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Technical Note

Exact solutions of the Hairsine–Rose precipitation-driven erosion model for a uniform grain-sized soil

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SUMMARY

Hairsine and Rose developed a mechanistic, one-dimensional, precipitation-driven erosion model that, since its appearance, has been validated by several sets of experimental results. The model allows any sediment particle to be present in one of three zones, viz., the flow zone, the deposited layer, or the original soil. The model has the general form of a two-region model, in which advection is the only transport process. For the special case of a soil composed of a single particle size and for overland flow that occurs at a steady rate and with a uniform depth, it is possible to derive fully explicit analytical solutions to the model. Details of the solutions for a slightly generalized mathematical form of the model are provided. The Goldstein *J* function, which appears commonly in two-region model solutions, was modified to accommodate some of the solutions presented. The form of the model analyzed indicated that, based only on sediment concentrations in runoff water, it is not possible to distinguish one mechanistic feature of the Hairsine–Rose model, i.e., that raindrop-induced detachment of the undisturbed soil moves directly into the flowing water. From the point of view of the model, it is equally plausible for raindrop impact to move sediment directly into the deposited layer.

1. Introduction

Hairsine and Rose (1991) presented a model, hereafter termed the HR model, for erosion where the only mechanism causing detachment of soil particles from the bed is impact by raindrops. Their model further developed the original theory of Rose et al. (1983a), which was successfully applied to sediment discharge data from the Walnut Gulch experimental watershed by Rose et al. (1983b). An essential development leading to the HR model was the incorporation of a mechanistic description of the shielding effect of eroded soil particles that settle out of the flow and form a deposited layer on top of the original soil surface. Erosion of the original soil is then moderated by the existence of the deposited layer, or shield, since its presence requires the removal of this sediment before any of the original soil can be accessed. Consequently, the energy of the raindrop impact is then partitioned between eroding both the shield and, depending on the shield thickness, the original soil.

As the ability of raindrop impact to cause erosion decreases with the overland flow depth and because flow-driven erosion mechanisms are neglected, the HR model only applies to shallow flows that are below the threshold streampower for sediment entrainment. Hairsine and Rose (1991) considered soil particles to be present in one of three locations, viz., in the original soil layer, in the deposited layer or in the water. The particles are motionless in each of the two possible soil layers, and are advected when in the water. Raindrop impact provides the only means of dislodging particles. The model is presented and described further in Section 2.

Several analytical studies and experimental analyses of the HR model have appeared. In their original paper, Hairsine and Rose (1991) provided the steady-state solution for the suspended sediment concentration under conditions of a constant excess rainfall rate. They assumed that the kinematic approximation to overland

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Nomenclature

a	ν/D T^{-1}
a_o	coefficient of detachability of the original soil ML^{-3}
a_d	coefficient of detachability of the deposited soil ML^{-3}
b	$P(a_d - a_o)/M_{dT}$ T^{-1}
c	ν/D T^{-1}
d	Pa_d/M_{dT} T^{-1}
D	overland flow depth (constant) L
f	Pa_o $ML^{-2}T^{-1}$
$\mathscr{F}$	defined by Eq. (27)
g	Pa_o $ML^{-2}T^{-1}$
H	heaviside function
HR	model of Hairsine and Rose (1991)
I_n	modified Bessel function of the first kind of order n
J	Goldstein J function
J_{mod}	modified Goldstein J function
L	flume length L
$\mathscr{L}^{-1}$	inverse Laplace transform operator
M_d	mass per unit area of sediment in the deposited layer ML^{-2}
M_{dT}	mass of redeposited soil per unit area sufficient to prevent erosion of the original soil ML^{-2}
M_s	mass per unit area of sediment in the water column ML^{-2}
p	M_s ML^{-2}
P	precipitation LT^{-1}
q	M_d ML^{-2}
q_f	volumetric water flux per unit width (constant) L^2T^{-1}
s	Laplace transform variable T^{-1}
sgn	sign function
t	time T
u	q_f/D LT^{-1}
w	several definitions, used in convolution integral solutions
x	position L
Greek	
α	defined by Eq. (18)
α_1	defined by Eq. (25)
α_2	defined by Eq. (25)
β	$\sqrt{bc\frac{x}{u}\left(t-\frac{x}{u}\right)}$
δ	dirac delta function
ν	particle setting velocity LT^{-1}

