phi(z) \sim e^{-i(zx+z^2t)\sigma_3}, \qquad x \to \pm\infty. \tag{2.2}$$

By making a transformation

$$\mu(z) = \Phi(z)e^{i(zx+z^2t)\sigma_3}, \tag{2.3}$$

we find the matrix function μ has the following asymptotic behavior

$$\mu(z) \sim I, \qquad x \to \pm\infty$$

and satisfies the following Lax pair

$$\mu_x + iz[\sigma_3, \mu] = Q_1\mu, \tag{2.4}$$

$$\mu_t + iz^2[\sigma_3, \mu] = Q_2\mu. \tag{2.5}$$

This Lax pair (2.4)-(2.5) can be written by fully differential form

$$d\left(e^{i(zx+z^2t)\widehat{\sigma}_3}\mu\right) = e^{i(zx+z^2t)\widehat{\sigma}_3}[(Q_1dx + Q_2dt)\mu]. \tag{2.6}$$

We expand μ into a Taylor series at infinity and prove that

$$\mu(z) \sim I, \qquad z \to \infty, \tag{2.7}$$

$$q(x,t) = 2i\lim_{z\to\infty}(z\mu)_{12}. \tag{2.8}$$

By integrating the equation in two directions parallel to the real axis, two eigenvalue functions can be obtained

$$\mu^{-}(z;x,t) = I + \int_{-\infty}^{x} e^{-iz(x-y)\widehat{\sigma}_3} Q_1(z;y,t)\mu^{-}(z;y,t)dy, \tag{2.9}$$

$$\mu^{+}(z;x,t) = I - \int_{x}^{+\infty} e^{-iz(x-y)\widehat{\sigma}_3} Q_1(z;y,t)\mu^{+}(z;y,t)dy. \tag{2.10}$$

From the relation (2.3), we know that

$$\Phi^{\pm}(z) = \mu^{\pm}(z)e^{-i(zx+z^2t)\sigma_3} \tag{2.11}$$

are two linear correlation matrix solutions of the Lax pair (2.1), which means there is a matrix $S(z) = (s_{ik}(z))_{i,k=1}^{2}$ satisfying the condition

$$\Phi^{-}(z) = \Phi^{+}(z)S(z). \tag{2.12}$$

Therefore, we obtain

$$\mu^{-}(z) = \mu^{+}(z)e^{-i(zx+z^2t)\sigma_3}S(z), \tag{2.13}$$

where the matrix function $S(z)$ is called the spectral matrix and $s_{ik}(z), i,k = 1,2$ is called the scattering data. Direct calculation shows that [7]

$$s_{11}(z) = \det\left(\mu_1^{-}, \mu_2^{+}\right) = 1 + \int_{-\infty}^{\infty} \bar{q}(y)\mu_{12}^{+}(y)dy = 1 + \int_{-\infty}^{\infty} q(y)\mu_{21}^{-}(y)dy, \tag{2.14}$$

$$s_{21}(z) = \det\left(\mu_1^{+}, \mu_1^{-}\right) = -\int_{-\infty}^{\infty} \bar{q}(y)e^{-2izy}\mu_{11}^{+}(y)dy = -\int_{-\infty}^{\infty} q(y)e^{-2izy}\mu_{22}^{-}(y)dy, \tag{2.15}$$

where we denote

$$\mu^{\pm} = (\mu_1^{\pm}, \mu_2^{\pm}) = \begin{pmatrix} \mu_{11}^{\pm} & \mu_{12}^{\pm} \\ \mu_{21}^{\pm} & \mu_{22}^{\pm} \end{pmatrix}.$$

When $q(x) \in L^1(\mathbb{R})$, by constructing iterative sequence and Neumann series, we can prove that μ_1^{-}, μ_2^{+}, s_{11} are analytic in the upper half complex plane; μ_2^{-}, μ_1^{+}, s_{22} are analytic in the lower half complex plane; s_{12} and s_{21} are not analytic in the upper and lower half complex plane but are continuous on the real axis.

In addition, we can find symmetries of $\mu^{\pm}$ and $S(z)$

$$\mu^{\pm}(z) = -\sigma\overline{\mu^{\pm}(\bar{z})}\sigma = \sigma_2\overline{\mu^{\pm}(\bar{z})}\sigma_2, \tag{2.16}$$

$$S(z) = -\sigma\overline{S(\bar{z})}\sigma = \sigma_2\overline{S(\bar{z})}\sigma_2, \tag{2.17}$$

where

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Here we give the definitions of several important concepts: the reflection coefficient $r(z) = s_{21}(z)/s_{11}(z)$ and the transmission coefficient $\tau(z) = 1/s_{11}(z)$. In particular, for $z \in \mathbb{R}$, we have $s_{11}(z) = \overline{s_{22}(z)}$, $s_{12}(z) = -\overline{s_{21}(z)}$, and $1 + |r(z)|^2 = |\tau(z)|^2$.

For simplicity, we assume that the initial data $q_0 \in H^{1,1}(\mathbb{R})$ and the corresponding scattering data satisfy the following conditions: $s_{11}(z)$ has no zeros on $\mathbb{R}$; $s_{11}(z)$ only has finite double roots; $s_{11}(z), r(z) \in H^{1,1}(\mathbb{R})$. The following lemma will illustrate the rationality of the above assumption.

Lemma 1. *The zeros of $s_{11}(z)$ in $\mathbb{C}^+$ are finite but not necessarily simple in the case $q_0 \in H^{1,1}(\mathbb{R})$.*

Proof. For $\Phi^{\pm}(z) = \left(\Phi_1^{\pm}(z), \Phi_2^{\pm}(z)\right)$, applying (2.11) to (2.14) gives

$$s_{11}(z) = \det\left(\Phi_1^{-}(z), \Phi_2^{+}(z)\right). \tag{2.18}$$

Suppose that $z_k \in \mathbb{C}^+(k = 1, ..., N)$ are the zeros of $s_{11}(z)$. From (2.18), we know the pair $\Phi_1^-(z_k)$ and $\Phi_2^+(z_k)$ are linearly related, which is there exists a constant $\gamma_k \in \mathbb{C}$ such that

$$\Phi_1^-(z_k) = \gamma_k \Phi_2^+(z_k). \tag{2.19}$$

Then, we consider the partial derivative of $s_{11}(z)$

$$\left.\frac{\partial s_{11}(z)}{\partial z}\right|_{z=z_k} = \det\left(\partial_z \Phi_1^-, \Phi_2^+\right) + \left.\det\left(\Phi_1^-, \partial_z \Phi_2^+\right)\right|_{z=z_k}. \tag{2.20}$$

Using (2.1), we find that

$$\frac{\partial}{\partial x}\det\left(\partial_z \Phi_1^-, \Phi_2^+\right) = -i \det\left(\sigma_3 \Phi_1^-, \Phi_2^+\right), \tag{2.21}$$

$$\frac{\partial}{\partial x}\det\left(\Phi_1^-, \partial_z \Phi_2^+\right) = -i \det\left(\Phi_1^-, \sigma_3 \Phi_2^+\right). \tag{2.22}$$

The equations (2.9), (2.9) and (2.11) tell us that

$$\Phi_1^-(z;x) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2it\theta(z;x)}, \quad x \to -\infty, \tag{2.23}$$

$$\Phi_2^+(z;x) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2it\theta(z;x)}, \quad x \to +\infty, \tag{2.24}$$

$$\partial_z \Phi_1^-(z;x) \sim \begin{pmatrix} -i\left(x + 2zt\right) \\ 0 \end{pmatrix} e^{-2it\theta(z;x)}, \quad x \to -\infty. \tag{2.25}$$

Then, we obtain

$$\det\left(\partial_z \Phi_1^-, \Phi_2^+\right) = -i\gamma_k \int_{-\infty}^{x} \det\left(\sigma_3 \Phi_2^+(z_k; s), \Phi_2^+(z_k; s)\right) ds, \tag{2.26}$$

$$\det\left(\Phi_1^-, \partial_z \Phi_2^+\right) = -i\gamma_k \int_{x}^{\infty} \det\left(\sigma_3 \Phi_2^+(z_k; s), \Phi_2^+(z_k; s)\right) ds. \tag{2.27}$$

Putting the above two terms together, we have

$$\left.\frac{\partial s_{11}(z)}{\partial z}\right|_{z=z_k} = -2i\gamma_k \int_{-\infty}^{\infty} \Phi_{12}^+(z_k;s)\Phi_{22}^+(z_k;s)ds. \tag{2.28}$$

Therefore, we can find: When the condition $\int_{-\infty}^{\infty} \Phi_{12}^+(z_k;s)\Phi_{22}^+(z_k;s)ds = 0$ is satisfied, the zero z_k is not simple. That means z_k might be a multiple zero of $s_{11}(z)$.

To prove that the number of zeros of s_{11} is finite, we first suppose that $s_{11}(z)$ has no zeros on $\mathbb{R}$. Using the asymptotic behavior of $s_{11}(z)$ ($s_{11}(z) \to 1$ as $z \to \infty$), we can give the finiteness of the number of zeros of s_{11}. $\square$

3. Discrete spectrum with high-order poles

Now we suppose that $s_{11}(z)$ has N double zeros, which is $s_{11}(z_k) = s'_{11}(z_k) = 0$ and $s''_{11}(z_k) \neq 0$, in the upper half complex plane $\mathbb{C}^+$ and denote them by $z_k(k = 1, ..., N)$. By the symmetry of the eigenfunction, we know that $\bar{z}_k \in \mathbb{C}^-$ $(k = 1, ..., N)$ are the double zeros of $s_{22}(z)$. Denote

$$\mathcal{Z} = \{z_k | s_{11}(z_k) = s'_{11}(z_k) = 0, s''_{11}(z_k) \neq 0\},$$
$$\bar{\mathcal{Z}} = \{\bar{z}_k | s_{22}(\bar{z}_k) = s'_{22}(\bar{z}_k) = 0, s''_{22}(\bar{z}_k) \neq 0\},$$

which are the sets of the zeros of $s_{11}(z)$ and $s_{22}(z)$ respectively.

From the relation (2.13) and $s_{11}(z_k) = s'_{11}(z_k) = 0$, we deduce that there are norming constants b_k and d_k that are independent of x and t such that

$$\mu_1^-(z_k) = b_k e^{2it\theta(z_k)}\mu_2^+(z_k), \tag{3.1}$$

$$\left(\mu_1^-\right)'(z_k) = e^{2it\theta(z_k)}\left(\left(2it\theta'(z_k)b_k + d_k\right)\mu_2^+(z_k) + b_k\left(\mu_2^+\right)'(z_k)\right). \tag{3.2}$$

where $\theta(z) = z^2 + xz/t$. Similarly,

$$\mu_2^-(\bar{z}_k) = \widehat{b}_k\theta(\bar{z}_k)e^{-2it\theta(\bar{z}_k)}\mu_1^+(\bar{z}_k) \tag{3.3}$$

$$\left(\mu_2^-\right)'(\bar{z}_k) = e^{-2it\theta(\bar{z}_k)}\left(\left(\widehat{d}_k - 2it\theta'(\bar{z}_k)\widehat{b}_k\right)\mu_1^+(\bar{z}_k) + \widehat{b}_k\left(\mu_1^+\right)'(\bar{z}_k)\right), \tag{3.4}$$

where $\widehat{b}_k = -\bar{b}_k$ and $\widehat{d}_k = -\bar{d}_k$ according to the symmetry of $S(z)$.

Notice that μ_1^- is analytic in the upper half plane $\mathbb{C}^+$ and z_k is the double zero of $s_{11}(z)$, then let μ_1^- and $s_{11}(z)$ do Taylor expansion at point z_k

$$\frac{\mu_1^-(z)}{s_{11}(z)} = \frac{\mu_1^-(z_k) + \left(\mu_1^-\right)'(z_k)(z-z_k) + \left(\mu_1^-\right)''(z_k)(z-z_k)^2/2 + ...}{s''_{11}(z_k)(z-z_k)^2/2 + s'''_{11}(z_k)(z-z_k)^3/6 + ...} \tag{3.5}$$

$$= \frac{2\mu_1^-(z_k)}{s''_{11}(z_k)}(z-z_k)^{-2} + \left(\frac{2\left(\mu_1^-\right)'(z_k)}{s''_{11}(z_k)} - \frac{2\mu_1^-(z_k)s'''_{11}(z_k)}{3s''_{11}(z_k)^2}\right)(z-z_k)^{-1} + ... \tag{3.6}$$

The above equations (3.1), (3.2) and (3.5) yield the residue condition and the coefficient of $(z-z_k)^{-2}$ in the Laurent expansion of $\frac{\mu_1^-(z)}{s_{11}(z)}$

$$\operatorname*{Res}_{z=z_k}\left[\frac{\mu_1^-(z)}{s_{11}(z)}\right] = \frac{2\left(\mu_1^-\right)'(z_k)}{s''_{11}(z_k)} - \frac{2\mu_1^-(z_k)s'''_{11}(z_k)}{3s''_{11}(z_k)^2} \tag{3.7}$$

$$= A_k e^{2it\theta(z_k)}\left(\left(\mu_2^+\right)'(z_k) + \mu_2^+(z_k)\left(B_k + 2it\theta'(z_k)\right)\right), \tag{3.8}$$

$$\underset{z=z_k}{P_{-2}}\left[\frac{\mu_1^-(z)}{s_{11}(z)}\right]=\frac{2\mu_1^-(z_k)}{s_{11}''(z_k)}=A_k e^{2it\theta(z_k)}\mu_2^+(z_k), \tag{3.9}$$

where

$$A_k=\frac{2b_k}{s_{11}''(z_k)},\quad B_k=\frac{d_k}{b_k}-\frac{s_{11}'''(z_k)}{3s_{11}''(z_k)}. \tag{3.10}$$

Likewise, as $z=\bar{z}_k$ is the double zero of $s_{22}(z)$, by equations (3.3), (3.4) and (3.5), we obtain

$$\underset{z=\bar{z}_k}{\mathrm{Res}}\left[\frac{\mu_2^-(z)}{s_{22}(z)}\right]=\frac{2\left(\mu_2^-\right)'(\bar{z}_k)}{s_{22}''(\bar{z}_k)}-\frac{2\mu_2^-(\bar{z}_k)s_{22}'''(\bar{z}_k)}{3s_{22}''(\bar{z}_k)^2} \tag{3.11}$$

$$=\widehat{A}_k e^{2it\theta(\bar{z}_k)}\left(\left(\mu_1^+\right)'(\bar{z}_k)+\mu_1^+(\bar{z}_k)\left(\widehat{B}_k-2it\theta'(\bar{z}_k)\right)\right), \tag{3.12}$$

$$\underset{z=\bar{z}_k}{P_{-2}}\left[\frac{\mu_2^-(z)}{s_{22}(z)}\right]=\frac{2\mu_2^-(\bar{z}_k)}{s_{22}''(\bar{z}_k)}=\widehat{A}_k e^{-2it\theta(\bar{z}_k)}\mu_1^+(\bar{z}_k), \tag{3.13}$$

where

$$\widehat{A}_k=\frac{2\widehat{b}_k}{s_{22}''(\bar{z}_k)},\quad \widehat{B}_k=\frac{\widehat{d}_k}{\widehat{b}_k}-\frac{s_{22}'''(\bar{z}_k)}{3s_{22}''(\bar{z}_k)}. \tag{3.14}$$

Moreover, it is easy to find that

$$\widehat{A}_k=-\bar{A}_k,\quad \widehat{B}_k=\bar{B}_k. \tag{3.15}$$

4. The RH problem with high-order poles

In our situation, we introduce the meromorphic matrices

$$m(z)=m(z;x,t)=\begin{cases}\left(\frac{\mu_1^-(z)}{s_{11}(z)},\mu_2^+(z)\right), & \text{as } \mathrm{Im}z>0,\\ \left(\mu_1^+(z),\frac{\mu_2^-(z)}{s_{22}(z)}\right), & \text{as } \mathrm{Im}z<0,\end{cases} \tag{4.1}$$

which satisfies the following RH problem.

