



# On the mean value of the index of composition of an integral ideal (II) <sup>☆</sup>

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## ABSTRACT

**Text.** For each integral ideal  $\mathfrak{A}$ , let  $\lambda(\mathfrak{A}) := \frac{\log N(\mathfrak{A})}{\log \gamma(\mathfrak{A})}$  be the index of composition of  $\mathfrak{A}$ , where  $\gamma(\mathfrak{A}) = \prod_{\mathfrak{P}|\mathfrak{A}} N(\mathfrak{P})$  and  $N(\mathfrak{A})$  is the norm of ideal  $\mathfrak{A}$ . In this paper, we obtain a new asymptotic formula of the sum  $\sum_{N(\mathfrak{A}) \leq x} \lambda^{-k}(\mathfrak{A})$ . Furthermore, we improve the error term under the Generalized Riemann Hypothesis.

**Video.** For a video summary of this paper, please click [here](#) or visit <http://www.youtube.com/watch?v=ly3QwXo5GeU>.

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## 1. Introduction

Motivated by the works [5,6,17], in [18] we studied the index of composition of an integral ideal on number fields. Let  $K$  be an algebraic number field of degree  $d$  and  $O_K$  be the ring of integers of  $K$ . For each integral ideal  $\mathfrak{A} \in O_K$ ,  $\mathfrak{A} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_g^{e_g}$ , where the  $\mathfrak{P}_i$  ( $i = 1, \dots, g$ ) are prime ideals of  $O_K$ , and this expression is unique up to the order of the factors. We define

$$\lambda(\mathfrak{A}) := \frac{\log N(\mathfrak{A})}{\log \gamma(\mathfrak{A})}, \quad \gamma(\mathfrak{A}) := \prod_{\mathfrak{P}|\mathfrak{A}} N(\mathfrak{P})$$

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be the *index of composition* of  $\mathfrak{A}$ , where  $N(\mathfrak{A})$  is the norm of ideal  $\mathfrak{A}$ . The index of composition of an integral ideal measures the multiplicity of its prime ideal factors. If  $\mathfrak{A} = \mathcal{O}_K$ , we write  $\lambda(\mathfrak{A}) = \gamma(\mathfrak{A}) = 1$ .

Suppose the Dedekind zeta-function for the field  $K$

$$\zeta_K(s) = \sum_{\mathfrak{A} \neq 0} \frac{1}{N^s(\mathfrak{A})} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \operatorname{Re} s > 1,$$

where  $a_n$  is the number of integral ideals of  $K$  with norm  $n$ .  $\zeta_K(s)$  can be analytically continued to the whole complex plane,  $s = 1$  is a simple pole with residue

$$\rho_K = \frac{2^{r_1} (2\pi)^{r_2} R_K}{\omega_K \sqrt{|d(K)|}},$$

where  $r_1, r_2$  denote the number of real and complex places respectively,  $R_K$  is the regulator of  $K$ ,  $d(K)$  is the discriminant of  $K$  and  $\omega_K$  is the order of the group of units. It is well known that  $a_n \leq (\tau(n))^d$ , where  $\tau(n)$  is the number of divisors of  $n$ . Let

$$C(x) := \sum_{n \leq x} a_n = \rho_K x + \Delta(x). \quad (1.1)$$

Then (see [4,9,10])

$$\Delta(x) = O(x^{\theta_d + \varepsilon}), \quad (1.2)$$

where

$$\theta_d := \begin{cases} \frac{131}{416}, & d = 2, \\ \frac{43}{96}, & d = 3, \\ 1 - \frac{2}{d} + \frac{8}{d(5d+2)}, & d = 4, 5, 6, \\ 1 - \frac{2}{d} + \frac{3}{2d^2}, & d \geq 7. \end{cases} \quad (1.3)$$

If the Riemann Hypothesis for  $\zeta_K(s)$  (or even the weaker Lindelöf hypothesis of  $\zeta_K(s)$ ) is true, then we can take  $\theta_d = 1/2$ . In [18] we proved a series of results about the mean value of  $\lambda(\mathfrak{A})$ . In particular, we got the following results

$$\sum_{N(\mathfrak{A}) \leq x} \lambda^k(\mathfrak{A}) = \rho_K x + \rho_K \sum_{j=1}^r \binom{j+k-1}{k-1} (-1)^j H^{(j)}(0) \int_2^x \log^{-j} z \, dz + O\left(\frac{x}{\log^{r+1} x}\right), \quad (1.4)$$

where

$$H(z) = \prod_{\mathfrak{P}} \left(1 - \frac{1}{N(\mathfrak{P})}\right) \left(1 + \frac{1}{N(\mathfrak{P}) - N^{-z}(\mathfrak{P})}\right) \quad (-1 < \operatorname{Re} z < 1) \quad (1.5)$$

and

$$\sum_{N(\mathfrak{A}) \leq x} \lambda^{-k}(\mathfrak{A}) = \rho_K x + \rho_K \sum_{j=1}^k \binom{k}{j} H^{(j)}(0) \int_2^x \log^{-j} z \, dz + O(x^{\vartheta+\varepsilon}), \quad (1.6)$$

where

$$\vartheta = \begin{cases} \frac{1}{2}, & d = 2, 3, \\ 1 - \frac{2}{d} + \frac{8}{d(5d+2)}, & d = 4, 5, 6, \\ 1 - \frac{2}{d} + \frac{3}{2d^2}, & d \geq 7. \end{cases} \quad (1.7)$$

The formulas (1.4) and (1.6) imply that the average order of  $\lambda(\mathfrak{A})$  is  $\rho_K$ .

In this paper, we shall further improve (1.6) for the quadratic and cubic number fields. We first prove the following

**Theorem 1.1.** *Let  $k \geq 1$  be a fixed integer and  $K$  a quadratic or cubic number field. Then we have*

$$\begin{aligned} \sum_{N(\mathfrak{A}) \leq x} \lambda^{-k}(\mathfrak{A}) &= \rho_K x + \rho_K \sum_{j=1}^k \binom{k}{j} H^{(j)}(0) \int_2^x \frac{1}{\log^j z} \, dz + \rho_K \sum_{j=1}^k C_{k,j} \int_2^x \frac{z^{-1/2}}{\log^j z} \, dz \\ &\quad + O_K(x^{\frac{1}{2}} \exp(-c' \log^{\frac{1}{3}} x (\log \log x)^{-\frac{1}{3}})), \end{aligned} \quad (1.8)$$

where  $C_{k,j}$  was defined in (5.11),  $c' > 0$  is a positive constant depending on  $K$ .

It is very difficult to improve the exponent  $1/2$  in the error term of (1.8) unless we have substantial progress in the study of the zero-free region of  $\zeta_K(s)$ . Therefore it is reasonable to get a better improvement by assuming the truth of the Riemann Hypothesis (RH) for  $\zeta_K(s)$ . In such a case we have the following

**Theorem 1.2.** *Suppose  $K$  is a quadratic or cubic number field and  $k \geq 1$  is a fixed integer. If the Riemann Hypothesis for  $\zeta_K(s)$  is true, then we have*

$$\begin{aligned} \sum_{N(\mathfrak{A}) \leq x} \lambda^{-k}(\mathfrak{A}) &= \rho_K x + \rho_K \sum_{j=1}^k \binom{k}{j} H^{(j)}(0) \int_2^x \frac{1}{\log^j z} \, dz \\ &\quad + \sum_{j=1}^k C_{K,j} \int_2^x \frac{z^{-1/2}}{\log^j z} \, dz + O(x^{\eta_K+\varepsilon}), \end{aligned} \quad (1.9)$$

where

$$\eta_K = \begin{cases} \frac{5}{12}, & d = 2, \\ \frac{73}{156}, & d = 3. \end{cases} \quad (1.10)$$

**Remark.** By more efforts, the exponent  $5/12$  in Theorem 2 can be improved  $227/562 \approx 0.40391 \dots$  (see Section 6). This exponent is near to  $2/5$ , which seems to be the limit of our approach.

## 2. Some properties of $\zeta_K(s)$

Suppose  $K$  is a quadratic or cubic number field. The following lemmas are needed in our proof, which are important properties of the Dedekind zeta-function  $\zeta_K(s)$ . The constants  $c_1, c_2, c_3, \dots$ , depend on  $K$ .

**Lemma 2.1.** *Let  $\varphi(t) = \log^{2/3}(|t|) \log \log^{1/3}(|t|)$ . There exist two positive constants  $c_1$  and  $c_2$  such that  $\zeta_K(s)$  has no zeros in the region*

$$\sigma \geq 1 - \frac{c_1}{\varphi(t)}, \quad |t| \geq c_2. \quad (2.1)$$

There exists a positive constant  $c_3$  such that in the region

$$\sigma \geq 1 - \frac{c_3}{\varphi(t)}, \quad |t| \geq 3c_2, \quad (2.2)$$

we have the estimates

$$\frac{\zeta'_K(s)}{\zeta_K(s)} \ll \varphi(t), \quad (2.3)$$

$$\frac{1}{\zeta_K(s)} \ll \varphi(t), \quad (2.4)$$

and

$$\zeta_K(s) \ll (\log(|t|))^{c_4}, \quad (2.5)$$

where  $c_4 > 0$  is a positive constant.

**Proof.** (2.1), (2.3) and (2.5) are results of Mitsui [8]. We only prove (2.4). If

$$1 - c_3\varphi^{-1}(t) \leq \sigma \leq 1 + \varphi^{-1}(t),$$

then

$$\begin{aligned} \log \frac{1}{|\zeta_K(s)|} &= -\operatorname{Re} \log \zeta_K(s) \\ &= -\operatorname{Re} \log \zeta_K(1 + \varphi^{-1}(t) + it) + \int_{\sigma}^{1+\varphi^{-1}(t)} \operatorname{Re} \frac{\zeta'_K(u+it)}{\zeta_K(u+it)} du \\ &\leq \log \zeta_K(1 + \varphi^{-1}(t)) + \int_{\sigma}^{1+\varphi^{-1}(t)} \varphi(t) du \ll \log \varphi(t). \end{aligned}$$

For larger  $\sigma$  it follows trivially from

$$\left| \frac{\zeta'_K(s)}{\zeta_K(s)} \right| \leq -\frac{\zeta'_K(\sigma_1)}{\zeta_K(\sigma_1)} \ll (\sigma_1 - 1)^{-1} \quad (\sigma \geq \sigma_1 > 1). \quad \square$$

**Lemma 2.2.** *Let  $K$  be a quadratic or cubic number field. Then the estimate*

$$\int_1^T |\zeta_K(\sigma + it)| dt \ll T \log^3 T, \quad (2.6)$$

holds uniformly for  $1/2 \leq \sigma \leq 1$ .

