Journal of Number Theory 129 (2009) 1034–1043

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Cyclotomic factors of Coxeter polynomials

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ARTICLE INFO

Article history:
Received 2 August 2007
Available online 6 March 2009
Communicated by Fernando Rodriguez Villegas

Keywords:
Salem polynomials
Adjacency matrices

ABSTRACT

In this paper we show that the cyclotomic factors of the E_n Coxeter polynomials depend only on the value of $n \bmod 360$, and come exclusively from spherical subdiagrams.

Contents

1. Introduction

In this paper we determine which roots of unity are zeros of the E_n Coxeter polynomial. We show these roots come exclusively from splittings of E_n into spherical subdiagrams; in particular they always have order 2, 3, 5, 8, 12, 18, or 30, and they only depend on the value of $n \bmod 360$ (provided we exclude the special case $n = 9$).

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1 Research supported in part by the NSF.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2008.09.021

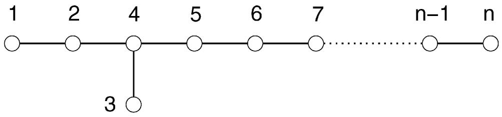

Fig. 1. The E_n diagram.

Table 2
Coxeter polynomials for small n.

n	h_n	Coxeter polynomial E_n	Factorization
3	6	$1+2x+2x^2+x^3$	$\Phi_2(x)\Phi_3(x)$
4	5	$1+x+x^2+x^3+x^4$	$\Phi_5(x)$
5	8	$1+x+x^4+x^5$	$\Phi_2(x)\Phi_8(x)$
6	12	$1+x-x^3+x^5+x^6$	$\Phi_3(x)\Phi_{12}(x)$
7	18	$1+x-x^3-x^4+x^6+x^7$	$\Phi_2(x)\Phi_{18}(x)$
8	30	$1+x-x^3-x^4-x^5+x^7+x^8$	$\Phi_{30}(x)$
9	∞	$1+x-x^3-x^4-x^5-x^6+x^8+x^9$	$\Phi_1(x)\Phi_2(x)^2\Phi_3(x)\Phi_5(x)$
10	∞	$1+x-x^3-x^4-x^5-x^6-x^7+x^9+x^{10}$	$S_{10}(x)$

The proof uses Mann's theorem on linear relations between roots of unity, and generalizes to other sequences of Coxeter diagrams where nodes are added to a separating edge.

The E_n diagram. Coxeter systems are a useful source of Salem numbers, Pisot numbers and other interesting algebraic integers. For example, the smallest known Salem number arises from the Coxeter system E_{10}.

The E_n Coxeter diagram, defined for $n \geqslant 3$, is shown in Fig. 1. Note that $E_3 \cong A_2 \oplus A_1$. The E_n diagram determines a quadratic form B_n on $\mathbb{Z}^n$, and a reflection group $W_n \subset O(\mathbb{Z}^n, B_n)$ (see Section 3). The product of the generating reflections is a *Coxeter element* $w_n \in W_n$; it is well-defined up to conjugacy, since E_n is a tree [Hum, §8.4].

The *Coxeter number* h_n is the order of the Coxeter element $w_n \in W_n$, and its characteristic polynomial

$$E_n(x) = \det(xI - w_n)$$

is the *Coxeter polynomial*. Explicitly, for $n \geqslant 3$ we have

$$E_n(x) = \frac{x^{n-2}Q(x) + R(x)}{(x-1)},$$

where $Q(x) = x^3 - x - 1$ and $R(x) = x^3 + x^2 - 1$. (See e.g. [MRS, Lemma 5], [Hir2, §4.2] or Corollary 4.3 below.)

We can write $E_n(x)$ uniquely as a product of monic integral polynomials

$$E_n(x) = C_n(x) S_n(x),$$

where the zeros of the *cyclotomic factor* $C_n(x)$ are roots of unity, and those of the *Salem factor* $S_n(x)$ are not. Table 2 lists $E_n(x)$ for $n \leqslant 10$, along with its factorization into irreducibles and the Coxeter number h_n. Here $\Phi_k(x)$ is the cyclotomic polynomial for the primitive kth roots of unity.

The spherical and affine cases. Since E_i is a spherical diagram (B_i is positive definite) when $3 \leqslant i \leqslant 8$, we have $E_i(x) = C_i(x)$ (and $S_i(x) = 1$) in this range.