RHP1. Find a matrix-valued function $m(z)$ which satisfies

(a) $m(z)$ is meromorphic in $\mathbb{C}\setminus\mathbb{R}$ and has double poles;
(b) $m(z)$ satisfies the jump condition $m_+(z)=m_-(z)v(z),\ z\in\mathbb{R}$, where

$$v(z)=\begin{pmatrix}1+|r(z)|^2 & e^{-2it\theta(z)}\overline{r(z)}\\ e^{2it\theta(z)}r(z) & 1\end{pmatrix}; \tag{4.2}$$

(c) The asymptotic behavior of $m(z)$ at infinity is

$$m(z)=I+\mathcal{O}(z^{-1}),\qquad z\to\infty;$$

(d) $m(z)$ satisfies the residue and the coefficient of negative second power term in the Laurent expansion conditions at double zeros $z_k\in\mathcal{Z}$ and $\bar{z}_k\in\bar{\mathcal{Z}}$:

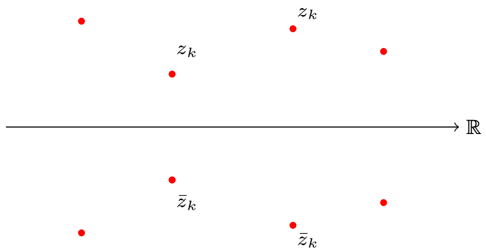

Fig. 1. The jump contour and poles for $m(z)$.

$$\operatorname*{Res}_{z=z_k} m^+(z) = \lim_{z\to z_k}\left[m'(z)\begin{pmatrix}0 & 0\\ A_k e^{2it\theta(z_k)} & 0\end{pmatrix} + m(z)\begin{pmatrix}0 & 0\\ A_k\left(B_k+2it\theta'(z_k)\right)e^{2it\theta(z_k)} & 0\end{pmatrix}\right], \tag{4.3}$$

$$\operatorname*{Res}_{z=\bar{z}_k} m^-(z) = \lim_{z\to \bar{z}_k}\left[m'(z)\begin{pmatrix}0 & \widehat{A}_k e^{-2it\theta(\bar{z}_k)}\\ 0 & 0\end{pmatrix} + m(z)\begin{pmatrix}0 & \widehat{A}_k\left(\widehat{B}_k-2it\theta'(\bar{z}_k)\right)e^{-2it\theta(\bar{z}_k)}\\ 0 & 0\end{pmatrix}\right], \tag{4.4}$$

$$\operatorname*{P_{-2}}_{z=z_k} m^+(z) = \lim_{z\to z_k} m(z)\begin{pmatrix}0 & 0\\ A_k e^{2it\theta(z_k)} & 0\end{pmatrix}, \tag{4.5}$$

$$\operatorname*{P_{-2}}_{z=\bar{z}_k} m^-(z) = \lim_{z\to \bar{z}_k} m(z)\begin{pmatrix}0 & \widehat{A}_k e^{-2it\theta(\bar{z}_k)}\\ 0 & 0\end{pmatrix}. \tag{4.6}$$

See Fig. 1. The existence and uniqueness of the above RHP1 can be given by Liouville's theorem and the vanishing lemma [1]. Plugging the asymptotic expansion $m = I + m_1/z + o(z^{-1})$ into the formula (2.1), we obtain that

$$m(z) = I + \frac{1}{2iz}\begin{pmatrix}-\int_x^\infty |q|^2 dx & q\\ \bar{q} & \int_x^\infty |q|^2 dx\end{pmatrix} + o(z^{-1}). \tag{4.7}$$

Thus, the solution $q(x,t)$ of initial value problem for NLS equation can be expressed by the above RHP1

$$q(x,t) = 2i\lim_{z\to\infty}(zm)_{12}. \tag{4.8}$$

5. Continuous extensions to a mixed $\bar{\partial}$-RH problem

In this section, we make factorizations of the jump matrix $v(z)$ and continuously extend each factor off $\mathbb{R}$. The idea of continuous extensions comes from [10,23,24]. Before doing continuous extensions, we renormalize the RH problem of $m(z)$ so that it is well-behaved at infinity. Then, we deform the jump matrix onto new contours on which they decay and obtain a new $\bar{\partial}$-RH problem by extensions.

5.1. Factorizations of jump matrix

We first consider the oscillatory term in the jump matrix (4.2)

$$e^{2it\theta(z)} = e^{2t\varphi(z)}, \quad \varphi(z) = i\theta(z) = i(z^2 + xz/t).$$

Differentiating $\varphi(z)$ with respect to z yields a stationary phase point and four paths of steepest descent

$$z_0 = -\frac{x}{2t}, \quad \Sigma_k = \left\{z_0 + e^{i(2k-1)\pi/4}\mathbb{R}_+\right\}, k = 1,2,3,4. \tag{5.1}$$

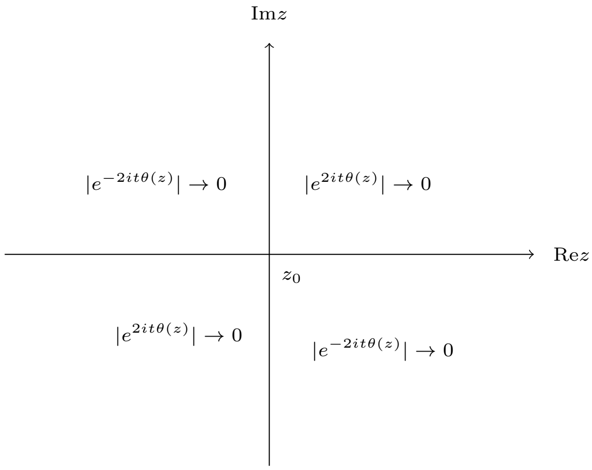

Fig. 2. Exponential decaying domains.

From $\theta(z) = z^2 - 2z_0 z = (z - z_0)^2 - z_0^2$, we get

$$\mathrm{Re}(i\theta) = -2\mathrm{Im}z(\mathrm{Re}z - z_0). \tag{5.2}$$

Therefore, we can divide the complex plane into two classes of domains according to the exponential decay $e^{2it\theta(z)}$. See Fig. 2.

From the above analysis, the jump matrix (4.2) admits two compositions

$$v(z) = \begin{cases} \begin{pmatrix} 1 & \bar{r}e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ re^{2it\theta} & 1 \end{pmatrix}, \quad z > z_0, \\ \\ \begin{pmatrix} 1 & 0 \\ \frac{r}{1+|r|^2} e^{2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1+|r|^2 & 0 \\ 0 & \frac{1}{1+|r|^2} \end{pmatrix} \begin{pmatrix} 1 & \frac{\bar{r}}{1+|r|^2} e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, \quad z < z_0. \end{cases}$$

To remove the intermediate matrix of the second decomposition, we introduce the following scalar RH problem.

RHP2. Find a scalar function $\delta(z)$ which satisfies

(a) $\delta(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$;
(b) $\delta_+(z) = \delta_-(z)(1 + |r|^2), \quad z < z_0$;
(c) $\delta(z) \sim I, \quad z \to \infty$.

By the Plemelj formula, we prove that this RH problem has a unique solution

$$\delta(z) = \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{z_0} \frac{\log(1 + |r(s)|^2)ds}{s - z} \right) = \exp\left(i \int_{-\infty}^{z_0} \frac{\nu(s)ds}{s - z} \right), \tag{5.3}$$

where $\nu(s) = -\frac{1}{2\pi} \log(1 + |r(s)|^2)$.

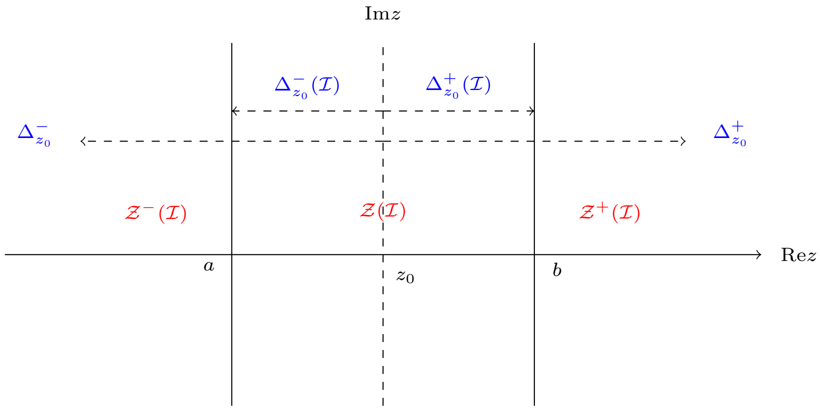

Fig. 3. Different spectrum sets.

5.2. Renormalizations of the RH problem for $m(z)$

For convenience, we introduce some notations

$$\Delta^+_{z_0} = \{k \in \{1, ..., N\} \, ||z_k| > z_0\} \, ,$$
$$\Delta^-_{z_0} = \{k \in \{1, ..., N\} \, ||z_k| < z_0\} \, .$$

For a real interval $I = [a, b]$, we define

$$\mathcal{Z}(I) = \{z_k \in \mathcal{Z} ||z_k| \in I\} \, ,$$
$$\mathcal{Z}^-(I) = \{z_k \in \mathcal{Z} ||z_k| < a\} \, ,$$
$$\mathcal{Z}^+(I) = \{z_k \in \mathcal{Z} ||z_k| > b\} \, .$$

For a fixed point $z_0 \in I$, we define

$$\Delta^-_{z_0}(I) = \{k \in \{1, ..., N\} \, |a \le |z_k| < z_0\} \, ,$$
$$\Delta^+_{z_0}(I) = \{k \in \{1, ..., N\} \, |z_0 < |z_k| \le b\} \, .$$

See the corresponding domains for different spectrum sets in Fig. 3.

Then, we introduce the function

$$T(z) = \prod_{k \in \Delta^-_{z_0}} \left(\frac{z - \bar{z}_k}{z - z_k} \right)^2 \delta(z) \tag{5.4}$$

$$= \prod_{k \in \Delta^-_{z_0}} \left(\frac{z - \bar{z}_k}{z - z_k} \right)^2 (z - z_0)^{i\nu(z_0)} e^{i\beta(z, z_0)}, \tag{5.5}$$

where

$$\beta(z, z_0) = -\nu(z_0) \log(z - z_0 + 1) + \int_{-\infty}^{z_0} \frac{\nu(s) - \chi(s)\nu(z_0)}{s - z} ds, \tag{5.6}$$

here $\chi(s)$ is the characteristic function of the interval $(z_0 - 1, z_0)$, and log takes the analytic branch along the cut $(-\infty, z_0 - 1]$.

Proposition 1. *The function T has the following properties*

(a) T is meromorphic in $\mathbb{C}\setminus(-\infty,z_0]$. For each $k\in\Delta^-_{z_0}$, $T(z)$ has double poles at z_k and double zeros at $\bar{z}_k$;
(b) For $z\in\mathbb{C}\setminus(-\infty,z_0]$, $\overline{T(\bar{z})}=1/T(z)$;
(c) For $z\in(-\infty,z_0]$,

$$T_+(z)=T_-(z)(1+|r(z)|^2); \tag{5.7}$$

(d) As $|z|\to\infty$ with $|arg(z)|\le c<\pi$,

$$T(z)=1+\frac{i}{z}\left[4\sum_{k\in\Delta^-_{z_0}}\mathrm{Im}(z_k)-\int_{-\infty}^{z_0}\nu(s)ds\right]+\mathcal{O}(z^{-2}); \tag{5.8}$$

(e) Along the ray $z=z_0+e^{i\phi}\mathbb{R}_+$ where $|\phi|<\pi$, as $z\to z_0$

$$|T(z,z_0)-T_0(z_0)(z-z_0)^{i\nu(z_0)}|\le c|z-z_0|^{1/2}, \tag{5.9}$$

where c is a fixed constant.

Proof. The proof of above properties is similar to the proof of Proposition 3.1 provided by Borghese et al. [7]. □

Next, we construct a new transformation

$$m^{(1)}(z)=m(z)T(z)^{-\sigma_3}. \tag{5.10}$$

From this transformation, we can achieve the following two goals:

- Renormalize m such that $m^{(1)}$ is well behaved as $t\to\infty$ along arbitrary characteristic;
- Split the residue coefficients into two sets according to signature of $\mathrm{Re}(i\theta)$.

In addition, $m^{(1)}(z)$ satisfies the following RH problem.

