Journal of Number Theory 132 (2012) 2700–2723

Contents lists available at SciVerse ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Hecke eigenvalues and relations for degree 2 Siegel Eisenstein series

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A R T I C L E   I N F O

Article history:
Received 19 April 2012
Accepted 7 May 2012
Available online 21 July 2012
Communicated by J. Brian Conrey

Keywords:
Siegel modular form
Eisenstein series
Hecke operators

MSC:
11F41

A B S T R A C T

We evaluate the action of Hecke operators on Siegel Eisenstein series of degree 2, square-free level $\mathcal{N}$ and arbitrary character χ, without using knowledge of their Fourier coefficients. From this we construct a basis of simultaneous eigenforms for the full Hecke algebra, and we compute their eigenvalues. As well, we obtain Hecke relations among the Eisenstein series. Using these Hecke relations, we discuss how to generate the Fourier series of Eisenstein series in a basis from the Fourier series of one basis element.

1. Introduction

Modular forms are of central interest in number theory, particularly because Hecke theory tells us that their Fourier coefficients carry number theoretic information. Eisenstein series are fundamental examples of modular forms and play an important role in the theory of modular forms. In the case of elliptic modular forms (i.e. Siegel degree 1), the Eisenstein series are well-understood; for instance, we have explicit formulas for their Fourier coefficients, we know that the space of Eisenstein series can be simultaneously diagonalised with respect to the Hecke operators attached to primes not dividing the level, and when the level is square-free, the space can be simultaneously diagonalised with respect to the full Hecke algebra. In the case of Siegel degree $n > 1$, the situation is less well understood, but some parallel results have been established. Freitag [3] has shown that the space of Eisenstein series can be simultaneously diagonalised with respect to the Hecke operators attached to primes not dividing the level. Many authors have worked on computing Fourier coefficients of Siegel Eisenstein series with $n > 1$. We do not try to give a comprehensive list of all the work that has contributed

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0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2012.05.019

to this, but rather give a sampling. For level 1, see [8,9] for degree 2; [5] for degree 3; [1,2,6,7] for arbitrary degree. For degree 2, level $\mathcal{N}$ and primitive character modulo $\mathcal{N}$, Fourier coefficients for 1 of the Eisenstein series $\mathbb{E}_{(\mathcal{N},1,1)}$ have been computed in [10] when $\mathcal{N}$ is odd and square-free, and in [11] for arbitrary $\mathcal{N}$.

In this work, without using any knowledge of Fourier coefficients, we evaluate the action of Hecke operators on the natural basis $\{\mathbb{E}_\rho\}$ for the space of degree 2 Siegel Eisenstein series of square-free level $\mathcal{N}$ and arbitrary character χ (Propositions 3.3–3.10); here $\rho = (\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)$ varies so that $\mathcal{N}_0\mathcal{N}_1\mathcal{N}_2 = \mathcal{N}$ and $\chi^2_{\mathcal{N}_1} = 1$. This evaluation reveals Hecke relations among these Eisenstein series in the case that χ^2 is not primitive. Using these relations, we construct another basis $\{\widetilde{\mathbb{E}}_\rho\}$ consisting of eigenforms for the full Hecke algebra (which is generated by $\{T(p), T_1(p^2)\colon\ p \text{ prime}\}$); when χ^2 is primitive, $\widetilde{\mathbb{E}}_\rho = \mathbb{E}_\rho$. Then for any prime p, the eigenvalues of $\widetilde{\mathbb{E}}_\rho$ for $T(p)$ and $T_1(p^2)$ are

$$\big(\chi_{\mathcal{N}_0}(p)p^{k-1} + \chi_{\mathcal{N}_1\mathcal{N}_2}(p)\big)\big(\chi_{\mathcal{N}_0\mathcal{N}_1}(p)p^{k-2} + \chi_{\mathcal{N}_2}(p)\big)$$

and

$$\big(p + \chi_{\mathcal{N}_1}(p^2)\big)\big(\chi_{\mathcal{N}_0}(p^2)p^{2k-3} + \chi(p)p^{k-3}(p-1) + \chi_{\mathcal{N}_2}(p^2)\big)$$

(Theorem 3.11). In the case that χ^2 is not primitive, these Hecke relations also allow us to generate some of the other Eisenstein series from $\mathbb{E}_{(\mathcal{N},1,1)}$ by applying particular elements of the Hecke algebra; in particular, when $\chi = 1$, we can generate a basis from $\mathbb{E}_{(\mathcal{N},1,1)}$ (Theorem 3.12). In the remark following this theorem, we briefly discuss how we can use [4] and the Fourier coefficients of the degree 2, level 1 Eisenstein series $\mathbb{E}$ to generate the Fourier coefficients of all the degree 2, level $\mathcal{N}$ Eisenstein series in the case that $\mathcal{N}$ is square-free and the character $\chi = 1$.

2. Preliminaries

Here we set notation and define degree 2 Siegel Eisenstein series and Hecke operators. We begin by fixing square-free $\mathcal{N} \in \mathbb{Z}_+$. With $Sp_2(\mathbb{Z})$ the group of 4×4 integral symplectic matrices, we set

$$\Gamma_\infty = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in Sp_2(\mathbb{Z}) \right\},$$

$$\Gamma_0(\mathcal{N}) = \left\{ \gamma \in Sp_2(\mathbb{Z})\colon\ \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\ (\mathcal{N}) \right\}.$$

The 0-dimensional cusps for $\Gamma_0(\mathcal{N})$ correspond to the elements of the double coset $\Gamma_\infty \setminus Sp_2(\mathbb{Z})/\Gamma_0(\mathcal{N})$. For $k \in \mathbb{Z}_+$ and χ a Dirichlet character modulo $\mathcal{N}$, we have one Siegel Eisenstein series for each cusp, defined as follows. For $\gamma_0 \in Sp_2(\mathbb{Z})$, the Eisenstein series associated to the cusp $\Gamma_\infty\gamma_0\Gamma_0(\mathcal{N})$ is

$$\mathbb{E}_{\gamma_0}(\tau) = \sum \overline{\chi}(\det D_{\gamma_0^{-1}\gamma})1|\gamma(\tau)$$

where $\Gamma_\infty\gamma$ varies over the $\Gamma_0(\mathcal{N})$-orbit of $\Gamma_\infty\gamma_0$,

$$\tau \in \mathcal{H}_{(2)} = \big\{X + iY\colon\ X, Y \in \mathbb{R}^{2,2}_{\text{sym}},\ Y > 0\big\}$$

where $Y > 0$ denotes that Y is the matrix for a positive definite quadratic form, and

$$1\bigg|\begin{pmatrix} A & B \\ C & D \end{pmatrix}(\tau) = \det(C\tau + D)^{-k}.$$

This sum is well-defined provided $\chi_q^2 = 1$ whenever q is a prime dividing $\mathcal{N}$ and $\operatorname{rank}_q M_0 = 1$ where $\gamma_0 = \begin{pmatrix} * & * \\ M_0 & N_0 \end{pmatrix}$ and $\operatorname{rank}_q M_0$ denotes the rank of M_0 modulo q. When well-defined, the sum is non-zero provided $\chi(-1) = (-1)^k$, and it is absolutely uniformly convergent on compact regions provided $k \geqslant 4$ (and hence it is analytic, meaning analytic in each variable of τ). For $\gamma' \in \Gamma_0(\mathcal{N})$, $\Gamma_\infty \gamma \gamma'$ varies over the $\Gamma_0(\mathcal{N})$-orbit of $\Gamma_\infty \gamma_0$ as $\Gamma_\infty \gamma$ does, and hence $\mathbb{E}_{\gamma_0} | \gamma' = \chi(\det D_{\gamma'}) \mathbb{E}_{\gamma_0}$. As noted in [3], these Eisenstein series are linearly independent, and the 0th Fourier coefficient of $\mathbb{E}_{\gamma_0}$ is 0 unless $\gamma_0 \in \Gamma_0(\mathcal{N})$, in which case it is 1.

A pair of 2×2 matrices $(M\ N)$ is called symmetric if $M\,{}^tN = N\,{}^tM$ with ${}^tN$ denoting the transpose of N; it is called a coprime pair if M, N are integral and $(GM\ GN)$ is integral only if G is. Note that $(M\ N)$ is a coprime pair if and only if, for each prime p, $\operatorname{rank}_p(M\ N) = 2$. It is well known that for $\gamma, \gamma' \in Sp_2(\mathbb{Z})$, γ and γ' lie in the same coset in $\Gamma_\infty \backslash Sp_2(\mathbb{Z})$ if and only if $\gamma = \begin{pmatrix} * & * \\ M & N \end{pmatrix}$, $\gamma' = \begin{pmatrix} * & * \\ GM & GN \end{pmatrix}$ for some $G \in GL_2(\mathbb{Z})$. Thus these cosets can be parameterised by $GL_2(\mathbb{Z})$-equivalence classes of coprime symmetric pairs; so $\mathbb{E}_{\gamma_0}$ is supported on a set of $GL_2(\mathbb{Z})$-equivalence class representatives for the $\Gamma_0(\mathcal{N})$-orbit of $GL_2(\mathbb{Z})(M_0\ N_0)$.

For each prime p, we have Hecke operators $T(p)$ and $T_1(p^2)$ that act on degree 2 Siegel modular forms, and $\{T(p), T_1(p^2):\ p \text{ prime}\}$ generates the Hecke algebra. For f a degree 2 Siegel modular form of weight k, level $\mathcal{N}$, and character χ, and for $\gamma' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we set

$$\gamma' \circ \tau = (A\tau + B)(C\tau + D)^{-1}$$

and

$$f(\tau)\big|\gamma' = \left(\det \gamma'\right)^{k/2} \det(C\tau + D)^{-k} f\left(\gamma' \circ \tau\right).$$

Then

$$f|T(p) = p^{k-3} \sum_{\gamma} \overline{\chi}(\det D_\gamma) f\big|\delta^{-1}\gamma$$

where $\delta = \begin{pmatrix} pI_2 & \\ & I_2 \end{pmatrix}$ and γ varies over a set of coset representatives for

$$\left(\delta \Gamma_0(\mathcal{N}) \delta^{-1} \cap \Gamma_0(\mathcal{N})\right) \backslash \Gamma_0(\mathcal{N}).$$

Somewhat similarly,

$$f|T_1\left(p^2\right) = p^{k-3} \sum_{\gamma} \overline{\chi}(\det D_\gamma) f\big|\delta_1^{-1}\gamma$$

where $\delta_1 = \begin{pmatrix} X & \\ & X^{-1} \end{pmatrix}$, $X = \begin{pmatrix} p & \\ & 1 \end{pmatrix}$, and γ varies over a set of coset representatives for

$$\left(\delta_1 \Gamma_0(\mathcal{N}) \delta_1^{-1} \cap \Gamma_0(\mathcal{N})\right) \backslash \Gamma_0(\mathcal{N}).$$

In Propositions 2.1 and 3.1 of [4] we computed an explicit set of upper triangular block matrices giving the action of the Hecke operators, and we will use these here in evaluating the action of Hecke operators on Eisenstein series. (Note that in [4] we did not introduce the normalisation of $T_1(p^2)$ until we averaged the Hecke operators to produce an alternative basis for the Hecke algebra.)

Given $Q \in \mathbb{Z}_{\text{sym}}^{2,2}$ and $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$, p prime, we can think of Q as a quadratic form on $V = \mathbb{F}x_1 \oplus \mathbb{F}x_2$. We say a non-zero vector $v \in V$ is isotropic if $Q(v) = 0$ (in $\mathbb{F}$). Suppose p is odd. Then Q is a $GL_2(\mathbb{F})$ conjugate of $\mathbb{H} = \langle 1, -1 \rangle$ or of $\mathbb{A} = \langle 1, -\omega \rangle$ where $(\frac{\omega}{p}) = -1$ and $\langle *, * \rangle$ denotes a diagonal matrix; we write $V \simeq \mathbb{H}$ or $V \simeq \mathbb{A}$ accordingly. Note that when $V \simeq \mathbb{H}$, V contains 2 isotropic lines, and when

$V \simeq \mathbb{A}$, V contains no isotropic lines. Now suppose $p=2$; then either Q is a $GL_2(\mathbb{F})$ conjugate of I or (over $\mathbb{F}$) of $Q = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (which is stabilised under conjugation by $GL_2(\mathbb{F})$). When $V \simeq I$, V contains 1 isotropic line; when $V \simeq \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, all 3 lines in V are isotropic.

3. Action of Hecke operators on Eisenstein series of square-free level

Throughout this section, we assume $k \in \mathbb{Z}_+$ with $k \geqslant 4$, and that $\mathcal{N}$ is square-free. We first show that we can parameterise the $\Gamma_0(\mathcal{N})$ cusps by multiplicative partitions $\rho = (\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)$ where $\mathcal{N}_0\mathcal{N}_1\mathcal{N}_2 = \mathcal{N}$: For such ρ, fix diagonal $M_\rho \in \mathbb{Z}^{2,2}$ so that

$$M_\rho \equiv \begin{cases} 0 & (\mathcal{N}_0), \\ \left(\begin{smallmatrix} 1 & \\ & 0 \end{smallmatrix}\right) & (\mathcal{N}_1), \\ I & (\mathcal{N}_2). \end{cases}$$

Then set $\gamma_\rho = \left(\begin{smallmatrix} I & 0 \\ M_\rho & I \end{smallmatrix}\right)$. For $\gamma = \left(\begin{smallmatrix} K & L \\ M & N \end{smallmatrix}\right)$, $\gamma' = \left(\begin{smallmatrix} K' & L' \\ M' & N' \end{smallmatrix}\right) \in Sp_2(\mathbb{Z})$, we know that $\Gamma_\infty\gamma = \Gamma_\infty\gamma'$ if and only if $G(M\ N) = (M'\ N')$ for some $G \in GL_2(\mathbb{Z})$. Thus it suffices to show that for a coprime symmetric pair $(M\ N)$, we have $GL_2(\mathbb{Z})(M\ N)$ in the $\Gamma_0(\mathcal{N})$-orbit of $GL_2(\mathbb{Z})(M_\rho\ I)$ if and only if $\operatorname{rank}_q M = \operatorname{rank}_q M_\rho$ for all primes $q|\mathcal{N}$.

To do this, suppose first that $(M\ N)$ is a coprime symmetric pair so that $(M\ N) \equiv (M_\rho\ I)\ (\mathcal{N})$. So there is some $\left(\begin{smallmatrix} K & L \\ M & N \end{smallmatrix}\right) \in Sp_2(\mathbb{Z})$. Since $N \equiv I(\mathcal{N})$, L is symmetric modulo $\mathcal{N}$ so we can choose symmetric $W \equiv -L(\mathcal{N})$. Then

$$\begin{pmatrix} K' & L' \\ M & N \end{pmatrix} = \begin{pmatrix} I & W \\ 0 & I \end{pmatrix} \begin{pmatrix} K & L \\ M & N \end{pmatrix} \in Sp_2(\mathbb{Z})$$

with $L' \equiv 0\ (\mathcal{N})$; so $K' \equiv I\ (\mathcal{N})$. Thus

$$\begin{pmatrix} I & 0 \\ -M_\rho & I \end{pmatrix} \begin{pmatrix} K' & L' \\ M & N \end{pmatrix} \in \Gamma_0(\mathcal{N}),$$

and hence $(M\ N) \in (M_\rho\ I)\Gamma_0(\mathcal{N})$.