The diagram E_9 is the affine version of E_8; its Coxeter element has infinite order, but still $E_9(x) = C_9(x)$. This is the only case where $E_n(x)$ has a multiple root (see Lemma 2.4 below).

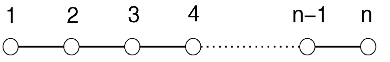

Fig. 3. The A_n diagram.

The hyperbolic case. For $n \geqslant 10$, the diagram E_n is hyperbolic; that is, the signature of B_n is $(n-1,1)$. By [A'C] this implies that the factor $S_n(x)$ is a *Salem polynomial*: it is an irreducible, reciprocal polynomial, with a unique root $\lambda > 1$ outside the unit disk. For $n = 10$, $E_n(x)$ coincides with *Lehmer's polynomial*, and its root $\lambda \approx 1.1762808 > 1$ is the smallest known Salem number.

We can now state our main result on the Coxeter polynomials $E_n(x)$.

Theorem 1.1. *For all $n \neq 9$:*

1. *The cyclotomic factor $C_n(x)$ is the least common multiple of the polynomials $\Phi_2(x)$, $\Phi_3(x)$ and $E_i(x)$, $3 \leqslant i \leqslant 8$, that divide $E_n(x)$;*
2. *$E_n(x)$ is divisible by $E_i(x)$, $3 \leqslant i \leqslant 8$, iff $n \equiv i \bmod h_i$; and*
3. *$E_n(x)$ is divisible by $\Phi_2(x)$ iff $n = 1 \bmod 2$, and by $\Phi_3(x)$ iff $n = 0 \bmod 3$.*

Corollary 1.2. *The cyclotomic factor $C_n(x)$ only depends on $n \bmod 360$.*

Corollary 1.3. *The Salem factor $S_n(x)$ satisfies $n - 15 \leqslant \deg(S_n) \leqslant n$.*

The value $n - 15$ is first attained when $n = 349$.

Corollary 1.4. *For $n \geqslant 10$, the polynomial $E_n(x)$ is irreducible (and hence $E_n(x) = S_n(x)$) iff $n \equiv 2, 10, 16, 20, 22, 26$ or $28 \bmod 30$.*

Joins of diagrams and periodicity. This behavior of E_n can be understood as a consequence of two general phenomena.

For the first, recall that the A_n diagram (Fig. 3) has Coxeter polynomial

$$A_n(x) = \frac{x^{n+1} - 1}{x - 1} = 1 + x + \cdots + x^n.$$

In Section 3 we will show

Theorem 1.5. *Let F be the Coxeter diagram obtained by joining together diagrams $F_1, \ldots, F_n$ at a single new vertex t. Then any zero of two or more of the Coxeter polynomials $F_i(x)$ is also a zero of $F(x)$.*

Noting that E_n is a join of E_i and A_{n-i-1}, we obtain

Corollary 1.6. *$E_n(x)$ is divisible by $\gcd(E_i(x), A_{n-i-1}(x))$ for $3 \leqslant i < n - 1$.*

This result explains why the spherical Coxeter polynomials $E_i(x)$, $3 \leqslant i \leqslant 8$, occur as factors of $E_n(x)$. For example, E_{38} is the join of E_8 and A_{29}. The zeros of $A_{29}(x)$ are the 30th roots of unity (save $\zeta = 1$); thus they include the zeros of $E_8(x)$, and consequently $E_8(x)$ divides $E_{38}(x)$. It also explains the occurrence of the cyclotomic factors Φ_2, Φ_3 and their product; these can occur as $\gcd(E_3, A_{n-4})$, depending on the value of $n \bmod 6$.

The second phenomenon underlying the behavior of E_n is the following periodicity result, proved in Section 4.

Theorem 1.7. *Let F_n be a sequence of Coxeter diagrams obtained by adjoining two fixed diagrams to the ends of A_n. Assume $F_n(x) \in \mathbb{Z}[x]$. Then either*

(i) *the cyclotomic factor of $F_n(x)$ is periodic for all $n \gg 0$, or*
(ii) *the diagram F_n is spherical or affine for all n.*

In case (ii), F_n (if connected) must be a re-indexing of one of the well-known spherical or affine series $A_n, B_n, D_n, \widetilde{B_n}, \widetilde{C_n}$ or $\widetilde{D_n}$.