RHP3. Find a matrix-valued function $m^{(1)}(z)=m^{(1)}(z;x,t)$ such that

(a) $m^{(1)}(z)$ is analytic in $\mathbb{C}\setminus\left(\mathbb{R}\cup\mathcal{Z}\cup\bar{\mathcal{Z}}\right)$;
(b) $m^{(1)}(z)$ has the following jump condition $m^{(1)}_+(z)=m^{(1)}_-(z)v^{(1)}(z),\quad z\in\mathbb{R}$, where

$$v^{(1)}(z)=\begin{pmatrix}1 & \overline{r(z)}T(z)^2e^{-2it\theta(z)}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ r(z)T^{-2}(z)e^{2it\theta(z)} & 1\end{pmatrix},\qquad z\in(z_0,+\infty); \tag{5.11}$$

$$=\begin{pmatrix}1 & 0\\ \frac{r(z)}{1+|r(z)|^2}T_-^{-2}(z)e^{2it\theta(z)} & 1\end{pmatrix}\begin{pmatrix}1 & \frac{\overline{r(z)}}{1+|r(z)|^2}T_+^{-2}(z)e^{-2it\theta(z)}\\ 0 & 1\end{pmatrix},\qquad z\in(-\infty,z_0); \tag{5.12}$$

(c) $m^{(1)}(z)=I+\mathcal{O}(z^{-1}),\quad \text{as } z\to\infty;$

(d) $m^{(1)}(z)$ satisfies the following residue conditions at double poles $z_k \in \mathcal{Z}$ and $\bar{z}_k \in \bar{\mathcal{Z}}$:

$$\operatorname*{Res}_{z=z_k} m^{(1)}(z) = \lim_{z\to z_k} [m_z^{(1)}(z)N_{1,k}(z_k) + m^{(1)}(z)N_{2,k}(z_k)], \quad k \in \Delta_{z_0}^-;$$

$$\operatorname*{Res}_{z=z_k} m^{(1)}(z) = \lim_{z\to z_k} [m_z^{(1)}(z)N_{3,k}(z_k) + m^{(1)}(z)N_{4,k}(z_k)], \quad k \in \Delta_{z_0}^+;$$

$$\operatorname*{Res}_{z=\bar{z}_k} m^{(1)}(z) = \lim_{z\to \bar{z}_k} [m_z^{(1)}(z)\sigma_2\overline{N_{1,k}(\bar{z}_k)}\sigma_2 + m^{(1)}(z)\sigma_2\overline{N_{2,k}(\bar{z}_k)}\sigma_2], \quad k \in \Delta_{z_0}^-;$$

$$\operatorname*{Res}_{z=\bar{z}_k} m^{(1)}(z) = \lim_{z\to \bar{z}_k} [m_z^{(1)}(z)\sigma_2\overline{N_{3,k}(\bar{z}_k)}\sigma_2 + m^{(1)}(z)\sigma_2\overline{N_{4,k}(\bar{z}_k)}\sigma_2], \quad k \in \Delta_{z_0}^+;$$

Moreover, the coefficients of the negative second-order term are

$$\mathop{P_{-2}}_{z=z_k} m^{(1)}(z) = \begin{cases} \lim_{z\to z_k} m^{(1)}(z)N_{1,k}(z_k), & k \in \Delta_{z_0}^-; \\ \lim_{z\to z_k} m^{(1)}(z)N_{2,k}(z_k)), & k \in \Delta_{z_0}^+; \end{cases}$$

$$\mathop{P_{-2}}_{z=\bar{z}_k} m^{(1)}(z) = \begin{cases} \lim_{z\to \bar{z}_k} m^{(1)}(z)\sigma_2\overline{N_{3,k}(\bar{z}_k)}\sigma_2, & k \in \Delta_{z_0}^+; \\ \lim_{z\to \bar{z}_k} m^{(1)}(z)\sigma_2\overline{N_{4,k}(\bar{z}_k)}\sigma_2, & k \in \Delta_{z_0}^+, \end{cases}$$

where

$$N_{1,k}(z_k) = \begin{pmatrix} 0 & 4A_k^{-1}\left((T^{-1})''(z_k)\right)^{-2} e^{-2it\theta(z_k)} \\ 0 & 0 \end{pmatrix},$$

$$N_{1,k}(z_k) = \begin{pmatrix} 0 & -4A_k^{-1}\left((T^{-1})''(z_k)\right)^{-2}\left[B_k + 2it\theta'(z_k) + \frac{2(T^{-1})'''(z_k)}{3(T^{-1})''(z_k)}\right] e^{-2it\theta(z_k)} \\ 0 & 0 \end{pmatrix},$$

$$N_{3,k}(z_k) = \begin{pmatrix} 0 & 0 \\ A_kT^{-2}(z_k)e^{2it\theta(z_k)} & 0 \end{pmatrix},$$

$$N_{4,k}(z_k) = \begin{pmatrix} 0 & 0 \\ A_kT^{-2}(z_k)\left[B_k + 2it\theta'(z_k) - \frac{2T'(z_k)}{T(z_k)}\right] e^{2it\theta(z_k)} & 0 \end{pmatrix}.$$

Proof. The analyticity, jump condition and asymptotic behavior of $m^{(1)}(z)$ are easily to be proven. The difficulty lies in the calculation of residue conditions. We first consider poles $z_k \in \mathcal{Z}$ in the upper half complex plane and denote $m(z) = (m_1(z), m_2(z))$, then

$$m^{(1)}(z) = \left(m_1^{(1)}(z), m_2^{(1)}(z)\right) = \left(m_1(z)T^{-1}(z), m_2(z)T(z)\right) = \left(\frac{\mu_1^-(z)}{s_{11}(z)}T^{-1}(z), \mu_2^+(z)T(z)\right).$$

(i) For $k \in \Delta_{z_0}^-$ and $z_k \in \mathcal{Z}$, z_k is the double poles of $m_1^{(1)}$ and T, but $m_2^{(1)}$ and T^{-1} are analytic at the point z_k with $T^{-1}(z_k) = 0$, then

$$m^{(1)}(z_k) = \frac{1}{2}A_k e^{2it\theta(z_k)} m_2(z_k)(T^{-1})''(z_k), \tag{5.13}$$

$$\operatorname*{Res}_{z=z_k} m_2^{(1)}(z) = m_2'(z_k)2\left(T^{-1}\right)''(z_k) - m_2(z_k)\frac{2\left(T^{-1}\right)'''(z_k)}{3\left(\left(T^{-1}\right)''(z_k)\right)^2}, \tag{5.14}$$

where $\widehat{T}(z) = T(z)(z-z_k)^2$. Next, we calculate the derivative of $m_1^{(1)}$

$$\left(m_1^{(1)}\right)'(z_k) = \left(\mu_1^-\right)'(z_k)\frac{T^{-1}(z_k)}{s_{11}(z_k)} + \mu_1^-(z_k)\left(\frac{T^{-1}}{s_{11}}\right)'(z_k). \tag{5.15}$$

From the Taylor expansion, we find

$$\frac{T^{-1}(z)}{s_{11}(z)} = \frac{\left(T^{-1}\right)''(z_k)}{s_{11}''(z_k)} + \left(\frac{\left(T^{-1}\right)'''(z_k)}{3s_{11}''(z_k)} - \frac{\left(T^{-1}\right)''(z_k)s_{11}'''(z_k)}{3(s_{11}''(z_k))^2}\right)(z-z_k) + \cdots \tag{5.16}$$

Thus, we know

$$\left(\frac{T^{-1}}{s_{11}}\right)'(z_k) = \frac{\left(T^{-1}\right)'''(z_k)}{3s_{11}''(z_k)} - \frac{(T^{-1})''(z_k)s_{11}'''(z_k)}{3\left(s_{11}''(z_k)\right)^2}. \tag{5.17}$$

Combing (3.1), (3.2) and (5.15) with (5.17), we obtain

$$\begin{aligned}\left(m_1^{(1)}\right)'(z_k) &= b_k\frac{\left(T^{-1}\right)''(z_k)}{s_{11}''(z_k)}e^{2it\theta(z_k)}\left(\mu_2^+\right)'(z_k)\\ &+\left[\frac{(T^{-1})''(z_k)}{s_{11}''(z_k)}\left(2it\theta'(z_k)b_k+d_k\right)+b_k\left(\frac{(T^{-1})'''(z_k)}{3s_{11}''(z_k)}-\frac{(T^{-1})''(z_k)s_{11}'''(z_k)}{3(s_{11}''(z_k))^2}\right)\right]e^{2it\theta(z_k)}\mu_2^+(z_k).\end{aligned} \tag{5.18}$$

Substituting (5.13) and (5.18) into (5.14), we find

$$\begin{aligned}\operatorname*{Res}_{z=z_k} m_2^{(1)}(z) &= 4A_k^{-1}\left((T^{-1})''(z_k)\right)^{-2}e^{-2it\theta(z_k)}\left(m_2^{(1)}\right)'(z_k)\\ &- 4A_k^{-1}\left((T^{-1})''(z_k)\right)^{-2}\left[B_k + 2it\theta'(z_k) + \frac{2(T^{-1})'''(z_k)}{3(T^{-1})''(z_k)}\right]e^{-2it\theta(z_k))}m_2^{(1)}(z_k),\end{aligned} \tag{5.19}$$

where we have used the fact $A_k = \frac{2b_k}{s_{11}''(z_k)}$. Finally, we obtain the corresponding residue condition for $m^{(1)}(z)$.

Then, we calculate the coefficient of $(z-z_k)^{-2}$ in the Laurent expansion of $m^{(1)}$. We still consider this condition according to the order of the columns of $m^{(1)}$.

$$\operatorname*{P_{-2}}_{z=z_k} m_1^{(1)}(z) = 0, \tag{5.20}$$

$$\operatorname*{P_{-2}}_{z=z_k} m_2^{(1)}(z) = m_2(z_k)\operatorname*{P_{-2}}_{z=z_k} T(z) = 2m_2(z_k)\left((T^{-1})''(z_k)\right)^{-1}. \tag{5.21}$$

Plugging (5.13) into (5.21), it is straightforward to find

$$\operatorname*{P_{-2}}_{z=z_k} m^{(1)}(z) = \left(0, 4A_k^{-1}\left((T^{-1})''(z_k)\right)^{-2}e^{-2it\theta(z_k)}\right). \tag{5.22}$$

(ii) For $k \in \Delta_{z_0}^+$ and $z_k \in \mathcal{Z}$, T and T^{-1} is analytic at the point z_k. In this case, the residue condition of the first column of $m^{(1)}$ is

$$\begin{aligned}\operatorname*{Res}_{z=z_k} m_1^{(1)}(z) &= \operatorname*{Res}_{z=z_k} m_1(z)\cdot T^{-1}(z_k) + \operatorname*{Res}_{z=z_k} m_1(z)\left(T^{-1}(z) - T^{-1}(z_k)\right)\\ &= \operatorname*{Res}_{z=z_k} m_1(z)\cdot T^{-1}(z_k) + \lim_{z\to z_k} m_1\left(T^{-1}\right)'(z_k)(z-z_k)^2.\end{aligned} \tag{5.23}$$

Since

$$\lim_{z\to z_k} m_1 \left(T^{-1}\right)'(z_k)(z-z_k)^2 = 2\mu_1^-(z_k)\left(T^{-1}\right)'(z_k)/s_{11}''(z_k), \tag{5.24}$$

$$\left(m_2^{(1)}\right)'(z) = m_2'(z)T(z) + m_2(z)T'(z), \tag{5.25}$$

we give the expression

$$\begin{aligned}\operatorname*{Res}_{z=z_k} m_1^{(1)}(z) &= A_k T^{-2}(z_k) e^{2it\theta(z_k)} (m_2^{(1)})'(z_k) \\ &\quad + A_k T^{-2}(z_k)\left[B_k + 2it\theta'(z_k) - \frac{2T'(z_k)}{T(z_k)}\right] e^{2it\theta(z_k)} m_2^{(1)}(z_k),\end{aligned} \tag{5.26}$$

in which we have used equations (3.1) and (3.10). Because of the analyticity of $m_2(z)$ and $T(z)$ at the point z_k, $\mathrm{Res}_{z=z_k} m_2^{(1)}(z) = 0$. Thus, we find the expression of the residue condition in this case.
In addition, we can obtain the coefficient of $(z-z_k)^{-2}$ in the Laurent expansion of $m^{(1)}$ directly because T^{-1} is analytic at the point z_k

$$\operatorname*{P_{-2}}_{z=z_k} m^{(1)}(z) = \operatorname*{P_{-2}}_{z=z_k} m(z)\cdot T^{-1}(z_k) = \lim_{z\to z_k} m^{(1)} \begin{pmatrix} 0 & 0 \\ A_k T^{-2}(z_k) e^{2it\theta(z_k)} & 0 \end{pmatrix}. \tag{5.27}$$

Using the same method, we can obtain the corresponding conditions for $\bar{z}_k \in \bar{\mathcal{Z}}$ in the lower half plane. $\square$

5.3. Continuous extensions of jump matrix

In this section, we make continuous extension to the scattering data of the jump matrix $v^{(1)}$ and construct a new transformation from $m^{(1)}$ to $m^{(2)}$. The transformed $m^{(2)}$ satisfies the following properties:

- $m^{(2)}(z)$ has no jump on $\mathbb{R}$ and matches $m^{(pc)}(z)$ model, which is given and analyzed in Section 6, on a new contour $\Sigma^{(2)}$ which is defined in (5.28).
- The norm of the function, which is introduced by this transformation, has been controlled so that the $\bar{\partial}$-contribution to the long-time asymptotics of $q(x,t)$ can be ignored.
- The residues are unaffected by the transformation.

To make continuous extension, we first define a new contour $\Sigma^{(2)}$

$$\Sigma^{(2)} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4, \tag{5.28}$$

where Σ_k are given in (5.1). Then, the real axis $\mathbb{R}$ and the contour $\Sigma^{(2)}$ separate complex plane $\mathbb{C}$ into six open sectors denoted by Ω_k, $k = 1, ..., 6$, depicted in Fig. 4.

Second, let

$$\rho = \frac{1}{2} \min_{\lambda,\mu\in\mathcal{Z}\cup\bar{\mathcal{Z}};\lambda\neq\mu} |\lambda-\mu|. \tag{5.29}$$

For any point $z_k = x_k + iy_k \in \mathcal{Z}$, we have $\bar{z}_k = x_k - iy_k \in \bar{\mathcal{Z}}$. Thus, $\mathrm{dist}(\mathcal{Z},\mathbb{R}) \geq \rho$. Suppose that $\chi_{\mathcal{Z}} \in C_0^\infty(\mathbb{C},[0,1])$ is the characteristic function defined in the neighborhood of discrete spectrum

$$\chi_{\mathcal{Z}}(z) = \begin{cases} 1 & \mathrm{dist}\left(z, \mathcal{Z}\cup\bar{\mathcal{Z}}\right) < \rho/3, \\ 0 & \mathrm{dist}\left(z, \mathcal{Z}\cup\bar{\mathcal{Z}}\right) > 2\rho/3. \end{cases} \tag{5.30}$$

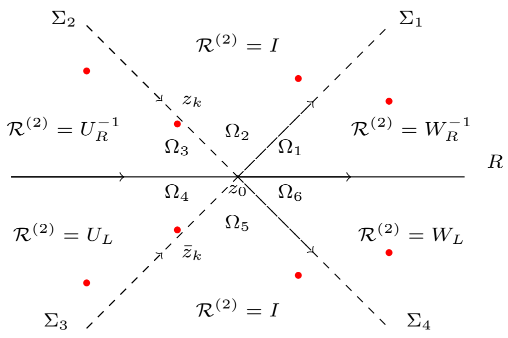

Fig. 4. Definition of $\mathcal{R}^{(2)}$ in different domains.

Finally, we introduce a transformation $\mathcal{R}^{(2)}$ to obtain a mixed $\bar{\partial}$-RH problem.

$$m^{(2)}(z) = m^{(1)}(z)\mathcal{R}^{(2)}(z), \tag{5.31}$$

where $\mathcal{R}^{(2)}(z)$ is defined as follows:

$$\mathcal{R}^{(2)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ R_1(z)e^{2it\theta} & 1 \end{pmatrix}^{-1} = W_R^{-1}, & z \in \Omega_1, \\ \begin{pmatrix} 1 & R_3(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}^{-1} = U_R^{-1}, & z \in \Omega_3, \\ \begin{pmatrix} 1 & 0 \\ R_4(z)e^{2it\theta} & 1 \end{pmatrix} = U_L, & z \in \Omega_4, \\ \begin{pmatrix} 1 & R_6(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix} = W_L, & z \in \Omega_6, \\ I, & z \in \Omega_2 \cup \Omega_5; \end{cases} \tag{5.32}$$

where the function R_j, $j = 1, 3, 4, 6$, is defined in following proposition, depicted in Fig. 4.