**Proof.** For the case  $d = 2$ , by the abelian class field theory, we can write  $\zeta_K(s)$  as a product of the Riemann zeta-function and a Dirichlet L-function, i.e.,

$$\zeta_K(s) = \zeta(s)L(s, \chi),$$

where  $L(s, \chi)$  is a primitive Dirichlet L-function. By Cauchy's inequality, we have

$$\int_1^T |\zeta_K(\sigma + it)| dt \ll \left( \int_1^T |\zeta(\sigma + it)|^2 dt \right)^{1/2} \left( \int_1^T |L(\sigma + it, \chi)|^2 dt \right)^{1/2} \ll T \log T,$$

where we have used the well-known estimates

$$\int_1^T |\zeta(\sigma + it)|^2 dt \ll T \log T \quad \text{and} \quad \int_1^T |L(\sigma + it, \chi)|^2 dt \ll T \log T, \quad (2.7)$$

which hold uniformly for  $1/2 \leq \sigma \leq 1$ .

For the case  $d = 3$ , we use Lemma 1 of [9]. If  $K$  is a cubic normal extension over  $\mathbb{Q}$ , then we have

$$\zeta_K(s) = \zeta(s)L(s, \chi)\overline{L(s, \chi)}.$$

By Hölder's inequality, we have that

$$\begin{aligned} \int_1^T |\zeta_K(\sigma + it)| dt &\ll \left( \int_1^T |\zeta(\sigma + it)|^2 dt \right)^{1/2} \left( \int_1^T |L(\sigma + it, \chi)|^4 dt \right)^{1/4} \\ &\quad \cdot \left( \int_1^T |\overline{L(\sigma + it, \chi)}|^4 dt \right)^{1/4} \\ &\ll T \log^3 T, \end{aligned}$$

where we used

$$\int_1^T |L(\sigma + it, \chi)|^4 dt \ll T \log^5 T \quad (2.8)$$

holds uniformly for  $1/2 \leq \sigma \leq 1$ . If  $K$  is not a normal extension, then  $d \neq 1$  and

$$\zeta_K(s) = \zeta(s)L(s, \chi_2),$$

where  $L(s, \chi_2)$  is a Dirichlet L-series over the quadratic field  $\Omega = \mathbb{Q}(\sqrt{d})$ , which is a degree 2 L-function. Hence we have

$$\int_1^T |L(\sigma + it, \chi_2)|^2 dt \ll T \log^5 T.$$

By Cauchy's inequality, we have

$$\int_1^T |\zeta_K(\sigma + it)| dt \ll \left( \int_1^T |\zeta(\sigma + it)|^2 dt \right)^{1/2} \left( \int_1^T |L(\sigma + it, \chi_2)|^2 dt \right)^{1/2} \ll T \log^3 T. \quad \square$$

**Lemma 2.3.** *If  $\sigma > 1$ , then*

$$\zeta_K(s) \ll \log(|t| + 2). \quad (2.9)$$

*Uniformly for  $1/2 \leq \sigma \leq 1$ , we have*

$$\zeta_K(s) \ll (|t| + 2)^{(\frac{d}{3} + \varepsilon)(1 - \sigma)} \log(|t| + 2). \quad (2.10)$$

**Proof.** Heath-Brown [3] applied an  $n$ -dimensional variant of van der Corput's method to establish the bound

$$\zeta_K(1/2 + it) \ll t^{d/6 + \varepsilon}.$$

By (2.9) and Phragmén–Lindelöf Principal, we can easily get (2.10).  $\square$

**Lemma 2.4.** *Write  $\zeta_K(s) = \chi(s)\zeta_K(1 - s)$ , where*

$$\chi(s) = B^{2s-1} \frac{A(1-s)}{A(s)},$$

*with*

$$B = 2^{r_2} \pi^{\frac{d}{2}} (d(K))^{-\frac{1}{2}}, \quad A(s) = \Gamma^{r_1} \left( \frac{1}{2} s \right) \Gamma^{r_2}(s),$$

*where  $r_1, r_2$  denote the number of real and complex places respectively and  $d(K)$  is the discriminant of  $K$ . Then*

$$\chi(s) \ll (|t| + 2)^{d(1/2 - \sigma)}. \quad (2.11)$$

**Proof.** See Chapter XIII of S. Lang [7].  $\square$

**Lemma 2.5.** *Write*

$$\frac{1}{\zeta_K(s)} = \sum_{n=1}^{\infty} \frac{\mu_K(n)}{n^s} \quad (\operatorname{Re} s = \sigma > 1).$$

*For any  $y > 1$  define*

$$g_{y,K}(s) = \frac{1}{\zeta_K(s)} - \sum_{n \leq y} \frac{\mu_K(n)}{n^s} \quad (\sigma > 1). \quad (2.12)$$

If the Riemann Hypothesis for  $\zeta_K(s)$  is true, then  $g_{y,K}(s)$  can be continued analytically to  $\sigma > 1/2 + \varepsilon$  and we have the estimate

$$g_{y,K}(s) \ll y^{1/2-\sigma+\varepsilon} (|t|+2)^\varepsilon \quad (\sigma > 1/2 + \varepsilon). \quad (2.13)$$

**Proof.** The proof is the same as Titchmarsh [14]. So we omit its details.  $\square$

**Lemma 2.6.** Let  $\Delta(x)$  be defined by (1.1). Then we have

$$\Delta(x) = C_K x^{\frac{d-1}{2d}} \sum_{l \leq L_0} \frac{a_l}{l^{\frac{d+1}{2d}}} \cos\left(2\pi d \left(\frac{lx}{D_K}\right)^{1/d} + \frac{\pi l_K}{4}\right) + O_K\left(\frac{x^{\frac{d-1}{d}+\varepsilon}}{L_0^{1/d}}\right), \quad (2.14)$$

where  $1 \leq L_0 \ll x$ ,  $C_K = (\pi d)^{-1/2} D_K^{\frac{1}{2d}}$  is a real number,  $D_K = d(K)$  and  $l_K = d - 3$  are two integers.

**Proof.** See Friedlander and Iwaniec [1].  $\square$

**Lemma 2.7.** We have

$$\int_1^T \Delta(x) dx \ll T^{1+\delta_K+\varepsilon}, \quad (2.15)$$

where

$$\delta_K = \begin{cases} 0, & d = 2, \\ \frac{1}{6}, & d = 3. \end{cases} \quad (2.16)$$

**Proof.** It suffices to prove that

$$\int_T^{2T} \Delta(x) dx \ll T^{1+\delta_K+\varepsilon}. \quad (2.17)$$

Firstly we consider the case  $d = 2$ . Taking  $N = T$  in Lemma 2.6, we get

$$\begin{aligned} \int_T^{2T} \Delta(x) dx &= c_K \sum_{n \leq T} \frac{a_n}{n^{\frac{3}{4}}} \int_T^{2T} x^{\frac{1}{4}} \cos\left(4\pi \left(\frac{nx}{D_K}\right)^{1/2} - \frac{\pi}{4}\right) dx + O(T^{1+\varepsilon}) \\ &\ll \sum_{n \leq T} \frac{a_n}{n^{\frac{3}{4}}} \cdot \frac{T^{3/4}}{n^{1/2}} + T^{1+\varepsilon} \ll T^{1+\varepsilon}. \end{aligned}$$

For the case  $d = 3$ , Lemma 2.6 gives only

$$\int_T^{2T} \Delta(x) dx \ll T^{\frac{4}{3}+\varepsilon}.$$

So we use another approach. By Perron's formula (see for example, Chapter 4 of Pan and Pan [12]), we have for  $T \leq x \leq 2T$  that

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{1-\varepsilon-iT}^{1+\varepsilon+iT} \zeta_K(s) \frac{x^s}{s} ds + O(T^\varepsilon).$$

Moving the integral line to  $\sigma = 1 - \varepsilon$ , we get

$$\Delta(x) = \frac{1}{2\pi i} \int_{1-\varepsilon-iT}^{1-\varepsilon+iT} \zeta_K(s) \frac{x^s}{s} ds + O(T^\varepsilon).$$

So

$$\begin{aligned} \int_T^{2T} \Delta(x) dx &= \frac{1}{2\pi i} \int_{1-\varepsilon-iT}^{1-\varepsilon+iT} \frac{\zeta_K(s)}{s} \left( \int_T^{2T} x^s dx \right) ds + O(T^{1+\varepsilon}) \\ &= \frac{1}{2\pi i} \int_{1-\varepsilon-iT}^{1-\varepsilon+iT} \frac{\chi(s) \zeta_K(1-s) (2^{s+1} - 1) T^{s+1}}{s(s+1)} ds + O(T^{1+\varepsilon}), \end{aligned} \quad (2.18)$$

where the functional equation was used. Moving the integral line in the last integral of (2.18) to  $\sigma = 1/6 + \varepsilon$ , we get

$$\int_T^{2T} \Delta(x) dx \ll T^{1+\frac{1}{6}+\varepsilon} \int_{-T}^T \frac{\zeta_K(\frac{5}{6}+\varepsilon) \chi(\frac{1}{6}+\varepsilon)}{(|t|+2)^2} dt + O(T^{1+\varepsilon}).$$

By Lemmas 2.2 and 2.4 we have

$$\begin{aligned} \int_T^{2T} \Delta(x) dx &\ll T^{\frac{7}{6}+\varepsilon} \int_1^T \frac{\zeta_K(\frac{5}{6}+\varepsilon)}{|t|+2} dt \\ &\ll T^{\frac{7}{6}+\varepsilon} \max_{1 \leq T_1 \leq T} \left\{ T_1^{-1} \int_{T_1/2}^{T_1} \left| \zeta_K\left(\frac{5}{6}+\varepsilon\right) \right| dt \right\} \ll T^{\frac{7}{6}+\varepsilon}. \quad \square \end{aligned}$$



### 3. Mean value of $\gamma^z(\mathfrak{A})$ (I)

From now on, we suppose  $\varepsilon > 0$  is a small positive constant,  $z$  is a complex number such that  $|z| \leq \varepsilon$ . For each fixed  $z$ ,  $\gamma^z(\mathfrak{A})$  is a multiplicative function of  $N(\mathfrak{A})$ . For each  $\mathfrak{A}$ ,  $\gamma^z(\mathfrak{A})$  is an analytic function of  $z$  in the region  $|z| \leq \varepsilon$ .