flow applied and used a steady-state water flux which increased linearly with position along the flow path. A form of the steady-state solution (assuming a uniform overland flow depth) was applied by Proffitt et al. (1991) in order to deduce model parameters. Sander et al. (1996) assumed that the flow depth and downgradient water flux were both constant and that spatial variations were negligible in comparison to temporal variations, i.e., they dropped the spatial derivative in the model. Their analytical approximation was able to reproduce the experimental data of Proffitt et al. (1991). The solution of Sander et al. (1996) involved a numerical element in that the problem was converted to a system of ordinary differential equations, which was solvable analytically, but required the numerical calculation of eigenvalues and eigenvectors. Under the same assumptions as Sander et al. (1996), Parlange et al. (1999) derived approximations for short and long time behavior of the model, which were both straightforward to compute and in good agreement with numerical simulations. Hairsine et al. (1999) extended the approach of Hairsine and Rose (1991) and provided an event-based (i.e., no spatial dependence) description of sediment sorting due to the erosion process. The HR model assumes that rainfall detachment of sediment particles is not particle-size selective. However, the model predicts that sorting occurs due to finer sediments settling out of the water column more slowly than coarse sediments and hence are transported further, although at steady-state the settling velocity distribution of the deposited and original soil were predicted to be identical (Hairsine and Rose, 1991). Hogarth et al. (2004a) presented an asymptotic space–time approximation motivated by a Laplace transform-based expansion that is increasingly valid for larger times. Their approximation was shown to compare well with the accurate numerical solutions of Hogarth et al. (2004b). Hogarth et al. (2004b) also clearly demonstrated the important role played by particle settling velocities in model prediction. Laboratory, i.e., small scale, experiments validating the HR model have been reported (e.g., Heilig et al., 2001; Gao et al., 2003), with good agreement found. Tromp-van Meerveld et al. (2008) modified slightly the analytical approximations of Parlange et al. (1999) to account for the effects of infiltration on deposition rates and analyzed data sets collected using the EPFL erosion flume. This brief survey shows that the HR model has been investigated in detail theoretically and has been validated using different experimental data sets.

Despite the numerous studies on or making use of the HR model, there have been no exact solutions published which are valid for all space and time. In this paper we present the first exact solutions to their model. The assumptions required to simplify the model so as to obtain these are (i) steady overland flow, (ii) constant water depth, and (iii) that the soil consists of a single particle size.

2. Theory

The HR model has been described previously, so only a brief summary is presented here. From the outset, the simplification of a single particle size is applied, since this is the main assumption that leads to the analytical solutions presented below.

The form of the HR model presented by Lisle et al. (1998) is convenient since it uses the mass per unit area of sediment in the water, M_s [ML^{-2}], and the mass per unit area of sediment in the deposited layer, M_d [ML^{-2}], as dependent variables. Note again that the model considers the development of a deposited layer, as time passes, which moderates the level of erosion of the underlying original soil layer. The model's governing equations are:

$$\frac{\partial M_s}{\partial t} + \frac{q_f}{D}\frac{\partial M_s}{\partial x} = -\frac{\nu}{D}M_s + \frac{a_d - a_o}{M_{dT}}PM_d + a_oP, \tag{1}$$

$$\frac{\partial M_d}{\partial t} = \frac{\nu}{D}M_s - \frac{a_d}{M_{dT}}PM_d, \tag{2}$$

where t [T] is the time, x [L] the position, q_f [L^2T^{-1}] the total volumetric flux per unit width of the domain, D [L] the depth of the overland flow, ν [LT^{-1}] the particle setting velocity, a_d [ML^{-3}] the coefficient of detachability of the deposited soil, a_o [ML^{-3}] the detachability of the original soil, M_{dT} [ML^{-2}] the mass of redeposited soil per unit area needed to block completely erosion of the original soil layer, and P [LT^{-1}] is the precipitation rate. The water advection rate, q_f/D, is assumed to be constant as both q_f and D are taken as constants. The erodible soil is in the region $x > 0$. Water flows into this zone from $x < 0$, where the soil bed is considered to be non-erodible. Indeed, the model assumes in addition that at time zero there is a water layer flowing at a steady rate across the soil surface, the latter becoming erodible for times greater than zero.

The second term on the right side of Eq. (1) vanishes if $a_d = a_o$, however the values will be different if the cohesion of the original soil and that of the deposited layer are different. Given the processes that take place such as bed compaction or disturbance by means other than rainfall impact, it is reasonable to expect $a_d \neq a_o$ under many circumstances. This inequality is fundamental to the HR model. During an erosion/deposition event, erosion of the original soil is halted at any locations where M_d/M_{dT} attains unity.

Eqs. (1) and (2) are solved subject to:

$$M_s(0,t) = 0, \tag{3}$$
$$M_s(x,0) = 0, \tag{4}$$
$$M_d(x,0) = 0. \tag{5}$$

Eqs. (3)–(5) mean that there is initially no deposited layer or suspended sediment in the overland flow, and that sediment-free water enters the region containing the erodible soil, beginning at $x = 0$.

To ease the clutter of notation that would otherwise appear in the solution, Eqs. (1)–(5) are replaced (and slightly generalized) by a version with simpler notation:

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} = -ap + bq + fH(x), \tag{6}$$
$$\frac{\partial q}{\partial t} = cp - dq + gH(x), \tag{7}$$
$$p(0,t) = 0, \tag{8}$$
$$p(x,0) = 0, \tag{9}$$
$$q(x,0) = 0. \tag{10}$$

where H is the Heaviside function, defined as:

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geqslant 0. \end{cases} \tag{11}$$

This function in Eqs. (6) and (7) forces the solution to zero where $x < 0$. Definitions of the variables follow directly given that Eqs. (1)–(5) correspond, respectively, to Eqs. (6)–(10). Eqs. (1) and (2) have $a = c$. In mathematical terms no significant simplification comes from enforcing this condition, so it is relaxed to generalize the solution slightly.