Proposition 2. *There exists a function R_j: $\bar{\Omega}_j \to C$, $j = 1, 3, 4, 6$ such that*

$$R_1(z) = \begin{cases} r(z)T(z)^{-2}, & z \in (z_0, \infty), \\ r(z_0)T_0(z_0)^{-2}(z - z_0)^{-2i\nu(z_0)}(1 - \chi_{\mathcal{Z}}(z)), & z \in \Sigma_1, \end{cases} \tag{5.33}$$

$$R_3(z) = \begin{cases} \dfrac{\overline{r(z)}T_+(z)^2}{1 + |r(z)|^2}, & z \in (-\infty, z_0), \\ \dfrac{\overline{r(z_0)}T_0(z_0)^2}{1 + |r(z_0)|^2}(z - z_0)^{2i\nu(z_0)}(1 - \chi_{\mathcal{Z}}(z)), & z \in \Sigma_2, \end{cases} \tag{5.34}$$

$$R_4(z) = \begin{cases} \dfrac{r(z)T_-(z)^{-2}}{1 + |r(z)|^2}, & z \in (-\infty, z_0), \\ \dfrac{r(z_0)T_0(z_0)^{-2}}{1 + |r(z_0)|^2}(z - z_0)^{-2i\nu(z_0)}(1 - \chi_{\mathcal{Z}}(z)), & z \in \Sigma_3, \end{cases} \tag{5.35}$$

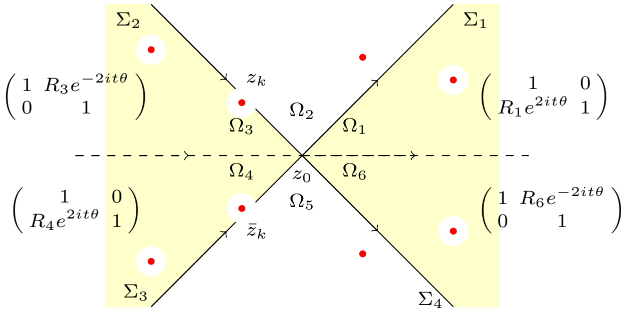

Fig. 5. Jump matrix $v^{(2)}$. Yellow parts support $\bar{\partial}$ derivative: $\bar{\partial}\mathcal{R}^{(2)} \neq 0$; White parts don't support $\bar{\partial}$ derivative: $\bar{\partial}\mathcal{R}^{(2)} = 0$.

$$R_6(z) = \begin{cases} \overline{r(z)}T(z)^2, & z \in (z_0, \infty), \\ \overline{r(z_0)}T_0(z_0)^2(z-z_0)^{2i\nu(z_0)}(1-\chi_{\mathcal{Z}}(z)), & z \in \Sigma_4, \end{cases} \tag{5.36}$$

and R_j admit estimates

$$|R_j(z)| \lesssim \sin^2(\arg(z-z_0)) + \langle \mathrm{Re}(z)\rangle^{-1/2}, \tag{5.37}$$

$$|\bar{\partial}R_j(z)| \lesssim |\bar{\partial}\chi_{\mathcal{Z}}(z)| + |r'(\mathrm{Re} z)| + |z-z_0|^{-1/2}, \tag{5.38}$$

$$\bar{\partial}R_j(z) = 0, \qquad \textit{if } z \in \Omega_2 \cup \Omega_5 \textit{ or } \mathrm{dist}(z, \mathcal{Z} \cup \bar{\mathcal{Z}}) < \rho/3. \tag{5.39}$$

The proof of above proposition is the same as that in [7] because the form of residue condition doesn't affect this transform $\mathcal{R}^{(2)}(z)$.

Therefore, $m^{(2)}(z)$ satisfies the mixed $\bar{\partial}$-RH problem as follows:

RHP4. Find a matrix-valued function $m^{(2)}(z) = m^{(2)}(z; x, t)$ which satisfies

(a) $m^{(2)}(z)$ is continuous in $\mathbb{C} \setminus \left(\Sigma^{(2)} \cup \mathcal{Z} \cup \bar{\mathcal{Z}}\right)$.

(b) $m^{(2)}(z)$ has the following jump condition $m_+^{(2)}(z) = m_-^{(2)}(z)v^{(2)}(z), \quad z \in \Sigma^{(2)}$, where

$$\begin{aligned} v^{(2)}(z) &= \left(\mathcal{R}_-^{(2)}\right)^{-1}(z)v^{(1)}(z)\mathcal{R}_+^{(2)}(z) \\ &= \begin{cases} \begin{pmatrix} 1 & 0 \\ R_1(z)e^{2it\theta} & 1 \end{pmatrix}, & z \in \Sigma_1, \\ \begin{pmatrix} 1 & R_3(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_2, \\ \begin{pmatrix} 1 & 0 \\ R_4(z)e^{2it\theta} & 1 \end{pmatrix}, & z \in \Sigma_3, \\ \begin{pmatrix} 1 & R_6(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_4. \end{cases} \end{aligned} \tag{5.40}$$

See Fig. 5.

(c) $m^{(2)}(z) = I + \mathcal{O}(z^{-1}), \qquad \text{as } z \to \infty.$

(d) For $\mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{Z} \cup \bar{\mathcal{Z}})$,

$$\bar{\partial}m^{(2)}(z) = m^{(2)}(z)\bar{\partial}\mathcal{R}^{(2)}(z), \tag{5.41}$$

where

$$\bar{\partial}\mathcal{R}^{(2)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\bar{\partial}R_1(z)e^{2it\theta} & 1 \end{pmatrix}, & z \in \Omega_1, \\ \\ \begin{pmatrix} 1 & -\bar{\partial}R_3(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Omega_3, \\ \\ \begin{pmatrix} 1 & 0 \\ \bar{\partial}R_4(z)e^{2it\theta} & 1 \end{pmatrix}, & z \in \Omega_4, \\ \\ \begin{pmatrix} 1 & \bar{\partial}R_6(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Omega_6, \\ \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & z \in \Omega_2 \cup \Omega_5. \end{cases} \tag{5.42}$$

(e) The $m^{(2)}(z)$ has the same residue conditions with $m^{(2)}(z)$.

Proof. (a)-(d) are easy to be checked, so here we only give a brief proof to (e). It is sufficient to prove the case where $k \in \Delta_{z_0}^{+}$ and $z_k \in \mathbb{C}^{+}$, since the proof of others is similar. In this case, any $z_k \in \Omega_1$ is not the pole of $\mathcal{R}^{(2)}$, so

$$\begin{aligned} \operatorname*{Res}_{z=z_k} m^{(2)}(z) &= \lim_{z\to z_k} \left(m^{(1)}\right)' \mathcal{R}^{(2)} \left(\mathcal{R}^{(2)}\right)^{-1} \begin{pmatrix} 0 & 0 \\ A_k T^{-2}(z_k) e^{2it\theta(z_k)} & 0 \end{pmatrix} \mathcal{R}^{(2)} \\ &+ \lim_{z\to z_k} m^{(1)}\mathcal{R}^{(2)} \left(\mathcal{R}^{(2)}\right)^{-1} \begin{pmatrix} 0 & 0 \\ A_k T^{-2}(z_k) \left[B_k + 2it\theta'(z_k) - \frac{2T'(z_k)}{T(z_k)}\right] e^{2it\theta(z_k)} & 0 \end{pmatrix} \mathcal{R}^{(2)}. \end{aligned}$$

Substitute $\left(m^{(2)}\right)' = \left(m^{(1)}\right)' \mathcal{R}^{(2)} + m^{(1)} \left(\mathcal{R}^{(2)}\right)'$ and $\left(\mathcal{R}^{(2)}\right)' = 0$ into the above equation, and we finish the proof. □

6. Decomposition of the mixed $\bar{\partial}$-RH problem

In this section, we will find the solution of the mixed $\bar{\partial}$-RH problem $m^{(2)}(z)$ as follows:

Step 1. Separate zero and non-zero parts of $\bar{\partial}\mathcal{R}^{(2)}(z)$. Thus, we decompose $m^{(2)}(z)$ into a pure Riemann-Hilbert problem with $\bar{\partial}\mathcal{R}^{(2)}(z) = 0$, which we denote by $m^{(2)}_{RHP}(z)$, and a pure $\bar{\partial}$ problem with $\bar{\partial}\mathcal{R}^{(2)}(z) \neq 0$, which we denote by $m^{(3)}(z)$.

$$m^{(2)}(z) = \begin{cases} \bar{\partial}\mathcal{R}^{(2)}(z) = 0 \to \ m^{(2)}_{RHP}(z), \\ \\ \bar{\partial}\mathcal{R}^{(2)}(z) \neq 0 \to \ m^{(3)}(z) = m^{(2)}(z) \left(m^{(2)}_{RHP}(z)\right)^{-1}. \end{cases} \tag{6.1}$$

The RH problem for the $m^{(2)}_{RHP}(z)$ is as follows:

RHP5. Find a matrix-valued function $m^{(2)}_{RHP}(z)$ which satisfies

(a) $m^{(2)}_{RHP}(z)$ is analytic in $\mathbb{C} \setminus \left(\Sigma^{(2)} \cup \mathcal{Z} \cup \bar{\mathcal{Z}}\right)$;

(b) $m_{RHP}^{(2)}(z)$ has the following jump condition $m_{RHP+}^{(2)}(z) = m_{RHP-}^{(2)}(z)v^{(2)}(z), \quad z \in \Sigma^{(2)}$, where $v^{(2)}(z)$ has been given by (5.40);
(c) $m_{RHP}^{(2)}(z) \to I, \quad$ as $z \to \infty$;
(d) $\bar{\partial}\mathcal{R}^{(2)}(z) = 0$, for $z \in \mathbb{C}$;
(e) The residue condition and the coefficient of the negative twice power of the Laurent expansion have the same form as $m^{(2)}(z)$ with $m_{RHP}^{(2)}(z)$ replacing $m^{(2)}(z)$.

Step 2. To prove the existence of the solution $m_{RHP}^{(2)}(z)$, we separate the jump line from the pole. Suppose $\mathcal{U}_{z_0} = \{z : |z - z_0| < \rho/2\}$. Let $m_{RHP}^{(2)}(z)$ be further decomposed into two parts:

$$m_{RHP}^{(2)}(z) = \begin{cases} E(z)m^{(out)}(z), & z \notin \mathcal{U}_{z_0}, \\ E(z)m^{(z_0)}(z), & z \in \mathcal{U}_{z_0}. \end{cases} \tag{6.2}$$

The outer model $m^{(out)}(z)$ is constructed by ignoring the jumps in RHP5, which can be approximated by solitons on the discrete spectrum. The inner model $m^{(z_0)}(z)$ has the same jump with RHP5, which can be approximated by the parabolic cylinder model in continuous spectrum.

Step 3. Find the solution and its asymptotic behavior of the pure $\bar{\partial}$ problem $m^{(3)}(z)$.

6.1. The pure RH problem and constructions of its solution

6.1.1. The construction of outer model

By definition (6.2), $m^{(out)}(z)$ is the solution of $m^{(2)}(z)$ in the soliton region, which satisfies the following RH problem.

RHP6. Find a matrix-valued function $m^{(out)}(z)$ which satisfies

(a) $m^{(out)}(z)$ is analytic in $\mathbb{C} \setminus \left(\mathcal{Z} \cup \bar{\mathcal{Z}}\right)$;
(b) $m^{(out)}(z) = I + O(z^{-1}), \quad z \to \infty$;
(c) $m^{(out)}(z)$ has double poles at each $z_k \in \mathcal{Z}$ and $\bar{z}_k \in \bar{\mathcal{Z}}$, which satisfies the residue relations in (e) of RHP4 with $m^{(out)}(z)$ replacing $m^{(2)}(z)$.

In order to show the existence and uniqueness of solution of $m^{(out)}(z)$, we first consider the reflectionless case of the RHP1. In this case, $r(z) = 0$ and $v(z) = I$, then $m_+ = m_-$. Thus, RHP1 of NLS equation has no jumps in the whole plane and is analytic in $\mathbb{C}$ except for $z_k \in \mathcal{Z}$ and $\bar{z}_k \in \bar{\mathcal{Z}}$. The RHP1 can be equivalently rewritten as the following solvable RH problem:

RHP7. Given discrete data $\sigma_d = \{(z_k, c_{i,k}), i = 0, 1, z_k \in \mathcal{Z}\}_{k=1}^N$. Find a matrix-valued function $m(z|\sigma_d)$ which has the following properties:

(a) $m(z|\sigma_d)$ is analytical in $\mathbb{C} \setminus \left(\mathcal{Z} \cup \bar{\mathcal{Z}}\right)$;
(b) $m(z|\sigma_d) = I + O(z^{-1}), \quad z \to \infty$;
(c) $m(z|\sigma_d)$ satisfies the following relations at each double pole $z_k \in \mathcal{Z}$ and $\bar{z}_k \in \bar{\mathcal{Z}}$

$$\operatorname*{Res}_{z=z_k} m(z|\sigma_d) = \lim_{z \to z_k} \left[m(z|\sigma_d)n_{0,k} + m'(z|\sigma_d)n_{1,k}\right], \tag{6.3}$$

$$\operatorname*{Res}_{z=\bar{z}_k} m(z|\sigma_d) = \lim_{z \to \bar{z}_k} \left[m(z|\sigma_d)\sigma_2\overline{n_{0,k}}\sigma_2 + m'(z|\sigma_d)\sigma_2\overline{n_{1,k}}\sigma_2\right], \tag{6.4}$$

$$\underset{z=z_k}{P_{-2}} m(z|\sigma_d) = \lim_{z \to z_k} m(z|\sigma_d)n_{1,k}, \tag{6.5}$$

$$\underset{z=\bar{z}_k}{P_{-2}} m(z|\sigma_d) = \lim_{z\to\bar{z}_k} m(z|\sigma_d)\sigma_2\overline{n_{1,k}}\sigma_2, \tag{6.6}$$

where

$$n_{0,k} = \begin{pmatrix} 0 & 0 \\ \gamma_{0,k}(x,t) & 0 \end{pmatrix}, \; n_{1,k} = \begin{pmatrix} 0 & 0 \\ \gamma_{1,k}(x,t) & 0 \end{pmatrix}, \tag{6.7}$$

with

$$\gamma_{0,k}(x,t) = c_{0,k}e^{2it\theta(z_k)}, \tag{6.8}$$

$$\gamma_{1,k}(x,t) = c_{1,k}e^{2it\theta(z_k)}, \tag{6.9}$$

$$c_{0,k} = A_k\left(B_k + 2it\theta'(z_k)\right), \tag{6.10}$$

$$c_{1,k} = A_k. \tag{6.11}$$

Proposition 3. *Given discrete data $\sigma_d = \{(z_k, c_{i,k}), i = 0, 1, z_k \in \mathcal{Z}\}_{k=1}^N$, there exists a unique solution of RHP7 for each $(x,t) \in \mathbb{R}^2$, as $t \to \infty$,*

$$q_{sol}(x,t;\sigma_d) = 2i\lim_{z\to\infty}\left(zm(z|\sigma_d)\right)_{12}. \tag{6.12}$$

Proof. The proof includes two parts. One is for the uniqueness, and the other is for the existence. The proof of uniqueness is relatively simple, here we only briefly introduce the steps and mainly prove the existence.