This result, made effective, reduces Theorem 1.1 to a finite computation.

It would be interesting to find a general condition to insure that the cyclotomic factors of $F_n(x)$ come exclusively from its spherical subdiagrams, as is the case for $E_n(x)$.

Notes and references. For background on Coxeter systems, see e.g. [Bou] and [Hum]. More on the relationship between Coxeter systems, Salem numbers and Pisot numbers can be found in [Mc,MRS, Hir1,MS]. A version of Theorem 1.1 was proved independently, and by different arguments, by Bedford and Kim [BK, Theorem 2.4].

2. Roots of unity

Let ζ_k denote the primitive kth root of unity $\exp(2\pi i/k)$. In this section we formulate Mann's theorem, and use it to prove:

Theorem 2.1. *Let $Q, R \in \mathbb{Z}[x]$ be polynomials, not both zero, such that*

$$\zeta_k^n Q(\zeta_k) + R(\zeta_k) = 0$$

for some $k \geqslant 1$ and $n \in \mathbb{Z}$. Then either $Q(x) = \pm x^i R(x)$ for some $i \in \mathbb{Z}$, or we have

$$k \leqslant 2s \max(\deg Q, \deg R),$$

where s is the product of the primes $p \leqslant \ell(Q) + \ell(R)$.

Here $\ell(P)$ denotes the number of terms in the polynomial P (see below).

We then deduce Theorem 1.1 on the cyclotomic factor of $E_n(x)$.

Polar rational polygons. Let $\mathrm{Div}(\mathbb{C})$ denote the group of finite divisors on the complex plane. Any $D \in \mathrm{Div}(\mathbb{C})$ can be expressed as $D = \sum_I a_i \cdot z_i$ where each coefficient $a_i \in \mathbb{Z}$ is non-zero and $\operatorname{supp} D = \{z_i\colon\ i \in I\}$ is a set of distinct points forming the *support* of D. There is a natural evaluation map $\mathrm{Div}(\mathbb{C}) \to \mathbb{C}$ defined by

$$D \mapsto \sigma(D) = \sum a_i z_i.$$

We say D is *effective* if its coefficients are positive.

A *polar rational polygon* (prp) is an effective divisor $D = \sum a_i \cdot z_i$ such that each z_i is a root of unity and $\sigma(D) = 0$. For each ordering of I, D determines an (immersed) polygon in the plane with vertices $v_i = \sum_{j<i} a_j z_j$; its angles are rational multiples of π, and its sides are of integral length.

The *length* of a prp is given by $\ell(D) = |\operatorname{supp} D|$. Its *order* is the cardinality $o(D)$ of the subgroup of $\mathbb{C}^*$ generated by the roots of unity $\{z_i/z_j\colon\ i, j \in I\}$.

A prp is *primitive* if it cannot be expressed as a sum $D = D' + D''$ of two other non-zero prp's. Every prp is a sum of primitive prp's.

We can now state the main result of [Man]:

Theorem 2.2 *(Mann). Let D be a primitive prp. Then the order $o(D)$ divides the product of the primes p less than or equal to the length $\ell(D)$.*

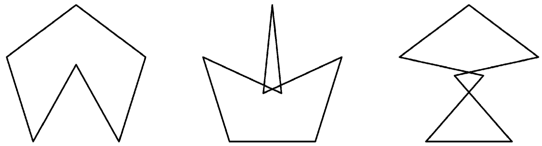

Fig. 4. Three primitive polar rational polygons.

Examples. The regular p-gons are primitive prp's whenever p is prime. The smallest primitive prp other than these has length 6 and order 15; it is given by

$$D = \zeta_5 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4 + \zeta_6 + \zeta_6^{-1}.$$

The corresponding hexagon (for a suitable ordering of the terms in the prp), with sides of length one, is shown at the left in Fig. 4. Two other primitive prp's, of length 7 and order 30, are shown in the center and at the right. Together with the regular p-gons for $p = 3, 5, 7$, these are (up to rotation) all the primitive prp's of length < 8 [Man].