Eqs. (6)–(10) have the form of the so-called two-region (mobile–immobile) model, although without the diffusion term normally found in models of this type (e.g., Coats and Smith, 1964; Lindstrom and Stone, 1974; Mironenko and Pachepsky, 1984; Li et al., 1994; Haggerty and Gorelick, 1995; Griffioen et al., 1998; Choi et al., 2000; Ekberli, 2006; Lu et al., 2009; Silva et al., 2009). Apart from the lack of diffusion, the other main characteristic of the model in Eqs. (6) and (7) that distinguishes it from the two-region model is that the coefficients a, b, c and d are not equal. Thus, existing solutions for two-region models cannot directly be applied to the present problem. Generalized solutions that consider an arbitrary transport operator, e.g., Walker (1987), provide solutions in the form of integrals that solve Eqs. (6)–(10). While examples of such integral solutions to (6)–(10) are given within this paper, so too are fully explicit series solutions.

In Eq. (7) an additional term not present in Eq. (2), $gH(x)$, has been added. The HR model assumes that all sediment particles eroded from the original bed enter the water column. This additional term, not present in the HR model, models the transition of particles from the original bed directly to the deposited layer. Depending on the energy transmitted by a raindrop impact and the density of the sediment, for instance, motion of any given particle could be minute, such that this modification to the HR model would be reasonable. This question is returned to briefly in Section 3.

Because of superposition, there is no need to solve Eqs. (6)–(10) with both f and g non-zero simultaneously, so two problems are solved in the following, setting g and f in turn to zero. In terms of the HR model, setting g to zero (f non-zero) means that material eroded from the original soil moves into the water phase only, whereas setting f to zero (g non-zero) means that this material moves only into the deposited layer. In reality, probably both these situations occur simultaneously.

The Laplace transform method is used to obtain the solutions. Let s be the Laplace transform variable (transform with respect to t) and let transformed functions be denoted by an overbar. The solution to Eqs. (6)–(10) in the Laplace space is:

$$\bar{p}(x,s) = \left[\frac{bg}{s(s+d)} + \frac{f}{s}\right]\frac{H(x)}{\bar{h}(s)}\left\{1 - \exp\left[-\frac{x}{u}\bar{h}(s)\right]\right\}, \tag{12}$$
$$\bar{q}(x,s) = \frac{c}{s+d}\bar{p}(x,s) + \frac{gH(x)}{s(s+d)}, \tag{13}$$

where

$$\bar{h}(s) = s + a - \frac{bc}{s+d}. \tag{14}$$

2.1. Solution for g = 0

2.1.1. Solution in the form of an integral

For $g = 0$, Eqs. (12) and (13) become, respectively:

$$\frac{\bar{p}(x,s)}{fH(x)} = \frac{1}{s\bar{h}(s)}\left\{1 - \exp\left[-\frac{x}{u}\bar{h}(s)\right]\right\}, \tag{15}$$
$$\frac{\bar{q}(x,s)}{cfH(x)} = \frac{1}{s(s+d)\bar{h}(s)}\left\{1 - \exp\left[-\frac{x}{u}\bar{h}(s)\right]\right\}. \tag{16}$$

Consider first the solution for Eq. (15). The Laplace transform inverse ($\mathscr{L}^{-1}$) of $1/[s\bar{h}(s)]$ is:

$$w(t) = \mathscr{L}^{-1}\left[\frac{1}{s\bar{h}(s)}\right] = \frac{d}{ad - bc}\left\{1 + \exp\left[-\frac{t}{2}(a+d)\right]\right.$$
$$\left.\times\left[\frac{a^2 - d^2 - \alpha^2}{2\alpha d}\sinh\left(\alpha\frac{t}{2}\right) - \cosh\left(\alpha\frac{t}{2}\right)\right]\right\}, \tag{17}$$

where

$$\alpha^2 = (d-a)^2 + 4bc. \tag{18}$$

The inverse Laplace transform of $\exp\left[-\frac{x}{u}\bar{h}(s)\right]$ is given by Eq. (48) in Appendix A. Appendix A contains a table of several forward and inverse Laplace transforms that are used throughout this paper. Therefore, by the convolution theorem for products of functions (e.g., Spiegel, 1965):

$$\mathscr{L}^{-1}\left\{\frac{1}{s\bar{h}(s)}\exp\left[-\frac{x}{u}\bar{h}(s)\right]\right\} = \left\{\int_{\frac{x}{u}}^{t} w(t-\tau)\exp\left[-d\left(\tau - \frac{x}{u}\right)\right]\right.$$
$$\left.\times\sqrt{\frac{bc}{\tau - \frac{x}{u}}\frac{x}{u}}\, I_1\left[2\sqrt{bc\frac{x}{u}\left(\tau - \frac{x}{u}\right)}\right]d\tau + w\left(t - \frac{x}{u}\right)\right\}H\left(t - \frac{x}{u}\right)\exp\left(-a\frac{x}{u}\right). \tag{19}$$