Let  $s = \sigma + it$  be a complex number with  $\operatorname{Re}(s - z) > 1$ . Define

$$G(s, z) := \sum_{\mathfrak{A}} \gamma^z(\mathfrak{A}) N^{-s}(\mathfrak{A}).$$

**Lemma 3.1.** For  $|z| \leq \varepsilon$ , we have

$$G(s, z) = \frac{\zeta_K(s - z) \zeta_K(2s - z)}{\zeta_K(2s - 2z)} G_1(s, z), \quad (3.1)$$

where  $G_1(s, z)$  can be expanded into a Dirichlet series of  $s$ , which is absolutely convergent for  $\sigma > 1/3 + 2\varepsilon$ .

**Proof.** Lemma 3.1 follows easily from Lemma 5.1 of [18].  $\square$

**Proposition A.** Suppose that  $z$  is a complex number such that  $|z| \leq (\log x)^{-2/3-\varepsilon}$ . Then

$$\sum_{N(\mathfrak{A}) \leq x} \gamma^z(\mathfrak{A}) = \rho_K H_1(z) x^{1+z} + \rho_K H_2(z) x^{\frac{1+z}{2}} + O\left(x^{\frac{1}{2}} \exp(-c \log^{\frac{1}{3}} x (\log \log x)^{-\frac{1}{3}})\right), \quad (3.2)$$

where  $c > 0$  is an absolute constant, and

$$H_1(z) := \frac{\zeta_K(2+z) G_1(1+z, z)}{(1+z) \zeta_K(2)}, \quad H_2(z) := \frac{\zeta_K(\frac{1+z}{2}) G_1(\frac{1+z}{2}, z)}{(1+z) \zeta_K(1-z)}.$$

**Proof.** By Perron's formula (see for example, Chapter 4 of Pan and Pan [12]), we have

$$\sum_{N(\mathfrak{A}) \leq x} \gamma^z(\mathfrak{A}) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} G(s, z) \frac{x^s}{s} ds + O\left(\frac{x^{1+2|\operatorname{Re} z| \log x}}{T}\right), \quad (3.3)$$

where  $b = 1 + 2|\operatorname{Re} z| + \frac{1}{\log x}$  and  $T = x^{2/3}$ .

From Lemma 2.1, we see that there exists an absolute constant  $c_1 > 0$  such that  $\zeta_K(s)$  is zero-free in the region

$$D(T): |t| \leq 3T, \quad \sigma \geq 1 - \frac{c_1}{(\log T)^{2/3} (\log \log T)^{1/3}}.$$

Furthermore in  $D(T)$  we have the estimates

$$\zeta_K(s) \ll (\log T)^{c_4}, \quad c_4 > 0, \quad (3.4)$$

and

$$\zeta_K^{-1}(s) \ll (\log T)^{2/3} (\log \log T)^{1/3}. \quad (3.5)$$

Let

$$z_0 = \frac{c_1}{1000(\log x)^{2/3}(\log \log x)^{1/3}},$$

and move the integral line of the integral in (3.3) to  $\operatorname{Re} s = \frac{1}{2} - 2z_0$ . By Cauchy's residue theorem we get from (3.3) that

$$\sum_{N(\mathfrak{A}) \leq x} \gamma^z(\mathfrak{A}) = \rho_K H_1(z) x^{1+z} + \rho_K H_2(z) x^{\frac{1+z}{2}} + \int_1 + \int_2 - \int_3 + O(x^{1/3+\varepsilon}), \quad (3.6)$$

where

$$\begin{aligned} \int_1 &= \frac{1}{2\pi i} \int_{\frac{1}{2}-2z_0-iT}^{\frac{1}{2}-z_0+iT} G(s, z) \frac{x^s}{s} ds, \\ \int_2 &= \frac{1}{2\pi i} \int_{\frac{1}{2}-2z_0+iT}^{b+iT} G(s, z) \frac{x^s}{s} ds, \\ \int_3 &= \frac{1}{2\pi i} \int_{\frac{1}{2}-2z_0-iT}^{b-iT} G(s, z) \frac{x^s}{s} ds. \end{aligned}$$

We consider  $\int_1$  first. By the definition of  $T$  and  $z_0$ , we see that for any  $s = \frac{1}{2} - 2z_0 + it$  with  $|t| \leq T$ , we must have  $2s - z \in D(T)$  and  $2s - 2z \in D(T)$  if noticing  $|z| \leq (\log x)^{-2/3-\varepsilon}$ . Hence from Lemma 3.1 and (3.4), (3.5) we get

$$\begin{aligned} \int_1 &\ll x^{\frac{1}{2}-2z_0} + x^{\frac{1}{2}-2z_0} \int_1^T |G(s, z)| \frac{dt}{t} \\ &\ll x^{\frac{1}{2}-2z_0} + x^{\frac{1}{2}-2z_0} (\log T)^{\frac{2}{3}+c_4} (\log \log T)^{\frac{1}{3}} \int_1^T \left| \zeta_K \left( \frac{1}{2} - 2z_0 - z + it \right) \right| \frac{dt}{t}. \end{aligned} \quad (3.7)$$

Applying the functional equation of  $\zeta_K(s)$  and by Lemma 2.4 we have

$$\begin{aligned} \int_1 &\ll x^{\frac{1}{2}-2z_0} + x^{\frac{1}{2}-2z_0} (\log x)^{c_5} \int_1^T t^{2z_0+2\operatorname{Re} z} \left| \zeta_K \left( \frac{1}{2} + 2z_0 + z - it \right) \right| \frac{dt}{t} \\ &\ll x^{\frac{1}{2}-2z_0} + x^{\frac{1}{2}-2z_0} (\log x)^{c_5} T^{2z_0+2\operatorname{Re} z} \int_1^T \left| \zeta_K \left( \frac{1}{2} + 2z_0 + z - it \right) \right| \frac{dt}{t}, \end{aligned} \quad (3.8)$$

where  $c_5 = c_4 + 1$ . By Lemma 2.2, we get

$$\int_1^T \left| \zeta_K \left( \frac{1}{2} + 2z_0 + z - it \right) \right| \frac{dt}{t} \ll \log T \max_{T_1 \leq T} \left\{ T_1^{-1} \int_{T_1/2}^{T_1} \left| \zeta_K \left( \frac{1}{2} + 2z_0 + z - it \right) \right| dt \right\} \ll \log^6 T.$$

Consequently, we have

$$\begin{aligned} \int_1^\infty &\ll x^{\frac{1}{2}-2z_0} + x^{\frac{1}{2}-2z_0} T^{2z_0+2\operatorname{Re} z} (\log x)^{c_6} \\ &\ll x^{\frac{1}{2}} \exp(-c \log^{\frac{1}{3}} x (\log \log x)^{-\frac{1}{3}}), \end{aligned} \quad (3.9)$$

where  $c = 2c_2/3000$  is an absolute constant.

Now we estimate  $J_2$ . By Lemma 3.1 and (3.4), (3.5) we get

$$\begin{aligned} \int_2^\infty &\ll \int_{\frac{1}{2}-2z_0}^b |G(\sigma + it, z)| \frac{x^\sigma}{T} d\sigma \\ &\ll \frac{(\log x)^{c_5}}{T} \int_{\frac{1}{2}-2z_0}^b |\zeta_K(\sigma + it - z)| x^\sigma d\sigma. \end{aligned} \quad (3.10)$$

We write the last integral in the above formula as

$$\int_{\frac{1}{2}-2z_0}^b = \int_1^b + \int_{\frac{1}{2}}^1 + \int_{\frac{1}{2}-2z_0}^{\frac{1}{2}}.$$

For the order of  $\zeta_K(\sigma + it - z)$ , we have from Lemmas 2.3 and 2.4 that

$$\zeta_K(\sigma + iT - z) \ll \begin{cases} (\log T)^{c_4}, & 1 \leq \sigma \leq b, \\ T^{\frac{d(1-\sigma)}{3}} \log T, & 1/2 \leq \sigma \leq 1, \\ T^{\frac{d}{2} - \frac{2d\sigma}{3} + \frac{2d\operatorname{Re} z}{3}} \log T, & 1/2 - 2z_0 \leq \sigma \leq 1/2. \end{cases}$$

From the above formulas we get ( $d = 2, 3$ )

$$\begin{aligned} \int_2^\infty &\ll \frac{(\log x)^{c_6}}{T} \int_1^b x^\sigma d\sigma + \frac{(\log x)^{c_6}}{T} \int_{\frac{1}{2}}^1 T^{\frac{d(1-\sigma)}{3}} x^\sigma d\sigma \\ &\quad + \frac{(\log x)^{c_6}}{T} \int_{\frac{1}{2}-2z_0}^{\frac{1}{2}} T^{\frac{d}{2} - \frac{2d\sigma}{3} + \frac{2d\operatorname{Re} z}{3}} x^\sigma d\sigma \\ &\ll \frac{x^{1+2|\operatorname{Re} z|} \log^{c_6} x}{T} + \frac{T^{1/2+2\operatorname{Re} z}}{x^{1/6-z_0}} \ll x^{1/3+\varepsilon}. \end{aligned} \quad (3.11)$$

Similarly, we have

$$\int_3 \ll x^{1/3+\varepsilon}. \quad (3.12)$$

Now Proposition A follows from (3.6), (3.9), (3.11) and (3.12).  $\square$

#### 4. Mean value of $\gamma^z(\mathfrak{A})$ (II)

In this section we study the mean value of  $\gamma^z(\mathfrak{A})$  under RH of  $\zeta_K(s)$ . Throughout this section we suppose  $z$  is a complex number with  $|z| \leq \varepsilon$ . We shall prove

**Proposition B.** Suppose that RH for  $\zeta_K(s)$  is true. If  $|z| \leq \varepsilon$ , then we have

$$\sum_{N(\mathfrak{A}) \leq x} \gamma^z(\mathfrak{A}) = \rho_K H_1(z) x^{1+z} + \rho_K H_2(z) x^{\frac{1+z}{2}} + O(x^{\eta_K+\varepsilon}), \quad (4.1)$$

where  $\eta_K$  was defined in Theorem 1.2, and  $H_1(z)$  and  $H_2(z)$  were defined in Proposition A.