Uniqueness: To prove the uniqueness of this solution, we first need to introduce a transformation to remove singularity of $m(z|\sigma_d)$ and then use Liouville's theorem to provide the uniqueness. Existence: We rewrite $\mathrm{Res}_{z=z_k} m(z|\sigma_d)$ and $\underset{z=z_k}{P_{-2}} m(z|\sigma_d)$ into the following form:

$$\operatorname*{Res}_{z=z_k} m(z|\sigma_d) = a^{(0)}(z_k)n_{0,k} + a^{(1)}(z_k)n_{1,k} = \begin{pmatrix} a_{12}^{(0)}(z_k)\gamma_{0,k} + a_{12}^{(1)}(z_k)\gamma_{1,k} & 0 \\ a_{22}^{(0)}(z_k)\gamma_{0,k} + a_{22}^{(1)}(z_k)\gamma_{1,k} & 0 \end{pmatrix} \triangleq \begin{pmatrix} \alpha_{1,k} & 0 \\ \beta_{1,k} & 0 \end{pmatrix}, \tag{6.13}$$

$$\underset{z=z_k}{P_{-2}} m(z|\sigma_d) = a^{(2)}(z_k)n_{1,k} = \begin{pmatrix} a_{12}^{(2)}(z_k)\gamma_{1,k} & 0 \\ a_{22}^{(2)}(z_k)\gamma_{1,k} & 0 \end{pmatrix} \triangleq \begin{pmatrix} \alpha_{2,k} & 0 \\ \beta_{2,k} & 0 \end{pmatrix}. \tag{6.14}$$

From the symmetry $m(z|\sigma_d) = \sigma_2\overline{m(\bar{z}|\sigma_d)}\sigma_2$, we know

$$\operatorname*{Res}_{z=\bar{z}_k} m(z|\sigma_d) = \sigma_2\left[\overline{a^{(0)}(z_k)n_{0,k} + a^{(1)}(z_k)n_{1,k}}\right]\sigma_2 = \begin{pmatrix} 0 & -\bar{\beta}_{1,k} \\ 0 & \bar{\alpha}_{1,k} \end{pmatrix}, \tag{6.15}$$

$$\underset{z=\bar{z}_k}{P_{-2}} m(z|\sigma_d) = \sigma_2\overline{a^{(2)}(z_k)n_{2,k}}\sigma_2 = \begin{pmatrix} 0 & -\bar{\beta}_{2,k} \\ 0 & \bar{\alpha}_{2,k} \end{pmatrix}. \tag{6.16}$$

Notice that when $r(z) = 0$, $v(z) = I$. Therefore, the above RH problem for $m(z|\sigma_d)$ has the following solution

$$\begin{aligned} m(z|\sigma_d) = I + \sum_{k=1}^N \Big[&\frac{1}{z-z_k}\begin{pmatrix} \alpha_{1,k} & 0 \\ \beta_{1,k} & 0 \end{pmatrix} + \frac{1}{(z-z_k)^2}\begin{pmatrix} \alpha_{2,k} & 0 \\ \beta_{2,k} & 0 \end{pmatrix} \\ &+ \frac{1}{z-\bar{z}_k}\begin{pmatrix} 0 & -\bar{\beta}_{1,k} \\ 0 & \bar{\alpha}_{1,k} \end{pmatrix} + \frac{1}{(z-\bar{z}_k)^2}\begin{pmatrix} 0 & -\bar{\beta}_{2,k} \\ 0 & \bar{\alpha}_{2,k} \end{pmatrix}\Big]. \end{aligned} \tag{6.17}$$

Substituting (6.17) into (6.3) and (6.5) respectively, we get the following linear equations after normalization

$$\alpha_{1,j}+\sum_{k=1}^{N}\left[\gamma_{0,j}\left(\frac{\bar\beta_{1,k}}{z_j-\bar z_k}+\frac{\bar\beta_{2,k}}{(z_j-\bar z_k)^2}\right)-\gamma_{1,j}\left(\frac{\bar\beta_{1,k}}{(z_j-\bar z_k)^2}+\frac{2\bar\beta_{2,k}}{(z_j-\bar z_k)^3}\right)\right]=0, \tag{6.18}$$

$$\bar\beta_{1,j}-\sum_{k=1}^{N}\left[\bar\gamma_{0,j}\left(\frac{\alpha_{1,k}}{\bar z_j-z_k}+\frac{\alpha_{2,k}}{(\bar z_j-z_k)^2}\right)-\bar\gamma_{1,j}\left(\frac{\alpha_{1,k}}{(\bar z_j-z_k)^2}+\frac{2\alpha_{2,k}}{(\bar z_j-z_k)^3}\right)\right]=\bar\gamma_{0,j}, \tag{6.19}$$

$$\alpha_{2,j}+\sum_{k=1}^{N}\left[\gamma_{2,j}\left(\frac{\bar\beta_{1,k}}{z_j-\bar z_k}+\frac{\bar\beta_{2,k}}{(z_j-\bar z_k)^2}\right)\right]=0, \tag{6.20}$$

$$\bar\beta_{2,j}-\sum_{k=1}^{N}\left[\bar\gamma_{2,j}\left(\frac{\alpha_{1,k}}{\bar z_j-z_k}+\frac{\alpha_{2,k}}{(\bar z_j-z_k)^2}\right)\right]=\bar\gamma_{2,j}. \tag{6.21}$$

Next, we transform the above linear equations (6.18)-(6.21) into matrix form. Let

$$\begin{aligned}
&\alpha_1=(\alpha_{1,1},\cdots,\alpha_{1,N})^T,\ \alpha_2=(\alpha_{2,1},\cdots,\alpha_{2,N})^T,\\
&\bar\beta_1=(\bar\beta_{1,1},\cdots,\bar\beta_{1,N})^T,\ \bar\beta_2=(\bar\beta_{2,1},\cdots,\bar\beta_{2,N})^T,\\
&A=(a_{ij})_{N\times N},\ a_{ij}=\frac{\gamma_{0,i}}{z_i-\bar z_j}-\frac{\gamma_{1,i}}{(z_i-\bar z_j)^2},\ i,j=1,\cdots,N,\\
&B=(b_{ij})_{N\times N},\ b_{ij}=\frac{\gamma_{0,i}}{(z_i-\bar z_j)^2}-\frac{2\gamma_{1,i}}{(z_i-\bar z_j)^3},\ i,j=1,\cdots,N,\\
&C=(c_{ij})_{N\times N},\ c_{ij}=\frac{\gamma_{1,i}}{z_i-\bar z_j},\ i,j=1,\cdots,N,\\
&D=(d_{ij})_{N\times N},\ d_{ij}=\frac{\gamma_{1,i}}{(z_i-\bar z_j)^2},\ i,j=1,\cdots,N.
\end{aligned}$$

Thus, the above linear equations (6.18)-(6.21) are equivalent to the following partitioned matrix equation

$$\begin{pmatrix} I_N & 0 & A & B\\ 0 & I_N & C & D\\ -A^* & -B^* & I_N & 0\\ -C^* & -D^* & 0 & I_N\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \beta_1\\ \beta_2\end{pmatrix}=\begin{pmatrix}0\\0\\ \bar\gamma_0\\ \bar\gamma_1\end{pmatrix}. \tag{6.22}$$

Moreover, we can prove the coefficient matrix of the above equations is positive definite as $t\to\infty$. According to the Cramer's rule, the solution of the (6.22) exists and is unique as $t\to\infty$. □

Remark 1. RHP7 is a special case in Zhang's paper [37] with $n_1=n_2=\cdots=n_N=2$, but the method of proof in his article is completely different from ours.

For convenience, let $\Delta\subsetneqq\{1,2,\cdots,N\}$, $\nabla=\Delta^c=\{1,2,\cdots,N\}\setminus\Delta$ and define

$$a_\Delta(z)=\prod_{k\in\Delta}\left(\frac{z-z_k}{z-\bar z_k}\right)^2,\ a_\nabla(z)=\frac{s_{11}}{a_\Delta(z)}\prod_{k\in\nabla}\left(\frac{z-z_k}{z-\bar z_k}\right)^2. \tag{6.23}$$

Then we make a transform

$$m^\Delta\left(z|\sigma_d^\Delta\right)=m\left(z|\sigma_d\right)a_\Delta(z)^{\sigma_3}. \tag{6.24}$$

As we can see from the above expression, the transformation (6.24) splits the poles between the columns of $m^{\Delta}\left(z|\sigma_d^{\Delta}\right)$ according to the choice of Δ, and it satisfies the following nonreflective RH problem.

RHP8. Given discrete data $\sigma_d^{\Delta} = \left\{\left(z_k, c_{i,k}^{\Delta}\right), i=0,1, z_k \in \mathcal{Z}\right\}_{k=1}^{N}$. Find a matrix-valued function $m^{\Delta}\left(z|\sigma_d^{\Delta}\right)$ which has the following properties:

(a) $m^{\Delta}\left(z|\sigma_d^{\Delta}\right)$ is analytical in $\mathbb{C}\setminus\left(\mathcal{Z}\cup\bar{\mathcal{Z}}\right)$;
(b) $m^{\Delta}\left(z|\sigma_d^{\Delta}\right) = I + O(z^{-1}), \quad z\to\infty$;
(c) $m^{\Delta}\left(z|\sigma_d^{\Delta}\right)$ has the following relations at discrete spectrum $\mathcal{Z}\cup\bar{\mathcal{Z}}$

$$\operatorname*{Res}_{z=z_k} m^{\Delta}\left(z|\sigma_d^{\Delta}\right) = \lim_{z\to z_k}\left[m^{\Delta}\left(z|\sigma_d^{\Delta}\right) n_{0,k}^{\Delta} + \left(m^{\Delta}\right)'\left(z|\sigma_d^{\Delta}\right) n_{1,k}^{\Delta}\right], \tag{6.25}$$

$$\operatorname*{Res}_{z=\bar{z}_k} m^{\Delta}\left(z|\sigma_d^{\Delta}\right) = \lim_{z\to \bar{z}_k}\left[m^{\Delta}\left(z|\sigma_d^{\Delta}\right) \sigma_2\overline{n_{0,k}^{\Delta}}\sigma_2 + \left(m^{\Delta}\right)'\left(z|\sigma_d^{\Delta}\right) \sigma_2\overline{n_{1,k}^{\Delta}}\sigma_2\right], \tag{6.26}$$

$$\operatorname*{P_{-2}}_{z=z_k} m^{\Delta}\left(z|\sigma_d^{\Delta}\right) = \lim_{z\to z_k} m^{\Delta}\left(z|\sigma_d^{\Delta}\right) n_{1,k}^{\Delta}, \tag{6.27}$$

$$\operatorname*{P_{-2}}_{z=\bar{z}_k} m^{\Delta}\left(z|\sigma_d^{\Delta}\right) = \lim_{z\to \bar{z}_k} m^{\Delta}\left(z|\sigma_d^{\Delta}\right) \sigma_2\overline{n_{1,k}^{\Delta}}\sigma_2, \tag{6.28}$$

where

$$n_{0,k}^{\Delta} = \begin{cases} \begin{pmatrix} 0 & \gamma_{0,k}^{\Delta}(x,t) \\ 0 & 0 \end{pmatrix}, & k\in\Delta, \\ \begin{pmatrix} 0 & 0 \\ \gamma_{0,k}^{\Delta}(x,t) & 0 \end{pmatrix}, & k\in\nabla, \end{cases} \tag{6.29}$$

$$n_{1,k}^{\Delta} = \begin{cases} \begin{pmatrix} 0 & \gamma_{1,k}^{\Delta}(x,t) \\ 0 & 0 \end{pmatrix}, & k\in\Delta, \\ \begin{pmatrix} 0 & 0 \\ \gamma_{1,k}^{\Delta}(x,t) & 0 \end{pmatrix}, & k\in\nabla, \end{cases} \tag{6.30}$$

with

$$\gamma_{0,k}^{\Delta}(x,t) = \begin{cases} c_{0,k}^{\Delta} e^{-2it\theta(z_k)}, & k\in\Delta, \\ c_{0,k}^{\Delta} e^{2it\theta(z_k)}, & k\in\nabla, \end{cases} \tag{6.31}$$

$$c_{0,k}^{\Delta} = \begin{cases} -4A_k^{-1} a_{\Delta}''(z_k)^{-2}\left[2it\theta'(z_k) + B_k + \frac{2a_{\Delta}'''(z_k)}{3a_{\Delta}''(z_k)}\right], & k\in\Delta, \\ 2b_k s_{11}''(z_k) a_{\nabla}''(z_k)^{-2}\left[2it\theta'(z_k) + B_k - \frac{2a_{\nabla}'''(z_k)}{3a_{\nabla}''(z_k)}\right], & k\in\nabla, \end{cases} \tag{6.32}$$

$$\gamma_{1,k}^{\Delta}(x,t) = \begin{cases} c_{1,k}^{\Delta} e^{-2it\theta(z_k)}, & k\in\Delta, \\ c_{1,k}^{\Delta} e^{2it\theta(z_k)}, & k\in\nabla, \end{cases} \tag{6.33}$$

$$c_{1,k}^{\Delta} = \begin{cases} 4A_k^{-1} a_{\Delta}''(z_k)^{-2}, & k\in\Delta, \\ b_k s_{11}''(z_k) a_{\nabla}''(z_k)^{-2}, & k\in\nabla. \end{cases} \tag{6.34}$$

The proof is similar to the proof of RHP3.

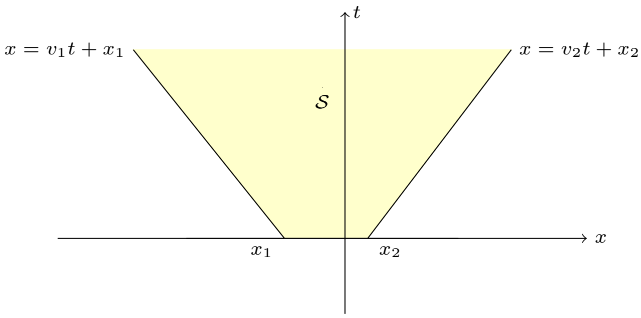

Fig. 6. Space-time $\mathcal{S}(x_1, x_2, v_1, v_2)$ on the half-plane $-\infty < x < \infty,\ t \geq 0$.

