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Mean values of L-functions and Dedekind sums

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Text. For arbitrary non-negative integers $a_1, \ldots, a_d$ and $m_1, \ldots, m_d$, we introduce and investigate the mean value of the product

$$\overline{\chi}_1(a_1)\ldots\overline{\chi}_d(a_d)L(m_1+1,\chi_1)\ldots L(m_d+1,\chi_d),$$

such that $m_1, \ldots, m_d$ have the same parity and $\chi_i(-1) = (-1)^{m_i+1}$, $i = 1, \ldots, d$. Using recent results of the authors on Dedekind reciprocity law we give explicit formulae for this mean. Our studies recover and improve the previous works of Walum, Louboutin, Liu and Zhang.

Video. For a video summary of this paper, please click here or visit http://www.youtube.com/watch?v=FG2aZBD3VS8.

1. Introduction and preliminaries

1.1. Introduction

Let q be a positive integer $\geqslant 2$, fixed. Let χ be a character modulo q, and $L(s,\chi)$ be the Dirichlet L-function corresponding to χ: $L(s,\chi) = \sum_{n\geqslant 1}\frac{\chi(n)}{n^s}$, where $\Re e(s) > 0$ if χ is non-principal and $\Re e(s) > 1$ if χ is the principal character. Let $m_1, \ldots, m_d$ be non-negative integers. We shall here be interested by the study of the mean values

$$S_d(\vec{a},\vec{m},q) := \sum_{(\chi_1,\ldots,\chi_d)}^{*}\prod_{i=1}^{d}\overline{\chi}_i(a_i)L(m_i+1,\chi_i),$$

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where $\sum^*$ denotes summation over all characters $\chi_1, \dots, \chi_d \pmod q$ such that: $\chi_1 \dots \chi_d = 1$ and $\chi_1(-1) = \dots = \chi_d(-1) = (-1)^{m_1+1} = \dots = (-1)^{m_d+1}$.

In the case $d = 2$, $m_1 = m_2 = 0$ and $\chi_2 = \overline{\chi}_1$, Walum [9] showed that for prime $q = p \geqslant 3$,

$$\sum_{\substack{\chi_1 \pmod p \\ \chi_1(-1)=-1}} \left|L(1, \chi_1)\right|^2 = \frac{\pi^2 (p-1)^2 (p-2)}{12 p^2}. \tag{1.1.1}$$

This result has been extended by Louboutin [3,4] and Zhang [10] to any positive integer $q \geqslant 2$ by the formula

$$\sum_{\substack{\chi_1 \pmod q \\ \chi_1(-1)=-1}} \left|L(1, \chi_1)\right|^2 = \frac{\pi^2}{12} \frac{\varphi^2(q)}{q^2} \left(q \prod_{\substack{p|q \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right) - 3 \right) \tag{1.1.2}$$

where $\varphi(q)$ is the Euler function. Louboutin [5] has considered the case $d = 2$, $m_1 = m_2 = k$ and proved the formula

$$\frac{2}{\varphi(q)} \sum_{\substack{\chi_1 \pmod q \\ \chi_1(-1)=-1}} \left|L(k, \chi_1)\right|^2 = \frac{(2\pi)^{2k}}{2((k-1)!)^2} \sum_{l=0}^{2k} r_{k,l} \varphi_l(q) q^{l-2k}, \tag{1.1.3}$$

where

$$\varphi_l(q) := \prod_{\substack{p|q \\ p \text{ prime}}} \left(1 - \frac{1}{p^l}\right),$$

and the coefficients $r_{k,l}$ are real numbers that were not given explicitly. In 2006, Liu and Zhang [2] treated the mean values of $L(m, \chi_1)L(n, \overline{\chi}_1)$ at positive integers $m, n \geqslant 1$,