Now suppose $(M\ N)$ is a coprime symmetric pair so that for each prime $q|\mathcal{N}$, we have $\operatorname{rank}_q M = \operatorname{rank}_q M_\rho$. We want to show that for some $E \in GL_2(\mathbb{Z})$ and $\gamma \in \Gamma_0(\mathcal{N})$ we have $E(M\ N)\gamma \equiv (M_\rho\ I)\ (\mathcal{N})$. We know $SL_2(\mathbb{Z})$ projects onto $SL_2(\mathbb{Z}/\mathcal{N}\mathbb{Z})$. Also,

$$SL_2(\mathbb{Z}/\mathcal{N}\mathbb{Z}) \simeq SL_2(\mathbb{Z}/\mathcal{N}_0\mathbb{Z}) \times SL_2(\mathbb{Z}/\mathcal{N}_1\mathbb{Z}) \times SL_2(\mathbb{Z}/\mathcal{N}_2\mathbb{Z})$$

via the map $G\ (\mathcal{N}) \mapsto (G\ (\mathcal{N}_0), G\ (\mathcal{N}_1), G\ (\mathcal{N}_2))$, which is well-defined with inverse map

$$\big(G\ (\mathcal{N}_0), G\ (\mathcal{N}_1), G\ (\mathcal{N}_2)\big) \mapsto r\mathcal{N}_1\mathcal{N}_2 G_0 + s\mathcal{N}_0\mathcal{N}_2 G_1 + t\mathcal{N}_1\mathcal{N}_1 G_2 \quad (\mathcal{N})$$

where $r, s, t \in \mathbb{Z}$ so that $r\mathcal{N}_1\mathcal{N}_2 + s\mathcal{N}_0\mathcal{N}_2 + t\mathcal{N}_0\mathcal{N}_1 = 1$. Thus we can choose $E, G \in SL_2(\mathbb{Z})$ so that

$$EMG \equiv \begin{cases} \left(\begin{smallmatrix} 1 & \\ & 0 \end{smallmatrix}\right) & (\mathcal{N}_1), \\ \left(\begin{smallmatrix} 1 & \\ & m \end{smallmatrix}\right) & (\mathcal{N}_2) \end{cases}$$

where m is a unit modulo $\mathcal{N}_2$, and $EN\,{}^tG^{-1} \equiv I$ $(\mathcal{N}_0)$. Write $EN\,{}^tG^{-1} = \left(\begin{smallmatrix} n_1 & n_2 \\ n_3 & n_4 \end{smallmatrix}\right)$; then by symmetry, $n_3 \equiv mn_2$ $(\mathcal{N}_2)$, $n_3 \equiv 0$ $(\mathcal{N}_1)$, and since M, N are coprime, n_4 is a unit modulo $\mathcal{N}_1$. Now choose symmetric W so that

$$W \equiv \begin{cases} \left(\begin{smallmatrix} 1-n_1 & -n_2 \\ -n_2 & 0 \end{smallmatrix}\right) & (\mathcal{N}_1), \\ \left(\begin{smallmatrix} 1-n_1 & -n_2 \\ -n_2 & \overline{m}^2-\overline{m}n_4 \end{smallmatrix}\right) & (\mathcal{N}_2) \end{cases}$$

and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ so that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{cases} I & (\mathcal{N}_0), \\ \left(\begin{smallmatrix} n_4 & \\ & \bar{n}_4 \end{smallmatrix}\right) & (\mathcal{N}_1), \\ \left(\begin{smallmatrix} \overline{m} & \\ & m \end{smallmatrix}\right) & (\mathcal{N}_2). \end{cases}$$

Set

$$\gamma = \begin{pmatrix} G & \\ & {}^tG^{-1} \end{pmatrix} \begin{pmatrix} I & W \\ & I \end{pmatrix} \begin{pmatrix} 1 & & 0 & \\ & a & & b \\ 0 & & 1 & \\ & c & & d \end{pmatrix}.$$

Then $\gamma \in \Gamma_0(\mathcal{N})$ and $E(M\ N)\gamma \equiv (M_\rho\ I)$ $(\mathcal{N})$. Hence with the previous paragraph, we have that $GL_2(\mathbb{Z})(M\ N)$ is in the $\Gamma_0(\mathcal{N})$-orbit of $GL_2(\mathbb{Z})(M_\rho\ I)$, as claimed. (So the $GL_2(\mathbb{Z})$-equivalence classes of two coprime symmetric pairs $(M\ N)$, $(M'\ N')$ are in the same $\Gamma_0(\mathcal{N})$-orbit if and only if $\operatorname{rank}_q M = \operatorname{rank}_q M'$ for all primes $q|\mathcal{N}$.)

Note that

$$GL_2(\mathbb{Z})(M_\rho\ I) = SL_2(\mathbb{Z})(M_\rho\ I) \cup SL_2(\mathbb{Z})(M_\rho\ I) \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix},$$

so we can identify the cusp with $SL_2(\mathbb{Z})(M_\rho\ I)\Gamma_0(\mathcal{N})$. We use $\mathbb{E}_\rho$ to denote $\mathbb{E}_{\gamma_\rho}$. To ease the discussions during our computations we consider $2\mathbb{E}_\rho$ to be supported on a set of representatives for the $SL_2(\mathbb{Z})$-equivalence classes in the $\Gamma_0(\mathcal{N})$-orbit of $(M_\rho\ I)$.

For q prime, we say $(M\ N)$ has q-type i if $(M\ N)$ is a coprime symmetric pair with $\operatorname{rank}_q M = i$. For $(M\ N)$ of q-type i, choose $E \in GL_2(\mathbb{Z})$ so that q divides the lower $2-i$ rows of EM; then we say $(M\ N)$ (or simply M) is q^2-type i, j where $j = \operatorname{rank}_q \left(\begin{smallmatrix} I_i & \\ & \frac{1}{q}I_{2-i} \end{smallmatrix}\right) EM$. Given square-free $\mathcal{N}$ and a partition $\rho = (\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)$ of $\mathcal{N}$, we say $(M\ N)$ has $\mathcal{N}$-type ρ if $(M\ N)$ is a coprime symmetric pair and, for each prime $q|\mathcal{N}_i$, $\operatorname{rank}_q M = i$.

Given a character χ modulo $\mathcal{N}$, and $(M\ N) = (M_\rho\ I)\gamma$ where $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0(\mathcal{N})$, we can describe $\chi(\det D)$ in terms of M, N, ρ as follows. For each prime $q|\mathcal{N}_0$, we have $N \equiv D$ (q), so $\chi_q(\det D) = \chi_q(\det N)$. For each prime $q|\mathcal{N}_2$, we have $M \equiv A \equiv {}^t\overline{D}$ (q), so $\chi_q(\det D) = \overline{\chi}_q(\det M)$. Now take a prime $q|\mathcal{N}_1$; write $D = \left(\begin{smallmatrix} d_1 & d_2 \\ d_3 & d_4 \end{smallmatrix}\right)$. Thus modulo q we have

$$A \equiv \overline{\det D} \begin{pmatrix} d_4 & -d_3 \\ -d_2 & d_1 \end{pmatrix},$$

so consequently modulo q we have

$$M \equiv \overline{\det D}\begin{pmatrix} d_4 & -d_3 \\ 0 & 0 \end{pmatrix}, \qquad N \equiv \begin{pmatrix} * & * \\ d_3 & d_4 \end{pmatrix}.$$

We know $q \nmid (d_3\ d_4)$, so $\chi_q(\det D) = \overline{\chi}_q(m_1)\chi_q(n_4)$ or $\overline{\chi}_q(-m_2)\chi_q(n_3)$, whichever is non-zero. Take $E \in SL_2(\mathbb{Z})$ so that q divides row 2 of EM; thus $E \equiv \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \overline{\alpha} \end{smallmatrix}\right)$ (q), and with

$$EM = \begin{pmatrix} m_1' & m_2' \\ qm_3' & qm_4' \end{pmatrix}, \qquad EN = \begin{pmatrix} n_1' & n_2' \\ n_3' & n_4' \end{pmatrix},$$

we have

$$\overline{\chi}_q(m_1')\chi_q(n_4') = \overline{\chi}_q(m_1)\chi_q(n_4), \qquad \overline{\chi}_q(-m_2')\chi_q(n_3') = \overline{\chi}_q(-m_2)\chi_q(n_3)$$

provided $\chi_q^2 = 1$. So when $\chi_q^2 = 1$ and $(M\ N)$ has q-type 1, we can choose $E \in SL_2(\mathbb{Z})$ so that $EM = \left(\begin{smallmatrix} m_1 & m_2 \\ qm_3 & qm_4 \end{smallmatrix}\right)$, $EN = \left(\begin{smallmatrix} n_1 & n_2 \\ n_3 & n_4 \end{smallmatrix}\right)$; set $\chi_{(1,q,1)}(M,N) = \chi_q(m_1)\chi_q(n_4)$ or $\chi_q(-m_2)\chi_q(n_3)$ (whichever is non-zero). Then set

$$\chi_\rho(M,N) = \prod_{q|\mathcal{N}_0}\chi_q(\det N)\prod_{q|\mathcal{N}_1}\chi_{(1,q,1)}(M,N)\prod_{q|\mathcal{N}_2}\overline{\chi}_q(\det M).$$

Hence $2\mathbb{E}_\rho(\tau) = \sum \overline{\chi}_\rho(M,N)\det(M\tau+N)^{-k}$ where $(M\ N)$ varies over a set of $SL_2(\mathbb{Z})$-equivalence class representatives for pairs of $\mathcal{N}$-type ρ. Also note that for $G \in SL_2(\mathbb{Z})$, $\chi_\rho(GM, GN) = \chi_\rho(M,N)$, and since $\left(\begin{smallmatrix} G & \\ & {}^tG^{-1} \end{smallmatrix}\right) \in Sp_2(\mathbb{Z})$, we have $\chi_\rho(MG, N\,{}^tG^{-1}) = \chi_\rho(M,N)$.

For the remainder of this section, fix a partition $\rho = (\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)$ of $\mathcal{N}$, and fix a character χ modulo $\mathcal{N}$. We decompose χ as $\chi_{\mathcal{N}_0}\chi_{\mathcal{N}_1}\chi_{\mathcal{N}_2}$ where $\chi_{\mathcal{N}_i}$ has modulus $\mathcal{N}_i$; we assume χ has been chosen so that $\chi_{\mathcal{N}_1}^2 = 1$. For p prime, let $\mathcal{G}_1(p)$ be a set of representatives for

$$\left\{\gamma \in SL_2(\mathbb{Z}):\ \gamma \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}\ (p)\right\} \backslash SL_2(\mathbb{Z});$$

note that for $p \nmid \mathcal{N}$, we can take the elements in $\mathcal{G}_1(p)$ to be congruent modulo $\mathcal{N}$ to I.

When evaluating the action of the Hecke operators, we often use the following simple lemmas.

Lemma 3.1. *Say p is a prime and $(M\ N)$ is p-type 1, $M = \left(\begin{smallmatrix} m_1 & m_2 \\ pm_3 & pm_4 \end{smallmatrix}\right)$, and $N = \left(\begin{smallmatrix} n_1 & n_2 \\ n_3 & n_4 \end{smallmatrix}\right)$. Then $p|m_1$ if and only if $p|n_4$, and $p|m_2$ if and only if $p|n_3$.*

Proof. Say $p|m_1$. Then $p \nmid m_2$ since $\operatorname{rank}_p M = 1$; by symmetry, $p|m_2n_4$ and hence $p|n_4$. The other arguments needed to prove the lemma are essentially identical to this. □

Lemma 3.2. *Let $V = \mathbb{F}x_1 \oplus \mathbb{F}x_2$ where $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$, p prime. For $G \in \mathcal{G}_1(p)$, let $(x_1'\ x_2') = (x_1\ x_2)\,{}^tG$. Then as G varies over $\mathcal{G}_1(p)$, $\mathbb{F}x_1'$ varies over all lines in V.*

Proof. Representatives for $\mathcal{G}_1(p)$ are $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$ where α varies modulo p. Thus $\mathbb{F}x_1'$ varies as claimed. □

Proposition 3.3. *For p a prime not dividing $\mathcal{N}$, we have*

$$\mathbb{E}_\rho|T(p) = \left(\chi_{\mathcal{N}_0}(p^2)\chi_{\mathcal{N}_1}(p)p^{2k-3} + \chi_{\mathcal{N}_0\mathcal{N}_2}(p)p^{k-2}(p+1) + \chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}(p^2)\right)\mathbb{E}_\rho.$$

Proof. Decompose $\mathbb{E}_\rho$ as $\mathbb{E}_0+\mathbb{E}_1+\mathbb{E}_2$ where $\mathbb{E}_i$ is supported on pairs of $\mathcal{N}$-type ρ and p-type i.

Using the matrices for $T(p)$ as described in Proposition 3.1 of [4], we can describe $\mathbb{E}_\rho|T(p)$ as follows. First, let $\mathcal{G}_1=\mathcal{G}_1(p)$. Then

$$2\mathbb{E}_\rho(\tau)|T(p)=p^{2k-3}\sum_{\substack{0\leqslant r\leqslant 2\\ G_r,Y_r}}\chi\left(p^{2-r}\right)\overline{\chi}_\rho(M,N)\det\left(MD_rG_r^{-1}\tau+pND_r^{-1}\,{}^tG_r+MY_r\,{}^tG_r\right)^{-k}$$

where $(M\ N)$ varies over a set of $SL_2(\mathbb{Z})$-equivalence class representatives for pairs of $\mathcal{N}$-type ρ, $D_r=\left(\begin{smallmatrix}I_r&\\&pI_{2-r}\end{smallmatrix}\right)$, $G_0=G_2=I$, G_1 varies over $\mathcal{G}_1$, $Y_0=0$, $Y_1=\left(\begin{smallmatrix}y&\\&0\end{smallmatrix}\right)$ with y varying modulo p, and $Y_2\in\mathbb{Z}^{2,2}_{\mathrm{sym}}$ varying modulo p. For convenience, we choose $Y_i\equiv 0\ (\mathcal{N})$.

Case I. Say $r=0$. We take $(M'\ N')=D_\ell^{-1}(pM\ N)$ where $\ell=\operatorname{rank}_p N$, and the $SL_2(\mathbb{Z})$-equivalence class representative $(M\ N)$ is chosen so that p divides the lower $2-\ell$ rows of N.

First suppose $\ell=2$. Thus $\operatorname{rank}_p M'=0$, and

$$\chi\left(p^2\right)\chi_\rho\left(M',N'\right)=\chi_{\mathcal{N}_0}\left(p^2\right)\chi_{\mathcal{N}_1}(p)\chi_\rho(M,N).$$

So the contribution from these terms is $\chi_{\mathcal{N}_0}(p^2)\chi_{\mathcal{N}_1}(p)p^{2k-3}\mathbb{E}_0$.