Define the function  $h(n, z)$  by the following

$$\frac{\zeta_K(s-z)\zeta_K(2s-z)}{\zeta_K(2s-2z)} = \sum_{n=1}^{\infty} \frac{h(n, z)}{n^s}, \quad \operatorname{Re}(s-z) > 1. \quad (4.2)$$

Obviously by Lemma 3.1 and the convolution approach, Proposition B follows easily from the following Lemma 4.1.

**Lemma 4.1.** Suppose that RH for  $\zeta_K(s)$  is true. If  $|z| \leq \varepsilon$ , then

$$\sum_{n \leq x} h(n, z) = \rho_K H_1^*(z) x^{1+z} + \rho_K H_2^*(z) x^{\frac{1+z}{2}} + O(x^{\eta_K+\varepsilon}), \quad (4.3)$$

where

$$H_1^*(z) = \frac{\zeta_K(2+z)}{(1+z)\zeta_K(2)}, \quad H_2^*(z) = \frac{\zeta_K(\frac{1-z}{2})}{(1+z)\zeta_K(1-z)}.$$

##### 4.1. Several lemmas

Define the function  $f(n, z)$  by the following

$$\zeta_K(s-z)\zeta_K(2s-z) = \sum_{n=1}^{\infty} \frac{f(n, z)}{n^s}, \quad \operatorname{Re}(s-z) > 1. \quad (4.4)$$

We first prove the following

**Lemma 4.2.** We have uniformly for  $|z| \leq \varepsilon$  that

$$\sum_{n \leq x} f(n, z) = \frac{\rho_K \zeta_K(2+z)}{1+z} x^{1+z} + \frac{\rho_K \zeta_K(\frac{1-z}{2})}{1+z} x^{\frac{1+z}{2}} + \Delta(x, z), \quad (4.5)$$

where

$$\begin{aligned} \Delta(x, z) &= x^z \sum_{m \leq x^{1/3}} a_m m^{-z} \Delta\left(\frac{x}{m^2}\right) + x^{\frac{z}{2}} \sum_{n \leq x^{1/3}} a_n n^{\frac{z}{2}} \Delta\left(\sqrt{\frac{x}{n}}\right) \\ &\quad + O\left(x^{(1+\delta_K)/3+\varepsilon}\right) \end{aligned} \quad (4.6)$$

with  $\delta_K$  defined in Lemma 2.7.

**Proof.** When  $\operatorname{Re}(s - z) > 1$  we have

$$\zeta_K(s - z) \zeta_K(2s - z) = \sum_{n=1}^{\infty} \frac{a_n n^z}{n^s} \cdot \sum_{m=1}^{\infty} \frac{a_m m^z}{m^{2s}}.$$

So by the hyperbolic approach (as Dirichlet did for the Dirichlet divisor problem) we get

$$\begin{aligned} \sum_{n \leq x} f(n, z) &= \sum_{nm^2 \leq x} a_n n^z a_m m^z \\ &= \sum_{m \leq x^{1/3}} a_m m^z \sum_{n \leq \frac{x}{m^2}} a_n n^z + \sum_{n \leq x^{1/3}} a_n n^z \sum_{m \leq \sqrt{\frac{x}{n}}} a_m m^z \\ &\quad - \left( \sum_{m \leq x^{1/3}} a_m m^z \right)^2. \end{aligned} \quad (4.7)$$

Define for any  $u > 1$

$$\Delta_1(u, z) := \sum_{n \leq u} a_n n^z - \frac{\rho_K}{1+z} u^{1+z}.$$

Hence (4.7) becomes

$$\sum_{n \leq x} f(n, z) = \sum_1 + \sum_2 + \sum_3 + \sum_4 - \left( \frac{\rho_K}{1+z} x^{\frac{1+z}{3}} + \Delta_1(x^{1/3}, z) \right)^2, \quad (4.8)$$

where

$$\begin{aligned} \sum_1 &= \frac{\rho_K x^{1+z}}{1+z} \sum_{m \leq x^{1/3}} \frac{a_m}{m^{2+z}}, & \sum_2 &= \sum_{m \leq x^{1/3}} a_m m^z \Delta_1\left(\frac{x}{m^2}, z\right), \\ \sum_3 &= \frac{\rho_K x^{\frac{1+z}{2}}}{1+z} \sum_{n \leq x^{1/3}} \frac{a_n}{n^{\frac{1-z}{2}}}, & \sum_4 &= \sum_{n \leq x^{1/3}} a_n n^z \Delta_1\left(\sqrt{\frac{x}{n}}, z\right). \end{aligned}$$

We first evaluate the sum  $\sum_1$  and  $\sum_3$ . Suppose  $s = \sigma + it$  is a complex number such that  $\sigma > 1$  and  $|t| \leq 1$ . By the definition of  $\zeta_K(s)$  and partial summation we have for any  $u \geq 2$  that

$$\begin{aligned}
\zeta_K(s) &= \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n \leq u} \frac{a_n}{n^s} + \int_u^{\infty} \frac{1}{w^s} d\left(\sum_{n \leq w} a_n\right) \\
&= \sum_{n \leq u} \frac{a_n}{n^s} + \rho_K \int_u^{\infty} \frac{dw}{w^s} + \int_u^{\infty} \frac{d\Delta(w)}{w^s} \\
&= \sum_{n \leq u} \frac{a_n}{n^s} + \frac{\rho_K}{s-1} u^{1-s} - \frac{\Delta(u)}{u^s} + s \int_u^{\infty} \frac{\Delta(w)}{w^{s+1}} dw.
\end{aligned} \tag{4.9}$$

Let

$$I_{\Delta}(V) := \int_1^V \Delta(w) dw.$$

By Lemma 2.7 we have

$$I_{\Delta}(V) \ll V^{1+\delta_K+\varepsilon}. \tag{4.10}$$

By partial integration we get

$$\int_u^{\infty} \frac{\Delta(w)}{w^{s+1}} dw = \int_u^{\infty} \frac{dI_{\Delta}(w)}{w^{s+1}} = -\frac{I_{\Delta}(u)}{u^{s+1}} + (s+1) \int_u^{\infty} \frac{I_{\Delta}(w)}{w^{s+2}} dw. \tag{4.11}$$

From (4.10) we see that the integral  $\int_u^{\infty} I_{\Delta}(w) w^{-s-2} dw$  is absolutely convergent for  $\sigma > \delta_K$ ,  $|t| \leq 1$ . From (4.9)–(4.11) we get for any  $\sigma > \delta_K$  and  $|t| \leq 1$  that

$$\sum_{n \leq u} \frac{a_n}{n^s} = \zeta_K(s) - \frac{\rho_K}{s-1} u^{1-s} + \frac{\Delta(u)}{u^s} + O(u^{\delta_K-\sigma+\varepsilon}). \tag{4.12}$$

Taking  $s = 2 + z$  in (4.12) we get that

$$\begin{aligned}
\sum_1 &= \frac{\rho_K x^{1+z}}{1+z} \left( \zeta_K(2+z) - \frac{\rho_K}{1+z} x^{-\frac{1+z}{3}} + \frac{\Delta(x^{1/3})}{x^{\frac{2+z}{3}}} + O(x^{\frac{\delta_K-2-\operatorname{Re} z}{3}+\varepsilon}) \right) \\
&= \frac{\rho_K \zeta_K(2+z)}{1+z} x^{1+z} - \frac{\rho_K^2}{(1+z)^2} x^{\frac{2+2z}{3}} + \frac{\rho_K \Delta(x^{1/3})}{1+z} x^{\frac{1+2z}{3}} + O(x^{\frac{1+\delta_K}{3}+\varepsilon}).
\end{aligned} \tag{4.13}$$

Taking  $s = \frac{1-z}{2}$  in (4.12) we get that

$$\sum_3 = \frac{\rho_K \zeta_K(\frac{1-z}{2})}{1+z} x^{\frac{1+z}{2}} + \frac{2\rho_K^2}{(1+z)^2} x^{\frac{2+2z}{3}} + \frac{\rho_K \Delta(x^{1/3})}{1+z} x^{\frac{1+2z}{3}} + O(x^{\frac{1+\delta_K}{3}+\varepsilon}). \tag{4.14}$$