The combination of Eqs. (15), (17), and (19) gives the solution for p as:

$$\frac{p(x,t)}{fH(x)} = w(t) - H\left(t-\frac{x}{u}\right)\exp\left(-a\frac{x}{u}\right) \times \left\{ w\left(t-\frac{x}{u}\right) + \int_{\frac{x}{u}}^{t} w(t-\tau)\exp\left[-d\left(\tau-\frac{x}{u}\right)\right] \times \sqrt{\frac{bc}{\tau-\frac{x}{u}}\frac{x}{u}}\, I_1\left[2\sqrt{bc\frac{x}{u}\left(\tau-\frac{x}{u}\right)}\right]d\tau\right\}. \quad (20)$$

The solution for q, from Eq. (16) is constructed in the same manner as p. In this case function w is given by:

$$w(t) = \frac{1}{2\alpha(ad-bc)}\left\{2\alpha + (a+d-\alpha)\exp\left[-\frac{t}{2}(a+d+\alpha)\right] - \left(a+d+\alpha\right)\exp\left[-\frac{t}{2}(a+d-\alpha)\right]\right\}. \quad (21)$$

Thus, the solution for $q/[cfH(x)]$ is given by the right side of Eq. (20), with w defined in this case by Eq. (21).

Case of ad = bc

It can be seen from Eq. (17) that this case leads to division by zero, and so it must be considered explicitly as a special case. Consider p first. The function w in Eq. (17) simplifies to:

$$w(t) = \frac{1}{a+d}\left\{dt + \frac{a}{a+d}[1-\exp(-t(a+d))]\right\}, \quad (22)$$

while Eq. (19) essentially remains the same, except that bc is replaced by ad. The solution is given by Eq. (20) with bc replaced by ad, and with w defined by Eq. (22).

For $q/[cfH(x)]$, the solution is given by the right side of Eq. (20), with bc replaced by ad, and with w given by:

$$w(t) = \frac{1}{a+d}\left\{t - \frac{1-\exp[-t(a+d)]}{a+d}\right\}. \quad (23)$$

2.1.2. Series solution

The denominator, $s\bar{h}(s)$, of Eq. (15) can be expanded in partial fractions as:

$$\frac{1}{s\bar{h}(s)} = \frac{d}{\alpha_1\alpha_2 s} + \frac{\alpha_1-d}{\alpha_1(\alpha_2-\alpha_1)(s+\alpha_1)} + \frac{\alpha_2-d}{\alpha_2(\alpha_1-\alpha_2)(s+\alpha_2)}, \quad (24)$$

where

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \left(a+d\begin{bmatrix}+\\ -\end{bmatrix}\alpha\right)\Big/2 \quad (25)$$

factorize Eq. (18). The Laplace inversion of the second term in Eq. (15) thus reduces to the inversion of the three terms on the right side of Eq. (24), with each term multiplied by $\exp[-x\bar{h}(s)/u]$. By then setting $x = 0$ in these inversions results in the inverse of the first term in Eq. (15). This approach to obtaining fully explicit solutions is used repeatedly below.

As is evident from Eq. (24), only a single inverse Laplace transform is needed, i.e., that given by Eq. (50) in Appendix A. Then, the inversion of Eq. (15) is:

$$\frac{p(x,t)}{fH(x)} = \frac{d}{ad-bc}\mathscr{F}(-d) - \frac{\alpha-d+a}{\alpha(\alpha+d+a)}\mathscr{F}(a+\alpha) - \frac{\alpha+d-a}{\alpha(d+a-\alpha)}\mathscr{F}(a-\alpha), \quad (26)$$

where

$$\mathscr{F}(m) = \exp\left[-\frac{t}{2}(d+m)\right] - H\left(t-\frac{x}{u}\right)\exp\left[-\frac{x}{u}\left(a-\frac{2bc}{d-m}\right) - \frac{d+m}{2}\left(t-\frac{x}{u}\right)\right]J_{mod}\left[\frac{2bc}{d-m}\frac{x}{u}, \frac{d-m}{2}\left(t-\frac{x}{u}\right)\right]. \quad (27)$$

The inverse transform of $\bar{q}$ is calculated from Eq. (16) in the same manner. The required partial fraction expansion is:

$$\frac{1}{s(s+d)\bar{h}(s)} = \frac{1}{\alpha_1\alpha_2 s} + \frac{1}{\alpha_1(\alpha_2-\alpha_1)(s+\alpha_1)} + \frac{1}{\alpha_2(\alpha_1-\alpha_2)(s+\alpha_2)}, \quad (28)$$

so, again, Eq. (50) is utilized. The final result is:

$$\frac{q(x,t)}{cfH(x)} = \frac{1}{ad-bc}\mathscr{F}(-d) + \frac{2}{\alpha(\alpha+d+a)}\mathscr{F}(a+\alpha) + \frac{2}{\alpha(\alpha-d-a)}\mathscr{F}(a-\alpha). \quad (29)$$

Case of ad = bc

The special case of $ad = bc$ is considered next. The solution is presented in terms of a and d rather than b and c. Eqs. (18) and (25) give, for this case:

$$\alpha_1 = a+d \text{ and } \alpha_2 = 0. \quad (30)$$

The partial fraction expansion in Eq. (24) becomes:

$$\frac{s+d}{s^2(s+a+d)} = \frac{a}{(a+d)^2 s} - \frac{a}{(a+d)^2(s+a+d)} + \frac{d}{(a+d)s^2}. \quad (31)$$