Next suppose $\ell=1$. So p divides row 2 of N and hence does not divide row 2 of M, and hence $\operatorname{rank}_p M'=1$ with p dividing row 1 of M', p not dividing row 1 of N' (and so $(M',N')=1$). We have $\chi(p^2)\chi_\rho(M',N')=\chi_{\mathcal{N}_0\mathcal{N}_2}(p)\chi_\rho(M,N)$. Reversing, take $(M'\ N')$ of $\mathcal{N}$-type ρ, p-type 1. We need to count the equivalence classes $SL_2(\mathbb{Z})(M\ N)$ so that $\left(\begin{smallmatrix}1&\\&\frac{1}{p}\end{smallmatrix}\right)(pM\ N)\in SL_2(\mathbb{Z})(M'\ N')$. For any $E\in SL_2(\mathbb{Z})$, we have $\left(\begin{smallmatrix}1&\\&p\end{smallmatrix}\right)E\left(\begin{smallmatrix}1&\\&\frac{1}{p}\end{smallmatrix}\right)\in SL_2(\mathbb{Z})$ if and only if $E\equiv\left(\begin{smallmatrix}*&0\\ *&*\end{smallmatrix}\right)\ (p)$; thus we need to count the integral, coprime pairs $(M\ N)=\left(\begin{smallmatrix}1&\\&p\end{smallmatrix}\right)E(M'/p\ N')$ where E varies over $\mathcal{G}_1$. (Note that we automatically have $M\,{}^tN$ symmetric since $M'\,{}^tN'$ is symmetric.) We can assume p divides row 2 of M'. To have M integral, we need p dividing row 1 of EM'; there is 1 choice of E so that this is the case. Thus p does not divide row 2 of M or row 1 of N, so $(M,N)=1$. So the contribution from these terms is $\chi_{\mathcal{N}_0\mathcal{N}_2}(p)p^{k-3}\mathbb{E}_1$.

Now suppose $\ell=0$. Thus $\operatorname{rank}_p M=2=\operatorname{rank}_p M'$. We have

$$\chi\left(p^2\right)\chi_\rho\left(M',N'\right)=\chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}\left(p^2\right)\chi_\rho(M,N).$$

So the contribution from these terms is $\chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}(p^2)p^{-3}\mathbb{E}_2$.

Case II. $r=1$. Here we take

$$\left(M'G\ N'\,{}^tG^{-1}\right)=D_\ell^{-1}\left(M\begin{pmatrix}1&\\&p\end{pmatrix}\ N\begin{pmatrix}p&\\&1\end{pmatrix}+MY\right),$$

where $\ell=\operatorname{rank}_p\left(M\left(\begin{smallmatrix}1&\\&p\end{smallmatrix}\right)\ N\left(\begin{smallmatrix}p&\\&1\end{smallmatrix}\right)\right)$, $G\in\mathcal{G}_1$, $Y=\left(\begin{smallmatrix}y&\\&0\end{smallmatrix}\right)$, and the equivalence class representative $(M\ N)$ is adjusted so that $(M'\ N')$ is integral.

Suppose $\ell=2$. Then $(M',N')=1$, $\operatorname{rank}_p M'=1$, and $\chi(p)\chi_\rho(M',N')=\chi_{\mathcal{N}_0}(p^2)\chi_{\mathcal{N}_1}(p)\chi_\rho(M,N)$. Reversing,

$$(M\ N)=\left(M'G\begin{pmatrix}1&\\&\frac{1}{p}\end{pmatrix}\ \left(N'\,{}^tG^{-1}-M'GY\right)\begin{pmatrix}\frac{1}{p}&\\&1\end{pmatrix}\right).$$

We can assume p divides row 2 of M'; to have M integral, we need to choose the unique G so that $q|m_2$ where $M'G=\left(\begin{smallmatrix}m_1&m_2\\pm_3&pm_4\end{smallmatrix}\right)$. Then $p\nmid m_1$, and by symmetry, $p\nmid n_4$ where $N'\,{}^tG^{-1}=\left(\begin{smallmatrix}n_1&n_2\\n_3&n_4\end{smallmatrix}\right)$. To have

N integral, we need to choose the unique y modulo p so that $n_1 \equiv m_1 y\ (p)$. So M, N are integral and coprime, and the contribution from these terms is $\chi_{\mathcal{N}_0}(p^2)\chi_{\mathcal{N}_1}(p)p^{2k-3}\mathbb{E}_1$.

Say $\ell = 1$. Then we must have $M = \left(\begin{smallmatrix} m_1 & m_2 \\ pm_3 & m_4 \end{smallmatrix}\right)$, $N = \left(\begin{smallmatrix} n_1 & n_2 \\ n_3 & pn_4 \end{smallmatrix}\right)$ with $p \nmid (m_1\ n_2)$; since $(M, N) = 1$, we must also have $p \nmid (m_4\ n_3)$. Thus $(M', N') = 1$ with $\operatorname{rank}_p M' = 0, 1,$ or 2, and

$$\chi(p)\chi_\rho\big(M', N'\big) = \chi_{\mathcal{N}_0\mathcal{N}_2}(p)\chi_\rho(M, N).$$

Reversing,

$$(M\ N) = \begin{pmatrix} 1 & \\ & p \end{pmatrix} E\left(M'G\begin{pmatrix} 1 & \\ & \frac{1}{p} \end{pmatrix}\ \big(N'\,{}^tG^{-1} - M'GY\big)\begin{pmatrix} \frac{1}{p} & \\ & 1 \end{pmatrix}\right),$$

$E \in \mathcal{G}_1$.

Still assuming $\ell = 1$, suppose $\operatorname{rank}_p M' = 0$. Then to have N integral, for each E we must choose the unique G so that $q | n_1$ where $EN'\,{}^tG^{-1} = \left(\begin{smallmatrix} n_1 & n_2 \\ n_3 & n_4 \end{smallmatrix}\right)$. Thus $p \nmid n_2 n_3$, and hence $\operatorname{rank}_p N = 2$ for all choices of y. So the contribution from these terms is $\chi_{\mathcal{N}_0\mathcal{N}_2}(p)p^{k-3} \cdot p(p+1)\mathbb{E}_0$.

Continuing with the assumption $\ell = 1$, suppose $\operatorname{rank}_p M' = 1$; we can assume p divides row 2 of M'. To have M, N integral, we need $p | m_2$, $n_1 \equiv m_1 y\ (p)$ where $EM'G = \left(\begin{smallmatrix} m_1 & m_2 \\ m_3 & m_4 \end{smallmatrix}\right)$, $EN'\,{}^tG^{-1} = \left(\begin{smallmatrix} n_1 & n_2 \\ n_3 & n_4 \end{smallmatrix}\right)$. Say p divides row 2 of EM' (this is the case for q choices of E); then we need to choose the unique G so that $p | m_2$. Then $p \nmid m_1$, and by symmetry, $p | n_3$. But then p divides row 2 of both M and N, so $(M, N) \neq 1$. So take the unique E so that p does not divide row 2 of EM'; thus, by our choice of representatives in $\mathcal{G}_1$, we have p dividing row 1 of EM'. To have N integral, we need to choose the unique G so that $p | n_1$. Then $p \nmid n_2$, and by symmetry, $p | m_4$. Since $\operatorname{rank}_p M' = 1$, $p \nmid m_3$. To have $(M, N) = 1$, we need to choose y so that $n_3 \not\equiv m_3 y\ (p)$; so we have $p - 1$ choices for y. The contributions from these terms is $\chi_{\mathcal{N}_0\mathcal{N}_2}(p)p^{k-3}(p-1)\mathbb{E}_1$.

Now assume $\ell = 1$, $\operatorname{rank}_p M' = 2$. Then for each E, choose the unique G so that $p | m_2$ where $EM'G = \left(\begin{smallmatrix} m_1 & m_2 \\ m_3 & m_4 \end{smallmatrix}\right)$; then M is integral. Also, $p \nmid m_1 m_3$ so $\operatorname{rank}_p M = 2$. Choose the unique y so that $n_1 \equiv m_1 y\ (p)$; so M, N are integral and coprime. The contributions from these terms is $\chi_{\mathcal{N}_0\mathcal{N}_2}(p)p^{k-3}(p+1)\mathbb{E}_2$.

Now assume $\ell = 0$; since $(M, N) = 1$, we must have $M = \left(\begin{smallmatrix} pm_1 & m_2 \\ pm_3 & m_4 \end{smallmatrix}\right)$, $N = \left(\begin{smallmatrix} n_1 & pn_2 \\ n_3 & pn_4 \end{smallmatrix}\right)$ with $p \nmid m_2 n_3 - n_1 m_4$. Thus $(M', N') = 1$, with $\operatorname{rank}_p M' \geqslant 1$, and

$$\chi(p)\chi_\rho\big(M', N'\big) = \chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}\big(p^2\big)\chi_\rho(M, N).$$

Reversing,

$$(M\ N) = p\left(M'G\begin{pmatrix} 1 & \\ & \frac{1}{p} \end{pmatrix}\ \big(N'\,{}^tG^{-1} - M'GY\big)\begin{pmatrix} \frac{1}{p} & \\ & 1 \end{pmatrix}\right).$$

Say $\operatorname{rank}_p M' = 1$; we can assume p divides row 2 of M'. So to have $(M, N) = 1$ we need to choose G so that $p \nmid m_2$ where

$$M'G = \begin{pmatrix} m_1 & m_2 \\ pm_3 & pm_4 \end{pmatrix};$$

this gives us p choices for G. Write

$$N'\,{}^tG^{-1} = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix};$$

by symmetry, if $p|n_3$ then we must have $p|n_4$, and consequently $(M', N') \neq 1$. So $p \nmid n_3$, and $(M, N) = 1$ for all choices of y. The contribution from these terms is $\chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}(p^2)p^{-1}\mathbb{E}_1$.

Now say $\operatorname{rank}_p M' = 2$; write

$$M'G = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}, \qquad N'\,{}^tG^{-1} = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}.$$

Then $(M, N) = 1$ unless $\binom{n_1}{n_3} - \binom{m_1}{m_3} \in \operatorname{span}_p \binom{m_2}{m_4}$. This gives us $p - 1$ choices for y, and the contribution is $\chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}(p^2)p^{-3}(p^2 - 1)\mathbb{E}_2$.

Case III. $r = 2$. So $(M'\ N') = D_\ell^{-1}(M\ pN + MY)$ where $Y \in \mathbb{Z}^{2,2}_{\mathrm{sym}}$ varies modulo p, and $\ell = \operatorname{rank}_p M$ and we assume p divides the lower $2 - \ell$ rows of M.

Suppose $\ell = 2$. So $(M\ N) = (M'\ (N' - M'Y)/p)$; there is a unique Y so that $N' \equiv M'Y\ (p)$. The contribution is $\chi_{\mathcal{N}_0}(p^2)\chi_{\mathcal{N}_1}(p)p^{2k-3}\mathbb{E}_2$.

Now suppose $\ell = 1$; so $\operatorname{rank}_p M' \geqslant 1$, and $\chi_\rho(M', N') = \chi_{\mathcal{N}_0\mathcal{N}_2}(p)\chi_\rho(M, N)$. Reversing,

$$(M\ N) = \begin{pmatrix} 1 & \\ & p \end{pmatrix} E\big(M'\ \big(N' - M'Y\big)/p\big)$$

where E varies over $\mathcal{G}_1$. Say $\operatorname{rank}_p M' = 2$; write

$$EM' = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}, \qquad EN' = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}.$$

To have N integral, we need $(n_1\ n_2) \equiv (m_1\ m_2)Y\ (p)$, and to have $(M, N) = 1$, we need $(n_3\ n_4) \not\equiv (m_3\ m_4)Y\ (p)$. Thus we have $p - 1$ choices for Y, and the contribution from these terms is $\chi_{\mathcal{N}_0\mathcal{N}_2}(p)p^{k-3}(p^2 - 1)\mathbb{E}_2$.

Say $\ell = 1$ and $\operatorname{rank}_p M' = 1$; assume p divides row 2 of M' (so p does not divide row 2 of N'). To have N integral, we need p not dividing row 1 of EM' and $(n_1\ n_2) \equiv (m_1\ m_2)Y\ (p)$; this gives us p choices for E and p choices for Y. Then p does not divide row 1 of M or row 2 of N, so $(M, N) = 1$. The contribution from these terms is $\chi_{\mathcal{N}_0\mathcal{N}_2}(p)p^{k-3} \cdot p^2\mathbb{E}_1$.

Say $\ell = 0$. So $\operatorname{rank}_p M' = 0, 1,$ or 2, and

$$\chi_\rho\big(M', N'\big) = \chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}\big(p^2\big)\chi_\rho(M, N).$$

Reversing, we have

$$(M\ N) = p\big(M'\ \big(N' - M'Y\big)/p\big).$$

Say $\operatorname{rank}_p M' = 2$. We need to choose Y so that $\operatorname{rank}_p(\overline{M}'N' - Y) = 2$; as Y varies over $\mathbb{Z}^{2,2}_{\mathrm{sym}}$ modulo p, so does $\overline{M}'N' - Y$, and $p^2(p - 1)$ of these matrices have p-rank 2. Thus the contribution from these terms is $\chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}(p^2)p^{-1}(p - 1)\mathbb{E}_2$.

Now say $\ell = 0$ and $\operatorname{rank}_p M' = 1$; we can assume p divides row 2 of M'. To have $(M, N) = 1$, we need $(n_1\ n_2) - (m_1\ m_2)Y \notin \operatorname{span}_p(n_3\ n_4)$. For each α modulo p, we have p choices of Y so that $(n_1\ n_2) - (m_1\ m_2)Y \equiv \alpha(n_3\ n_4)\ (p)$. Thus we have $p^2(p - 1)$ choices for Y so that $(M, N) = 1$. The contribution from these terms is $\chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}(p^2)p^{-1}(p - 1)\mathbb{E}_1$.

Finally, say $\ell = 0$ and $\operatorname{rank}_p M' = 0$. Thus $\operatorname{rank}_p N' = 2 = \operatorname{rank}_p N$ for all choices of Y. So the contribution from these terms is $\chi_{\mathcal{N}_1}(p)\chi_{\mathcal{N}_2}(p^2)\mathbb{E}_0$.

Combining all the terms yields the result. □

Proposition 3.4. *For p a prime not dividing $\mathcal{N}$,*

$$\mathbb{E}_\rho|T_1(p^2) = (p+1)\big(\chi_{\mathcal{N}_0}(p^2)p^{2k-3} + \chi(p)p^{k-3}(p-1) + \chi_{\mathcal{N}_2}(p^2)\big)\mathbb{E}_\rho.$$

Proof. Let $\mathcal{G}_1 = \mathcal{G}_1(p)$, and let $\mathbb{E}_0$, $\mathbb{E}_1$, $\mathbb{E}_2$ be defined as in the proof of Proposition 3.3. Decompose $\mathbb{E}_1, \mathbb{E}_2$ as $\mathbb{E}_1 = \mathbb{E}_{1,1} + \mathbb{E}_{1,2}$, $\mathbb{E}_0 = \mathbb{E}_{0,0} + \mathbb{E}_{0,1} + \mathbb{E}_{0,2}$ where $\mathbb{E}_{i,j}$ is supported on pairs $(M\ N)$ of p^2-type i, j. Further, we split $\mathbb{E}_{0,2}$ as $\mathbb{E}_{0,2,+} + \mathbb{E}_{0,2,-}$ where, for p odd, $\mathbb{E}_{0,2,\nu}$ is supported on pairs $(M\ N)$ so that

$$\nu\left(\frac{\det(MN/p)}{p}\right) = 1,$$

and for $p = 2$, $\mathbb{E}_{0,2,+}$ is supported on pairs $(M\ N)$ where $\frac{1}{2}M\,{}^tN \simeq I$ (2) and $\mathbb{E}_{0,2,-}$ on pairs $(M\ N)$ where $\frac{1}{2}M\,{}^tN \equiv \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ (2). When p is odd, set $\epsilon = (\frac{-1}{p})$.