Polynomials. Any polynomial $P(x) \in \mathbb{Z}[x]$ can be uniquely expressed in the form

$$P(x) = \sum_{i \in I} \epsilon_i a_i x^i,$$

where $a_i > 0$ and $\epsilon_i = \pm 1$. The *length* $\ell(P) = |I|$ is the number of terms in P.

Given $\zeta \in \mathbb{C}$, let $DP(\zeta)$ denote the effective divisor

$$DP(\zeta) = \sum_{i \in I} a_i \cdot \left(\epsilon_i \zeta^i\right).$$

If ζ is a root of unity and $P(\zeta) = 0$, then $DP(\zeta)$ is a prp.

Proof of Theorem 2.1. Let $P(x) = x^n Q(x) + R(x)$. Then there are finite sums $Q(x) = \sum Q_j(x)$ and $R(x) = \sum R_j(x)$ such that

$$DP(\zeta_k) = \sum_j DP_j(\zeta_k) = \sum_j \zeta_k^n DQ_j(\zeta_k) + DR_j(\zeta_k)$$

gives a decomposition of $DP(\zeta_k)$ into primitive prps.

If $\ell(Q_j) > 1$ for some j, then we have $o(DP_j(\zeta_k)) \geqslant k/(2\deg(Q))$, since the ratio of any two roots of unity occurring in $DQ_j(\zeta_k)$ has the form $\pm\zeta_k^e$ with $1 \leqslant e \leqslant \deg(Q)$. By Mann's theorem, $o(DP_j(\zeta_k))$ is bounded above by the product of the primes less than or equal to $\ell(P_j) \leqslant \ell(Q) + \ell(R)$, and so the desired upper bound for k follows. The same argument applies if $\ell(R_j) > 1$ for some j.

Now assume $\ell(Q_j) = \ell(R_j) = 1$ for all j, but the desired bound on k fails. Then $k > 4m$, where $m = \max(\deg(Q), \deg(R))$. Writing $Q_j(x) = a_j x^{e_j}$ and $R_j(x) = b_j x^{f_j}$, we have

$$\zeta^n Q_j(\zeta_k) + R_j(\zeta_k) = a_j \zeta_k^{n+e_j} + b_j \zeta_k^{f_j} = 0$$

for all j. Consequently $\zeta_k^{f_j - e_j} = \pm\zeta_k^n$ for all j. This implies $f_j - e_j$ is constant $\bmod k$ or $\bmod(k/2)$ (depending on the parity of k). But $k > 4m$ and $(f_j - e_j) \in [-m, m]$, so the difference of exponents $i = f_j - e_j$ is also constant in $\mathbb{Z}$. We then have

$$a_j\zeta_k^{n-i+f_j}+b_j\zeta_k^{f_j}=0$$

for all j; thus $\epsilon=\zeta_k^{n-i}=\pm 1$ and $\epsilon a_j+b_j=0$, which gives $\epsilon x^i Q_j(x)+R_j(x)=0$ and hence $Q(x)=\pm x^{-i}R(x)$. $\square$

Application to E_n. Now recall that for $n\geqslant 3$ we have

$$E_n(x)(x-1)=x^{n-2}\big(x^3-x-1\big)+\big(x^3+x^2-1\big)=x^{n-2}Q(x)+R(x).$$

Since $\deg(Q)=\deg(R)=3$ and $\ell(Q)+\ell(R)=6$, the theorem above implies

Corollary 2.3. *If $E_n(\zeta_k)=0$, then $k\leqslant 180$.*

Lemma 2.4. *The polynomial $E_n(x)$ is separable for all $n\neq 9$.*

Proof. The only possible multiple roots of $E_n(x)$ are in its cyclotomic factor $C_n(x)$. But for $|x|=1$ we have

$$\big|\big(E_n(x)(x-1)\big)'\big|>(n-2)\big|Q(x)\big|-\big|Q'(x)\big|-\big|R'(x)\big|>0.3(n-2)-9,$$

so $E_n(x)$ is separable once $n\geqslant 32$. The remaining cases are easily checked individually. $\square$

Proof of Theorem 1.1. It is straightforward to verify that the theorem is correct for $3\leqslant n\leqslant 182$. Thus $E_n(\zeta_k)=0$ for some n in this range, $n\neq 9$, iff $k\in\{2,3,5,8,12,18,30\}=K$.