For the relation between  $\Delta(u)$  and  $\Delta_1(u, z)$ , by partial summation and Lemma 2.7 we get

$$\begin{aligned}
\sum_{n \leq u} a_n n^z &= \int_{1^-}^u w^z d\left(\sum_{n \leq w} a_n\right) = \int_{1^-}^u w^z d(\rho_K w + \Delta(w)) \\
&= \rho_K \int_{1^-}^u w^z dw + \int_{1^-}^u w^z d\Delta(w) \\
&= \frac{\rho_K}{1+z} u^{1+z} + u^z \Delta(u) - z \int_1^u \Delta(w) w^{z-1} dw \\
&= \frac{\rho_K}{1+z} u^{1+z} + u^z \Delta(u) + O(u^{\delta_K + \varepsilon}),
\end{aligned}$$

which implies that

$$\Delta_1(u, z) = u^z \Delta(u) + O(u^{\delta_K + \varepsilon}). \quad (4.15)$$

Inserting (4.15) into  $\sum_2$  and  $\sum_4$ , we get

$$\sum_2 = x^z \sum_{m \leq x^{1/3}} a_m m^{-z} \Delta\left(\frac{x}{m^2}\right) + O(x^{\frac{1+\delta_K}{3} + \varepsilon}) \quad (4.16)$$

and

$$\sum_4 = x^{z/2} \sum_{n \leq x^{1/3}} a_n n^{z/2} \Delta\left(\sqrt{\frac{x}{n}}\right) + O(x^{\frac{1+\delta_K}{3} + \varepsilon}). \quad (4.17)$$

From (4.15) and the estimate (1.2) we have

$$\begin{aligned}
\left(\frac{\rho_K}{1+z} x^{\frac{1+z}{3}} + \Delta_1(x^{1/3}, z)\right)^2 &= \left(\frac{\rho_K}{1+z} x^{\frac{1+z}{3}} + x^{z/3} \Delta(x^{1/3}) + O(x^{\frac{\delta_K}{3} + \varepsilon})\right)^2 \\
&= \frac{\rho_K^2}{(1+z)^2} x^{\frac{2+2z}{3}} + \frac{2\rho_K \Delta(x^{1/3})}{1+z} x^{\frac{1+2z}{3}} + O(x^{\frac{1+\delta_K}{3} + \varepsilon}).
\end{aligned} \quad (4.18)$$

Now Lemma 4.2 follows from (4.8) and (4.13)–(4.18).  $\square$

Define

$$\Delta^*(x, z) := \sum_{n \leq x} h(n, z) - \rho_K H_1^*(z) x^{1+z} - \rho_K H_2^*(z) x^{\frac{1+z}{2}}.$$

Then we have the following Lemma 4.3.

**Lemma 4.3.** Suppose RH of  $\zeta_K(s)$  is true. If  $|z| \leq \varepsilon$ , then

$$\Delta^*(x, z) = \sum_{m \leq y} \mu_K(m) m^{2z} \Delta\left(\frac{x}{m^2}, z\right) + O(x^{1/2+\varepsilon} y^{-1/2}), \quad (4.19)$$

where  $1 < y < \sqrt{x}$  is a parameter.

**Proof.** From the definitions of  $h(n, z)$ ,  $f(n, z)$  and  $\mu_K(n)$  we have for  $\operatorname{Re}(s - z) > 1$  that

$$\sum_{n=1}^{\infty} \frac{h(n, z)}{n^s} = \frac{\zeta_K(s - z)\zeta_K(2s - z)}{\zeta_K(2s - 2z)} = \sum_{n=1}^{\infty} \frac{f(n, z)}{n^s} \cdot \sum_{m=1}^{\infty} \frac{\mu_K(m)m^{2z}}{m^{2s}}.$$

So we have for  $1 < y < \sqrt{x}$  that

$$\begin{aligned} \sum_{n \leq x} h(n, z) &= \sum_{nm^2 \leq x} f(n, z)\mu_K(m)m^{2z} \\ &= \sum_{m \leq y} \mu_K(m)m^{2z} \sum_{n \leq x/m^2} f(n, z) + \sum_{\substack{nm^2 \leq x \\ m > y}} f(n, z)\mu_K(m)m^{2z} \\ &= \sum_5 + \sum_6, \end{aligned} \quad (4.20)$$

say. By Lemma 4.2 we get

$$\begin{aligned} \sum_5 &= \frac{\rho_K \zeta_K(2 + z)x^{1+z}}{1 + z} \sum_{m \leq y} \frac{\mu_K(m)}{m^2} + \frac{\rho_K \zeta_K(\frac{1-z}{2})x^{\frac{1+z}{2}}}{1 + z} \sum_{m \leq y} \frac{\mu_K(m)}{m^{1-z}} \\ &\quad + \sum_{m \leq y} \mu_K(m)m^{2z} \Delta\left(\frac{x}{m^2}, z\right). \end{aligned} \quad (4.21)$$

With the help of Lemma 2.5 we get by the analytic approach (see for example Nowak [11] or Zhai [16]) that

$$\begin{aligned} \sum_6 &= \frac{\rho_K \zeta_K(2 + z)x^{1+z}}{1 + z} \sum_{m > y} \frac{\mu_K(m)}{m^2} + \frac{\rho_K \zeta_K(\frac{1-z}{2})x^{\frac{1+z}{2}}}{1 + z} \sum_{m > y} \frac{\mu_K(m)}{m^{1-z}} \\ &\quad + O(x^{1/2+\varepsilon}y^{-1/2}). \end{aligned} \quad (4.22)$$

Combining the above estimates, the proof of Lemma 4.3 is now complete.  $\square$

In order to prove Lemma 4.1 we use the following Lemmas 4.4 and 4.5. Lemma 4.4 is Theorem 1 of Robert and Sargos [13] and Lemma 4.5 is Lemma 2.4 of Graham and Kolesnik [2].

**Lemma 4.4.**

$$\begin{aligned} &\sum_{H_1 < h \leq 2H_1} \sum_{N_1 < n \leq 2N_1} a(h, n) \sum_{M_1 < m \leq 2M_1} b(m) e\left(X \frac{h^\beta n^\gamma m^\alpha}{H_1^\beta N_1^\gamma M_1^\alpha}\right) \\ &\ll (H_1 N_1 M_1)^{1+\varepsilon} \left( \left( \frac{X}{H_1 N_1 M_1^2} \right)^{1/4} + \frac{1}{(H_1 N_1)^{1/4}} + \frac{1}{M_1^{1/2}} + \frac{1}{X^{1/2}} \right), \end{aligned}$$

where  $a(h, n) \ll 1$ ,  $b(m) \ll 1$  are complex numbers,  $X > 1$ ,  $\alpha, \beta, \gamma$  are fixed real numbers such that  $\alpha(\alpha - 1)\beta\gamma \neq 0$ .



**Lemma 4.5.** Suppose that

$$L(H) = \sum_{i=1}^m A_i H^{a_i} + \sum_{j=1}^n B_j H^{-b_j},$$

where  $A_i, B_j, a_i, b_j$  are positive. Assume that  $H_1 \leq H_2$ . Then there exists some  $H$  with  $H_1 \leq H \leq H_2$  that

$$L(H) \ll \sum_{i=1}^m \sum_{j=1}^n (A_i^{b_j} B_j^{a_i})^{1/(a_i+b_j)} + \sum_{i=1}^m A_i H_1^{a_i} + \sum_{j=1}^n B_j H_2^{-b_j}.$$

Now we prove Lemma 4.1. Suppose that  $x^\varepsilon \ll y \ll x^{1/5}$  is a parameter to be determined. By Lemma 4.2 and Lemma 4.3 we have

$$\Delta^*(x, z) = \sum_7 + \sum_8 + O(x^{(1+\delta_K)/3+\varepsilon} y^{(1-2\delta_K)/3} + x^{1/2+\varepsilon} y^{-1/2}), \quad (4.23)$$

where

$$\begin{aligned} \sum_7 &= x^z \sum_{m \leq y} \mu_K(m) \sum_{\substack{n \leq \frac{x^{1/3}}{m^{2/3}}} } a_n n^{-z} \Delta\left(\frac{x}{m^2 n^2}\right), \\ \sum_8 &= x^{z/2} \sum_{m \leq y} \mu_K(m) m^z \sum_{\substack{n \leq \frac{x^{1/3}}{m^{2/3}}} } a_n n^{z/2} \Delta\left(\sqrt{\frac{x}{m^2 n}}\right). \end{aligned}$$

From now on we shall prove Lemma 4.1 for  $d = 2$  and  $d = 3$  separately.

#### 4.2. Proof of Lemma 4.1 for $d = 2$

We estimate  $\sum_7$  first. By a splitting argument we get that

$$\sum_7 \ll |\Sigma_7(M, N)| x^\varepsilon \log^2 x, \quad (4.24)$$

for some  $1 \ll M \ll y$  and  $1 \ll N \ll x^{1/3} M^{-2/3}$ , where

$$\Sigma_7(M, N) = \sum_{M < m \leq 2M} \mu_K(m) \sum_{\substack{N < n \leq 2N \\ n \leq \frac{x^{1/3}}{m^{2/3}}} } a_n n^{-z} \Delta\left(\frac{x}{m^2 n^2}\right).$$

Taking  $L_0 = y$  in Lemma 2.6 and then using a splitting argument, we get

$$\begin{aligned} \Sigma_7(M, N) &\ll \left| \sum_{M < m \leq 2M} \mu_K(m) \sum_{\substack{N < n \leq 2N \\ n \leq \frac{x^{1/3}}{m^{2/3}}} } a_n n^{-z} \frac{x^{1/4}}{(mn)^{1/2}} \sum_{l \leq L} \frac{a_l}{l^{3/4}} e\left(\frac{2x^{\frac{1}{2}} l^{\frac{1}{2}}}{D_K^{\frac{1}{2}} mn}\right) \right| + \frac{x^{1/2+\varepsilon}}{y^{1/2}} \\ &\ll |\Sigma_7^*(M, N, L)| \log x + x^{1/2+\varepsilon} y^{-1/2}, \end{aligned} \quad (4.25)$$

for some  $1 \ll L \ll y$ , where

$$\Sigma_7^*(M, N, L) = \frac{x^{\frac{1}{4}}}{L^{\frac{3}{4}} M^{\frac{1}{2}} N^{\frac{1}{2}}} \sum_{M < m \leq 2M} \frac{\mu_K(m) M^{\frac{1}{2}}}{m^{\frac{1}{2}}} \sum_{\substack{N < n \leq 2N \\ n \leq \frac{x^{1/3}}{m^{2/3}}}} \frac{a_n N^{\frac{1}{2}}}{n^{\frac{1}{2}+z}} \sum_{L < l \leq 2L} \frac{a_l L^{\frac{3}{4}}}{l^{\frac{3}{4}}} e\left(\frac{2x^{\frac{1}{2}} l^{\frac{1}{2}}}{D_K^{\frac{1}{2}} m n}\right).$$