Using Proposition 2.1 of [4], the action of $T_1(p^2)$ is given by matrices parameterised by $r_0, r_1, r_2 \in \mathbb{Z}_{\geqslant 0}$ where $r_0 + r_1 + r_2 = 1$.

Case I. $r_0 = 1$. Here the summands are

$$p^{k-3}\chi_\rho(M,N)\det\left(M\begin{pmatrix}\frac{1}{p}&\\&1\end{pmatrix}G^{-1}\tau + N\begin{pmatrix}p&\\&1\end{pmatrix}{}^tG + M\begin{pmatrix}\frac{1}{p}&\\&1\end{pmatrix}Y\,{}^tG\right)^{-k}$$

where $(M\ N)$ varies over pairs of $\mathcal{N}$-type ρ, G varies over $\mathcal{G}_1$, and $Y = \left(\begin{smallmatrix}y_1&y_2\\y_2&0\end{smallmatrix}\right)$ with y_1 varying modulo p^2 and y_2 varying modulo p.

Case Ia. Say $M\left(\begin{smallmatrix}\frac{1}{p}&\\&1\end{smallmatrix}\right)$ is not integral. Then we can adjust the equivalence class representative for $(M'\ N')$ so that

$$\big(M'G\ N'\,{}^tG^{-1}\big) = \begin{pmatrix}p&\\&1\end{pmatrix}\left(M\begin{pmatrix}\frac{1}{p}&\\&1\end{pmatrix}\ N\begin{pmatrix}p&\\&1\end{pmatrix} + M\begin{pmatrix}\frac{1}{p}&\\&1\end{pmatrix}Y\right)$$

is coprime with $\operatorname{rank}_p M' = 1$ or 2. So $\chi_\rho(M', N') = \chi_{\mathcal{N}_0}(p^2)\chi_\rho(M,N)$. Using the techniques used in the proof of Proposition 3.2, we find that in the case $\operatorname{rank}_p M' = 2$ the contribution is $\chi_{\mathcal{N}_0}(p^2)p^{k-3}p^k(p+1)\mathbb{E}_2$, and in the case $\operatorname{rank}_p M' = 1$ the contribution is $\chi_{\mathcal{N}_0}(p^2)p^{k-3}p^{k+1}\mathbb{E}_1$.

Case Ib. Suppose $M\left(\begin{smallmatrix}\frac{1}{p}&\\&1\end{smallmatrix}\right)$ is integral and that $(M', N') = 1$ where

$$\big(M'G\ N'\,{}^tG^{-1}\big) = \left(M\begin{pmatrix}\frac{1}{p}&\\&1\end{pmatrix}\ N\begin{pmatrix}p&\\&1\end{pmatrix} + M\begin{pmatrix}\frac{1}{p}&\\&1\end{pmatrix}Y\right).$$

Note that $\operatorname{rank}_p M' \geqslant 1$, else $\operatorname{rank}_p N' \leqslant 1$ and hence $(M', N') \neq 1$. Also, note that $\chi_\rho(M', N') = \chi(p)\chi_\rho(M,N)$.

Reversing, when $\operatorname{rank}_p M' = 2$, we find that for each G, there are $p-1$ choices for Y so that $(M\ N)$ is an integral, coprime pair; the contribution from these terms is $\chi(p)p^{k-3}(p^2-1)\mathbb{E}_2$. When $\operatorname{rank}_p M' = 1$, we can assume p divides row 2 of M'; to have N integral we need to choose the unique $G \in \mathcal{G}_1$ so that p divides the $2,1$-entry of $N'\,{}^tG^{-1}$, and then we have $p(p-1)$ choices for Y so that $(M,N) = 1$. So the contribution from these terms is $\chi(p)p^{k-3} \cdot p(p-1)\mathbb{E}_1$.

Case Ic. Suppose $M\begin{pmatrix}\frac{1}{p} & \\ & 1\end{pmatrix}$ is integral and that

$$\operatorname{rank}_p\left(M\begin{pmatrix}\frac{1}{p} & \\ & 1\end{pmatrix} N\begin{pmatrix}p & \\ & 1\end{pmatrix}\right)<2;$$

adjusting the equivalence class representative, we have $(M',N')=1$ where

$$\left(M'G\ N'\,{}^tG^{-1}\right)=\begin{pmatrix}1 & \\ & \frac{1}{p}\end{pmatrix}\left(M\begin{pmatrix}\frac{1}{p} & \\ & 1\end{pmatrix} N\begin{pmatrix}p & \\ & 1\end{pmatrix}+M\begin{pmatrix}\frac{1}{p} & \\ & 1\end{pmatrix}Y\right).$$

Also, $\chi_\rho(M',N')=\chi_{\mathcal{N}_2}(p^2)\chi_\rho(M,N)$.

Reversing, take a coprime pair $(M'\ N')$. Then arguing as above, we find that when $\operatorname{rank}_p M'=2$, the contribution is $\chi_{\mathcal{N}_2}(p^2)p^{-2}(p^2-1)(p+1)\mathbb{E}_2$. When $\operatorname{rank}_p M'=1$, the contribution is $\chi_{\mathcal{N}_2}(p^2)p^{-1}(p^2+p-1)\mathbb{E}_1$. When $\operatorname{rank}_p M'=0$, the contribution is $\chi_{\mathcal{N}_2}(p^2)p^{k-3}\cdot p^{3-k}(p+1)\mathbb{E}_0$.

Case II. $r_1=1$. Here the summands are

$$p^{k-3}\chi(p)\overline{\chi}_\rho(M,N)\det\left(MG^{-1}\tau+N\,{}^tG+M\begin{pmatrix}\frac{y}{p} & \\ & 0\end{pmatrix}{}^tG\right)^{-k}$$

where $(M\ N)$ varies over pairs of $\mathcal{N}$-type ρ, y varies modulo p with $p\nmid y$, G varies over ${}^t\mathcal{G}_1$.

Case IIa. Say $M\begin{pmatrix}\frac{1}{p} & \\ & 1\end{pmatrix}$ is not integral. Adjust the representative $(M\ N)$

$$\left(M'G\ N'\,{}^tG^{-1}\right)=\begin{pmatrix}p & \\ & 1\end{pmatrix}\left(M\ N+M\begin{pmatrix}\frac{y}{p} & \\ & 0\end{pmatrix}\right)$$

is integral. Then $(M',N')=1$ with $(M'\ N')$ of p^2-type $1,2$ or $0,1$ or $0,2$. Also, $\chi(p)\chi_\rho(M',N')=\chi_{\mathcal{N}_0}(p^2)\chi_\rho(M,N)$.

Reversing, when $(M'\ N')$ is p^2-type $1,2$, we can assume p divides row 2 of M'; we find there are unique choices for G,Y so that N is integral, and then we get $(M,N)=1$. Thus the contribution from these terms is $\chi_{\mathcal{N}_0}(p^2)p^{k-3}\cdot p^k\mathbb{E}_{1,2}$.

Next suppose M' is p^2-type $0,1$; we can assume p^2 divides row 2 of M'. For 1 choice of E we have $p^2|(m_1\ m_2)$; but then we cannot have N integral and coprime to M. For the other p choices of E we have $p^2|(m_3\ m_4)$. Choose the unique G so that $p|n_2$; so $p\nmid n_1n_4$ and $p^2\nmid m_1$, else by symmetry $p^2|m_2$, contradicting that M' is p^2-type $(0,1)$. Then choose the unique $y\not\equiv 0\ (p)$ so that $n_1\equiv m_1\frac{y}{p}\ (p)$; we get p not dividing row 1 of M or row 2 of N, so $(M,N)=1$. The contribution from these terms is $\chi_{\mathcal{N}_0}(p^2)p^{k-3}\cdot p^k\cdot p\mathbb{E}_{0,1}$.

Now suppose $(M'\ N')$ is p^2-type $0,2$. For each choice of $E\in\mathcal{G}_1$, we choose the unique $G\in{}^tG^{-1}$ so that $EN'G=\begin{pmatrix}n_1 & pn_2\\ n_3 & n_4\end{pmatrix}$ (thus $p\nmid n_1n_4$). To have N integral, we need to choose $y\not\equiv 0\ (p)$ so that $n_1\equiv m_1\frac{y}{p}\ (p)$; this is possible if and only if $p^2\nmid m_1$. Let $V=\mathbb{F}x_1\oplus\mathbb{F}x_2$ be equipped with the quadratic form $\frac{1}{p}M'\,{}^tN'$ relative to the basis $(x_1\ x_2)$. Then with $(x_1'\ x_2')=(x_1\ x_2)\,{}^tE$, $\mathbb{F}x_1'$ varies over all lines in V, and $\frac{1}{p}EM'\,{}^tN'\,{}^tE$ represents the quadratic form relative to $(x_1'\ x_2')$. When p is odd and $V\simeq\mathbb{H}$, there are 2 choices of E so that $\mathbb{F}x_1'$ is isotropic; equivalently, when $V\simeq\mathbb{H}$ there are 2 choices of E so that $p^2|m_1$ (since the value of the quadratic form on x_1' is $m_1n_1/p\in\mathbb{F}$). When p is odd and $V\not\simeq\mathbb{H}$, we have $p^2\nmid m_1$ for all $p+1$ choices of E. So the contribution when p is odd is

$$\chi_{\mathcal{N}_0}(p^2)p^{k-3}\cdot p^k(p-1)\mathbb{E}_{0,2,\epsilon}+\chi_{\mathcal{N}_0}(p^2)p^{k-3}\cdot p^k(p+1)\mathbb{E}_{0,2,-\epsilon}.$$

When $p=2$ and $V \simeq \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we have $\mathbb{F}x_1'$ isotropic for all E; when $V \simeq I$ we have $\mathbb{F}x_1'$ isotropic for 1 choice of E. So the contribution when $p=2$ is $2\chi_{\mathcal{N}_0}(p^2)p^{2k-3}\mathbb{E}_{0,2,+}$.

Case IIb. Suppose $M\left(\begin{smallmatrix} \frac{1}{p} & \\ & 1 \end{smallmatrix}\right)$ is integral, and

$$\left(M'G\ N'\,{}^tG^{-1}\right) = \left(M\ N + M\begin{pmatrix} \frac{y}{p} & \\ & 0 \end{pmatrix}\right)$$

is a coprime symmetric pair with $\operatorname{rank}_p M' \leqslant 1$. Note that $\chi(p)\chi_\rho(M',N') = \chi(p)\chi_\rho(M,N)$.

Reversing, first suppose that $\operatorname{rank}_p M' = 1$, and assume p divides row 2 of M'. To have N integral, choose the unique G so that $p|m_1$ (so $p \nmid m_2$). To have $(M,N)=1$, we need p not dividing row 2 of N. If M' is p^2-type $(1,1)$, we can assume p^2 divides row 2 of M' and then for any choice of $y \not\equiv 0\ (p)$ we have p not dividing row 2 of N. If M' is p^2-type $(1,2)$ then $p^2 \nmid m_3$ so there are $p-2$ choices for $y \not\equiv 0\ (p)$ so that p does not divide row 2 of N. Thus the contribution from these terms is $\chi(p)p^{k-3}(p-1)\mathbb{E}_{1,1} + \chi(p)p^{k-3}(p-2)\mathbb{E}_{1,2}$.

Now suppose M' is p^2-type $(0,0)$; then N is invertible modulo p for all choices of G and y. Hence the contribution is $\chi(p)p^{k-3}(p^2-1)\mathbb{E}_{0,0}$.

Suppose M' is p^2-type $(0,1)$; then we can assume p^2 divides row 2 of M'. For 1 choice of G we have $p^2|m_1$ and then for any y we have $\operatorname{rank}_p N = 2$. Say we take any of the other p choices for G so that $p \nmid m_1$. By symmetry, $p \nmid n_4$. So there are $p-2$ choices of $y \not\equiv 0\ (p)$ so that $\operatorname{rank}_p N = 2$. Thus the contribution from these terms is $\chi(p)p^{k-3}(p^2-p-1)\mathbb{E}_{0,1}$.

Suppose M' is p^2-type $(0,2)$. Let $V = \mathbb{F}x_1 \oplus \mathbb{F}x_2$ be equipped with the quadratic form $\frac{1}{p}\overline{N}'M'$ relative to $(x_1\ x_2)$. Then with $(x_1'\ x_2') = (x_1\ x_2)G$, $\mathbb{F}x_1'$ varies over all lines in V as G varies over ${}^t\mathcal{G}_1$, and the value of the quadratic form on x_1' is $\overline{\det N'}(m_1n_4 - m_3n_2)/p$. We have $\operatorname{rank}_2 N = 2$ if and only if $\det N' \not\equiv y(m_1n_4 - m_3n_2)/p\ (p)$. When p is odd and $V \simeq \mathbb{H}$, there are 2 choices of G so that x_1' is isotropic, and then $\operatorname{rank}_p N = 2$ for all $y \not\equiv 0\ (p)$; for a choice of G so that x_1' is anisotropic, there are $p-2$ choices of $y \not\equiv 0\ (p)$ so that $\operatorname{rank}_p N = 2$. Hence the contribution from these terms when p is odd is $\chi(p)p^{k-3}\cdot p(p-1)\mathbb{E}_{0,2,\epsilon} + \chi(p)p^{k-3}\cdot(p+1)(p-2)\mathbb{E}_{0,2,-\epsilon}$. When $p=2$, we have $y \equiv 1\ (p)$; when $V \simeq \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\mathbb{F}x_1'$ is isotropic for all 3 G, and when $V \simeq I$ there is 1 choice of G so that $\mathbb{F}x_1'$ is isotropic. So the contribution when $p=2$ is $\chi(p)p^{k-3}\mathbb{E}_{0,2,+} + 3\chi(p)p^{k-3}\mathbb{E}_{0,2,-}$.