$$\begin{aligned} &\frac{2}{\varphi(q)} \sum_{\substack{\chi_1 \pmod q \\ \chi_1(-1)=-1}} L(m, \chi_1) L(n, \overline{\chi}_1) \\ &\quad = \frac{(-1)^{\frac{m-n}{2}} (2\pi)^{m+n}}{2(m!n!)} \left(\sum_{l=0}^{m+n} r_{m,n,l} \varphi_l(q) q^{l-m-n} - \frac{\epsilon_{m,n}}{q} B_m B_n \varphi_{m+n-1}(q) \right), \end{aligned} \tag{1.1.4}$$

where

$$r_{m,n,l} = B_{m+n-l} \sum_{a=0}^{m} \sum_{b=0}^{n} B_{m-a} B_{n-b} \frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{m+n-l}}{a+b+1},$$

and B_m is the m-th Bernoulli number defined, as well known, by the generating function

$$\frac{z}{e^z - 1} = \sum_{k \geqslant 0} \frac{B_k}{k!} z^k, \quad |z| \leqslant 2\pi. \tag{1.1.5}$$

We refer to [6] for some applications of these explicit formulae. In [6] the author gives explicit upper bounds on relative class numbers of cyclotomic fields by the study of mean values of L-functions of prime conductors.

In this paper we study these explicit formulae in general case. Namely, $d \geqslant 2$ and $m_1, \ldots, m_d$ are arbitrary non-negative integers. In particular, in the case $d = 2$ our results improve the previous works of Louboutin, Liu and Zhang. Especially, in this case our formulae are very simple and explicit.

1.2. Dedekind reciprocity

In this subsection we review the Dedekind reciprocity law for cotangents Dedekind sums [1]. We need it to illuminate the formulation of Theorems 2.1.1, 2.1.3 and 2.1.4. Let d, a_i be positive integers, $a_0, \ldots, \widehat{a_i}, \ldots, a_d$ be positive integers prime to a_i and $m_0, \ldots, m_d$ be non-negative integers. For $i = 0, \ldots, d$, we consider the multiple Dedekind–Rademacher sum defined by

$$C(a_i; a_0, \ldots, \widehat{a_i}, \ldots, a_d \mid m_i; m_0, \ldots, \widehat{m_i}, \ldots, m_d) := \begin{cases} \frac{1}{a_i^{m_i+1}} \sum_{k=1}^{a_i-1} \prod_{\substack{j=0\\ j\neq i}}^{d} \cot^{(m_j)}(\frac{\pi a_j k}{a_i}) & \text{if } a_i \geqslant 2, \\ 0 & \text{if } a_i = 1. \end{cases} \tag{1.2.6}$$

As usual, $\widehat{x_n}$ means we omit the term x_n and $\cot^{(m)}$ denotes the m-th derivative of the cotangent function.

Next we state the reciprocity law for these sums that allows us to compute them.

Theorem 1.2.1. *(See [1].) Let d be a positive integer, $a_0, \ldots, a_d$ be pairwise coprime positive integers and $m_0, \ldots, m_d$ be non-negative integers. Assume that the integer*

$$M = d + m_0 + \cdots + m_d \quad \textit{is even}.$$

Then we have

$$\begin{aligned} &\sum_{i=0}^{d} (-1)^{m_i} m_i! \sum_{\substack{\ell_0, \ldots, \widehat{\ell_i}, \ldots, \ell_d \geqslant 0\\ \ell_0 + \cdots + \widehat{\ell_i} + \cdots + \ell_d = m_i}} \left(\prod_{\substack{j=0\\ j\neq i}}^{d} \frac{a_j^{\ell_j}}{\ell_j!} \right) \\ &\quad \times C(a_i; a_0, \ldots, \widehat{a_i}, \ldots, a_d \mid m_i; m_0 + \ell_0, \ldots, \widehat{m_i + \ell_i}, \ldots, m_d + \ell_d) \\ &= \begin{cases} R + (-1)^{d/2} & \textit{if all } m_i \textit{ are zero}, \\ R & \textit{otherwise}, \end{cases} \end{aligned}$$

where

$$R = \frac{(-1)^{M/2} 2^M}{\prod_{i=0}^{d} a_i^{m_i+1}} \sum_{\substack{j_0, \ldots, j_d \geqslant 0\\ j_0 + \cdots + j_d = M/2}} \prod_{i=0}^{d} a_i^{2j_i} A_{i, j_i}, \tag{1.2.7}$$

and

$$A_{i, j_i} = \begin{cases} \frac{B_{2j_i}}{(2j_i - 1 - m_i)!(2j_i)} & \textit{if } j_i \textit{ is an integer} \geqslant (m_i + 1)/2, \\ (-1)^{m_i} m_i! & \textit{if } j_i = 0, \\ 0 & \textit{otherwise}. \end{cases}$$