Proposition 4. *For nonreflective scattering data* $\sigma_d^{\Delta} = \left\{\left(z_k, c_{i,k}^{\Delta}\right), i = 0, 1, z_k \in \mathcal{Z}\right\}_{k=1}^{N}$, *RHP8 owns a unique solution as* $t \to \infty$ *and*

$$q_{sol}\left(x, t; \sigma_d^{\Delta}\right) = 2i \lim_{z\to\infty} \left(zm^{\Delta}\left(z|\sigma_d^{\Delta}\right)\right)_{12} = 2i \lim_{z\to\infty} \left(zm\left(z|\sigma_d\right)\right)_{12} = q_{sol}\left(x, t; \sigma_d\right), \quad t \to \infty. \tag{6.35}$$

Proof. Since the transformation (6.24) is explicit, from Proposition 3, we know this RHP8 has a unique solution as $t \to \infty$. Using transformations (4.7) and (6.24), we can obtain

$$m^{\Delta}\left(z|\sigma_d^{\Delta}\right) = I + \frac{1}{2iz}\begin{pmatrix} -\int_x^{\infty} |q|^2 dx + 8\Sigma_{k\in\Delta}\mathrm{Im}z_k & q(x,t) \\ \bar{q}(x,t) & \int_x^{\infty} |q|^2 dx - 8\Sigma_{k\in\Delta}\mathrm{Im}z_k \end{pmatrix} + o\left(z^{-1}\right). \tag{6.36}$$

Hence, the formula (6.35) can be found. □

In order to establish the relationship between $m^{(out)}(z)$ and $m^{\Delta}\left(z|\sigma_d^{\Delta}\right)$, we take $\Delta = \Delta_{z_0}^{-}$ and replace the scattering data σ_d^{Δ} with scattering data

$$\sigma_d^{(out)} = \left\{\left(z_k, \tilde{c}_{i,k}\right), \tilde{c}_{i,k} = c_{i,k}\delta(z_k)^2,\ i = 0, 1\right\}_{k=1}^{N}. \tag{6.37}$$

Notice that the conditions defining $m^{(out)}(z)$ are identical to those defining $m^{\Delta_{z_0}^{-}}\left(z|\sigma_d^{(out)}\right)$, we can draw a conclusion.

Corollary 1. *There exists a unique solution for the RHP5 as* $t \to \infty$. *Moreover,*

$$m^{(out)}(z) = m^{\Delta_{z_0}^{-}}\left(z|\sigma_d^{(out)}\right), \tag{6.38}$$

where the scattering data $\sigma_d^{(out)}$ *is given by (6.37). In addition, the corresponding N-soliton solution satisfies*

$$q_{sol}\left(x, t; \sigma_d^{(out)}\right) = q_{sol}\left(x, t; \sigma_d^{\Delta_{z_0}^{-}}\right), \quad t \to \infty.$$

Next, we consider the large z behavior of the above solutions.

Proposition 5. *Given discrete scattering data* $\sigma_d = \{(z_k, c_{i,k}), i = 0, 1, z_k \in \mathcal{Z}\}_{k=1}^{N}$, *pairs of points* y_1, y_2 *with* $y_1 \leq y_2 \in \mathbb{R}$ *and velocities* v_1, v_2 *with* $v_1 \leq v_2 \in \mathbb{R}$, *we define the cone*

$$\mathcal{S}(y_1, y_2, v_1, v_2) := \left\{(y, t) \in R^2 | y = y_0 + vt,\ y_0 \in [y_1, y_2],\ v \in [v_1, v_2]\right\}. \tag{6.39}$$

See Fig. 6.

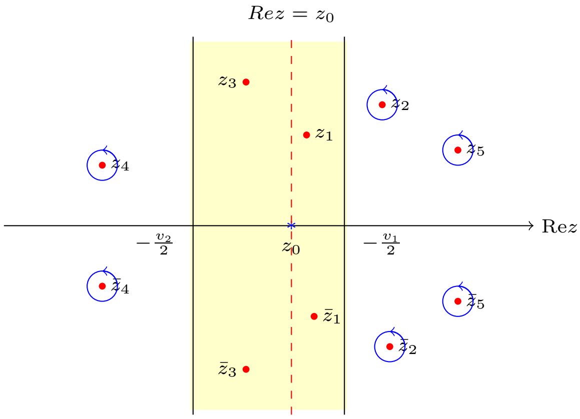

Fig. 7. Fix v_1 and v_2 so that $v_1 < v_2$. $I = [-v_2/2, -v_1/2]$. For example, the original data has five pairs of discrete spectrum, but inside cone $\mathcal{S}(x_1, x_2, v_1, v_2)$, the solution is asymptotically described by two double-pole solutions $q_{sol}(x, t; \sigma_d(I))$ with discrete spectrum in $\mathcal{Z}(I) = \{z_1, z_3\}$.

Take $I = [-v_2/2, -v_1/2]$. Then, when $t \to \infty$ with $(y, t) \in \mathcal{S}(y_1, y_2, v_1, v_2)$, we have

$$m^{\Delta^-_{z_0}}(z|\sigma_d) = \left(I + \mathcal{O}(e^{-2\mu|t|})\right) m^{\Delta^-_{z_0}(I)}\left(z; y, t|\sigma^-_d(I)\right) \tag{6.40}$$

where

$$\sigma^-_d(I) = \left\{(z_k, c_{i,k}(I)), i = 0, 1, z_k \in \Delta^-_{z_0}(I)\right\}^N_{k=1}, \tag{6.41}$$

$$\mu = \mu(I) = \min_{z_k \in \mathcal{Z} \setminus \mathcal{Z}(I)} \{\mathrm{Im}(z_k)\mathrm{dist}(\mathrm{Re}(z_k), I)\} = \min_{z_k \in \mathcal{Z} \setminus \mathcal{Z}(I), i=1,2} \{2\mathrm{Im}(z_k)|v_i - v_{z_k}|\}. \tag{6.42}$$

An example has been given in Fig. 7.

Proof. Let

$$\Delta^-(I) = \{k||z_k| < -v_2/2\}, \quad \Delta^+(I) = \{k||z_k| > -v_1/2\}.$$

For $t > 0$, $(x, t) \in \mathcal{S}(x_1, x_2, v_1, v_2)$, we have

$$-v_2/2 < z_0 + x_0/(2t) < -v_1/2,$$

and as $t \to \infty$, $x_0/(2t) \to 0$, $-v_2/2 < z_0 < -v_1/2$. By the residue condition and the coefficient of negative second power of Laurent expansion, it is easy to calculate that

$$\| n^{\Delta^\pm(I)}_{i,k} \| = \begin{cases} \mathcal{O}(1), & z_k \in \mathcal{Z}(I), \\ \mathcal{O}\left(e^{-4\mu|t|}\right), & z_k \in \mathcal{Z} \setminus \mathcal{Z}(I). \end{cases} \tag{6.43}$$

For each $z_k \in \mathcal{Z} \setminus \mathcal{Z}(I)$, we introduce small disks D_k whose radius are sufficiently small that they are non-overlapping. We define a function

$$\Upsilon(z) = \begin{cases} I - \frac{1}{z-z_k} n^{\Delta^-_{z_0}}_{0,k} - \frac{1}{(z-z_k)^2} n^{\Delta^-_{z_0}}_{2,k}, & z \in D_k, \\ I - \sigma_2 \left[\frac{1}{z-\bar{z}_k} \bar{n}^{\Delta^-_{z_0}}_{0,k} - \frac{1}{(z-\bar{z}_k)^2} \bar{n}^{\Delta^-_{z_0}}_{2,k}\right] \sigma_2, & z \in \bar{D}_k, \\ I, & \text{elsewhere}. \end{cases} \tag{6.44}$$

Then we introduce a transformation

$$\widehat{m}^{\Delta_{z_0}^-}\left(z|\sigma_d^{\Delta_{z_0}^-}\right)=m^{\Delta_{z_0}^-}\left(z|\sigma_d^{\Delta_{z_0}^-}\right)\Upsilon(z). \tag{6.45}$$

Furthermore, $\widehat{m}^{\Delta_{z_0}^-}\left(z|\sigma_d^{\Delta_{z_0}^-}\right)$ has jumps across each boundary of D_k and $\bar{D}_k$,

$$\widehat{m}_+^{\Delta_{z_0}^-}\left(z|\sigma_d^{\Delta_{z_0}^-}\right)=\widehat{m}_-^{\Delta_{z_0}^-}\left(z|\sigma_d^{\Delta_{z_0}^-}\right)\widehat{v}(z),\quad z\in\partial D_k\cup\partial\bar{D}_k, \tag{6.46}$$

with

$$\|\ \widehat{v}-I\ \|=\mathcal{O}\left(e^{-4\mu|t|}\right) \tag{6.47}$$

which can be given by the formula (6.43).

Take $\Delta=\Delta_{z_0}^-(I)$, then $m^{\Delta_{z_0}^-(I)}\left(z|\sigma_d^-(I)\right)$ has the same poles as $\widehat{m}^{\Delta_{z_0}^-}\left(z|\sigma_d^{\Delta_{z_0}^-}\right)$ with the same residue conditions and the coefficient of negative second power of Laurent expansion. Hence,

$$\varepsilon(z)=\widehat{m}^{\Delta_{z_0}^-}\left(z|\sigma_d^{\Delta_{z_0}^-}\right)\left[m^{\Delta_{z_0}^-(I)}\left(z|\sigma_d^-(I)\right)\right]^{-1} \tag{6.48}$$

has no poles but satisfies $\varepsilon_+(z)=\varepsilon_-(z)v_\varepsilon(z)$ with $\|\ v_\varepsilon-I\ \|=\mathcal{O}\left(e^{-4\mu|t|}\right)$. From the theory of small-norm Riemann-Hilbert problems, we know $\varepsilon(z)=I+\mathcal{O}\left(e^{-4\mu|t|}\right)$ as $|t|\to\infty$. Finally, by equations (6.45) and (6.48), we obtain

$$m^{\Delta_{z_0}^-}(z|\sigma_d)=\left(I+\mathcal{O}(e^{-2\mu|t|})\right)m^{\Delta_{z_0}^-(I)}\left(z;y,t|\sigma_d^-(I)\right).\quad\square \tag{6.49}$$

Therefore, we can obtain the following corollary.

Corollary 2. *Assume that* $q_{sol}\left(x,t;\sigma_d^{\Delta_{z_0}^-}\right)$ *is the* N*-soliton solution of the NLS equation with scattering data* $\sigma_d^{\Delta_{z_0}^-}=\left\{\left(z_k,c_{i,k}^{\Delta_{z_0}^-}\right),i=0,1,k\in\Delta_{z_0}^-\right\}_{k=1}^N$. *Then, as* $(x,t)\in\mathcal{S}(x_1,x_2,v_1,v_2)$, $t\to\infty$,

$$q_{sol}\left(x,t;\sigma_d^{(out)}\right)=q_{sol}\left(x,t;\sigma_d^{\Delta_{z_0}^-}\right)=q_{sol}\left(x,t;\sigma_d^-(I)\right)+\mathcal{O}\left(e^{-4\mu|t|}\right), \tag{6.50}$$

where $q_{sol}\left(x,t;\sigma_d^-(I)\right)$ *is the* $N(I)$*-soliton solution of the NLS equation with the scattering data* $\sigma_d^-(I)$.

6.1.2. The construction of local model

At the beginning of the construction of local model near the saddle point, we consider the jump matrix in the interior of the region.

Proposition 6.

$$\|\ v^{(2)}-I\ \|_{L^\infty(\Sigma^{(2)})}=\begin{cases}\mathcal{O}\left(|z-z_0|^{-1}t^{-1/2}\right), & z\in\Sigma^{(2)}\cap\epsilon_{z_0},\\ \mathcal{O}\left(e^{-t\rho^2/2}\right), & z\in\Sigma^{(2)}\setminus\mathcal{U}_{z_0}.\end{cases} \tag{6.51}$$

Proof. We prove the above proposition for the case $z \in \Sigma_1$, and other cases can be shown in a similar way. The jump line is $z - z_0 = |z - z_0|e^{i\pi/4}$ at Σ_1 and

$$\theta = (z - z_0)^2 - z_0^2 = i|z - z_0|^2 - z_0^2. \tag{6.52}$$

Using (5.37) and (5.40), we obtain

$$|R_1 e^{2it\theta(z)}| \leq \left(\frac{1}{2}c_1 + c_2 \langle \mathrm{Re}z\rangle^{-1/2}\right) e^{-2t|z-z_0|^2} \tag{6.53}$$

where $\langle \mathrm{Re}z\rangle^{-1/2} \leq c$. In the interior of $\mathcal{U}_{z_0}$, $m^{(2)}_{RHP}$ has no pole and

$$\| v^{(2)} - I \|_{L^\infty(\Sigma^{(2)})} \leq c|z - z_0|^{-1} t^{-1/2}, \quad z \in \Sigma^{(2)} \cap \epsilon_{z_0}. \tag{6.54}$$

Thus, it is clear that the jump $v^{(2)}$ is point-wise bounded, but not uniformly decayed to the identity matrix. Additionally, as $z \in \Sigma^{(2)} \cap \{|z - z_0| \geq \rho/2\}$,

$$\| v^{(2)} - I \|_{L^\infty(\Sigma^{(2)})} \leq ce^{-t\rho^2/2}. \quad \square \tag{6.55}$$

In order to achieve a uniformly small jump Riemann-Hilbert problem for the function $E(z)$ defined by (6.2), we establish a local model $m^{(z_0)}$ which matches $m^{(2)}_{RHP}$ on $\Sigma^{(2)} \cap \mathcal{U}_{z_0}$. For this reason, the translation scale transformation is defined by

$$\lambda = \lambda(z) = 2\sqrt{t}(z - z_0). \tag{6.56}$$

Notice that if we take $r_0 = r(z_0)T_0(z_0)^{-2}e^{2i(\nu(z_0)\log(2\sqrt{\mathrm{t}})-\mathrm{t}z_0^2)}$, the jump of $m^{(2)}_{RHP}$ is in accordance with that of the parabolic cylinder model problem $m^{(pc)}$, which satisfies the following RH problem, see more details in [17].

RHP9. Fix $r_0 \in \mathbb{R}$, find an analytic function $m^{(pc)}(\lambda, r_0)$ such that

(a) $m^{(pc)}(\lambda, r_0)$ is analytic in $\mathbb{C} \setminus \Sigma^{(2)}$.
(b) $m^{(pc)}(\lambda, r_0)$ has continuous boundary value $m^{(pc)}_{\pm}(\lambda, r_0)$ on $\Sigma^{(2)}$

$$m^{(pc)}_{+}(\lambda, r_0) = m^{(pc)}_{-}(\lambda, r_0) v^{(pc)}(\lambda, r_0), \quad \zeta \in \Sigma^{(2)}, \tag{6.57}$$

where

$$v^{(pc)}(\lambda, r_0) = \begin{cases} \begin{pmatrix} 1 & 0 \\ r_0 \lambda^{-2i\nu} e^{i\lambda^2/2} & 1 \end{pmatrix}, & \lambda = \Sigma_1, \\ \begin{pmatrix} 1 & \frac{\bar{r}_0}{1+|r_0|^2}\lambda^{2i\nu} e^{-i\lambda^2/2} \\ 0 & 1 \end{pmatrix}, & \lambda = \Sigma_2, \\ \begin{pmatrix} 1 & 0 \\ \frac{r_0}{1+|r_0|^2}\lambda^{-2i\nu} e^{i\lambda^2/2} & 1 \end{pmatrix}, & \lambda = \Sigma_3, \\ \begin{pmatrix} 1 & \bar{r}_0 \lambda^{2i\nu} e^{-i\lambda^2/2} \\ 0 & 1 \end{pmatrix}, & \lambda = \Sigma_4. \end{cases} \tag{6.58}$$

See Fig. 8.