Case IIc. Suppose $M\left(\begin{smallmatrix} \frac{1}{p} & \\ & 1 \end{smallmatrix}\right)$ is integral and

$$\operatorname{rank}_p\left(M\ N + M\begin{pmatrix} \frac{y}{p} & \\ & 0 \end{pmatrix}\right) = 1;$$

we can adjust the representative so that

$$\left(M'G\ N'\,{}^tG^{-1}\right) = \begin{pmatrix} 1 & \\ & \frac{1}{p} \end{pmatrix}\left(M\ N + M\begin{pmatrix} \frac{y}{p} & \\ & 0 \end{pmatrix}\right)$$

is an integral pair. Then $(M',N')=1$ with $\operatorname{rank}_p M' \geqslant 1$, and

$$\chi(p)\chi_\rho\left(M',N'\right) = \chi_{\mathcal{N}_2}\left(p^2\right)\chi_\rho(M,N).$$

Reversing, we need to count the equivalence classes $SL_2(\mathbb{Z})(M\ N)$ so that $\left(\begin{smallmatrix} 1 & \\ & \frac{1}{p} \end{smallmatrix}\right)\left(M\ N + M\left(\begin{smallmatrix} \frac{y}{p} & \\ & 0 \end{smallmatrix}\right)\right) \in SL_2(\mathbb{Z})(M'\ N')$. Thus we need to count the integral, coprime pairs

$$(M\ N) = \begin{pmatrix} 1 & \\ & p \end{pmatrix} E\left(M'G\ N'\,{}^tG^{-1} - M'G\begin{pmatrix} \frac{y}{p} & \\ & 0 \end{pmatrix}\right),$$

where $E \in \mathcal{G}_1$. Write $EM'G = \left(\begin{smallmatrix} m_1 & m_2 \\ m_3 & m_4 \end{smallmatrix}\right)$, $EN'\,{}^tG^{-1} = \left(\begin{smallmatrix} n_1 & n_2 \\ n_3 & n_4 \end{smallmatrix}\right)$.

Say $\mathrm{rank}_p\, M' = 2$. To have N integral, choose the unique G so that $p|m_1$ (and hence $p \nmid m_2m_3$). Then for any choice of y, $(M, N) = 1$. So the contribution from these terms is $\chi_{\mathcal{N}_2}(p^2)p^{k-3} \cdot p^{-k}(p^2-1)\mathbb{E}_2$.

Now say $\mathrm{rank}_p\, M' = 1$; we can assume p divides row 2 of M'. For p choices of E, we have p dividing row 2 of EM', and then p divides row 2 of $(M\ N)$ (meaning $(M, N) \neq 1$). So take the unique E so that p divides row 1 of EM' (and hence p does not divide row 1 of EN'). So to have $(M, N) = 1$, we need to choose G so that $p \nmid m_3$; this gives us p choices for G. Then for every $y \not\equiv 0\ (p)$, we have $(M, N) = 1$. So the contribution from these terms is $\chi_{\mathcal{N}_2}(p^2)p^{k-3} \cdot p^{-k} \cdot p(p-1)\mathbb{E}_1$.

Case III. $r_2 = 1$. Here the summands are

$$\chi(p^2)\overline{\chi}_\rho(M, N)p^{k-3}\det\left(M\begin{pmatrix} p & \\ & 1 \end{pmatrix}G^{-1}\tau + N\begin{pmatrix} \frac{1}{p} & \\ & 1 \end{pmatrix}{}^tG\right)^{-k},$$

where $(M\ N)$ varies over pairs of $\mathcal{N}$-type ρ, and G varies over ${}^t\mathcal{G}_1$.

Case IIIa. Suppose $N\left(\begin{smallmatrix} \frac{1}{p} & \\ & 1 \end{smallmatrix}\right)$ is not integral. We can adjust the representative so that $N = \left(\begin{smallmatrix} n_1 & n_2 \\ pn_3 & n_4 \end{smallmatrix}\right)$ (so $p \nmid n_1$). Write $M = \left(\begin{smallmatrix} m_1 & m_2 \\ m_3 & m_4 \end{smallmatrix}\right)$. Then $p \nmid (m_4\ n_4)$, else by symmetry, $p|m_3$ and $(M, N) \neq 1$. Thus with

$$\left(M'G\ N'\,{}^tG^{-1}\right) = \begin{pmatrix} p & \\ & 1 \end{pmatrix}\left(M\begin{pmatrix} p & \\ & 1 \end{pmatrix}\ N\begin{pmatrix} \frac{1}{p} & \\ & 1 \end{pmatrix}\right),$$

we have $(M', N') = 1$ and $\mathrm{rank}_p\, M' \leqslant 1$. When $\mathrm{rank}_p\, M' = 1$, we must have M' of p^2-type $1, 1$ with p dividing row 1 of M', and

$$\chi(p^2)\chi_\rho(M', N') = \chi_{\mathcal{N}_0}(p^2)\chi_\rho(M, N).$$

Reversing,

$$(M\ N) = \begin{pmatrix} \frac{1}{p} & \\ & 1 \end{pmatrix} E\left(M'G\begin{pmatrix} \frac{1}{p} & \\ & 1 \end{pmatrix}\ N'\,{}^tG^{-1}\begin{pmatrix} p & \\ & 1 \end{pmatrix}\right).$$

We know $\left(\begin{smallmatrix} \frac{1}{p} & \\ & 1 \end{smallmatrix}\right)E\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ if and only if $E \equiv \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)\ (p)$. Write $EM'G = \left(\begin{smallmatrix} m_1 & m_2 \\ m_3 & m_4 \end{smallmatrix}\right)$, $EN'\,{}^tG^{-1} = \left(\begin{smallmatrix} n_1 & n_2 \\ n_3 & n_4 \end{smallmatrix}\right)$.

First suppose M' is p^2-type $1, 1$ with p dividing row 1 of M'. We know p divides row 1 of EM' if and only if $E \equiv \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)\ (p)$; thus we only need to consider $E = I$, and we can assume p^2 divides row 1 of M'. To ensure N is integral, we need to choose the unique $G \in {}^t\mathcal{G}_1$ so that $p|n_2$; then by symmetry, $p|n_3$, and since $\mathrm{rank}_p\, M' = 1$, $p \nmid m_4$. Then M, N are integral and coprime. So the contribution from these terms is $\chi_{\mathcal{N}_0}(p^2)p^{k-3} \cdot p^k\mathbb{E}_{1,1}$.

Now suppose M' is p^2-type $0, 0$. Then for each $E \in \mathcal{G}_1$, there is a unique $G \in {}^t\mathcal{G}_1$ so that $p|n_2$ (and hence $p \nmid n_1n_4$). Thus $\mathrm{rank}_p\, N = 1$ so $(M, N) = 1$, and the contribution from these terms is $\chi_{\mathcal{N}_0}(p^2)p^{k-3} \cdot p^k(p+1)\mathbb{E}_{0,0}$.

Suppose M' is p^2-type $0, 1$; we can assume p^2 divides row 2 of EM'. Suppose p^2 divides row 2 of EM'; to have M integral, we choose the unique G so that $p^2|m_1$. Then $p^2 \nmid m_2$, and by symmetry, $p|n_4$. Hence $p \nmid n_2n_3$, and N is not integral. So choose E so that p^2 does not divide row 2 of EM'; we have

1 choice for E, and then p^2 divides row 1 of EM'. Choose the unique G so that $p|n_2$; then $p \nmid n_1 n_4$, and N is integral with $\operatorname{rank}_p N = 2$. So the contribution from these terms is $\chi_{\mathcal{N}_0}(p^2) p^{k-3} \cdot p^k \mathbb{E}_{0,1}$.

Suppose M' is p^2-type $0, 2$. For each $E \in \mathcal{G}_1$, choose the unique $G \in {}^t\mathcal{G}_1$ so that $q|n_2$. To have M integral, we need $p^2|m_1$. Let $V = \mathbb{F}x_1 \oplus \mathbb{F}x_2$ be equipped with the quadratic form given by $\frac{1}{p} M' {}^t N'$ relative to $(x_1\ x_2)$. Then with $(x_1'\ x_2') = (x_1\ x_2)\,{}^t E$, the quadratic form on V is given by $\frac{1}{p} E M' {}^t N' {}^t E$ relative to $(x_1'\ x_2')$. As E varies over $\mathcal{G}_1$, $\mathbb{F}x_1'$ varies over all lines in V. Suppose p is odd; then to have $p^2|m_1$, we need $V \simeq \mathbb{H}$, and then $p|m_1$ for 2 choices of E. Hence the contribution from these terms when p is odd is $2\chi_{\mathcal{N}_0}(p^2) p^{k-3} \cdot p^k \mathbb{E}_{0,2,\epsilon}$. In the case that $p = 2$, the contribution is $\chi_{\mathcal{N}_0}(p^2) p^{2k-3} \mathbb{E}_{0,2,+} + 3\chi_{\mathcal{N}_0}(p^2) p^{2k-3} \mathbb{E}_{0,2,-}$.

Case IIIb. Suppose $N\begin{pmatrix} \frac{1}{p} & \\ & 1 \end{pmatrix}$ is integral, and $\operatorname{rank}_p(M'\ N') = 2$ where

$$\left(M'G\ N'\,{}^tG^{-1}\right) = \left(M\begin{pmatrix} p & \\ & 1 \end{pmatrix}\ N\begin{pmatrix} \frac{1}{p} & \\ & 1 \end{pmatrix}\right).$$

So $\operatorname{rank}_p M \geqslant 1$, $\operatorname{rank}_p M' \leqslant 1$, and M' cannot be p^2-type $0, 0$. Also, when M' is p-type 1, we can assume p divides row 2 of $M\begin{pmatrix} p & \\ & 1 \end{pmatrix}$; then using symmetry, we see p divides row 2 of N, so we must have M' of p^2-type $1, 2$. Note that $\chi(p^2)\chi_\rho(M', N') = \chi(p)\chi_\rho(M, N)$.

Reversing, $(M\ N) = \left(M'G\begin{pmatrix} \frac{1}{p} & \\ & 1 \end{pmatrix}\ N'\,{}^tG^{-1}\begin{pmatrix} p & \\ & 1 \end{pmatrix}\right)$. Write

$$M'G = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}, \qquad N'\,{}^tG^{-1} = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}.$$

Suppose M' is p^2-type $1, 2$; assume p divides row 2 of M'. Choose the unique G so that $p|m_1$; thus $p \nmid m_2$, $p^2 \nmid m_3$. So $(M, N) = 1$ and the contribution from these terms is $\chi(p) p^{k-3} \mathbb{E}_{1,2}$.

Suppose M' is p^2-type $0, 1$; assume p^2 divides row 2 of M'. We have $(M, N) = 1$ if and only if $p^2 \nmid m_1 n_4$; by symmetry, $p^2|m_1$ if and only if $p|n_4$. So choose G so that $p|m_1$; we have p choices for G, and hence the contribution from these terms is $\chi(p) p^{k-3} \cdot p\mathbb{E}_{0,1}$.

Now suppose that $(M'\ N')$ is p^2-type $0, 2$. Let $V = \mathbb{F}x_1 \oplus \mathbb{F}x_2$ be equipped with the quadratic form given by $\frac{1}{p}\overline{N}'M'$ relative to $(x_1\ x_2)$. So relative to $(x_1'\ x_2') = (x_1\ x_2)G$, the quadratic form is given by $\frac{1}{p}\,{}^tG\overline{N}'M'G \equiv \bar{d}\begin{pmatrix} n_4 m_1/p & * \\ * & * \end{pmatrix}$ (p) where $d = \det N'$ (recall that $p^2|m_3$). We know $\mathbb{F}x_1'$ varies over all lines in V as G varies over ${}^t\mathcal{G}_1$. When p is odd, $p^2 \nmid m_1 n_4$ for $p - 1$ choices of G if $V \simeq \mathbb{H}$, and $p^2 \nmid m_1 n_4$ for $p + 1$ choices of G otherwise. So the contribution from these terms when p is odd is $\chi(p) p^{k-3}(p-1)\mathbb{E}_{0,2,\epsilon} + \chi(p) p^{k-3}(p+1)\mathbb{E}_{0,2,-\epsilon}$. When $p = 2$ the contribution is $2\chi(p) p^{k-3} \mathbb{E}_{0,2,+}$.

Case IIIc. Suppose $N\begin{pmatrix} \frac{1}{p} & \\ & 1 \end{pmatrix}$ is integral with

$$\operatorname{rank}_p\left(M\begin{pmatrix} p & \\ & 1 \end{pmatrix}\ N\begin{pmatrix} \frac{1}{p} & \\ & 1 \end{pmatrix}\right) = 1.$$

Adjust the equivalence class representative of $(M\ N)$ so that p divides row 2 of $\left(M\begin{pmatrix} p & \\ & 1 \end{pmatrix}\ N\begin{pmatrix} \frac{1}{p} & \\ & 1 \end{pmatrix}\right) = 1$. Set

$$\left(M'G\ N'\,{}^tG^{-1}\right) = \begin{pmatrix} 1 & \\ & \frac{1}{p} \end{pmatrix}\left(M\begin{pmatrix} p & \\ & 1 \end{pmatrix}\ N\begin{pmatrix} \frac{1}{p} & \\ & 1 \end{pmatrix}\right) = 1.$$

Since M', N' are integral and $(M, N) = 1$, we have $(M', N') = 1$. Also, $\operatorname{rank}_p M' \geqslant 1$ and $\chi(p^2)\chi_\rho(M', N') = \chi_{\mathcal{N}_2}(p^2)\chi_\rho(M, N)$.

Reversing, take $(M'\ N')$ of p-type 1 or 2, and set

$$(M\ N) = \begin{pmatrix} 1 & \\ & p \end{pmatrix} E\left(M'G \begin{pmatrix} \frac{1}{p} & \\ & 1 \end{pmatrix} N'\,{}^tG^{-1} \begin{pmatrix} p & \\ & 1 \end{pmatrix} \right)$$

where $E \in \mathcal{G}_1$. Write $EM'G = \left(\begin{smallmatrix} m_1 & m_2 \\ m_3 & m_4 \end{smallmatrix}\right)$, $EN'\,{}^tG^{-1} = \left(\begin{smallmatrix} n_1 & n_2 \\ n_3 & n_4 \end{smallmatrix}\right)$.

First suppose $\operatorname{rank}_p M' = 2$. For each E, choose the unique G so that $p|m_1$. Thus $p \nmid m_2m_3$, so $(M, N) = 1$. The contribution from these terms is $\chi_{\mathcal{N}_2}(p^2)p^{k-3} \cdot p^{-k} \cdot (p+1)\mathbb{E}_2$.

Finally, suppose $\operatorname{rank}_p M' = 1$; assume p divides row 2 of M'. Suppose p divides row 2 of EM'; choosing G so that $p|m_1$, by symmetry $p|n_4$ and hence $(M, N) \neq 1$. So choose the unique E so that p does not divide row 2 of EM'; then p divides row 1 of EM'. To have $(M, N) = 1$, choose G so that $p \nmid m_3$; we have p choices for G, and the contribution from these terms is $\chi_{\mathcal{N}_2}(p^2)p^{k-3} \cdot p^{1-k}\mathbb{E}_1$.

Combining all the contributions yields the result. $\square$

Now we determine the action on Eisenstein series of $T(q)$, $T_1(q^2)$ where q is a prime dividing $\mathcal{N}$. We let $\mathbb{F}$ denote $\mathbb{Z}/q\mathbb{Z}$; when q is odd, we let $\epsilon = (\frac{-1}{q})$, and we fix ω so that $(\frac{\omega}{q}) = -1$. Let $\mathcal{G}_1 = \mathcal{G}_1(q)$; note that we can choose the elements of $\mathcal{G}_1$ to be congruent modulo $\mathcal{N}/q$ to I.