Example 1. If the integer M is odd, since $\sum_{\substack{0\leqslant j\leqslant d\\ j\neq i}}(m_j+\ell_j+1)=M$, each term on the left-hand side of the reciprocity formula in Theorem 1.2.1 is zero.

Example 2. When all m_i are zero, we have $M=d$ and $A_{i,j_i}=\frac{(-1)^{j_i}2^{2j_i}B_{2j_i}}{(2j_i)!}$, hence the right member of the reciprocity formula in Theorem 1.2.1 becomes

$$R+(-1)^{d/2}=(-1)^{d/2}\left(1-\frac{2^d}{a_0\dots a_d}\sum_{\substack{j_0,\dots,j_d\geqslant 0\\ j_0+\dots+j_d=d/2}}\prod_{i=0}^{d}\frac{B_{2j_i}}{(2j_i)!}a_i^{2j_i}\right). \tag{1.2.8}$$

Example 3. The case $d=2$, and m_0, m_1, m_2 are arbitrary non-negative integers. Then we have $M=2+m_0+m_1+m_2$. We can write R as

$$R=2K(S_0+S_1+S_2)+4K(S_{0,1}+S_{0,2}+S_{1,2})$$

where

$$K=-\frac{(-1)^{(m_0+m_1+m_2)/2}2^{m_0+m_1+m_2-1}}{a_0^{m_0+1}a_1^{m_1+1}a_2^{m_2+1}},$$

$$S_0=(-1)^{m_0}m_0!\sum_{\substack{j_1\geqslant(m_1+1)/2\\ j_2\geqslant(m_2+1)/2\\ j_1+j_2=M/2}}\left(\frac{1}{(2j_1-m_1-1)!(2j_2-m_2-1)!}\frac{B_{2j_1}B_{2j_2}}{j_1j_2}a_1^{2j_1}a_2^{2j_2}\right),$$

$$S_{0,1}=(-1)^{m_0+m_1}m_0!m_1!\frac{2}{(1+m_0+m_1)!}\frac{B_M}{M}a_2^M.$$

The expression of S_1 (resp. S_2) is obtained by replacing symbolically in S_0 the indices 0, 1, 2 by 1, 2, 0 (resp. by 2, 0, 1). Similarly, the expression of $S_{1,2}$ (resp. $S_{0,2}$) is obtained by replacing symbolically in $S_{0,1}$ the index i by $i+1$ mod d (resp. by $i+2$ mod d).

2. Main results

We are now able to give the main results of this paper and their proofs.

2.1. Statement of the main results

Let d be an integer $\geqslant 1$. If $\vec{\mathbf{m}}=(m_1,\dots,m_d)$ is a d-tuple of positive integers, we use the notations: $|\vec{\mathbf{m}}|=\sum_{i=1}^d m_i$ and $\vec{\mathbf{m}}!=\prod_{1\leqslant i\leqslant d}m_i!$.

Theorem 2.1.1. *Let d be an integer $\geqslant 1$ and $\vec{\mathbf{m}}=(m_1,\dots,m_d)$ a d-tuple of non-negative integers such that $M:=d+|\vec{\mathbf{m}}|$ is even. Let q be an integer $\geqslant 2$. Let $a_1,\dots,a_{d-1}$ be positive such that $(a_i,q)=1$ ($i=1,\dots,d-1$). We set $a_d=1$. Then we have*

$$S_d(\vec{a},\vec{m},q)=A_q(\vec{\mathbf{m}})\sum_{\substack{b\mid q\\ b\neq 1}}\mu\left(\frac{q}{b}\right)\sum_{k=1}^{b-1}\prod_{i=1}^{d}\cot^{(m_i)}\left(\frac{\pi k a_i}{b}\right), \tag{2.1.9}$$

where $A_q(\vec{\mathbf{m}})=\frac{(-1)^d}{2^d(\vec{\mathbf{m}}!)}(\frac{\pi}{q})^M\varphi(q)^{d-1}$.