By separability, the cyclotomic factor only depends on the roots of unity where $E_n(\zeta_k)=0$. But the vanishing of $E_n(\zeta_k)$ only depends on the value of $n \bmod k$, so by Corollary 2.3 no new roots of unity can occur as zeros of $E_n(x)$ for $n>182$. So once the theorem is checked for all $n\leqslant 182$ it also holds for all larger values of n. $\square$

3. Joins

In this section we define the *join* of a collection of Coxeter systems, and establish the following more precise version of Theorem 1.5.

Theorem 3.1. *Let (W,S) be the join of Coxeter systems $(W_i,S_i)_{i=1}^m$, with bicolored Coxeter elements w_i. Suppose λ is an eigenvalue of w_i with multiplicity $m_i\geqslant 0$. Then λ occurs as an eigenvalue of the bicolored Coxeter element $w\in W$ with multiplicity at least $(\sum m_i)-1$.*

Coxeter systems. Recall that a *Coxeter system* (W,S) is an abstract group W with a distinguished set of generators S, such that the product $st\in W$ of two generators has finite order $m_{st}\geqslant 2$, the generators themselves have order 2, and these relations give a presentation for W.

The pair (W,S) determines a quadratic form B on $\mathbb{R}^S$ with matrix $B_{st}=-2\cos(\pi/m_{st})$, and a geometric representation $W\hookrightarrow O(\mathbb{R}^S,B)$ where the generators act by the reflections

$$s\cdot v=v-B(e_s,v)e_s. \tag{3.1}$$

The *Coxeter diagram* F of (W,S) is the (undirected) graph with vertex set S and an edge of weight $m_{st}-2$ joining s to t whenever $m_{st}>2$. By convention an unlabeled edge has weight one, and i parallel unlabeled edges indicate a single edge of weight i.

The product of the generators $w=s_1\cdots s_n$ of W, taken in any order, is a *Coxeter element* of (W,S). If the diagram F is a tree, then the conjugacy class of w is independent of the choice of ordering. If F

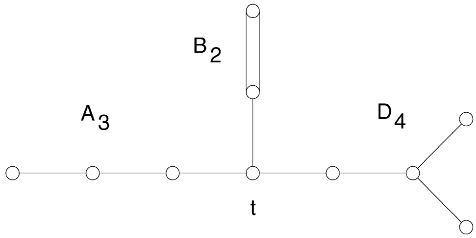

Fig. 5. The join of A_3, B_2 and D_4.

is bipartite (meaning we can write $S = S_0 \sqcup S_1$ with all edges connecting S_0 to S_1), then the *bicolored* Coxeter element

$$w = \prod S_0 \prod S_1$$

is well-defined up to conjugacy (cf. [Mc, §5]). Thus in Theorem 3.1 we implicitly assume the Coxeter systems (W_i, s_i) are bipartite.

The *Coxeter polynomial* of a bipartite Coxeter system (W, S) is the characteristic polynomial

$$F(x) = \det(xI - w)$$

of its bicolored Coxeter elements. We generally denote it using the same symbol as the diagram. Note that if the diagram F has no multiple edges, then W preserves the lattice $\mathbb{Z}^S$ and thus $F(x) \in \mathbb{Z}[x]$.

Pointed Coxeter systems. A *pointed* Coxeter system is a triple (W, S, s) with $s \in S$. It is determined up to isomorphism by a pointed diagram (F, s). By deleting s, we obtain a Coxeter subsystem $(\widehat{W}, \widehat{S})$ with Coxeter polynomial $\widehat{F}(x)$.

We let (A_n, i) and (E_n, i) denote the A_n and E_n diagrams with the ith vertex distinguished, using the numbering in Figs. 1 and 3.

Joins. The *join* (W, S) of pointed Coxeter systems $(W_i, S_i, s_i)_{i=1}^m$ is defined by taking an independent generator t, setting $S = \{t\} \cup S_i$, and setting

$$W = \big(W_1 * \cdots * W_m * \langle t\rangle\big) / \big\langle t^2 = (s_1 t)^3 = \cdots = (s_m t)^3 = \mathrm{id}\big\rangle.$$

The corresponding diagram F is obtained from $\sqcup F_i$ by adding a new vertex t and connecting it to each s_i with a single edge (see Fig. 5). If all the diagrams F_i are bipartite, so is F.