Proposition 3.5. *Suppose q is a prime dividing $\mathcal{N}_2$; then*

$$\mathbb{E}_\rho | T(q) = \chi_{\mathcal{N}_0}\big(q^2\big)\chi_{\mathcal{N}_1}(q)q^{2k-3}\mathbb{E}_\rho.$$

Proof. From Proposition 3.1 of [4], we know that

$$2\mathbb{E}_\rho(\tau)|T(q) = q^{2k-3} \sum \overline{\chi}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2/q)}(M, N)\chi_q(\det M)\det(M\tau + qN + MY)^{-k}$$

where $(M\ \ N)$ varies over a set of $SL_2(\mathbb{Z})$-equivalence class representatives for pairs of $\mathcal{N}$-type ρ, and Y varies over all symmetric matrices modulo q; recall that we can choose $Y \equiv 0\ (\mathcal{N}/q)$. Take $(M\ \ N)$ of $\mathcal{N}$-type ρ; set $(M'\ \ N') = (M\ \ qN + MY)$. Thus $(M'\ \ N')$ is $\mathcal{N}$-type ρ; also, $\chi_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2/q)}(M', N') = \chi_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2/q)}(M, N)$. Reversing, take $(M'\ \ N')$ of $\mathcal{N}$-type ρ; set $(M\ \ N) = (M'\ \ \frac{1}{q}(N' - M'Y))$. So we need to choose $Y \equiv \overline{M}'N'\ \ (q)$ to have N integral. Thus $\overline{\chi}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2/q)}(M', N') = \overline{\chi}_{\mathcal{N}_0}(q^2)\chi_{\mathcal{N}_1}(q)\overline{\chi}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2/q)}(M, N)$. $\square$

Proposition 3.6. *For q a prime dividing $\mathcal{N}_1$, let $\rho' = (\mathcal{N}_0, \mathcal{N}_1/q, q\mathcal{N}_2)$; then*

$$\mathbb{E}_\rho | T(q) = \begin{cases} \chi_{\mathcal{N}_0\mathcal{N}_2}(q)(q^{k-1}\mathbb{E}_\rho + q^{k-3}(q^2-1)\mathbb{E}_{\rho'}) & \text{if } \chi_q = 1, \\ \chi_{\mathcal{N}_0\mathcal{N}_2}(q)q^{k-1}\mathbb{E}_\rho & \text{if } \chi_q \neq 1. \end{cases}$$

Proof. Recall that we must have $\chi_q^2 = 1$ since $q|\mathcal{N}_1$. Take $(M\ N)$ of $\mathcal{N}$-type ρ so that q divides row 2 of M. Set

$$\big(M'\ N'\big) = \begin{pmatrix} 1 & \\ & \frac{1}{q} \end{pmatrix} (M\ qN + MY);$$

since q does not divide row 1 of M or row 2 of N, $(M', N') = 1$. Also, $\operatorname{rank}_q M' \geqslant 1$, and $\chi_{(\mathcal{N}_0,\mathcal{N}_1/q,\mathcal{N}_2)}(M', N') = \chi_{\mathcal{N}_0\mathcal{N}_2}(q)\chi_{(\mathcal{N}_0,\mathcal{N}_1/q,\mathcal{N}_2)}(M, N)$.

Reversing, take $(M'\ N')$ of q-type 1 or 2, $\mathcal{N}/q$-type $(\mathcal{N}_0, \mathcal{N}_1/q, \mathcal{N}_2)$; set

$$(M\ N) = \begin{pmatrix} 1 & \\ & q \end{pmatrix} E\big(M'\ \big(N' - M'Y\big)/q\big),$$

$E \in \mathcal{G}_1$. Write $EM' = \left(\begin{smallmatrix} m_1 & m_2 \\ m_3 & m_4 \end{smallmatrix}\right)$, $EN' = \left(\begin{smallmatrix} n_1 & n_2 \\ n_3 & n_4 \end{smallmatrix}\right)$.

First suppose $\operatorname{rank}_q M' = 2$. To have N integral, we need to choose Y so that $(n_1\ n_2) \equiv (m_1\ m_2)Y\ (q)$; then to have $(M, N) = 1$, we need $(n_3\ n_4) \not\equiv (m_3\ m_4)Y\ (q)$. If $q \nmid m_1$ then y_4 can be chosen freely; if $q|m_1$ then y_1 can be chosen freely. This gives us $q-1$ choices for Y. Summing over these Y, we have

$$\sum_Y \chi_{(1,q,1)}(M, N) = \begin{cases} q-1 & \text{if } \chi_q = 1, \\ 0 & \text{if } \chi_q \neq 1. \end{cases}$$

So the contribution from these terms is

$$\begin{cases} q^{2k-3} \cdot q^{-k}\big(q^2-1\big)\mathbb{E}_{\rho'} & \text{if } \chi_q = 1, \\ 0 & \text{if } \chi_q \neq 1. \end{cases}$$

Now suppose $\operatorname{rank}_q M' = 1$; assume q divides row 2 of M'. To have $\operatorname{rank}_q M = 1$, we cannot have q dividing row 1 of EM'; this leaves us q choices for E, and with these choices we have q dividing row 2 of EM'. Then choose Y so that $(n_1\ n_2) \equiv (m_1\ m_2)Y\ (q)$; we have q choices for Y. Then row 2 of N is congruent modulo q to $(n_3\ n_4)$, so $(M, N) = 1$ and

$$\sum_Y \chi_{(1,q,1)}(M, N) = (q-1)\chi_{(1,q,1)}\big(M', N'\big).$$

So the contribution from these terms is $\chi_{\mathcal{N}_0\mathcal{N}_2}(q)q^{k-1}\mathbb{E}_\rho$. $\square$

Proposition 3.7. *For q a prime dividing $\mathcal{N}_0$, set*

$$\rho' = (\mathcal{N}_0/q, q\mathcal{N}_1, \mathcal{N}_2), \qquad \rho'' = (\mathcal{N}_0/q, \mathcal{N}_1, q\mathcal{N}_2).$$

Then

$$\mathbb{E}_\rho|T(q) = \begin{cases} \chi_{\mathcal{N}_1}(q)\chi_{\mathcal{N}_2}(q^2)(\mathbb{E}_\rho + q^{-1}(q-1)\mathbb{E}_{\rho'} + q^{-1}(q-1)\mathbb{E}_{\rho''}) & \text{if } \chi_q = 1, \\ \chi_{\mathcal{N}_1}(q)\chi_{\mathcal{N}_2}(q^2)(\mathbb{E}_\rho + \epsilon q^{-2}(q-1)\mathbb{E}_{\rho''}) & \text{if } \chi_q^2 = 1,\ \chi_q \neq 1, \\ \chi_{\mathcal{N}_1}(q)\chi_{\mathcal{N}_2}(q^2)\mathbb{E}_\rho & \text{if } \chi_q^2 \neq 1. \end{cases}$$

Proof. Take $(M\ N)$ of $\mathcal{N}$-type ρ, and set

$$\big(M'\ N'\big) = \frac{1}{q}(M\ qN + MY).$$

Since $\operatorname{rank}_q N = 2$, we have $\operatorname{rank}_q(M'\ N') = \operatorname{rank}_q(M'\ N) = 2$; hence $(M', N') = 1$. Also,

$$\chi_{(\mathcal{N}_0/q, \mathcal{N}_1, \mathcal{N}_2)}\big(M', N'\big) = \chi_{\mathcal{N}_1}(q)\chi_{\mathcal{N}_2}\big(q^2\big)\chi_{(\mathcal{N}_0/q, \mathcal{N}_1, \mathcal{N}_2)}(M, N).$$

Reversing, take $(M'\ N')$ of $\mathcal{N}/q$-type $(\mathcal{N}_0/q, \mathcal{N}_1, \mathcal{N}_2)$. Set

$$(M\ N) = \big(qM'\ N' - M'Y\big).$$

Write $M' = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$, $N' = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}$. We need to choose Y so that $\operatorname{rank}_q N = 2$. Then we sum $\sum_Y \overline{\chi}_q(\det N)$; since $\chi_q(\det N) = 0$ when $\operatorname{rank}_q N < 2$, we can simply sum over all Y.

First suppose $\operatorname{rank}_q M' = 2$. Then $\overline{M}'N' - Y$ varies over all symmetric matrices modulo q as Y does (here $\overline{M}'M' \equiv I\ (q)$), and each invertible symmetric matrix is either in the $GL_2(\mathbb{F})$-orbit of I or of $J = \begin{pmatrix} 1 & \\ & \omega \end{pmatrix}$, where $(\frac{\omega}{q}) = -1$ and $GL_2(\mathbb{F})$ acts by conjugation. With q odd and G varying over $GL_2(\mathbb{F})$, we have

$$\sum_Y \overline{\chi}_q(\det N) = \frac{\chi_q(\det M')}{o(I)} \sum_G \overline{\chi}_q^2(\det G) + \frac{\chi_q(\omega \det M')}{o(J)} \sum_G \overline{\chi}_q^2(\det G)$$

where $o(T) = \#\{C \in GL_2(\mathbb{F})\colon\ {}^tCTC = T\}$; it is known that $o(I) = 2(q-\epsilon)$ and $o(J) = 2(q+\epsilon)$ when q is odd. Now, $\overline{\chi}_q^2 \circ \det$ is a character on $GL_2(\mathbb{F})$, and it is the trivial character if and only if $\chi_q^2 = 1$. Hence

$$\sum_Y \overline{\chi}_q(\det N) = \begin{cases} q^2(q-1) & \text{if } \chi_q = 1, \\ \epsilon q(q-1)\chi_q(\det M') & \text{if } \chi_q^2 = 1,\ \chi_q \neq 1, \\ 0 & \text{if } \chi_q^2 \neq 1. \end{cases}$$

So the contribution from these terms when q is odd is

$$\begin{cases} \chi_{\mathcal{N}_1}(q) q^{-1}(q-1)\mathbb{E}_{\rho''} & \text{if } \chi_q = 1, \\ \epsilon\chi_{\mathcal{N}_1}(q) q^{-2}(q-1)\mathbb{E}_{\rho''} & \text{if } \chi_q^2 = 1,\ \chi_q \neq 1, \\ 0 & \text{if } \chi_q^2 \neq 1. \end{cases}$$

When $q = 2$ we have $\sum_Y \overline{\chi}_q(\det N) = 4$ so the contribution is $\chi_{\mathcal{N}_1}(q) q^{-1}(q-1)\mathbb{E}_{\rho''}$.

Now say $\operatorname{rank}_q M' = 1$; we can assume q divides row 2 of M'. Choose $E \in SL_2(\mathbb{Z})$ so that $M'E \equiv \begin{pmatrix} m_1' & 0 \\ 0 & 0 \end{pmatrix}\ (q)$; by symmetry, $N'\,{}^tE^{-1} \equiv \begin{pmatrix} n_1' & n_2' \\ 0 & n_4' \end{pmatrix}\ (q)$. Clearly $E^{-1}Y\,{}^tE^{-1}$ varies over all symmetric matrices as Y does, so

$$\begin{aligned} \sum_Y \overline{\chi}_q(\det N) &= q^2 \sum_{u\ (q)} \overline{\chi}_q\big((n_1' - m_1'u)n_4'\big) \\ &= \begin{cases} q^2(q-1) & \text{if } \chi_q = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus the contribution from these terms is

$$\begin{cases} q^{-1}(q-1)\mathbb{E}_{\rho'} & \text{if } \chi_q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, suppose $\operatorname{rank}_q M' = 0$. Then $\sum_Y \overline{\chi}_q(\det N) = q^3 \overline{\chi}_q(\det N')$, so the contribution is $\chi_{\mathcal{N}_1}(q)\chi_{\mathcal{N}_2}(q^2)\mathbb{E}_\rho$. $\square$

Proposition 3.8. *For q a prime dividing $\mathcal{N}_2$, we have*

$$\mathbb{E}_\rho | T_1\big(q^2\big) = \chi_{\mathcal{N}_0}\big(q^2\big)(q+1)q^{2k-3}\mathbb{E}_\rho.$$

Proof. From Proposition 2.1 of [4], we know $2\mathbb{E}_\rho(\tau)|T_1(q^2)$ is the sum of terms

$$p^{k-3}\overline{\chi}_\rho(M,N)\det\left(M\begin{pmatrix}\frac{1}{q}&\\&1\end{pmatrix}G^{-1}\tau+N\begin{pmatrix}q&\\&1\end{pmatrix}{}^tG+M\begin{pmatrix}\frac{1}{q}&\\&1\end{pmatrix}Y\,{}^tG\right)^{-k}$$

where $(M\ N)$ varies over $SL_2(\mathbb{Z})$-equivalence class representatives of pairs of $\mathcal{N}$-type ρ, $G\in\mathcal{G}_1$, $Y=\left(\begin{smallmatrix}y_1&y_2\\y_2&0\end{smallmatrix}\right)$ with y_1 varying modulo q^2, y_2 modulo q. Recall that we can choose $G\equiv I\ (\mathcal{N}/q)$ and $Y\equiv 0\ (\mathcal{N}/q)$.

We know $M\left(\begin{smallmatrix}\frac{1}{q}&\\&1\end{smallmatrix}\right)$ is never integral. Adjust the equivalence class representative for $(M\ N)$ so that

$$\left(M'G\ N'\,{}^tG^{-1}\right)=\begin{pmatrix}1&\\&q\end{pmatrix}\left(M\begin{pmatrix}\frac{1}{q}&\\&1\end{pmatrix}\ N\begin{pmatrix}q&\\&1\end{pmatrix}+M\begin{pmatrix}\frac{1}{q}&\\&1\end{pmatrix}Y\right)$$

is an integral pair. Since $\operatorname{rank}_q M'=2$, we have $(M',N')=1$. Also,

$$\chi_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2/q)}\left(M',N'\right)=\chi_{\mathcal{N}_0\mathcal{N}_1}\left(q^2\right)\chi_{(\mathcal{N}_1,\mathcal{N}_1,\mathcal{N}_2/q)}(M,N),$$

and we know $\chi_{\mathcal{N}_1}^2=1$.

Reversing, choose $(M'\ N')$ of $\mathcal{N}$-type ρ, and set

$$(M\ N)=\begin{pmatrix}1&\\&\frac{1}{q}\end{pmatrix}E\left(M'G\begin{pmatrix}q&\\&1\end{pmatrix}\ \left(N'\,{}^tG^{-1}-M'GY\right)\begin{pmatrix}\frac{1}{q}&\\&1\end{pmatrix}\right),$$

where $E\in{}^t\mathcal{G}_1$. Write $EM'G=\left(\begin{smallmatrix}m_1&m_2\\m_3&m_4\end{smallmatrix}\right)$, $EN'\,{}^tG^{-1}=\left(\begin{smallmatrix}n_1&n_2\\n_3&n_4\end{smallmatrix}\right)$.