Here, an additional inverse transform is needed to invert the function that results from the final term on the right side of Eq. (31). The required inverse transform is given by Eq. (53) in Appendix A. The solution that results is:

$$\frac{p(x,t)}{fH(x)} = \frac{a}{(a+d)^2}[\mathscr{F}(-d) - \mathscr{F}(2a+d)] + \frac{d}{a+d}\left\{t - \frac{1}{d}H\left(t-\frac{x}{u}\right) \times \exp\left[-a\frac{x}{u} - d\left(t-\frac{x}{u}\right)\right]\sum_{n=1}^{\infty} n\left[\frac{d\left(t-\frac{x}{u}\right)}{a\frac{x}{u}}\right]^{\frac{n}{2}} I_n(2\beta)\right\}. \quad (32)$$

The partial fraction expansion arising in the inverse transform for q in Eq. (16) is:

$$\frac{1}{s^2(s+a+d)} = \frac{1}{(a+d)^2(s+a+d)} - \frac{1}{(a+d)^2 s} + \frac{1}{(a+d)s^2}. \quad (33)$$

This expression is very similar to that for p, Eq. (31), so that the inverse transform for q differs from Eq. (32) only in the coefficients of each term on the right side. The result is:

$$\frac{q(x,t)}{cfH(x)} = \frac{1}{(a+d)^2}[\mathscr{F}(2a+d) - \mathscr{F}(-d)] + \frac{1}{a+d}\left\{t - \frac{1}{d}H\left(t-\frac{x}{u}\right) \times \exp\left[-a\frac{x}{u} - d\left(t-\frac{x}{u}\right)\right]\sum_{n=1}^{\infty} n\left[\frac{d\left(t-\frac{x}{u}\right)}{a\frac{x}{u}}\right]^{\frac{n}{2}} I_n(2\beta)\right\}. \quad (34)$$

2.2. Solution for f = 0

The two Laplace-domain solutions, Eqs. (12) and (13) become, respectively:

$$\bar{p}(x,s) = \frac{bg}{s(s+d)}\frac{H(x)}{\bar{h}(s)}\left\{1-\exp\left[-\frac{x}{u}\bar{h}(s)\right]\right\}, \quad (35)$$

$$\bar{q}(x,s) = \frac{bcg}{s(s+d)^2}\frac{H(x)}{\bar{h}(s)}\left\{1-\exp\left[-\frac{x}{u}\bar{h}(s)\right]\right\} + \frac{gH(x)}{s(s+d)}. \quad (36)$$

2.2.1. Solution in the form of an integral

For $ad \neq bc$, and due to the equivalence of Eqs. (15) and (34), the solution for *p/gH(x)* is simply b/c times the right-hand side of Eq. (19) with $w(t)$ given by Eq. (21). For $ad = bc$ the same statement applies, but with w given by Eq. (23).

For q, as in Section 2.1.1, essentially all that changes is the w function used in Eq. (20). The functions needed to obtain the solution from Eq. (20) are given here (with a summary of all the solutions given in Table 1). Also, there is an additional term in the solution corresponding to the final term on the right side of Eq. (36). Solutions will be written with $q/[gH(x)]$ on the left side, so:

$$\mathscr{L}^{-1}\left[\frac{1}{s(s+d)}\right] = \frac{1-\exp(-dt)}{d}, \tag{37}$$

should be added to the right side of Eq. (20) for each of the two following solutions for q.

For $ad \neq bc$, the w function is found from the inverse of $bc/[s(s+d)^2\bar{h}(s)]$, with the result:

$$\begin{aligned} w(t) = &\frac{bc}{d(ad-bc)} + \frac{\exp(-dt)}{d} \\ &- \exp\left[-\frac{t}{2}(d+a-\alpha)\right]\frac{a-d+\alpha}{\alpha(d+a-\alpha)} \\ &+ \exp\left[-\frac{t}{2}(d+a+\alpha)\right]\frac{a-d-\alpha}{\alpha(d+a+\alpha)}, \end{aligned} \tag{38}$$

while for $ad = bc$, it is

$$w(t) = \frac{at}{a+d} + \frac{\exp(-dt)}{d} - \frac{d\exp[-(a+d)t]}{(a+d)^2} - \frac{a(a+2d)}{d(a+d)^2}. \tag{39}$$

2.2.2. Series solution

As in Section 2.2.1, the solution for $p(x,t)/[bgH(x)]$, as is apparent from Eqs. (12) and (13), is, for $ad \neq bc$, simply the right side of Eq. (29). Similarly, for $ad = bc$, the solution for $p(x,t)/[bgH(x)]$ is given by the right side of Eq. (34).