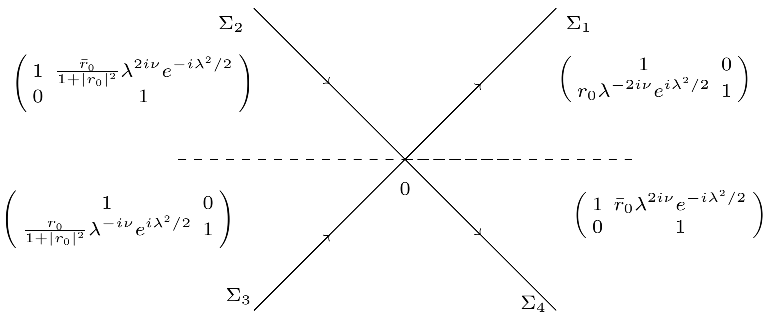

Fig. 8. Jump matrix $v^{(pc)}$.

(c) As $\lambda \to \infty$, $m^{(pc)}(\lambda, r_0) = I + \frac{m_1^{(pc)}(r_0)}{\lambda} + \mathcal{O}(\lambda^{-2})$.

Moreover, the asymptotic behavior of $m^{(pc)}(\lambda, r_0)$ has been verified in the paper [38], which is

$$m^{(pc)}(\lambda, r_0) = I + \frac{1}{\lambda}\begin{pmatrix} 0 & -i\beta_{12}(r_0) \\ i\beta_{21}(r_0) & 0 \end{pmatrix} + \mathcal{O}\left(\lambda^{-2}\right), \tag{6.59}$$

where

$$\beta_{12}(r_0) = \frac{\sqrt{2\pi}e^{i\pi/4}e^{-\pi\nu/2}}{r_0\Gamma(-i\nu)}, \quad \beta_{21}(r_0) = \frac{\sqrt{2\pi}e^{-i\pi/4}e^{-\pi\nu/2}}{\bar{r}_0\Gamma(i\nu)} = \frac{\nu}{\beta_{12}(r_0)}. \tag{6.60}$$

Therefore, it is convenient to define $m^{(z_0)}(z)$, which is given by (6.2), as follows

$$m^{(z_0)}(z) = m^{(out)}(z)m^{(pc)}(\lambda, r_0). \tag{6.61}$$

Further, $m^{(z_0)}(z)$ is a bounded function in $\mathcal{U}_{z_0}$ and fulfills the jump condition $v^{(2)}(z)$ of $m^{(2)}_{RHP}(z)$.

6.2. The small-norm RH problem for $E(z)$

In this section, we deal with the error function $E(z)$. By the definition (6.38) and (6.61), it is obvious that $E(z)$ meets the RH problem as below.

RHP10. Find a holomorphic function $E(z)$ such that

(a) $E(z)$ is analytical in $\mathbb{C} \setminus \Sigma^{(E)}$, where $\Sigma^{(E)} = \partial\mathcal{U}_{z_0} \cup \left(\Sigma^{(2)} \setminus \mathcal{U}_{z_0}\right)$, see Fig. 9.
(b) For $z \in \Sigma^{(E)}$, $E(z)$ has continuous boundary values $E_\pm(z)$ which satisfy

$$E_+(z) = E_-(z)v^{(E)}(z),$$

where

$$v^{(E)}(z) = \begin{cases} m^{(out)}(z)v^{(2)}(z)m^{(out)}(z)^{-1}, & z \in \Sigma^{(2)} \setminus \mathcal{U}_{z_0}, \\ m^{(out)}(z)m^{(pc)}\left(\lambda, r_0\right)m^{(out)}(z)^{-1}, & z \in \partial\mathcal{U}_{z_0}. \end{cases} \tag{6.62}$$

(c) For $z \in \infty$, $E(z) = I + \mathcal{O}\left(z^{-1}\right)$.

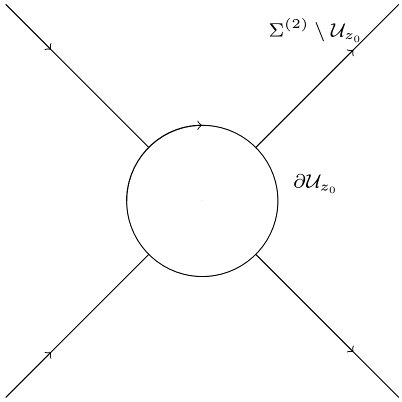

Fig. 9. Jump contour $\Sigma^{(E)}$ for error function $E(z)$.

Utilizing Proposition 6, the formula (6.62) and the boundedness of $m^{(out)}(z)$, we can obtain

$$\left|v^{(E)}(z) - I\right| = \begin{cases} \mathcal{O}\left(e^{-t\rho^2/2}\right), & z \in \Sigma^{(2)} \setminus \mathcal{U}_{z_0}, \\ \mathcal{O}\left(t^{-1/2}\right), & z \in \partial\mathcal{U}_{z_0}. \end{cases} \tag{6.63}$$

Proposition 7. *RHP10 has a unique solution.*

Proof. According to Beal-Cofiman theorem, the solution of RHP10 can be constructed by

$$E(z) = I + \frac{1}{2\pi i} \int\limits_{\Sigma^{(E)}} \frac{\mu_E(s)\left(v^{(E)}(s) - I\right)}{s - z} ds, \tag{6.64}$$

where $\mu_E \in L^2\left(\Sigma^{(E)}\right)$ satisfies

$$(1 - C_{w_E})\,\mu_E = I \tag{6.65}$$

with C_{w_E} being a integral operator defined by

$$C_{w_E}(f)(z) = C_-\left(f\left(v^{(E)}(z) - I\right)\right), \tag{6.66}$$

where C_- is the Cauchy projection operator

$$C_-(f)(s) = \lim_{z' \to z \in \Sigma^{(E)}} \frac{1}{2\pi i} \int\limits_{\Sigma^{(E)}} \frac{f(s)}{s - z'} ds. \tag{6.67}$$

Using the above formulas (6.66) and (6.67), we get

$$\|C_{w_E}\|_{L^2(\Sigma^{(E)})} \leq \|C_-\|_{L^2(\Sigma^{(E)})} \left\|v^{(E)} - I\right\|_{L^\infty(\Sigma^{(E)})} \lesssim \mathcal{O}\left(t^{-1/2}\right). \tag{6.68}$$

This means that $1 - C_{w_E}$ is invertible. Subsequently, μ_E and the solution of the RHP10 exist and are unique. □

By (6.63) and the process of proof of Proposition 7, it is straightforward to obtain the following corollary.

Corollary 3.

$$\left\|\langle\cdot\rangle\left(v^{(E)}-I\right)\right\|_{L^p}=\mathcal{O}\left(t^{-1/2}\right),\quad p\in[1,\infty),\ k\geq 0,\tag{6.69}$$

$$\|\mu_E-I\|_{L^2(\Sigma^{(E)})}=\mathcal{O}\left(t^{-1/2}\right).\tag{6.70}$$

Next, we consider the asymptotic expansion of $E(z)$

$$E(z)=I+\frac{E_1}{z}+\mathcal{O}\left(z^{-2}\right),\tag{6.71}$$

where

$$E_1=-\frac{1}{2\pi i}\int\limits_{\Sigma^{(E)}}\mu_E(s)\left(v^{(E)}-I\right)ds.\tag{6.72}$$

Proposition 8.

$$E_1(x,t)=\frac{1}{2i\sqrt{t}}m^{(out)}(z_0)m_1^{(pc)}(r_0)m^{(out)}(z_0)^{-1}+\mathcal{O}\left(t^{-1}\right).\tag{6.73}$$

Proof. Rewrite formula (6.72) as

$$E_1=-\frac{1}{2\pi i}\oint\limits_{\partial\mathcal{U}_{z_0}}\left(v^{(E)}-I\right)ds--\frac{1}{2\pi i}\int\limits_{\Sigma^{(E)}\setminus\mathcal{U}_{z_0}}\left(v^{(E)}-I\right)ds-\frac{1}{2\pi i}\int\limits_{\Sigma^{(E)}}(\mu_E(s)-I)\left(v^{(E)}-I\right)ds.\tag{6.74}$$

Then, the conclusion can be obtained by estimating the three integrals respectively. □

6.3. The pure $\bar\partial$-problem and its asymptotic behaviors

In this section, we acquire a pure $\bar\partial$-problem after removing the $\bar\partial$ component of $m^{(2)}$. We define

$$m^{(3)}(z):=m^{(2)}(z)m^{(2)}_{RHP}(z)^{-1}.\tag{6.75}$$

Then, $m^{(3)}(z)$ is continuous and has no jump in $\mathbb{C}$, which satisfies a pure $\bar\partial$-problem.

RHP11. Find a matrix-valued function $m^{(3)}(z)$ with the following properties.

(a) $m^{(3)}(z)$ is continuous in $\mathbb{C}\setminus\left(\mathbb{R}\cup\Sigma^{(2)}\right)$;
(b) For $z\in\mathbb{C}$, $\bar\partial m^{(3)}(z)=m^{(3)}(z)w^{(3)}(z)$ where

$$w^{(3)}(z)=m^{(2)}_{RHP}(z)\bar\partial\mathcal{R}^{(2)}m^{(2)}_{RHP}(z)^{-1};\tag{6.76}$$

(c) For $z\in\infty$, $m^{(3)}(z)=I+\mathcal{O}\left(z^{-1}\right)$.

Proof. By the definition of $m^{(3)}(z)$, it is obvious that $m^{(3}(z)$ is continuous in $\mathbb{C}\setminus\Sigma^{(2)}$ and satisfies the condition (b). At the same time, using formulas (5.40) and (6.75), we can find that

$$m^{(3)}_+(z)=m^{(3)}_-(z),\quad z\in\Sigma^{(2)}.$$

From the property (e) of RHP3, we prove the residue condition for the case $k \in \Delta^+_{z_0}$, $\mathrm{Im} z_k > 0$, because the proofs for the other cases are similar.

$$\operatorname*{Res}_{z=z_k} m^{(2)}(z) = \left(\gamma^{(2)}_{0,k} m^{(2)}_2(z_k), 0\right) + \left(\gamma^{(2)}_{1,k}\left(m^{(2)}_2\right)'(z_k), 0\right) = \lim_{z\to z_k}\left(m^{(2)} n^{(2)}_{0,k} + \left(m^{(2)}\right)' n^{(2)}_{1,k}\right), \quad (6.77)$$

$$\operatorname*{P_{-2}}_{z=z_k} m^{(2)}(z) = \left(\gamma^{(2)}_{1,k} m^{(2)}_2(z_k), 0\right) = \lim_{z\to z_k}\left(m^{(2)} n^{(2)}_{1,k}\right), \quad (6.78)$$

where

$$\gamma^{(2)}_{0,k} = A_k T^{-2}(z_k)\left[B_k + 2it\theta'(z_k) - \frac{2T'(z_k)}{T(z_k)}\right] e^{2it\theta(z_k)}, \quad (6.79)$$

$$\gamma^{(2)}_{1,k} = A_k T^{-2}(z_k) e^{2it\theta(z_k)}, \quad (6.80)$$

$$n^{(2)}_{0,k} = \begin{pmatrix} 0 & 0 \\ \gamma^{(2)}_{0,k} & 0 \end{pmatrix}, \quad n^{(2)}_{1,k} = \begin{pmatrix} 0 & 0 \\ \gamma^{(2)}_{1,k} & 0 \end{pmatrix}. \quad (6.81)$$

Since $\left(n^{(2)}_{i,k}\right)^2 = 0$ $(i = 0, 1)$, $n^{(2)}_{i,k}$ is the nilpotent matrix.

As z_k is the second-order pole of $m^{(2)}(z)$, $m^{(2)}(z)$ has the Laurent expansion with the following form

$$m^{(2)}(z) = \frac{\operatorname*{P_{-2}}_{z=z_k} m(z|\sigma_d)}{(z-z_k)^2} + \frac{\operatorname*{Res}_{z=z_k} m(z|\sigma_d)}{z-z_k} + a(z_k) + b(z_k)(z-z_k) + \mathcal{O}(z-z_k)^2. \quad (6.82)$$

Substituting (6.82) into (6.77) and (6.78) respectively, we acquire

$$\operatorname*{Res}_{z=z_k} m^{(2)}(z) = a(z_k) n^{(2)}_{0,k} + b(z_k) n^{(2)}_{1,k}, \quad (6.83)$$

$$\operatorname*{P_{-2}}_{z=z_k} m^{(2)}(z) = a(z_k) n^{(2)}_{1,k}. \quad (6.84)$$

And then, bring (6.83) and (6.84) back to (6.82), we have the Laurent expansion

$$m^{(2)}(z) = a(z_k)\left[I + \frac{n^{(2)}_{0,k}}{z-z_k} + \frac{n^{(2)}_{1,k}}{(z-z_k)^2}\right] + b(z_k)\frac{n^{(2)}_{1,k}}{z-z_k} + b(z_k)(z-z_k) + \mathcal{O}(z-z_k)^2. \quad (6.85)$$

Notice that $m^{(2)}$ and $m^{(2)}_{RHP}$ have the same residue relations and $\det m^{(2)}(z) = \det m^{(2)}_{RHP}(z) = 1$, it can be calculated directly

$$\left(m^{(2)}_{RHP}\right)^{-1}(z) = \left[I - \frac{n^{(2)}_{0,k}}{z-z_k} - \frac{n^{(2)}_{1,k}}{(z-z_k)^2}\right]\sigma_2 a(z_k)^T\sigma_2 + \left(I - \frac{n^{(2)}_{1,k}}{(z-z_k)^2}\right)\sigma_2 b(z_k)^T\sigma_2\,(z-z_k) + \mathcal{O}(z-z_k)^2. \quad (6.86)$$

Then,

$$m^{(2)}(z)\left(m^{(2)}_{RHP}\right)^{-1}(z) = \mathcal{O}(1), \quad (6.87)$$

in which we have used the property of nilpotent matrix $n^{(2)}_{i,k}$ $(i = 0, 1)$. Hence, $m^{(3)}(z)$ has only removable singularities at each z_k.

The property (b) can be obtained by $\bar\partial m^{(2)}_{RHP} = 0$. □

The solution of this RHP11 is constructed by the following integral equation

$$m^{(3)}(z) = I - \frac{1}{\pi} \iint_{\mathbb{C}} \frac{m^{(3)}(s)w^{(3)}(s)}{s-z} dA(s), \tag{6.88}$$

where $dA(s)$ is the Lebesgue measure on $\mathbb{C}$. Meanwhile, the equation (6.88) can also be represented by operators, which is

$$(I - C)m^{(3)}(z) = I, \tag{6.89}$$

where C is the Cauchy-Green integral operator,

$$C[f](z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(s)w^{(3)}(s)}{s-z} dA(s). \tag{6.90}$$

In addition, this operator C admits the following estimation.