For each choice of E, we need to choose the unique G so that $q|m_4$ to ensure M is integral. Thus $q\nmid m_2m_3$. Choose y_2 so that $n_4\equiv m_3y_2\ (q)$; then choose y_q so that $n_3\equiv m_3y_1+m_4y_2\ (q^2)$. By symmetry, $m_1n_3+m_2n_4\equiv m_3n_1\ (q)$. Thus

$$m_3n_1\equiv m_3m_3y_1+m_2m_3y_2\quad(q),$$

so $n_1\equiv m_1y_1+m_2y_2\ \ (q)$, and hence N is integral. Also, $\det M=\det M'$, so $\chi_\rho(M',N')=\chi_{\mathcal{N}_0}(q^2)\chi_\rho(M,N)$. $\square$

Proposition 3.9. *For q a prime dividing $\mathcal{N}_1$, set $\rho'=(\mathcal{N}_0,\mathcal{N}_1/q,q\mathcal{N}_2)$. Then*

$$\begin{aligned}&\mathbb{E}_\rho|T_1\left(q^2\right)\\&\quad=\begin{cases}(\chi_{\mathcal{N}_0}(q^2)q^{2k-2}+\chi_{\mathcal{N}_2}(q^2)q)\mathbb{E}_\rho+q^{-1}(\chi_{\mathcal{N}/q}(q)q^{k-2}+\chi_{\mathcal{N}_2}(q^2))(q^2-1)\mathbb{E}_{\rho'} & \text{if } \chi_q=1,\\(\chi_{\mathcal{N}_0}(q^2)q^{2k-2}+\chi_{\mathcal{N}_2}(q^2)q)\mathbb{E}_\rho & \text{if } \chi_q\neq 1.\end{cases}\end{aligned}$$

Proof. Recall that we must have $\chi_q^2=1$ since $q|\mathcal{N}_1$. Take $(M\ N)$ of $\mathcal{N}$-type ρ; assume q divides row 2 of M'.

First suppose $M\left(\begin{smallmatrix}\frac{1}{q}&\\&1\end{smallmatrix}\right)$ is not integral; thus by symmetry,

$$N=\begin{pmatrix}*&*\\ *&n_4\end{pmatrix}$$

where $q\nmid n_4$. Thus $(M',N')=1$ where

$$\left(M'G\ N'\,{}^tG^{-1}\right)=\begin{pmatrix} q & \\ & 1\end{pmatrix}\left(M\begin{pmatrix} \frac{1}{q} & \\ & 1\end{pmatrix}N\begin{pmatrix} q & \\ & 1\end{pmatrix}+M\begin{pmatrix} \frac{1}{q} & \\ & 1\end{pmatrix}Y\right).$$

We know $q\nmid M'$ and $q|\det M'$, so $\operatorname{rank}_q M'=1$. Also,

$$\chi_{(\mathcal{N}_0,\mathcal{N}_1/q,\mathcal{N}_2)}\left(M',N'\right)=\chi_{\mathcal{N}_0}\left(q^2\right)\chi_{(\mathcal{N}_0,\mathcal{N}_1/q,\mathcal{N}_2)}(M,N).$$

Reversing, take $(M'\ N')$ of $\mathcal{N}$-type ρ; assume q divides row 2 of M'. Set

$$(M\ N)=\begin{pmatrix} \frac{1}{q} & \\ & 1\end{pmatrix}E\left(M'G\begin{pmatrix} q & \\ & 1\end{pmatrix}\ \left(N'\,{}^tG^{-1}-M'GY\right)\begin{pmatrix} \frac{1}{q} & \\ & 1\end{pmatrix}\right)$$

where $E\in\mathcal{G}_1$. Write $EM'G=\left(\begin{smallmatrix} m_1 & m_2\\ m_3 & m_4\end{smallmatrix}\right)$, $EN'\,{}^tG^{-1}=\left(\begin{smallmatrix} n_1 & n_2\\ n_3 & n_4\end{smallmatrix}\right)$. For 1 choice of E, q divides row 1 of EM'; then to have N integral we need $q|(n_1\ n_2)$, which is impossible since $(M',N')=1$. So choose E so that q does not divide row 1 of EM' (q choices for E). Thus q divides row 2 of EM'; to have M integral, choose the unique G so that $q|m_2$; so $q\nmid m_1$, and by symmetry, $q|n_3$ (so $q\nmid n_4$). Choose the unique y_2 (q) so that $n_2\equiv m_1y_2$ (q) and choose the unique y_1 (q^2) so that $n_1\equiv m_1y_1+m_2y_2$ (q^2). Thus $(M,N)=1$, and $\chi_\rho(M',N')=\chi_\rho(M,N)$. So the contribution from these terms is $\chi_{\mathcal{N}_0}(q^2)q^{2k-2}\mathbb{E}_\rho$.

Now suppose $M\left(\begin{smallmatrix} \frac{1}{q} & \\ & 1\end{smallmatrix}\right)$ is integral with

$$\operatorname{rank}_q\left(M\begin{pmatrix} \frac{1}{q} & \\ & 1\end{pmatrix}N\begin{pmatrix} q & \\ & 1\end{pmatrix}\right)=2.$$

Thus $M\equiv\left(\begin{smallmatrix} m_1 & m_2\\ m_3 & m_4\end{smallmatrix}\right)$ (q) where $q|m_1,m_3,m_4$, $q\nmid m_2$. By symmetry, q divides row 2 of $N\left(\begin{smallmatrix} q & \\ & 1\end{smallmatrix}\right)$. So $\operatorname{rank}_q\left(M\left(\begin{smallmatrix} \frac{1}{q} & \\ & 1\end{smallmatrix}\right)N\left(\begin{smallmatrix} q & \\ & 1\end{smallmatrix}\right)\right)\geqslant 1$. In the case this rank is 2, we have $q^2\nmid m_3$, and $\operatorname{rank}_q M'=2$ where

$$\left(M'G\ N'\,{}^tG^{-1}\right)=\left(M\begin{pmatrix} \frac{1}{q} & \\ & 1\end{pmatrix}N\begin{pmatrix} q & \\ & 1\end{pmatrix}+M\begin{pmatrix} \frac{1}{q} & \\ & 1\end{pmatrix}Y\right).$$

Also, $\chi_{(\mathcal{N}_0,\mathcal{N}_1/q,\mathcal{N}_2)}(M',N')=\chi_{\mathcal{N}/q}(q)\chi_{(\mathcal{N}_0,\mathcal{N}_1/q,\mathcal{N}_2)}(M,N)$.

Reversing, first choose $(M'\ N')$ of $\mathcal{N}$-type $(\mathcal{N}_0,\mathcal{N}_1/q,q\mathcal{N}_2)$. Set

$$(M\ N)=\left(M'G\begin{pmatrix} q & \\ & 1\end{pmatrix}\ \left(N'\,{}^tG^{-1}-M'GY\right)\begin{pmatrix} \frac{1}{q} & \\ & 1\end{pmatrix}\right).$$

Write $M'G=\left(\begin{smallmatrix} m_1 & m_2\\ m_3 & m_4\end{smallmatrix}\right)$, $N'\,{}^tG^{-1}=\left(\begin{smallmatrix} n_1 & n_2\\ n_3 & n_4\end{smallmatrix}\right)$. Suppose $\operatorname{rank}_q M'=2$. Then for each G, adjust the equivalence class representative so that $q|m_4$ (so $q\nmid m_2m_3$). Take u,y_2 so that

$$\overline{M}'\begin{pmatrix} n_1\\ n_3\end{pmatrix}\equiv\begin{pmatrix} u\\ y_2\end{pmatrix}\quad(q);$$

set $y_1=u+qu'$ where u' varies modulo q. Then summing over corresponding Y, with $n_3'=(n_3-m_3u-m_4y_2)/q$, we have

$$\begin{aligned}\sum_Y\chi_{(1,q,1)}(M,N)&=\sum_{u'}\chi_q(-m_2)\chi_q\left(n_3'-m_3u'\right)\\&=\begin{cases}(q-1) & \text{if } \chi_q=1,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Thus the contribution from these terms is $\chi_{\mathcal{N}/q}(q)q^{k-3}(q^2-1)\mathbb{E}_{\rho'}$ if $\chi_q=1$, and 0 otherwise.

Suppose $M\left(\begin{smallmatrix}\frac{1}{q} & \\ & 1\end{smallmatrix}\right)$ is integral, $\operatorname{rank}_q\left(M\left(\begin{smallmatrix}\frac{1}{q} & \\ & 1\end{smallmatrix}\right)\ N\left(\begin{smallmatrix}q & \\ & 1\end{smallmatrix}\right)\right)=1$. Since $q \nmid m_2 n_3$, $(M'\ N')$ is an integral coprime pair where

$$\left(M'G\ N'\,{}^tG^{-1}\right)=\begin{pmatrix}1 & \\ & \frac{1}{q}\end{pmatrix}\left(M\begin{pmatrix}\frac{1}{q} & \\ & 1\end{pmatrix}\ N\begin{pmatrix}q & \\ & 1\end{pmatrix}+M\begin{pmatrix}\frac{1}{q} & \\ & 1\end{pmatrix}Y\right).$$

Note that $\operatorname{rank}_q M' \geqslant 1$. So

$$\chi_{(\mathcal{N}_0,\mathcal{N}_1/q,\mathcal{N}_2)}\left(M',N'\right)=\chi_{\mathcal{N}_2}\left(q^2\right)\chi_{(\mathcal{N}_0,\mathcal{N}_1/q,\mathcal{N}_2)}(M,N).$$

Reversing, take $(M'\ N')$ of $\mathcal{N}/q$-type $(\mathcal{N}_0,\mathcal{N}_1/q,\mathcal{N}_2)$, $\operatorname{rank}_q M' \geqslant 1$. Set

$$(M\ N)=\begin{pmatrix}1 & \\ & q\end{pmatrix}E\left(M'G\begin{pmatrix}q & \\ & 1\end{pmatrix}\ \left(N'\,{}^tG^{-1}-M'GY\right)\begin{pmatrix}\frac{1}{q} & \\ & 1\end{pmatrix}\right),$$

$E \in \mathcal{G}_1$. Write $EM'G=\left(\begin{smallmatrix}m_1 & m_2\\ m_3 & m_4\end{smallmatrix}\right)$, $EN'\,{}^tG^{-1}=\left(\begin{smallmatrix}n_1 & n_2\\ n_3 & n_4\end{smallmatrix}\right)$.

To have $\operatorname{rank}_q M=1$, we need to choose E,G so that $q \nmid m_2$, and to have N integral with $(M,N)=1$, we need to choose Y so that $n_1 \equiv m_1y_1+m_2y_2\ (q)$, $n_3 \not\equiv m_3y_1+m_4y_2\ (q)$.

First suppose $\operatorname{rank}_q M'=2$. To have $\operatorname{rank}_q M=1$, for each E we need to choose G so that $q \nmid m_2$ (for each E, this gives us q choices for G). If $q \nmid m_3$ then we choose y_1 freely; if $q | m_3$ then $q \nmid m_1m_4$, so we can choose y_2 freely (subject to $n_3 \not\equiv m_4y_2\ (q)$). In either case, we get

$$\begin{aligned}\sum_Y \chi_{(1,q,1)}(M,N)&=\chi_q(-m_2)\sum_Y \chi_q(n_3-m_3y_1-m_4y_2)\\ &=\begin{cases}q(q-1) & \text{if } \chi_q=1,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

So the contribution from these terms is $\chi_{\mathcal{N}_2}(q^2)q^{-1}(q^2-1)\mathbb{E}_{\rho'}$ if $\chi_q=1$, 0 otherwise.

Finally, suppose $\operatorname{rank}_q M'=1$; assume q divides row 2 of M'. We have q choices for E so that q does not divide row 1 of EM' (and then q divides row 2 of EM'); then we have q choices for G so that $q \nmid m_2$. By symmetry, $q \nmid n_3$. Choose y_1 freely, then choose y_2 so that $n_1 \equiv m_1y_1+m_2y_2\ (q)$. So the contribution from these terms is $\chi_{\mathcal{N}_2}(q^2)q\mathbb{E}_1$. $\square$

Proposition 3.10. *Let q be a prime dividing $\mathcal{N}_0$. Set*

$$\rho'=(\mathcal{N}_0/q,q\mathcal{N}_1,\mathcal{N}_2),\qquad \rho''=(\mathcal{N}_0/q,\mathcal{N}_1,q\mathcal{N}_2).$$

Then

$$\mathbb{E}_\rho|T_1\left(q^2\right)=\begin{cases}\chi_{\mathcal{N}_2}(q^2)(q+1)\mathbb{E}_\rho & \\ \quad+q^{-1}(\chi_{\mathcal{N}/q}(q)q^{k-1}+\chi_{\mathcal{N}_2}(q^2))(q-1)\mathbb{E}_{\rho'} & \\ \quad+\chi_{\mathcal{N}_2}(q^2)q^{-2}(q^2-1)\mathbb{E}_{\rho''} & \text{if } \chi_q=1,\\ \chi_{\mathcal{N}_2}(q^2)(q+1)\mathbb{E}_\rho+\epsilon\,\chi_{\mathcal{N}_2}(q^2)q^{-2}(q^2-1)\mathbb{E}_{\rho''} & \text{if } \chi_q^2=1,\ \chi_q\neq 1,\\ \chi_{\mathcal{N}_2}(q^2)(q+1)\mathbb{E}_\rho & \text{if } \chi_q^2\neq 1.\end{cases}$$

Proof. Take $(M\ N)$ of $\mathcal{N}$-type ρ. So $\operatorname{rank}_q M=0$, $\operatorname{rank}_q M\left(\begin{smallmatrix}\frac{1}{q} & \\ & 1\end{smallmatrix}\right)\leqslant 1$.