For q, considering $ad \neq bc$, it is seen from Eq. (36) that the partial fraction expansion of $[s(s+d)^2\bar{h}(s)]^{-1}$ is needed. It is:

$$\begin{aligned} \frac{1}{s(s+d)^2\bar{h}(s)} = &\frac{1}{d\alpha_1\alpha_2 s} - \frac{1}{\alpha_1(\alpha_1-d)(\alpha_1-\alpha_2)(s+\alpha_1)} \\ &- \frac{1}{d(\alpha_1-d)(\alpha_2-d)(s+d)} \\ &- \frac{1}{\alpha_2(\alpha_2-d)(\alpha_2-\alpha_1)(s+\alpha_2)}. \end{aligned} \tag{40}$$

From Eq. (40), it is apparent that the inverse of the corresponding exponential terms in Eq. (36) involve two entries in the transform pairs given in Appendix A, viz., Eqs. (49) and (50). Then, the inverse of Eq. (36) is:

Table 1
Summary of equations solved and analytical solutions.

Laplace transform equation solved	Solution	Section	Remarks
Eq. (15)	Eq. (20)	2.1.1	$g = 0$ (all entries to the partition below are for this case), integral solution for p, w from Eq. (17), $ad \neq bc$. Solution for the water phase sediment concentration. Physical interpretation for all solutions with $g = 0$: Sediment mobilized from the original soil moves only to the water phase, in accordance with the HR model.
Eq. (16)	Eq. (20)	2.1.1	$g = 0$, right side gives the integral solution for $q/[cfH(x)]$, w from Eq. (21), $ad \neq bc$. Solution for the deposited layer concentration. The physical interpretation corresponds to that given in the entry above for Eq. (15).
Eq. (15)	Eq. (20)	2.1.1, *Case of ad = bc*	$g = 0$, integral solution for p, w from Eq. (22), $ad = bc$. Solution for the water phase sediment concentration. This case is given for mathematical completeness. For the HR model, it corresponds to a non-erodible original soil, which is physically implausible.
Eq. (16)	Eq. (20)	2.1.1, *Case of ad = bc*	$g = 0$, right side gives the integral solution for $q/[cfH(x)]$, w from Eq. (23), $ad = bc$. Solution for the deposited layer concentration. Again, this solution is given for completeness as, in terms of the HR model, $ad = bc$ is physically implausible.
Eq. (15)	Eq. (26)	2.1.2	$g = 0$, series solution for p, $ad \neq bc$. Solution for the water phase sediment concentration. Same interpretation as given for Eq. (15) above (first entry in this table).
Eq. (16)	Eq. (29)	2.1.2	$g = 0$, series solution for q, $ad \neq bc$. Solution for the deposited layer concentration. Same interpretation as given for Eq. (16) above (second entry in this table).
Eq. (15)	Eq. (32)	2.1.2, *Case of ad = bc*	$g = 0$, series solution for p, $ad = bc$. Solution for the water phase sediment concentration. Same interpretation as the above entry for $ad = bc$.
Eq. (16)	Eq. (34)	2.1.2, *Case of ad = bc*	$g = 0$, series solution for q, $ad = bc$. Solution for the deposited layer concentration. Same interpretation as the above entry for $ad = bc$.
Eq. (35)	Eq. (20)	2.2.1	$f = 0$ (all the solutions to the end of the table are for this case), right side gives the integral solution for $p/[bfH(x)]$, w from Eq. (21), $ad \neq bc$. Solution for the water phase sediment concentration. Physical interpretation for all solutions with $f = 0$: Sediment mobilized from the original soil moves only to the deposited (shield) layer. This is in contrast to the HR model where sediment moves from the original soil only to the water phase.
Eq. (35)	Eq. (20)	2.2.1	$f = 0$, right side gives the integral solution for $p/[bfH(x)]$, w from Eq. (23), $ad = bc$. Solution for the water phase sediment concentration. As for the case above, this case is given for mathematical completeness. It corresponds to a non-erodible original soil, which is physically implausible.
Eq. (36)	Eq. (20)	2.2.1	$f = 0$, right side gives the integral solution for $q/[gH(x)]$, w from Eq. (38), $ad \neq bc$. Solution for the deposited layer concentration. The physical interpretation corresponds to that given in the entry two rows above for Eq. (35).
Eq. (36)	Eq. (20)	2.2.1	$f = 0$, right side gives the integral solution for $q/[gH(x)]$, w from Eq. (39), $ad = bc$. Solution for the deposited layer concentration. Physical explanation follows that given two rows above.
Eq. (35)	Eq. (29)	2.2.2	$f = 0$, the right side of Eq. (29) gives the series solution for $p(x,t)/[bgH(x)]$, $ad \neq bc$. Solution for the water phase sediment concentration. Same interpretation as given for Eq. (35) above (first entry in this sub-section of this table).
Eq. (35)	Eq. (34)	2.2.2	$f = 0$, the right side of Eq. (34) gives the series solution for $p(x,t)/[bgH(x)]$, $ad = bc$. This case is for the water phase sediment concentration, but is physically implausible as explained above.
Eq. (36)	Eq. (41)	2.2.2	$f = 0$, series solution for q, $ad \neq bc$. Deposited layer concentration for the case where the sediment from the original soil is transferred only to the deposited layer.
Eq. (36)	Eq. (43)	2.2.2	$f = 0$, series solution for q, $ad = bc$. Solution for the deposited layer concentration. Again, this solution is given for completeness as $ad = bc$ is physically implausible.