Proposition 9. *For $t \to \infty$,*

$$\| C \|_{L^\infty \to L^\infty} \lesssim |t|^{-1/4}, \tag{6.91}$$

which implies that $(I - C)^{-1}$ exists.

As $z \to \infty$, we consider the asymptotic expansion of $m^{(3)}(z)$

$$m^{(3)}(z) = I + \frac{m_1^{(3)}}{z} + \frac{1}{\pi} \iint_{\mathbb{C}} \frac{s m^{(3)}(s)w^{(3)}(s)}{z(s-z)} dA(s), \tag{6.92}$$

where

$$m_1^{(3)} = \frac{1}{\pi} \iint_{\mathbb{C}} m^{(3)}(s)w^{(3)}(s) dA(s). \tag{6.93}$$

To reconstruct the solution $q(x,t)$ of the fNLS equation with double poles, we need to determine the long time asymptotic behavior of $m_1^{(3)}$. We can testify the following property of $m_1^{(3)}$.

Proposition 10. *There is a constant c such that*

$$|m_1^{(3)}| \leq ct^{-3/4}. \tag{6.94}$$

Proof. The proofs of Propositions 9 and 10 are similar to the proof of Proposition 6.1 and Appendix D in [7] respectively. □

7. Long time asymptotics for fNLS equation

Now we begin to seek the long time asymptotic behavior of the solution for the fNLS equation.

Theorem 1. *Take $q_0(x) \in H^{1,1}(\mathbb{R})$ and suppose that the corresponding scattering data is $\sigma_d = \{(z_k, c_{i,k}), i = 0, 1, z_k \in \mathcal{Z}\}_{k=1}^{N}$ with z_k is double zeros of the scattering coefficient $s_{11}(z)$. Fix $x_1, x_2, v_1, v_2 \in \mathbb{R}$ with $x_1 \le x_2, v_1 \le v_2$. Take $I = [-v_2/2, -v_1/2]$ and $z_0 = -x/(2t)$. Take $q_{sol}\left(x, t; \sigma_d^-(I)\right)$ be the $N(I)$ soliton corresponding to the scattering data*

$$\sigma_d^-(I) = \left\{\left(z_k, c_{i,k}(I)\right), i = 0, 1, z_k \in \Delta_{z_0}^-(I)\right\}_{k=1}^{N}.$$

Then as $t \to \infty$ with $(x, t) \in \mathcal{S}(x_1, x_2, v_1, v_2)$, we have

$$q(x,t) = q_{sol}\left(x, t; \sigma_d^-(I)\right) + t^{-1/2} f + \mathcal{O}\left(t^{-3/4}\right), \tag{7.1}$$

where

$$f = (\eta_{11})^2\, \alpha(z_0) e^{i\left(x^2/(2t) - \nu(z_0)\log|4t|\right)} + (\eta_{12})^2 \overline{\alpha(z_0)} e^{-i\left(x^2/(2t) - \nu(z_0)\log|4t|\right)}, \tag{7.2}$$

with $|\alpha(z_0)|^2 = |v(z_0)|$, $v(z_0) = -\frac{1}{2\pi}\log(1 + |r(z_0)|^2)$ and

$$\arg\alpha(z_0) = \frac{\pi}{4} + \arg\Gamma\left(i\nu(z_0)\right) - \arg\gamma(z_0) - 4\sum_{k\in\Delta_{z_0}^-} \arg(z_0 - z_k) - 2\int_{-\infty}^{z_0} \ln|s - z_0| d\ln\left(1 + |r(s)|^2\right), \tag{7.3}$$

where η_{11}, η_{12} is the elements in the first row of $m^{\triangle_{z_0}^-(I)}\left(z; y, t|\sigma_d^-(I)\right)$.

Proof. Reviewing a series of transformations we have made in the process of solving the initial problem (1.1)-(1.2), which are (5.10), (5.31), (6.2) and (6.75), and backward pushing these transformation processes gives us

$$m(z) = m^{(3)}(z) E(z) m^{(out)} \left(\mathcal{R}^{(2)}(z)\right)^{-1} T^{\sigma_3}, \quad z \in \mathbb{C} \setminus \mathcal{U}_{z_0}. \tag{7.4}$$

In particular, we consider cases where z tends to infinity in the vertical direction of $z \in \Omega_2$ or Ω_5. In these cases, we have $\mathcal{R}^{(2)} = I$ and

$$m(z) = \left(I + \frac{m_1^{(3)}}{z} + \mathcal{O}\left(\frac{1}{z^2}\right)\right)\left(I + \frac{E_1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right)\right)\left(I + \frac{m_1^{(out)}}{z} + \mathcal{O}\left(\frac{1}{z^2}\right)\right)\left(I + \frac{T_1\sigma_3}{z} + \mathcal{O}(\frac{1}{z^2})\right). \tag{7.5}$$

After that, we get

$$m_1 = m_1^{(out)} + E_1 + m_1^{(3)} + T_1\sigma_3, \tag{7.6}$$

where m_1 is the coefficient of the z^{-1} in the Laurent expansion of m. Meanwhile, the equation (4.8) and Proposition 10 tell us that

$$q(x,t) = 2i\left(\left(m_1^{(out)}\right)_{12} + (E_1)_{12}\right) + \mathcal{O}\left(t^{-3/4}\right). \tag{7.7}$$

Let $m^{(out)} = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}$. Using Proposition 8, we have

$$(E_1)_{12} = \frac{1}{2i\sqrt{t}}\left(\beta_{12}(\eta_{11})^2 + \beta_{21}(\eta_{12})^2\right), \tag{7.8}$$

where

$$\beta_{12}(z_0) = \overline{\beta_{21}(z_0)} = \alpha(z_0) e^{i\left(x^2/(2t) - \nu(z_0)\log|4t|\right)}. \tag{7.9}$$

Bringing $2i\left(m_1^{(out)}\right)_{12} = q_{sol}\left(x, t; \sigma_d^{(out)}\right)$ and (7.8) back to (7.7), we find

$$q(x,t) = q_{sol}\left(x, t; \sigma_d^{(out)}\right) + t^{-1/2} f + \mathcal{O}\left(t^{-3/4}\right), \tag{7.10}$$

where f is given by (7.2). In addition, using Corollary 2, we obtain

$$q(x,t) = q_{sol}\left(x, t; \sigma_d^{-}(I)\right) + t^{-1/2} f + \mathcal{O}\left(t^{-3/4}\right), \tag{7.11}$$

where $(x,t) \in \mathcal{S}(x_1, x_2, v_1, v_2)$. □

Remark 2. Though the asymptotic result (7.1) has the same form with that obtained in [7], they have different meanings. For example, the first term $q_{sol}\left(x, t; \sigma_d^{-}(I)\right)$ demonstrates high-order pole solutions, while it denotes simple pole solutions in [7]; The second term $t^{-1/2} f$ is an interaction between high-order pole solutions and the dispersion term, while it denotes an interaction between simple pole solutions and the dispersion term in [7].

Remark 3. The asymptotic result (7.1) shows that the initial value problem of the fNLS equation with zero-boundary and double poles in scattering coefficient has the property of the soliton resolution, which is as $t \to \infty$, any solution of the fNLS equation can be decomposed into solitary wave part and dispersion part. Linear NLS equation $iq_t + q_{xx}/2$ is dispersive and any solution of this linear equation has the estimation $\| q \|_{L^\infty} \sim t^{-1/2}$. Therefore, the second term in the formula (7.1), which includes the $t^{-1/2}$, represents the contribution of the dispersion term. The multiple solitary wave solutions $q_{sol}\left(x, t; \sigma_d^{-}(I)\right)$ corresponding to the scattering data, which is superposed by a finite single soliton solution, appear in the long time asymptotic expansion when the NLS equation includes a nonlinear term $|q|^2 q$.

Remark 4. After modification on the residue conditions (4.3)-(4.6), we can show that the solutions of the Cauchy problem of the fNLS equation with high-order pole spectrum data still possess the property of soliton resolution like Theorem 1.

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References

[1] M. Ablowitz, D. Kaup, A. Newell, H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, Stud. Appl. Math. 53 (4) (1974) 249–315.
[2] G.P. Agrawal, Nonlinear Fiber Optics, Academic Press, New York, 2001.
[3] T. Aktosun, F. Demontis, C. VanDerMee, Exact solutions to the focusing nonlinear Schrödinger equation, Inverse Probl. 23 (2007) 2171–2195.
[4] R.F. Bikbaev, Complex Whitham deformations in problems with "integrable instability", Teor. Mat. Fiz. 104 (1995) 393–419 (Russian); English transl., Theor. Math. Phys. 104 (1996) 1078–1097.
[5] G. Biondini, Long-time asymptotics for the focusing nonlinear Schrodinger equation with nonzero boundary conditions at infinity and asymptotic stage of modulational instability, Commun. Pure Appl. Math. LXX (2017) 2300–2365.
[6] G. Biondini, S.T. Li, D. Mantzavinos, Long-time asymptotics for the focusing nonlinear Schrodinger equation with nonzero boundary conditions in the presence of a discrete spectrum, Commun. Math. Phys. 382 (2021) 1495–1577.

[7] M. Borghese, R. Jenkins, K.T.R. McLaughlin, Long-time asymptotic behavior of the focusing nonlinear Schrödinger equation, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 35 (2018) 887–920.
[8] J. Bourgain, Global Solutions of Nonlinear Schrödinger Equations, American Mathematical Society Colloquium Publications, vol. 46, American Mathematical Society, Providence, R.I., 1999.
[9] S.Y. Chen, Z.Y. Yan, B.L. Guo, Long-time asymptotics for the focusing Hirota equation with nonzero boundary conditions at infinity via the Deift-Zhou approach, Math. Phys. Anal. Geom. 24 (2021) 17, 37 pp.
[10] S. Cuccagna, R. Jenkins, On asymptotic stability of N-solitons of the defocusing nonlinear Schrödinger equation, Commun. Math. Phys. 343 (2016) 921–969.
[11] A.B. de Monvel, A. Its, V. Kotlyarov, Long-time asymptotics for the focusing NLS equation with time-periodic boundary condition on the half-line, Commun. Math. Phys. 290 (2009) 479–522.
[12] A.B. de Monvel, V. Kotlyarov, D. Shepelsky, Focusing NLS equation: long-time dynamics of step-like initial data, Int. Math. Res. Not. 2011 (2010) 1613–1653.
[13] A.B. de Monvel, J. Lenells, D. Shepelsky, The focusing NLS equation with step-like oscillating background: scenarios of long-time asymptotics, Commun. Math. Phys. 383 (2021) 893–952.
[14] P. Deift, X. Zhou, Long-Time Behavior of the Non-focusing Nonlinear Schrödinger Equation: A Case Study, Lectures in Mathematical Sciences, Graduate School of Mathematical Sciences, University of Tokyo, 1994.
[15] J. Garnier, K. Kalimeris, Inverse scattering perturbation theory for the nonlinear Schrödinger equation with non-vanishing background, J. Phys. A, Math. Theor. 45 (2012) 035202, 13 pp.
[16] E. Infeld, G. Rowlands, Nonlinear Waves, Solitons and Chaos, Cambridge University Press, Cambridge, 2000.
[17] A. Its, Asymptotic behavior of the solutions to the nonlinear Schrödinger equation, and isomonodromic deformations of systems of linear differential equations, Dokl. Akad. Nauk SSSR 261 (1981) 14–18.
[18] A.R. Its, V.P. Kotljarov, Explicit formulas for solutions of a nonlinear Schrödinger equation, Dokl. Akad. Nauk Ukr. SSR, Ser. A. Fiz.-Mat. Teh. Nauki (11) (1976) 965–968, p. 1051.
[19] R. Jenkins, J. Liu, P. Perry, C. Sulem, Soliton resolution for the derivative nonlinear Schrödinger equation, Commun. Math. Phys. 363 (2018) 1003–1049.
[20] S. Kamvissis, Long time behavior for the focusing nonlinear Schrodinger equation with real spectral singularities, Commun. Math. Phys. 180 (1996) 325–341.
[21] T. Kawata, H. Inoue, Inverse scattering method for the nonlinear evolution equations under nonvanishing conditions, J. Phys. Soc. Jpn. 44 (1978) 1722–1729.
[22] Y.C. Ma, The perturbed plane wave solutions of the cubic Schrödinger equation, Stud. Appl. Math. 60 (1979) 43–58.
[23] K.T.R. McLaughlin, P.D. Miller, The $\bar{\partial}$ steepest descent method and the asymptotic behavior of polynomials orthogonal on the unit circle with fixed and exponentially varying non-analytic weights, Int. Math. Res. Not. (2006) 48673.
[24] K.T.R. McLaughlin, P.D. Miller, The $\bar{\partial}$ steepest descent method for orthogonal polynomials on the real line with varying weights, Int. Math. Res. Not. (2008) 075.
[25] M. Pichler, G. Biondini, On the focusing non-linear Schrödinger equation with non-zero boundary conditions and double poles, IMA J. Appl. Math. 82 (2017) 131–151.
[26] L.P. Pitaevskii, S. Stringari, Bose-Einstein Condensation, Clarendon Press, Oxford, 2004.
[27] C. Schiebold, Asymptotics for the multiple pole solutions of the nonlinear Schrödinger equation, Nonlinearity 30 (2017) 2930–2981.
[28] C. Schiebold, Asymptotics for the multiple pole solutions of the nonlinear Schrödinger equation, Nonlinearity 30 (2017) 2930–2981.
[29] Y. Tsutsumi, $L2$-solutions for nonlinear Schrödinger equations and nonlinear groups, Funkc. Ekvacioj 30 (1987) 115–125.
[30] W.F. Weng, Z.Y. Yan, Inverse scattering and N-triple-pole soliton and breather solutions of the focusing nonlinear Schrödinger hierarchy with nonzero boundary conditions, Phys. Lett. A 407 (2021) 127472, 18 pp.
[31] J. Xu, Long-time asymptotics for the short pulse equation, J. Differ. Equ. 265 (2018) 3494–3532.
[32] J. Xu, E.G. Fan, Y. Chen, Long-time asymptotic for the derivative nonlinear Schrödinger equation with step-like initial value, Math. Phys. Anal. Geom. 16 (2013) 253–288.
[33] Y.L. Yang, E.G. Fan, Soliton resolution for the short-pulse equation, J. Differ. Equ. 280 (2021) 644–689.
[34] V.E. Zahkarov, A.B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP 34 (1) (1972) 62–69.
[35] V.E. Zahkarov, A.B. Shabat, Interaction between solitons in a stable medium, Sov. Phys. JETP 37 (5) (1973) 823–828.
[36] V.E. Zakharov, S.V. Manakov, Asymptotic behavior of nonlinear wave systems integrated by the inverse scattering method, Sov. Phys. JETP 44 (1976) 106–112.
[37] Y.S. Zhang, X.X. Tao, T.T. Yao, The regularity of the multiple higher-order poles solitons of the NLS equation, Stud. Appl. Math. 145 (4) (2020) 812–827.
[38] X. Zhou, P. Deift, A steepest descent method for oscillatory Riemann-Hilbert problems, Ann. Math. 137 (1993) 295–368.