First suppose $(M',N')=1$ where

$$\big(M'G\ N'\,{}^tG^{-1}\big)=\left(M\begin{pmatrix}\frac{1}{q}&\\&1\end{pmatrix}\ N\begin{pmatrix}q&\\&1\end{pmatrix}+M\begin{pmatrix}\frac{1}{q}&\\&1\end{pmatrix}Y\right).$$

Since $\operatorname{rank}_q N\left(\begin{smallmatrix}q&\\&1\end{smallmatrix}\right)=1$, we must have $\operatorname{rank}_q M'=1$. Also,

$$\chi_{(\mathcal{N}_0/q,\mathcal{N}_1,\mathcal{N}_2)}\big(M',N'\big)=\chi_{\mathcal{N}/q}(q)\chi_{(\mathcal{N}_0/q,\mathcal{N}_1,\mathcal{N}_2)}(M,N).$$

Reversing, take $(M'\ N')$ of $\mathcal{N}$-type ρ'; set

$$(M\ N)=\left(M'G\begin{pmatrix}q&\\&1\end{pmatrix}\ \big(N'\,{}^tG^{-1}-M'GY\big)\begin{pmatrix}\frac{1}{q}&\\&1\end{pmatrix}\right).$$

Write $M'G=\left(\begin{smallmatrix}m_1&m_2\\m_3&m_4\end{smallmatrix}\right)$, $N'\,{}^tG^{-1}=\left(\begin{smallmatrix}n_1&n_2\\n_3&n_4\end{smallmatrix}\right)$; we can assume that $q|(m_3\ m_4)$. To have $q|M$, choose the unique G so that $q|m_2$; then $q\nmid m_1$, and by symmetry $q|n_3$ (so $q\nmid n_4$). To have N integral, choose u so that $n_1\equiv m_1u\ (q)$ and set $y_1=u+qu'$. For each choice of u', y_2, we have $\det N\equiv x-m_1n_4u'\ (q)$ where x does not depend on u'. Hence, fixing y_2,

$$\sum_{u'\ (q)}\overline{\chi}_q\big(x-m_1n_4u'\big)=\begin{cases}(q-1)&\text{if }\chi_q=1,\\0&\text{otherwise.}\end{cases}$$

So, letting y_2 vary modulo q, we see the contribution from these terms is

$$\begin{cases}\chi_{\mathcal{N}/q}(q)q^{k-3}\cdot q(q-1)\mathbb{E}_{\rho'}&\text{if }\chi_q=1,\\0&\text{otherwise.}\end{cases}$$

Now say $\operatorname{rank}_q\big(M\left(\begin{smallmatrix}\frac{1}{q}&\\&1\end{smallmatrix}\right)\ N\left(\begin{smallmatrix}q&\\&1\end{smallmatrix}\right)\big)=1$; adjust the equivalence class representative $(M\ N)$ as necessary so that

$$\big(M'G\ N'\,{}^tG^{-1}\big)=\begin{pmatrix}1&\\&\frac{1}{q}\end{pmatrix}\left(M\begin{pmatrix}\frac{1}{q}&\\&1\end{pmatrix}\ N\begin{pmatrix}q&\\&1\end{pmatrix}+M\begin{pmatrix}\frac{1}{q}&\\&1\end{pmatrix}Y\right)$$

is integral. Then $(M',N')=1$ since $\operatorname{rank}_q N'=2$. Also,

$$\chi_{(\mathcal{N}_0/q,\mathcal{N}_1,\mathcal{N}_2)}\big(M',N'\big)=\chi_{\mathcal{N}_2}\big(q^2\big)\chi_{(\mathcal{N}_0/q,\mathcal{N}_1,\mathcal{N}_2)}(M,N).$$

Reversing, take $(M'\ N')$ of $\mathcal{N}/q$-type $(\mathcal{N}_0/q,\mathcal{N}_1,\mathcal{N}_2)$, and set

$$(M\ N)=\begin{pmatrix}1&\\&q\end{pmatrix}E\left(M'G\begin{pmatrix}q&\\&1\end{pmatrix}\ \big(N'\,{}^tG^{-1}-M'GY\big)\begin{pmatrix}\frac{1}{q}&\\&1\end{pmatrix}\right),$$

$E\in\mathcal{G}_1$. Write $EM'G=\left(\begin{smallmatrix}m_1&m_2\\m_3&m_4\end{smallmatrix}\right)$, $EN'\,{}^tG^{-1}=\left(\begin{smallmatrix}n_1&n_2\\n_3&n_4\end{smallmatrix}\right)$.

Say $\operatorname{rank}_q M'=2$. To have $q|M$, for each E we need to choose the unique G so that $q|m_2$; thus $q\nmid m_1m_4$. To have N integral, choose u so that $n_1\equiv m_1u\ (q)$; set $y_1=u+qu'$, u' varying modulo q. By symmetry, $m_1n_3\equiv m_3n_1+m_4n_2\ (q)$, so $n_3\equiv\overline{m}_1m_3n_1+\overline{m}_1m_4n_2\ (q)$; hence $\det N\equiv-m_1m_4(\overline{m}_1n_2-y_2)^2\ (q)$. Thus summing over Y where $y_1=u+qu'$,

$$\sum_Y \overline{\chi}_q(\det N) = q\overline{\chi}_q(-m_1m_4) \sum_{y_2\ (q)} \chi_q^2(\overline{m}_1 n_2 - y_2)$$
$$= \begin{cases} q(q-1)\chi_q(-m_1m_4) & \text{if } \chi_q^2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

So the contribution from these terms is

$$\begin{cases} \chi_{\mathcal{N}_2}(q^2)q^{-2}(q^2-1)\mathbb{E}_{\rho''} & \text{if } \chi_q = 1, \\ \chi_{\mathcal{N}_2}(q^2)q^{k-3} \cdot q^{-k} \cdot q(q^2-1)\epsilon\mathbb{E}_{\rho''} & \text{if } \chi_q \neq 1,\ \chi_q^2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now say $\operatorname{rank}_q M' = 1$; we can assume q divides row 2 of M'. For q choices of E we have q dividing row 2 of EM'. Then to have $q|M$, choose the unique G so that $q|m_2$; so $q \nmid m_1$ and by symmetry, $q|n_3$. But then q divides row 2 of $(M\ N)$, so $(M,N) \neq 1$. With the 1 other choice of E, we have q dividing row 1 of EM'. Then to have N integral, choose the unique G so that $q|n_1$. Then $q \nmid n_2$ (since $(M',N')=1$), so by symmetry, $q|m_4$ (and hence $q \nmid m_3$). Thus

$$\sum_Y \overline{\chi}_q(\det N) = \sum_Y \overline{\chi}_q\big(-n_2(n_3 - m_3 y_1)\big)$$
$$= \begin{cases} q^3 & \text{if } \chi_q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

So the contribution from these terms is

$$\begin{cases} \chi_{\mathcal{N}_2}(q^2)q^{k-3} \cdot q^{-k} \cdot q^2(q-1)\mathbb{E}_{\rho'} & \text{if } \chi_q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, say $\operatorname{rank}_q M' = 0$. So to have N integral, for each E we need to choose the unique G so that $q|n_1$. Then $q \nmid n_2n_3$, and for each choice of Y we have N integral with $\det N \equiv \det N'\ (q)$. Hence the contribution from these terms is $\chi_{\mathcal{N}_2}(q^2)q^{k-3} \cdot q^{-k} \cdot q^3(q+1)\mathbb{E}_\rho$. $\square$

With these results, we now construct a basis of simultaneous Hecke eigenforms.

Theorem 3.11. *Take square-free $\mathcal{N} \in \mathbb{Z}_+$ and a Dirichlet character χ modulo $\mathcal{N}$ so that $\chi(-1) = (-1)^k$. There is a basis*

$$\big\{\widetilde{\mathbb{E}}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}\colon \mathcal{N}_0\mathcal{N}_1\mathcal{N}_2 = \mathcal{N},\ \chi_{\mathcal{N}_1}^2 = 1\big\}$$

for the space $\mathcal{E}_k^{(2)}(\mathcal{N},\chi)$ of degree 2 Siegel Eisenstein series of weight k, level $\mathcal{N}$, character χ so that for any prime p, $\widetilde{\mathbb{E}}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}|T(p) = \lambda(p)\widetilde{\mathbb{E}}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}$ and $\widetilde{\mathbb{E}}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}|T_1(p^2) = \lambda_1(p^2)\widetilde{\mathbb{E}}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}$ where

$$\lambda(p) = \big(\chi_{\mathcal{N}_0}(p)p^{k-1} + \chi_{\mathcal{N}_1\mathcal{N}_2}(p)\big)\big(\chi_{\mathcal{N}_0\mathcal{N}_1}(p)p^{k-2} + \chi_{\mathcal{N}_2}(p)\big)$$

and

$$\lambda_1(p^2) = \big(p + \chi_{\mathcal{N}_1}(p^2)\big)\big(\chi_{\mathcal{N}_0}(p^2)p^{2k-3} + \chi(p)p^{k-3}(p-1) + \chi_{\mathcal{N}_2}(p^2)\big).$$

Proof. Fix a partition $\rho = (\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)$ of $\mathcal{N}$. For q an odd prime dividing $\mathcal{N}$, set $\epsilon = (\frac{-1}{q})$, and set

$$a_\rho(q) = \begin{cases} -\frac{\chi_{\mathcal{N}_1\mathcal{N}_2}(q)q^{-1}(q-1)}{\chi_{\mathcal{N}_0}(q)q^{k-1} - \chi_{\mathcal{N}_1\mathcal{N}_2}(q)} & \text{if } \chi_q = 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$b_\rho(q) = \begin{cases} -\frac{\chi_{\mathcal{N}_2}(q^2)q^{-1}(q-1)(\chi_{\mathcal{N}_0}(q)q^{k-3} - \chi_{\mathcal{N}_1\mathcal{N}_2}(q))}{(\chi_{\mathcal{N}_0}(q)q^{k-1} - \chi_{\mathcal{N}_1\mathcal{N}_2}(q))(\chi_{\mathcal{N}_0}(q^2)q^{2k-3} - \chi_{\mathcal{N}_2}(q^2))} & \text{if } \chi_q = 1, \\ -\frac{\epsilon\chi_{\mathcal{N}_2}(q^2)q^{-2}(q-1)}{\chi_{\mathcal{N}_0}(q^2)q^{2k-3} - \chi_{\mathcal{N}_2}(q^2)} & \text{if } \chi_q \neq 1,\ \chi_q^2 = 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$c_\rho(q) = \begin{cases} -\frac{\chi_{\mathcal{N}_2}(q)q^{-2}(q^2-1)}{\chi_{\mathcal{N}_0\mathcal{N}_1}(q)q^{k-2} - \chi_{\mathcal{N}_2}(q)} & \text{if } \chi_q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Extend these functions multiplicatively, and set

$$\widetilde{\mathbb{E}}_\rho = \sum_{\substack{Q_0 Q_0' | \mathcal{N}_0 \\ Q_1 | \mathcal{N}_1}} a_\rho(Q_0) b_\rho(Q_0') c_\rho(Q_1) \mathbb{E}_{(\mathcal{N}_0/(Q_0Q_0'), Q_0\mathcal{N}_1/Q_1, Q_0'Q_1\mathcal{N}_2)}.$$

Since $a_\rho(Q_0) = 0$ unless $\chi_{Q_0} = 1$, $b_\rho(Q_0') = 0$ unless $\chi_{Q_0'}^2 = 1$, and $c_\rho(Q_1) = 0$ unless $\chi_{Q_1} = 1$, Propositions 3.3, 3.4, 3.5, and 3.8 show that $\widetilde{\mathbb{E}}_\rho$ is an eigenform for all $T(p)$, $T_1(p^2)$ where $p \nmid \mathcal{N}_0\mathcal{N}_1$, with eigenvalues as claimed in the theorem. For a prime $q | \mathcal{N}_1$, write

$$\widetilde{\mathbb{E}}_\rho = \sum_{\substack{Q_0 Q_0' | \mathcal{N}_0 \\ Q_1 | \mathcal{N}_1/q}} a_\rho(Q_0) b_\rho(Q_0') c_\rho(Q_1) \cdot \left(\mathbb{E}_{(\mathcal{N}_0/(Q_0Q_0'), Q_0\mathcal{N}_1/Q_1, Q_0'Q_1\mathcal{N}_2)} + c_\rho(q)\mathbb{E}_{(\mathcal{N}_0/(Q_0Q_0'), Q_0\mathcal{N}_1/(qQ_1), qQ_0'Q_1\mathcal{N}_2)}\right).$$

Propositions 3.5, 3.6, 3.8 and 3.9 show that $\widetilde{\mathbb{E}}_\rho$ is an eigenform for $T(q)$ and $T_1(q^2)$, with eigenvalues as claimed. For $q | \mathcal{N}_0$, using Propositions 3.5 through 3.10 and a similar rearrangement of the sum defining $\widetilde{\mathbb{E}}_\rho$, we find $\widetilde{\mathbb{E}}_\rho$ is an eigenform for $T(q)$ and $T_1(q^2)$, with eigenvalues as claimed. □

Note that for q a prime dividing $\mathcal{N}$ with $\chi_q^2 = 1$, Propositions 3.6, 3.7, 3.9 and 3.10 give us Hecke relations among Eisenstein series. In particular, when $\operatorname{cond}\chi^2 < \mathcal{N}$ we can generate some of the Eisenstein series from $\mathbb{E}_{(\mathcal{N},1,1)}$. To see this, let q be a prime dividing $\mathcal{N}$ so that $\chi_q^2 = 1$. If $\chi_q = 1$, set $c(q) = \frac{q^2}{(q-1)(\chi_{\mathcal{N}/q}(q)q^k - 1)}$,

$$S_1(q) = c(q)\left[T_1(q^2) - q^{-1}(q+1)T(q) - q^{-1}(q^2-1)\right]$$

and

$$S_2(q) = c(q)\left[\left(\chi_{\mathcal{N}/q}(q)q^{k-1} + 1\right)T(q) - T_1(q^2) - q\left(\chi_{\mathcal{N}/q}(q)q^{k-2} - 1\right)\right];$$

if $\chi_q \neq 1$, set

$$S_2(q) = \frac{(\frac{-1}{q})q^2}{(q-1)}\left[T(q) - 1\right].$$

Extending these maps multiplicatively, we have the following.

Theorem 3.12. *Suppose $\mathcal{N}$ is square-free, $\mathcal{N}_0\mathcal{N}_1\mathcal{N}_2 = \mathcal{N}$, $\chi_{\mathcal{N}_1} = 1$, and $\chi^2_{\mathcal{N}_2} = 1$. Then*

$$\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)} = \mathbb{E}_{(\mathcal{N},1,1)}|S_1(\mathcal{N}_1)S_2(\mathcal{N}_2).$$

In particular, when $\chi = 1$,

$$\left\{\mathbb{E}_{(\mathcal{N},1,1)}|S_1(\mathcal{N}_1)S_2(\mathcal{N}_2)\colon \mathcal{N}_1\mathcal{N}_2|\mathcal{N}\right\}$$

is a basis for $\mathcal{E}_k^{(2)}(\mathcal{N},1)$.

Proof. When $\chi_q = 1$, we use Propositions 3.7, 3.10 to solve for $\mathbb{E}_{(\mathcal{N}_0/q,q\mathcal{N}_1,\mathcal{N}_2)}$ and for $\mathbb{E}_{(\mathcal{N}_0/q,\mathcal{N}_1,q\mathcal{N}_2)}$ in terms of $\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}$, presuming $q|\mathcal{N}_0$. When $\chi_q \neq 1$ we use Proposition 3.7 to get $\mathbb{E}_{(\mathcal{N}_0/q,\mathcal{N}_1,q\mathcal{N}_2)}$ in terms of $\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}$, again presuming $q|\mathcal{N}_0$. Now a simple induction argument yields the result. $\square$

Remarks. (1) When f is a Siegel modular form with Fourier coefficients $a(T)$, we have $a({}^tGTG) = a(T)$ for all $G \in GL_2(\mathbb{Z})$ if k is even, and for all $G \in SL_2(\mathbb{Z})$ if k is odd. Thus we can consider the Fourier series of f to be supported on lattices Λ equipped with a positive, semi-definite quadratic form given by T (relative to some basis), with Λ oriented if k is odd; for such Λ we set $a(\Lambda) = a(T)$. Then by Theorem 6.1 of [4], with QP square-free, the Λth coefficient of $f|T_1(Q^2)T(P)$ is

$$\sum_{\substack{Q\Lambda\subset\Omega\subset\Lambda\\ [\Lambda:\Omega]=Q}} a\left(\Omega^P\right),$$

where Ω^P denotes the lattice Ω whose quadratic form has been scaled by P. (Note that an orientation on Λ induces an orientation on $\Omega \subset \Lambda$.)

(2) With $\mathbb{E}$ the degree 2 Eisenstein series of level 1, we have

$$\mathbb{E}(\tau) = \sum_{\mathcal{N}_0\mathcal{N}_1\mathcal{N}_2=\mathcal{N}} \mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}.$$

Thus formulas for the Fourier coefficients of $\mathbb{E}$ together with our Hecke relations can be used to generate Fourier coefficients of all degree 2, square-free level $\mathcal{N}$ Eisenstein series with trivial character.

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