$$\frac{q(x,t)}{gH(x)} = \frac{1}{d}\left\{1 - H\left(t-\frac{x}{u}\right)\exp\left[-a\frac{x}{u} - d\left(t-\frac{x}{u}\right)\right]I_0(2\beta) + \frac{bc}{ad-bc}\mathscr{F}(-d) + \frac{4bcd}{\alpha(\alpha+d+a)(d-a-\alpha)}\mathscr{F}(a+\alpha) + \frac{4bcd}{\alpha(a-d-\alpha)(d+a-\alpha)}\mathscr{F}(a-\alpha)\right\}. \tag{41}$$

For $ad = bc$, Eq. (40) becomes:

$$\frac{1}{s(s+d)^2\bar{h}(s)} = \frac{1}{ad^2(s+d)} + \frac{1}{d(a+d)s^2} - \frac{1}{a(a+d)^2(s+d+a)} - \frac{2d+a}{d^2(a+d)^2 s}. \tag{42}$$

The inversions for the terms appearing on the right side of Eq. (42) are, respectively, Eqs. (49), (53), (50) and (50), respectively. The resulting expression for $q(x,t)$ is:

$$\frac{q(x,t)}{gh(x)} = \frac{1}{d}\left\{1 - H\left(t-\frac{x}{u}\right)\exp\left[-a\frac{x}{u} - d\left(t-\frac{x}{u}\right)\right]I_0(2\beta)\right\} + \frac{a}{d(a+d)}\left\{dt - H\left(t-\frac{x}{u}\right)\exp\left[-a\frac{x}{u} - d\left(t-\frac{x}{u}\right)\right] \times \sum_{n=1}^{\infty} n\left[\frac{d(t-\frac{x}{u})}{a\frac{x}{u}}\right]^{\frac{n}{2}} I_n(2\beta)\right\} - \frac{1}{(a+d)^2}\left[d\mathscr{F}(2a+d) + \frac{a}{d}(2d+a)\mathscr{F}(-d)\right]. \tag{43}$$

3. Discussion

The model in Eqs. (6) and (7) can aid in the question of identifiability of the HR model parameters and, indeed, how such a model is validated. Concerning validation, in particular, apart from very small scale laboratory experiments, soil erosion experiments are usually carried out using flumes set up to measure sediment and water fluxes at the end of the flume, i.e., in the notation used here experiments measure a quantity proportional to $up(L,t)$, where L is the flume length. Consider setting $f = 0$ in Eq. (6) and allow the constant g to become time-dependent such that Eqs. (6) and (7) become, respectively:

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} = -ap + bq, \tag{44}$$

$$\frac{\partial q}{\partial t} = cp - dq + \frac{f}{b}[\delta(t) + d]H(x) \tag{45}$$

Here, sediment is supplied from the original soil to the deposited layer, whereas in the HR model sediment is supplied only to the water phase. Thus, it is, in physical terms, quite different from the HR model. The Laplace-domain solution of Eqs. (44) and (45) subject to Eqs. (8)–(10) is:

$$\frac{\bar{p}(x,s)}{fH(x)} = \frac{1}{s\bar{h}(s)}\left\{1 - \exp\left[-\frac{x}{u}\bar{h}(s)\right]\right\}, \tag{46}$$

$$\frac{\bar{q}(x,s)}{fH(x)} = \frac{c}{s(s+d)\bar{h}(s)}\left\{1 - \exp\left[-\frac{x}{u}\bar{h}(s)\right]\right\} + \frac{1}{bs} \tag{47}$$

Observe that Eq. (46) is identical to Eq. (15), which was obtained for the case $g = 0$, i.e., sediment was supplied only to the water phase. But, Eq. (47) differs from Eq. (16) by the final term, i.e., $1/(bs)$. Solutions to Eq. (47) are therefore the same as those for Eq. (16), with an additional term $fH(x)/b$. This means that experiments that do not measure both p and q (i.e., for p, sediment concentrations exiting the flume and, for q, the deposited layer) are unable to say definitively, in the absence of other information, whether the HR model form is correct. In other words, a model validated based only on sediment concentrations in the runoff cannot distinguish whether the original soil sediment has been moved directly into the flow, or has been moved to the deposited layer, and from there to the flowing water.

4. Conclusion

In a mechanistic model, parameters are determined, ideally, independently, and the model used to make predictions. Soil erosion is a complex process, and is an extremely challenging system in which to make measurements, in consequence making model validation subject to uncertainty. Solutions for a slightly generalized HR model have been presented. The model generalization, however, makes clear that the mechanisms included in the model cannot be validated solely on sediment concentration data collected in runoff. Rather, experimental measurements of the deposited layer would provide an additional means to analyze whether the form of the model is correct. The reason for this is that the HR model assumes that, when considering the original soil, eroded sediment is transferred to the water phase and from there to the deposited layer. Certainly for large particles, this assumption would be open to question. An alternative model would be for the original soil sediment to move directly to the deposited layer. As an example, the extreme case where this is the only possibility was solved, with the solution revealing that the model prediction of the deposited layer changes, whereas the sediment concentrations in the runoff do not.

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Appendix A. Laplace transform pairs and the modified Goldstein *J* function

In this Appendix results used to derive the analytical solutions are listed. Some additional transform pairs are included for those interested in solving similar problems. A modification of the Goldstein *J* function was found to be necessary, as is discussed below. The function $\bar{h}(s)$ is defined in Eq. (14).