In the special case when $d=2$, the theorem gives immediately

Corollary 2.1.2. *Let m and n be positive having the same parity. Let a be a positive integer such that $(a,q)=1$. Then we have*

$$\sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^{m+1}}} \overline{\chi}(a)L(m+1,\chi)L(n+1,\overline{\chi}) = A\sum_{\substack{b|q \\ b\neq 1}} \mu\left(\frac{q}{b}\right)\sum_{k=1}^{b-1}\cot^{(m)}\left(\frac{\pi ka}{b}\right)\cot^{(n)}\left(\frac{\pi k}{b}\right),$$

where $A=\frac{\varphi(q)}{4m!n!}(\frac{\pi}{q})^{m+n+2}$.

For every real $\alpha>0$, let J_α be the Jordan's totient function defined for all positive integer n by

$$J_\alpha(n) := n^\alpha\sum_{m|n}\frac{\mu(m)}{m^\alpha} = n^\alpha\prod_{\substack{p|n \\ p\text{ prime}}}\left(1-\frac{1}{p^\alpha}\right), \quad \text{see [8]}$$

where μ is the Mobius function. For $\alpha=1$, this is, of course, Euler's function φ.

We obtain the following theorem:

Theorem 2.1.3. *Let q be an integer $\geqslant 2$. Let d be an integer $\geqslant 1$ and $\vec{\mathbf{m}}=(m_1,\ldots,m_d)$ a d-tuple of non-negative integers such that the number $M:=d+|\vec{\mathbf{m}}|$ is even. Then*

(i) *if $\vec{\mathbf{m}}\neq\vec{\mathbf{0}}$ we have*

$$S_d(\vec{1},\vec{m},q) = D_q(\vec{\mathbf{m}})\left(\sum_{j_0=1}^{M/2}\left(\sum_{\substack{j_1,\ldots,j_d\geqslant 0 \\ j_1+\cdots+j_d=M/2-j_0 \\ j_i=0 \text{ or } \geqslant(m_i+1)/2}}\prod_{i=1}^{d}A_{i,j_i}\right)\frac{B_{2j_0}}{(2j_0)!}J_{2j_0}(q)\right)$$

where

$$D_q(\vec{\mathbf{m}}) = (-1)^{M/2}2^M A_q(\vec{\mathbf{m}}).$$

(ii) *if $\vec{\mathbf{m}}=\vec{\mathbf{0}}$ we have*

$$S_d(\vec{1},\vec{m},q) = D_q(\vec{\mathbf{0}})\left(2^{-d}\varphi(q)-\sum_{j_0=1}^{d/2}\left(\sum_{\substack{j_1,\ldots,j_d\geqslant 0 \\ j_1+\cdots+j_d=d/2-j_0}}\prod_{i=1}^{d}\frac{B_{2j_i}}{(2j_i)!}\right)\frac{B_{2j_0}}{(2j_0)!}J_{2j_0}(q)\right)$$

where $D_q(\vec{\mathbf{0}})=(-1)^{d/2}(\frac{\pi}{q})^d\varphi(q)^{d-1}$.

As an immediate consequence, taking $d=2$, $m_1=m_2=0$ we obtain a sensitive improvement of Louboutin, Liu and Zhang results [4,5,10,2].