In Theorem 3.1, basepoints $s_i \in S_i$ must be chosen to make the join (W, S) well-defined, but the conclusion holds independent of the choice of basepoints.

Proof of Theorem 3.1. Let (W, S) be the join of $(W_i, S_i)_1^m$. By Eq. (3.1), a given reflection $s(v)$ only changes the coordinate v_s of a vector $v \in \mathbb{R}^S$. Thus we have natural inclusions $W_i \subset W$ compatible with the inclusions $\mathbb{R}^{S_i} \subset \mathbb{R}^S$.

Since $s, t \in S$ commute whenever they are not joined by an edge in the Coxeter diagram, we can write the bicolored Coxeter element $w \in W$ in the form

$$w = t w_1 \cdots w_m.$$

Let $E_i \subset \mathbb{C}^{S_i} \subset \mathbb{C}^S$ be the λ-eigenspaces for w_i, extended by zero in the remaining coordinates. By (3.1) we have $w_i | E_j = \mathrm{id}$ for $i \neq j$. Thus $\bigoplus E_i$ is a λ-eigenspace for $w_1 \cdots w_m$. Since $t(v)$ only

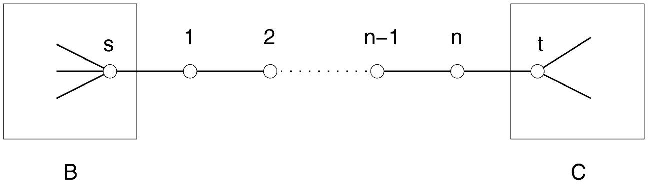

Fig. 6. The diagram F_n obtained by attaching (B, s) and (C, t) to the ends of A_n.

changes v_t, there is a codimension-one subspace $E \subset \bigoplus E_i$ such that $t|E = \mathrm{id}$. Consequently the multiplicity of λ as an eigenvalue for w is bounded below by

$$\dim(E) = \Bigl(\sum \dim(E_i)\Bigr) - 1 = \Bigl(\sum m_i\Bigr) - 1. \qquad \Box$$

The Coxeter polynomial of a join. Here is an alternative approach to the result above. When F is the join of $(F_i, s_i)_1^m$, a straightforward matrix computation yields the following useful formula for its Coxeter polynomial:

$$F(x) = F_1(x) \cdots F_m(x)\left((x+1) - x\sum_1^m \frac{\widehat{F}_i(x)}{F_i(x)}\right). \tag{3.2}$$

Cf. [CDS, Problem 9, p. 78], [MRS, Corollary 4].

By writing the Coxeter element of (W_i, S_i) with s_i at the end, one can verify that the order of vanishing of its Coxeter polynomial satisfies $\mathrm{ord}(P_i, \lambda) - 1 \leqslant \mathrm{ord}(\widehat{P}_i, \lambda)$. Thus Eq. (3.2) implies

$$\mathrm{ord}(F, \lambda) \geqslant -1 + \sum \mathrm{ord}(F_i, \lambda).$$

This inequality is equivalent to Theorem 3.1 when the quadratic form B of (W, S) is non-degenerate, as it is for E_n, $n \neq 9$.

4. Decorating A_n

In this section we generalize our results on E_n to more general diagrams F_n of the form shown in Fig. 6. Our main result is

Theorem 4.1. *Let F_n be the sequence of Coxeter diagrams obtained by attaching pointed diagrams (B, s) and (C, t) to the ends of A_n. Assume $F_n(x) \in \mathbb{Z}[x]$ for all n. Then either*

1. *the diagram F_n is spherical or affine for all n, or*
2. *the cyclotomic factor of $F_n(x)$ is periodic for $n \gg 0$.*

Coxeter polynomials. We begin by determining the Coxeter polynomial $F_n(x)$. First, by repeatedly applying Eq. (3.2) with $m = 1$, we obtain

Proposition 4.2. *The Coxeter polynomial of the diagram B_n obtained by attaching (B, s) to one end of A_n satisfies*:

$$B_n(x)(x-1) = x^{n+1}\bigl(B(x) - \widehat{B}(x)\bigr) + \bigl(x\widehat{B}(x) - B(x)\bigr).$$

Here is an example:

Corollary 4.3. *For $n \geqslant 4$, we have*

$$E_n(x)(x-1) = x^{n-2}\big(x^3 - x - 1\big) + \big(x^3 + x^2 - 1\big).$$

Proof. Take $(B,s) = (A_4, 2)$; then $B(x) = A_4(x)$, $\widehat{B}(x) = A_1(x)A_2(x)$, and $B_n(x) = E_{n+4}(x)$. Thus $B(x) - \widehat{B}(x) = x(x^3 - x - 1)$ and $x\widehat{B}(x) - B(x) = x^3 + x^2 - 1$, which gives

$$E_{n+4}(x)(x-1) = x^{n+2}\big(x^3 - x - 1\big) + \big(x^3 + x^2 - 1\big). \qquad \square$$

Since F_n is the join of B_{n-1} and C, by applying Eq. (3.2) once more we find

Proposition 4.4. *The Coxeter polynomials of F_n, (B,s) and (C,t) are related by $F_n(x)(x-1) = x^{n+1}Q(x) - R(x)$, where*

$$Q(x) = \big(B(x) - \widehat{B}(x)\big)\big(C(x) - \widehat{C}(x)\big) \quad \textit{and}$$
$$R(x) = \big(x\widehat{B}(x) - B(x)\big)\big(x\widehat{C}(x) - C(x)\big).$$

We will also need the following result. Let $\beta(F_n) \geqslant 1$ denote the largest real zero of $F_n(x)$; equivalently, the spectral radius of the bicolored Coxeter element for F_n.

Proposition 4.5 *(Hoffman–Smith). If $\beta(F_n) > 1$, then $\beta(F_n) \neq \beta(F_{n+1})$.*

Proof. Let $A_{st} = 2I - B_{st}$ denote the symmetric 'adjacency matrix' for the F_n diagram, and $\alpha(F_n)$ its spectral radius. Then since $\beta(F_n) > 1$, we have

$$\alpha(F_n) = \big(2 + \beta(F_n) + \beta(F_n)^{-1}\big)^{1/2} > 2$$

(see e.g. [Mc, Theorem 5.1]).

By [HS, Lemma 2.3, Proposition 2.4], the condition $\alpha(F_n) > 2$ implies that $\alpha(F_{n+1}) < \alpha(F_n)$ if we are adding nodes to an internal path, and that $\alpha(F_{n+1}) > \alpha(F_n)$ if we are adding nodes to an external path (i.e. if (B,s) or (C,t) is equal to $(A_i, 1)$.) (The proof in [HS] is given for graphs, but it applies without change to Coxeter diagrams, using the following key fact: if s is an endpoint of a maximal A_k embedded in F_n, either s is an endpoint of F_n, or $\sum_{t \neq s} A_{st} \geqslant 1 + \sqrt{2}$.)

In particular, we have $\alpha(F_{n+1}) \neq \alpha(F_n)$, and hence $\beta(F_n) \neq \beta(F_{n+1})$. $\square$

Proof of Theorem 4.1. Since $\deg B > \deg \widehat{B}$ and $\deg C > \deg \widehat{C}$, we have $Q(x) \neq 0$. By Theorem 2.1, either:

(i) $F_n(x)(x-1) = (x^n \pm x^i)Q(x)$, or
(ii) only finitely many k satisfy $F_n(\zeta_k) = 0$ for some n.

In case (i), the zeros of $F_n(x)$ outside the unit circle must be constant as n varies. This implies the spectral radius $\beta(F_n)$ of the bicolored Coxeter element is constant; hence $\beta(F_n) = 1$ by Proposition 4.5, which means F_n is spherical or affine by [A'C].

For case (ii), fix k such that $F_n(\zeta_k) = 0$. Clearly the values of $F_n(\zeta_k)$ are periodic in n. To complete the proof, we must show the order of vanishing of F_n at ζ_k is also periodic. For this we may assume $Q(x)$ and $R(x)$ are relatively prime. Then $Q(\zeta_k) \neq 0$, and hence for all $n \gg 0$, $F_n'(\zeta_k) \neq 0$, since the dominant term in the derivative is $(n+1)\zeta_k^n Q(\zeta_k)$. Consequently the cyclotomic zeros of F_n are simple for all $n \gg 0$, and the proof is complete. $\square$

Notes. For a survey of results on the largest eigenvalues of graphs, including the inequality of Hoffman and Smith used above, see [CR].

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