Taking  $(H_1, N_1, M_1) = (L, M, N)$  in Lemma 4.4 we get

$$\begin{aligned} x^{-\varepsilon} \Sigma_7^*(M, N, L) &\ll x^{\frac{3}{8}} N^{-\frac{1}{4}} L^{\frac{1}{8}} + x^{\frac{1}{4}} M^{\frac{1}{4}} N^{\frac{1}{2}} + x^{\frac{1}{4}} L^{\frac{1}{4}} M^{\frac{1}{2}} + MN \\ &\ll x^{\frac{3}{8}} y^{\frac{1}{8}} + x^{\frac{5}{12}} M^{-\frac{1}{12}} + x^{\frac{1}{4}} y^{\frac{3}{4}} + x^{\frac{1}{3}} y^{\frac{1}{3}}, \end{aligned} \quad (4.26)$$

if noticing that  $N \ll x^{1/3} M^{-2/3}$ ,  $L \ll y$ ,  $M \ll y$ . From (4.24)–(4.26) we get

$$x^{-\varepsilon} \sum_7 \ll x^{\frac{5}{12}} + x^{\frac{3}{8}} y^{\frac{1}{8}} + x^{\frac{1}{4}} y^{\frac{3}{4}} + x^{\frac{1}{3}} y^{\frac{1}{3}} + x^{\frac{1}{2}} y^{-\frac{1}{2}}. \quad (4.27)$$

Now we estimate  $\Sigma_8$ . By a splitting approach we have

$$\sum_8 \ll |\Sigma_8(M, N)| x^{\varepsilon} \log^2 x, \quad (4.28)$$

for some  $1 \ll M \ll y$  and  $1 \ll N \ll x^{1/3} M^{-2/3}$ , where

$$\Sigma_8(M, N) = \sum_{M < m \leq 2M} \mu_K(m) m^z \sum_{\substack{N < n \leq 2N \\ n \leq \frac{x^{1/3}}{m^{2/3}}}} a_n n^{z/2} \Delta\left(\frac{x^{1/2}}{mn^{1/2}}\right).$$

Taking  $L = L_0$  in Lemma 2.6 we get by a splitting argument that

$$\begin{aligned} \Sigma_8(M, N) &\ll \left| \sum_{M < m \leq 2M} \mu_K(m) m^z \sum_{\substack{N < n \leq 2N \\ n \leq \frac{x^{1/3}}{m^{2/3}}}} a_n n^{\frac{z}{2}} \frac{x^{\frac{1}{8}}}{m^{\frac{1}{4}} n^{\frac{1}{8}}} \sum_{l \leq L_0} \frac{a_l}{l^{\frac{3}{4}}} e\left(\frac{2x^{\frac{1}{4}} l^{\frac{1}{2}}}{D_K^{\frac{1}{2}} m^{\frac{1}{2}} n^{\frac{1}{4}}}\right) \right| \\ &\quad + x^{1/4+\varepsilon} M^{\frac{1}{2}} N^{\frac{3}{4}} L_0^{-1/2} \\ &\ll |\Sigma_8^*(M, N, L)| \log x + x^{1/4+\varepsilon} M^{\frac{1}{2}} N^{\frac{3}{4}} L_0^{-1/2}, \end{aligned} \quad (4.29)$$

where

$$\Sigma_8^*(M, N, L) = \frac{x^{\frac{1}{8}}}{L^{\frac{3}{4}} M^{\frac{1}{4}} N^{\frac{1}{8}}} \sum_{M < m \leq 2M} \frac{\mu_K(m) M^{\frac{1}{4}}}{m^{\frac{1}{4}-z}} \sum_{\substack{N < n \leq 2N \\ n \leq \frac{x^{1/3}}{m^{2/3}}}} \frac{a_n N^{\frac{1}{8}}}{n^{\frac{1}{8}-\frac{z}{2}}} \sum_{L < l \leq 2L} \frac{a_l L^{\frac{3}{4}}}{l^{\frac{3}{4}}} e\left(\frac{2x^{\frac{1}{4}} l^{\frac{1}{2}}}{D_K^{\frac{1}{2}} m^{\frac{1}{2}} n^{\frac{1}{4}}}\right).$$

By Lemma 4.4 with  $(H_1, N_1, M_1) = (L, M, N)$  we get

$$x^{-\varepsilon} \Sigma_8^*(M, N, L) \ll x^{\frac{3}{16}} L^{\frac{1}{8}} M^{\frac{3}{8}} N^{\frac{5}{16}} + x^{\frac{1}{8}} M^{\frac{1}{2}} N^{\frac{7}{8}} + x^{\frac{1}{8}} L^{\frac{1}{4}} M^{\frac{3}{4}} N^{\frac{3}{8}} + MN \\ \ll x^{\frac{3}{16}} L_0^{\frac{1}{8}} M^{\frac{3}{8}} N^{\frac{5}{16}} + x^{\frac{1}{8}} L_0^{\frac{1}{4}} M^{\frac{3}{4}} N^{\frac{3}{8}} + x^{\frac{1}{8}} M^{\frac{1}{2}} N^{\frac{7}{8}} + MN,$$

which combining (4.29) gives

$$x^{-\varepsilon} \Sigma_8(M, N) \ll x^{\frac{3}{16}} L_0^{\frac{1}{8}} M^{\frac{3}{8}} N^{\frac{5}{16}} + x^{\frac{1}{8}} L_0^{\frac{1}{4}} M^{\frac{3}{4}} N^{\frac{3}{8}} + x^{\frac{1}{8}} M^{\frac{1}{2}} N^{\frac{7}{8}} + MN + x^{1/4} M^{\frac{1}{2}} N^{\frac{3}{4}} L_0^{-1/2}.$$

Choosing a best  $L_0 \in (1, \frac{x^{1/2}}{MN^{1/2}})$  with Lemma 4.5 we get

$$x^{-\varepsilon} \Sigma_8(M, N) \ll x^{\frac{3}{16}} M^{\frac{3}{8}} N^{\frac{5}{16}} + x^{\frac{1}{8}} M^{\frac{3}{4}} N^{\frac{3}{8}} + x^{\frac{1}{8}} M^{\frac{1}{2}} N^{\frac{7}{8}} + x^{\frac{1}{5}} M^{\frac{2}{5}} N^{\frac{2}{5}} + x^{\frac{1}{6}} M^{\frac{2}{3}} N^{\frac{1}{2}} + MN.$$

Inserting the above estimate into (4.28) and noticing that  $N \ll x^{1/3} M^{-2/3}$  and  $M \ll y$  we get

$$x^{-\varepsilon} \sum_8 \ll x^{\frac{7}{24}} y^{\frac{1}{6}} + x^{\frac{5}{16}} y^{\frac{3}{8}} + x^{\frac{2}{5}} + x^{\frac{1}{3}} y^{\frac{1}{3}} \ll x^{\frac{2}{5}}. \quad (4.30)$$

From (4.23), (4.27) and (4.30) we get by taking  $y = x^{1/5}$  that

$$\Delta^*(x, z) \ll x^{\frac{5}{12} + \varepsilon}. \quad (4.31)$$

#### 4.3. Proof of Lemma 4.1 for $d = 3$

In this subsection we prove Lemma 4.1 for  $d = 3$ . We estimate  $\Sigma_7(M, N)$  first, which is defined the same as in (4.24). Taking  $L = L_0$  in Lemma 2.6 we get by a splitting argument that

$$\Sigma_7(M, N) \ll \left| \sum_{M < m \leq 2M} \mu_K(m) \sum_{\substack{N < n \leq 2N \\ n \leq \frac{x^{1/3}}{m^{2/3}}} a_n n^{-z} \frac{x^{1/3}}{(mn)^{2/3}} \sum_{l \leq L_0} \frac{a_l}{l^{2/3}} e\left(\frac{3x^{\frac{1}{3}} l^{\frac{1}{3}}}{D_K^{\frac{1}{3}} (mn)^{\frac{2}{3}}}\right) \right| \\ + x^{2/3 + \varepsilon} L_0^{-1/3} M^{-1/3} N^{-1/3} \\ \ll |\Sigma_7^{**}(M, N, L)| \log x + x^{2/3 + \varepsilon} L_0^{-1/3} M^{-1/3} N^{-1/3}, \quad (4.32)$$

where

$$\Sigma_7^{**}(M, N, L) = \frac{x^{\frac{1}{3}}}{L^{\frac{2}{3}} M^{\frac{2}{3}} N^{\frac{2}{3}}} \sum_{M < m \leq 2M} \frac{\mu_K(m) M^{\frac{2}{3}}}{m^{\frac{2}{3}}} \sum_{\substack{N < n \leq 2N \\ n \leq \frac{x^{1/3}}{m^{2/3}}} \frac{a_n N^{\frac{2}{3}}}{n^{\frac{2}{3} + z}} \sum_{L < l \leq 2L} \frac{a_l L^{\frac{2}{3}}}{l^{\frac{2}{3}}} e\left(\frac{3x^{\frac{1}{3}} l^{\frac{1}{3}}}{D_K^{\frac{1}{3}} (mn)^{2/3}}\right).$$