Theorem 2.1.4. *Let m and n be two positive integers having the same parity. Then*:

- *If* $(m,n)\neq(1,1)$, *we have*

$$\frac{2}{\varphi(q)}\sum_{\substack{\chi\\ \chi(-1)=(-1)^m}} L(m,\chi)L(n,\overline{\chi})=\frac{1}{2}(-1)^{\frac{m+n}{2}}\left(\frac{2\pi}{q}\right)^{m+n}(M_1+M_2+M_3)$$

where

$$M_1=\frac{B_{m+n}}{(m+n)!}J_{m+n}(q),$$

$$M_2=\frac{(-1)^{m-1}}{(n-1)!m!}\sum_{j=1}^{[m/2]}\binom{m}{2j}\frac{B_{m+n-2j}}{m+n-2j}B_{2j}J_{2j}(q),$$

$$M_3=\frac{(-1)^{n-1}}{(m-1)!n!}\sum_{j=1}^{[n/2]}\binom{n}{2j}\frac{B_{m+n-2j}}{m+n-2j}B_{2j}J_{2j}(q).$$

- *If* $m=n=1$, *we recover* (1.1.2)

$$\frac{2}{\varphi(q)}\sum_{\substack{\chi\\ \chi(-1)=-1}}\left|L(1,\chi)\right|^2=\frac{\pi^2}{6}\frac{\varphi(q)}{q^2}\left(q\prod_{\substack{p\mid q\\ p\text{ prime}}}\left(1+\frac{1}{p}\right)-3\right).$$

Remark 2.1.5. For any given positive integer a, in the recent preprint [7] S. Louboutin gives an explicit formula for the moment

$$\sum_{\substack{\chi\\ \chi(-1)=(-1)}}\chi(a)\left|L(1,\overline{\chi})\right|^2.$$

2.2. Proof of the main results

2.2.1. Proof of Theorem 2.1.1

To prove this theorem we need the following two lemmas.

Lemma 2.2.2. *Let χ be a character modulo $q\geqslant 2$. Let m be a positive integer such that $\chi(-1)=(-1)^{m+1}$. Then we have*

$$L(m+1,\chi)=\frac{(-1)^m\pi^{m+1}}{2q^{m+1}(m!)}\sum_{k=1}^{q-1}\chi(k)\cot^{(m)}\left(\frac{\pi k}{q}\right). \tag{2.2.10}$$

Let us set

$$\delta_v(u)=\begin{cases}1 & \text{if } u\equiv v \pmod{q};\\ 0 & \text{otherwise.}\end{cases}$$

Proof of Lemma 2.2.2. This lemma has been proved by Louboutin [5] for $q\geqslant 3$. The same proof works also for $q=2$. □

Lemma 2.2.3. *Let q be an integer $\geqslant 2$, u and v be two positive integers such that $(uv, q) = 1$. Let $\varepsilon \in \{-1, 1\}$, fixed. Then we have*

$$\sum_{\substack{\chi \pmod q \\ \chi(-1)=\varepsilon}} \chi(u)\overline{\chi}(v) = \frac{\varphi(q)}{2}\bigl(\delta_v(u) + \varepsilon\delta_{-v}(u)\bigr). \tag{2.2.11}$$

Proof. For $q \geqslant 3$ the proof of this lemma is given in [5]. So we omit it. For $q = 2$, we have only one character χ, which is given by $\chi(k) = 0$ if $2 \mid k$ and $\chi(k) = 1$ otherwise, and (2.2.11) is clearly satisfied. Thus, the lemma yields. $\square$

End of the proof of Theorem 2.1.1. Let $\sum$ denote the sum on the left of (2.1.9). Let $c_i = -\frac{\chi_i(-1)}{2(m_i!)}(\frac{\pi}{q})^{m_i+1}$ $(i = 1, \dots, d)$. By Lemma 2.2.2, we can write

$$\begin{aligned} S_d(\vec{a}, \vec{m}, q) &= \sum_{(\chi_1,\dots,\chi_d)}^{*} \prod_{i=1}^{d} \overline{\chi}_i(a_i)c_i \sum_{k_i=1}^{q-1} \chi_i(k_i)\cot^{(m_i)}\left(\frac{\pi k_i}{q}\right) \\ &= c\sum_{k_1,\dots,k_d=1}^{q-1} \cot^{(m_d)}\left(\frac{\pi k_d}{q}\right)\prod_{i=1}^{d-1}\cot^{(m_i)}\left(\frac{\pi k_i}{q}\right)\sum_{\chi_i}\chi_i(k_i)\overline{\chi}_i(a_ik_d), \end{aligned}$$