Laplace domain function	Real domain function	Equation
$\exp\left[-\bar{h}(s)\frac{x}{u}\right]$	$H(t-\frac{x}{u})\exp\left[-a\frac{x}{u} - d(t-\frac{x}{u})\right] \times\left[\sqrt{\frac{bc}{t-\frac{x}{u}}\frac{x}{u}}\,I_1(2\beta) + \delta(t-\frac{x}{u})\right]$	(48)
$\frac{\exp\left[-\bar{h}(s)\frac{x}{u}\right]}{s+d}$	$H(t-\frac{x}{u})\exp\left[-a\frac{x}{u} - d(t-\frac{x}{u})\right]I_0(2\beta)$	(49)
$\frac{\exp\left[-\bar{h}(s)\frac{x}{u}\right]}{s+B}$, $B\neq d$	$H(t-\frac{x}{u})\exp\left[-\frac{x}{u}(a-\frac{bc}{d-B}) - B(t-\frac{x}{u})\right]J_{mod}\left[\frac{bc}{d-B}\frac{x}{u}, (d-B)(t-\frac{x}{u})\right]$	(50)
$\frac{s\exp\left[-\bar{h}(s)\frac{x}{u}\right]}{s+B}$	$\mathscr{L}^{-1}\left\{\exp\left[-\bar{h}(s)\frac{x}{u}\right]\right\} - B\mathscr{L}^{-1}\left\{\frac{\exp\left[-\bar{h}(s)\frac{x}{u}\right]}{s+B}\right\}$	(51)
$\frac{\exp\left[-\bar{h}(s)\frac{x}{u}\right]}{(s+d)^2}$	$H(t-\frac{x}{u})\exp\left[-a\frac{x}{u} - d(t-\frac{x}{u})\right] \times\sqrt{\frac{t-\frac{x}{u}}{bc}\frac{u}{x}}\,I_1(2\beta)$	(52)
$\frac{\exp\left[-\bar{h}(s)\frac{x}{u}\right]}{(s+B)^2}$, $B\neq d$	$\frac{H(t-\frac{x}{u})}{d-B}\exp\left[-a\frac{x}{u} - d(t-\frac{x}{u})\right] \times\sum_{n=1}^{\infty} n(d-B)^n\left[\frac{(t-\frac{x}{u})}{bc\frac{x}{u}}\right]^{\frac{n}{2}}I_n(2\beta)$	(53)

In Eq. (50), J_{mod} is a modification of the Goldstein *J* function (Goldstein, 1953). Note that original *J* function arises naturally in two-region problems such as that in Eqs. (6)–(10), see, for example, Goltz and Roberts (1986), Barry and Parker (1987), Veling (2002), De Smedt et al. (2005). However, it is necessary to modify it as per the following definition:

$$J_{mod}(y,z) = \exp(-y-z)\sum_{n=0}^{\infty}[\mathrm{sgn}(y)]^n\left(\frac{z}{y}\right)^{\frac{n}{2}}I_n(2\sqrt{yz}), \tag{54}$$

where the sign function, sgn, is defined by:

$$\mathrm{sgn}(y) = \begin{cases} -1 & y<0 \\ 0, & y=0 \\ 1, & y>0. \end{cases} \tag{55}$$

The modified J function is necessary to account for negative arguments, which can occur in the solutions reported here. The J function of Goldstein (1953) is recovered by setting $\mathrm{sgn}(y) = 1$ in Eq. (54), i.e., it is defined by:

$$J(y,z) = \exp(-y-z)\sum_{n=0}^{\infty}\left(\frac{z}{y}\right)^{\frac{n}{2}}I_n(2\sqrt{yz}). \tag{56}$$

Goldstein (1953) also gave the alternative definition:

$$J(y,z) = 1-\exp(-y-z)\sum_{n=0}^{\infty}\left(\frac{y}{z}\right)^{\frac{n}{2}}I_n(2\sqrt{yz}). \tag{57}$$

The corresponding definition for $J_{mod}(y,z)$ is:

$$J_{mod}(y,z) = 1-\exp(-y-z)\sum_{n=1}^{\infty}[\mathrm{sgn}(z)]^n\left(\frac{y}{z}\right)^{\frac{n}{2}}I_n(2\sqrt{yz}). \tag{58}$$

The third J function definition of Goldstein (1953):

$$J(y,z) = 1-\exp(-z)\int_0^y \exp(-\bar{y})I_0(2\sqrt{\bar{y}z})d\bar{y} \tag{59}$$

remains unchanged, i.e., $J = J_{mod}$ in this case.

Limiting values of $J_{mod}(y,z)$ are, as for the J function:

$$J_{mod}(y,0) = \exp(-y), J_{mod}(0,z) = J_{mod}(y,\infty) = 1, J_{mod}(\infty,z) = 0. \tag{60}$$

To these limits, the following limits for $J_{mod}(y,z)$ can be added:

$$J_{mod}(y,-\infty) = \infty, J_{mod}(-\infty,z) = \infty. \tag{61}$$

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