Taking  $(H_1, N_1, M_1) = (L, M, N)$  in Lemma 4.4 we get

$$x^{-\varepsilon} \Sigma_7^{**}(M, N, L) \ll x^{\frac{5}{12}} L^{\frac{1}{6}} M^{-\frac{1}{12}} N^{-\frac{1}{3}} + x^{\frac{1}{3}} L^{\frac{1}{12}} M^{\frac{1}{12}} N^{\frac{1}{3}} + x^{\frac{1}{3}} L^{\frac{1}{3}} M^{\frac{1}{3}} N^{-\frac{1}{6}} + x^{\frac{1}{6}} L^{\frac{1}{6}} M^{\frac{2}{3}} N^{\frac{2}{3}},$$

which combining (4.32) gives (noting that  $L \ll L_0$ )

$$x^{-\varepsilon} \Sigma_7(M, N) \ll x^{\frac{5}{12}} L_0^{\frac{1}{6}} M^{-\frac{1}{12}} N^{-\frac{1}{3}} + x^{\frac{1}{3}} L_0^{\frac{1}{12}} M^{\frac{1}{12}} N^{\frac{1}{3}} + x^{\frac{1}{3}} L_0^{\frac{1}{3}} M^{\frac{1}{3}} N^{-\frac{1}{6}} \\ + x^{\frac{1}{6}} L_0^{\frac{1}{6}} M^{\frac{2}{3}} N^{\frac{2}{3}} + x^{\frac{2}{3}} L_0^{-\frac{1}{3}} M^{-\frac{1}{3}} N^{-\frac{1}{3}}.$$

Choosing a best  $L_0 \in (1, \frac{x}{M^2 N^2})$  with Lemma 4.5 and noting that  $N \ll x^{1/3} M^{-2/3}$  we get

$$\begin{aligned} x^{-\varepsilon} \Sigma_7(M, N) &\ll x^{\frac{5}{12}} M^{-\frac{1}{12}} N^{-\frac{1}{3}} + x^{\frac{1}{3}} M^{\frac{1}{12}} N^{\frac{1}{3}} + x^{\frac{1}{6}} M^{\frac{2}{3}} N^{\frac{2}{3}} \\ &\quad + x^{\frac{1}{3}} M^{\frac{1}{3}} N^{\frac{1}{3}} + x^{\frac{1}{2}} M^{-\frac{1}{6}} N^{-\frac{1}{3}} + x^{\frac{2}{5}} N^{\frac{1}{5}} + x^{\frac{1}{2}} N^{-\frac{1}{4}} \\ &\ll x^{\frac{7}{18}} M^{\frac{2}{9}} + x^{\frac{4}{9}} M^{\frac{1}{9}} + x^{\frac{7}{15}} M^{-\frac{2}{15}} + x^{\frac{1}{2}} M^{-\frac{1}{6}} N^{-\frac{1}{3}} + x^{\frac{1}{2}} N^{-\frac{1}{4}} \\ &\ll x^{\frac{7}{15}} + x^{\frac{1}{2}} M^{-\frac{1}{6}} N^{-\frac{1}{3}} + x^{\frac{1}{2}} N^{-\frac{1}{4}}. \end{aligned} \quad (4.33)$$

Using the estimate  $\Delta(x) \ll x^{43/96+\varepsilon}$  we get trivially

$$x^{-\varepsilon} \sum_7(M, N) \ll x^{\frac{43}{96}} (MN)^{\frac{10}{96}}. \quad (4.34)$$

From (4.33) and (4.34) we get

$$\begin{aligned} x^{-\varepsilon} \sum_7(M, N) &\ll x^{\frac{7}{15}} + \min(x^{\frac{43}{96}} (MN)^{\frac{10}{96}}, x^{\frac{1}{2}} M^{-\frac{1}{6}} N^{-\frac{1}{3}}) \\ &\quad + \min(x^{\frac{43}{96}} (MN)^{\frac{10}{96}}, x^{\frac{1}{2}} N^{-\frac{1}{4}}) \\ &\ll x^{\frac{7}{15}} + (x^{\frac{43}{96}} (MN)^{\frac{10}{96}})^{16/21} (x^{\frac{1}{2}} M^{-\frac{1}{6}} N^{-\frac{1}{3}})^{5/21} \\ &\quad + (x^{\frac{43}{96}} (MN)^{\frac{10}{96}})^{12/17} (x^{\frac{1}{2}} N^{-\frac{1}{4}})^{5/17} \\ &\ll x^{\frac{7}{15}} + x^{\frac{58}{126}} M^{\frac{5}{126}} + x^{\frac{63}{136}} M^{\frac{5}{68}} \\ &\ll x^{\frac{7}{15}} + x^{\frac{63}{136}} M^{\frac{5}{68}}, \end{aligned} \quad (4.35)$$

which implies that (noting that  $M \ll y$ )

$$x^{-\varepsilon} \sum_7 \ll x^{\frac{7}{15}} + x^{\frac{63}{136}} y^{\frac{5}{68}}. \quad (4.36)$$

Now we estimate  $\Sigma_8(M, N)$ , which is defined the same as in (4.28). Taking  $N = L_0$  in Lemma 2.6 and then using a splitting argument, we get

$$\begin{aligned} \Sigma_8(M, N) &\ll \left| \sum_{M < m \leq 2M} \mu_K(m) m^z \sum_{\substack{N < n \leq 2N \\ n \leq \frac{x^{1/3}}{m^{2/3}}}} a_n n^{\frac{z}{2}} \frac{x^{\frac{1}{6}}}{m^{\frac{1}{3}} n^{\frac{1}{6}}} \sum_{l \leq L_0} \frac{a_l}{l^{\frac{2}{3}}} e\left(\frac{3x^{\frac{1}{6}} l^{\frac{1}{3}}}{D_K^{\frac{1}{3}} m^{\frac{1}{3}} n^{\frac{1}{6}}}\right) \right| \\ &\quad + x^{1/3+\varepsilon} M^{\frac{1}{3}} N^{\frac{2}{3}} L_0^{-1/3} \\ &\ll |\Sigma_8^{**}(M, N, L)| \log x + x^{1/3+\varepsilon} M^{\frac{1}{3}} N^{\frac{2}{3}} L_0^{-1/3}, \end{aligned} \quad (4.37)$$

where

$$\Sigma_8^{**}(M, N, L) = \frac{x^{\frac{1}{6}}}{L^{\frac{2}{3}} M^{\frac{1}{3}} N^{\frac{1}{6}}} \sum_{M < m \leq 2M} \frac{\mu_K(m) M^{\frac{1}{3}}}{m^{\frac{1}{3}-z}} \sum_{\substack{N < n \leq 2N \\ n \leq \frac{x^{1/3}}{m^{2/3}}}} \frac{a_n N^{\frac{1}{6}}}{n^{\frac{1}{6}-\frac{z}{2}}} \sum_{L < l \leq 2L} \frac{a_l L^{\frac{2}{3}}}{l^{\frac{2}{3}}} e\left(\frac{3x^{\frac{1}{6}} l^{\frac{1}{3}}}{D_K^{\frac{1}{3}} m^{\frac{1}{3}} n^{\frac{1}{6}}}\right).$$

By Lemma 4.4 with  $(H_1, N_1, M_1) = (L, M, N)$  we get

$$x^{-\varepsilon} \Sigma_8^{**}(M, N, L) \ll x^{\frac{5}{24}} L^{\frac{1}{6}} M^{\frac{1}{3}} N^{\frac{7}{24}} + x^{\frac{1}{6}} L^{\frac{1}{12}} M^{\frac{5}{12}} N^{\frac{5}{6}} + x^{\frac{1}{6}} L^{\frac{1}{3}} M^{\frac{2}{3}} N^{\frac{1}{3}} + x^{\frac{1}{12}} L^{\frac{1}{6}} M^{\frac{5}{6}} N^{\frac{11}{12}},$$

which combining (4.37) gives

$$\begin{aligned} x^{-\varepsilon} \Sigma_8(M, N) &\ll x^{\frac{5}{24}} L_0^{\frac{1}{6}} M^{\frac{1}{3}} N^{\frac{7}{24}} + x^{\frac{1}{6}} L_0^{\frac{1}{12}} M^{\frac{5}{12}} N^{\frac{5}{6}} \\ &\quad + x^{\frac{1}{6}} L_0^{\frac{1}{3}} M^{\frac{2}{3}} N^{\frac{1}{3}} + x^{\frac{1}{12}} L_0^{\frac{1}{6}} M^{\frac{5}{6}} N^{\frac{11}{12}} + x^{1/3} M^{\frac{1}{3}} N^{\frac{2}{3}} L_0^{-1/3}. \end{aligned}$$

Choosing a best  $L_0 \in (1, \frac{x^{1/2}}{MN^{1/2}})$  with Lemma 4.5 and noting  $N \ll x^{1/3} M^{-2/3}$  we get

$$\begin{aligned} x^{-\varepsilon} \Sigma_8(M, N) &\ll x^{\frac{5}{24}} M^{\frac{1}{3}} N^{\frac{7}{24}} + x^{\frac{1}{6}} M^{\frac{5}{12}} N^{\frac{5}{6}} + x^{\frac{1}{6}} M^{\frac{2}{3}} N^{\frac{1}{3}} + x^{\frac{1}{12}} M^{\frac{5}{6}} N^{\frac{11}{12}} \\ &\quad + x^{\frac{1}{4}} M^{\frac{1}{3}} N^{\frac{5}{12}} + x^{\frac{1}{5}} M^{\frac{2}{5}} N^{\frac{4}{5}} + x^{\frac{1}{4}} M^{\frac{1}{2}} N^{\frac{1}{2}} + x^{\frac{1}{6}} M^{\frac{2}{3}} N^{\frac{5}{6}} \\ &\ll x^{\frac{1}{12}} M^{\frac{5}{6}} N^{\frac{11}{12}} + x^{\frac{1}{5}} M^{\frac{2}{5}} N^{\frac{4}{5}} + x^{\frac{1}{4}} M^{\frac{1}{2}} N^{\frac{1}{2}} + x^{\frac{1}{6}} M^{\frac{2}{3}} N^{\frac{5}{6}} \\ &\ll x^{\frac{7}{18}} M^{\frac{2}{9}} + x^{\frac{7}{15}} M^{-\frac{2}{15}} + x^{\frac{5}{12}} M^{\frac{1}{6}} + x^{\frac{4}{9}} M^{\frac{1}{9}} \\ &\ll x^{\frac{7}{15}}, \end{aligned}$$

which implies immediately that

$$x^{-\varepsilon} \sum_8 \ll x^{7/15}. \quad (4.38)$$

Now combining (4.23), (4.36) and (4.38) we get

$$x^{-\varepsilon} \Delta^*(x, z) \ll x^{\frac{7}{15}} + x^{\frac{63}{136}} y^{\frac{5}{68}} + x^{\frac{7}{18}} y^{\frac{2}{9}} + x^{\frac{1}{2}} y^{-\frac{1}{2}} \ll x^{\frac{73}{156}} \quad (4.39)$$

by choosing  $y = x^{5/78}$ .