where $c = \prod_{i=1}^{d} c_i$. We will use Lemma 2.2.3 and that the function $g_i : t \mapsto \cot^{(m_i)}(\pi t/q)$ is periodic with period q and verify: $g_i(-t) = (-1)^{m_i+1}g_i(t) = \chi_i(-1)g_i(t)$. We have

$$\begin{aligned} S_d(\vec{a}, \vec{m}, q) &= c\left(\frac{\varphi(q)}{2}\right)^{d-1} \sum_{\substack{k_d=1 \\ (k_d,q)=1}}^{q-1} \cot^{(m_d)}\left(\frac{\pi k_d}{q}\right) \sum_{I\subset\{1,\dots,d-1\}} \sum_{k_1,\dots,k_{d-1}=1}^{q-1} \\ &\quad \times \prod_{i\in I}\delta_{a_ik_d}(k_i)\cot^{(m_i)}\left(\frac{\pi k_i}{q}\right) \prod_{i\in\{1,\dots,d-1\}\smallsetminus I} \chi_i(-1)\delta_{-a_ik_d}(k_i)\cot^{(m_i)}\left(\frac{\pi k_i}{q}\right) \\ &= c\left(\frac{\varphi(q)}{2}\right)^{d-1} \sum_{\substack{k_d=1 \\ (k_d,q)=1}}^{q-1} \cot^{(m_d)}\left(\frac{\pi k_d}{q}\right) \sum_{I\subset\{1,\dots,d-1\}} \prod_{i=1}^{d-1}\cot^{(m_i)}\left(\frac{\pi k_d a_i}{q}\right) \\ &= c\varphi(q)^{d-1} \sum_{\substack{h=1 \\ (h,q)=1}}^{q-1} \prod_{i=1}^{d}\cot^{(m_i)}\left(\frac{\pi h a_i}{q}\right). \end{aligned}$$

To complete the proof of Theorem 2.1.1, notice that

$$\sum_{\substack{h=1 \\ (h,q)=1}}^{q-1} f\left(\frac{h}{q}\right) = \sum_{\substack{b\mid q \\ b\neq 1}} \mu\left(\frac{q}{b}\right)\sum_{k=1}^{b-1} f\left(\frac{k}{b}\right)$$

for any function $f : \mathbb{Q} \to \mathbb{C}$. $\square$

2.2.4. Proof of Theorem 2.1.3

From the definition (1.2.6), taking $a_0 = b > 1$ and $m_0 = 0$, we have

$$C(b; a_1, \dots, a_d \mid 0; m_1, \dots, m_d) = \frac{1}{b} \sum_{k=1}^{b-1} \prod_{i=1}^{d} \cot^{(m_i)}\left(\frac{\pi k a_i}{b}\right)$$

and for $b = 1$

$$C(b; a_1, \dots, a_d \mid 0; m_1, \dots, m_d) = 0.$$

Then we can write

$$\sum_{\substack{b \mid q \\ b \neq 1}} \mu\left(\frac{q}{b}\right) \sum_{k=1}^{b-1} \prod_{i=1}^{d} \cot^{(m_i)}\left(\frac{\pi k a_i}{b}\right) = \sum_{b \mid q} \mu\left(\frac{q}{b}\right) b C(b; a_1, \dots, a_d \mid 0; m_1, \dots, m_d).$$

By using Theorems 1.2.1, 2.1.1 and the well-known identities

$$\sum_{b \mid q} \mu\left(\frac{q}{b}\right) b^{2j} = J_{2j}(q) \quad \text{and} \quad \sum_{b \mid q} \mu\left(\frac{q}{b}\right) b = \varphi(q)$$

we obtain the desired theorem.

2.2.5. Proof of Theorem 2.1.4

Taking $d = 2$, $m_1 = m_2 = 0$ and by using Theorem 2.1.3, we obtain directly Theorem 2.1.4.

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Supplementary material

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