## 5. Proofs of the theorems

Now we prove Theorem 1.1. Suppose  $z$  is a complex number such that  $|z| \leq \log^{-2/3-\varepsilon} x$ . From Proposition A we get

$$\sum_{N(\mathfrak{A}) \leq x} \gamma^z(\mathfrak{A}) = \rho_K H_1(z) x^{1+z} + \rho_K H_2(z) x^{\frac{1+z}{2}} + E(x, z), \quad (5.1)$$

with

$$E(x, z) \ll x^{\frac{1}{2}} \exp(-c \log^{\frac{1}{3}} x (\log \log x)^{-\frac{1}{3}}). \quad (5.2)$$

Note that each term in the formula (5.1) is analytic in the region  $|z| \leq \log^{-2/3-\varepsilon} x$ . Taking the  $k$ -th derivative from both sides of (5.1) and then putting  $z = 0$ , we get

$$\sum_{N(\mathfrak{A}) \leq x} \log^k \gamma(\mathfrak{A}) = \rho_K (H_1(z) x^{1+z})^{(k)} \Big|_{z=0} + \rho_K (H_2(z) x^{\frac{1+z}{2}})^{(k)} \Big|_{z=0} + \frac{\partial^k}{\partial z^k} E(x, z) \Big|_{z=0}. \quad (5.3)$$

The function  $H_1(z)x^{1+z}$  can be written as

$$x \sum_{m=0}^{\infty} \frac{H_1^{(m)}(0)}{m!} z^m \sum_{n=0}^{\infty} \frac{\log^n x}{n!} z^n = x \sum_{m,n} \left( \frac{H_1^{(m)}(0)}{m!} \frac{\log^n x}{n!} \right) z^{m+n}.$$

Then we have

$$(H_1(z)x^{1+z})^{(k)} \Big|_{z=0} = x \sum_{m+n=k} \frac{k! H_1^{(m)}(0) \log^n x}{m! n!} = x \sum_{j=0}^k \binom{k}{j} H_1^{(j)}(0) \log^{k-j} x. \quad (5.4)$$

Write (note that  $\frac{1}{\zeta_K(1-z)} \rightarrow 0$ , when  $z \rightarrow 0$ )

$$H_2(z)x^{\frac{1+z}{2}} = x^{1/2} \sum_{m=1}^{\infty} \frac{H_2^{(m)}(0)}{m!} z^m \sum_{n=0}^{\infty} \frac{\log^n x}{2^n n!} z^n = x^{1/2} \sum_{m \geq 1, n \geq 0} \left( \frac{H_2^{(m)}(0)}{m!} \frac{\log^n x}{2^n n!} \right) z^{m+n}.$$

Hence

$$\begin{aligned} (H_2(z)x^{\frac{1+z}{2}})^{(k)} \Big|_{z=0} &= x^{1/2} \sum_{\substack{m+n=k \\ m \geq 1, n \geq 0}} \frac{k! H_2^{(m)}(0) \log^n x}{m! n! 2^n} \\ &= x^{1/2} \sum_{j=1}^k \binom{k}{j} \frac{H_2^{(j)}(0)}{2^{k-j}} \log^{k-j} x. \end{aligned} \quad (5.5)$$

By (5.2) and using Cauchy's derivative formula inside the circle  $|z| \leq \frac{1}{2} \log^{-2/3-\varepsilon} x$  we get

$$\begin{aligned} \frac{\partial^k}{\partial z^k} E(x, z) \Big|_{z=0} &\ll x^{\frac{1}{2}} \exp(-c \log^{\frac{1}{3}} x (\log \log x)^{-\frac{1}{3}}) R_0^{-k} \\ &\ll x^{\frac{1}{2}} \exp(-c \log^{\frac{1}{3}} x (\log \log x)^{-\frac{1}{3}}) (\log x)^{\frac{2}{3}k+k\varepsilon}. \end{aligned} \quad (5.6)$$

Inserting (5.4)–(5.6) into (5.3) we have

$$\sum_{2 \leq N(\mathfrak{A}) \leq x} \log^k \gamma(\mathfrak{A}) = M(x) + E(x), \quad (5.7)$$

where

$$\begin{aligned} M(x) &= \rho_K x \sum_{j=0}^k \binom{k}{j} H_1^{(j)}(0) \log^{k-j} x + \rho_K x^{1/2} \sum_{j=1}^k \binom{k}{j} \frac{H_2^{(j)}(0)}{2^{k-j}} \log^{k-j} x, \\ E(x) &= O\left(x^{\frac{1}{2}} \exp(-c' \log^{\frac{1}{3}} x (\log \log x)^{-\frac{1}{3}})\right). \end{aligned}$$

By partial integration we get

$$\begin{aligned} \sum_{2 \leq N(\mathfrak{A}) \leq x} \frac{\log^k \gamma(\mathfrak{A})}{\log^k N(\mathfrak{A})} &= \int_{2^-}^x \frac{1}{\log^k t} d\left( \sum_{2 \leq N(\mathfrak{A}) \leq t} \log^k \gamma(\mathfrak{A}) \right) \\ &= \int_{2^-}^x \frac{dM(t)}{\log^k t} + \int_{2^-}^x \frac{dE(t)}{\log^k t}. \end{aligned} \quad (5.8)$$

By the definition of  $M(x)$  we have

$$M'(t) = \rho_K \log^k x + \rho_K \sum_{j=1}^k C_{k,j} \log^{k-j} x + \rho_K \sum_{j=1}^k C'_{k,j} x^{-1/2} \log^{k-j} x, \quad (5.9)$$

where

$$C_{k,j} = \binom{k}{j} H_1^{(j)}(0) + (k-j+1) \binom{k}{j-1} H_1^{(j-1)}(0) = \binom{k}{j} H^{(j)}(0), \quad (5.10)$$

if we note that  $H(z) = H_1(z)(1+z)$ , and

$$C'_{k,j} = \binom{k}{j} \frac{H_2^{(j)}(0)}{2^{k-j+1}} + (k-j+1) \binom{k}{j-1} \frac{H_2^{(j-1)}(0)}{2^{k-j+1}}. \quad (5.11)$$

Then we have

$$\int_{2^-}^x \frac{dM(t)}{\log^k t} = \rho_K x + \sum_{j=1}^k C_{k,j} \int_2^x \frac{1}{\log^j t} dt + \sum_{j=1}^k C'_{k,j} \int_2^x \frac{t^{-1/2}}{\log^j t} dt. \quad (5.12)$$

By partial summation and (5.6) we get

$$\int_{2^-}^x \frac{dE(t)}{\log^k t} \ll x^{\frac{1}{2}} \exp(-c' \log^{\frac{1}{3}} x (\log \log x)^{-\frac{1}{3}}). \quad (5.13)$$

Theorem 1.1 follows from (5.8)–(5.13).

The proof of Theorem 1.2 is almost the same. So we omit the details.

## 6. A note

In this section we show that when  $K$  is a quadratic number field, the exponent  $\frac{5}{12}$  in the error term in (1.9) can be slightly improved to  $227/562$ . In this case  $\zeta_K(s) = \zeta(s)L(s, \chi)$ , where  $L(s, \chi)$  is a primitive character. So we have for  $\text{Re } s = \sigma > 1 + |z|$  that

$$\sum_{n=1}^{\infty} \frac{f(n, z)}{n^s} = \zeta(s-z)L(s-z, \chi)\zeta(2s-z)L(2s-z, \chi). \quad (6.1)$$

Hence we get

$$\sum_{n \leq x} f(n, z) = \sum_{n_1 n_2 m_1^2 m_2^2 \leq x} \chi(n_2) \chi(n_4) (n_1 n_2 n_3 n_4)^z, \quad (6.2)$$

which is an analogue of the divisor function  $d(1, 1, 2, 2; n)$  defined by

$$d(1, 1, 2, 2; n) = \sum_{n = n_1 n_2 m_1^2 m_2^2} 1.$$

In [15] Wu showed that

$$\sum_{n \leq x} d(1, 1, 2, 2; n) = C_1 x \log x + C_2 x + C_3 x^{1/2} \log x + C_4 x^{1/2} + O(x^{47/130+\varepsilon}).$$

Wu's approach can be used to  $\Delta(x, z)$  with slight modifications yields

$$\Delta(x, z) \ll x^{47/130+\varepsilon}. \quad (6.3)$$

Now we prove that

$$\Delta^*(x, z) \ll x^{227/562+\varepsilon}. \quad (6.4)$$

Take  $y = x^{1/5}$  in Lemma 4.3 and  $1 < y_0 < x^{1/5}$  is a parameter to be determined. By (6.3) we have

$$\sum_{m \leq y_0} \mu_K(m) m^{2z} \Delta\left(\frac{x}{m^2}, z\right) \ll x^{47/130+\varepsilon} y_0^{36/130}. \quad (6.5)$$

For  $\sum_{y_0 < m \leq x^{1/5}} \mu_K(m) m^{2z} \Delta\left(\frac{x}{m^2}, z\right)$ , we use the approach in Section 4. In this case the estimate (4.26) becomes (note that  $y = x^{1/5}$ )

$$\ll x^{5/12} M^{-1/12} + x^{2/5} \ll x^{5/12} y_0^{-1/12} + x^{2/5}$$

and the estimate (4.27) becomes

$$x^{-\varepsilon} \sum_7 \ll x^{5/12} y_0^{-1/12} + x^{2/5}. \quad (6.6)$$

From (4.23), (4.30), (6.5) and (6.6) we get (6.4) by choosing  $y_0 = x^{43/281}$ .

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## Supplementary material

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