Journal of Number Theory • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

A more accurate approximation for the gamma function

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ARTICLE INFO

Article history:
Received 12 August 2015
Accepted 9 November 2015
Available online xxxx
Communicated by David Goss

MSC:
primary 33B15
secondary 41A60, 26D15

Keywords:
Gamma function
Inequality
Approximation
Asymptotic expansion

ABSTRACT

In this paper, we establish a double inequality for the gamma function, from which we deduce the following approximation formula:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)^{x^2+\frac{53}{210}}, \quad x \to \infty,$$

which is more accurate than the Burnside, Gosper, Ramanujan, Windschitl, and Nemes formulas. We develop the previous approximation formula to produce an asymptotic expansion.

1. Introduction

Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad n \in \mathbb{N} := \{1, 2, \ldots\} \tag{1.1}$$

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http://dx.doi.org/10.1016/j.jnt.2015.11.007

has many applications in statistical physics, probability theory and number theory. Actually, it was first discovered in 1733 by the French mathematician Abraham de Moivre (1667–1754) in the form

$$n! \sim \text{constant} \cdot \sqrt{n}(n/e)^n$$

when he was studying the Gaussian distribution and the central limit theorem. Afterwards, the Scottish mathematician James Stirling (1692–1770) found the missing constant $\sqrt{2\pi}$ when he was trying to give the normal approximation of the binomial distribution.

Stirling's series for the gamma function is given (see [1, p. 257, Eq. (6.1.40)]) by

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)x^{2m-1}}\right) \tag{1.2}$$

as $x \to \infty$, where B_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) are the Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \qquad |z| < 2\pi.$$

The following asymptotic formula is due to Laplace

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51\,840x^3} - \frac{571}{2\,488\,320x^4} + \cdots\right) \tag{1.3}$$

as $x \to \infty$ (see [1, p. 257, Eq. (6.1.37)]). The expression (1.3) is sometimes incorrectly called Stirling's series (see [19, pp. 2–3]). Stirling's formula is in fact the first approximation to the asymptotic formula (1.3).

Inspired by (1.1), Burnside [10] found a slightly more accurate approximation than Stirling's formula as follows:

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}. \tag{1.4}$$

A much better approximation is the following the Gosper formula [20]:

$$n! \sim \sqrt{2\pi\left(n+\frac{1}{6}\right)}\left(\frac{n}{e}\right)^n. \tag{1.5}$$

The formulas (1.4) and (1.5) have motivated a large number of research papers; see [6, 7,9,17,26–29,31–39,41–43,51,52].

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Ramanujan (see [53, p. 339] and [5, pp. 117–118]) presented the following approximation formula for the gamma function:

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6}, \qquad x \to \infty. \tag{1.6}$$

The Ramanujan formula has been the subject of intense investigations and is reviewed in [8, p. 48, Question 754], and has motivated a large number of research papers (see, for example, [3,11,13,14,16,21–23,31,32,40,44–47,49]).

Windschitl (see [5, p. 128], [54, Eq. (42)] and [55]) presented that

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2}, \qquad x \to \infty. \tag{1.7}$$

Nemes' formula (see [50, Corollary 4.1])

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{12x^2-\frac{1}{10}}\right)^{x}, \qquad x \to \infty \tag{1.8}$$

has the same number of exact digits as (1.7) but is much simpler. The formulas (1.7) and (1.8) are stronger than the formula (1.6). Some inequalities and asymptotic expansions related to the Windschitl and Nemes formulas were established in [4,12,13,18,30,31,48].

In this paper, we prove that for $x \geq 2$,

$$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{12x^3+\frac{24}{7}x-\frac{1}{2}}\right)^{x^2+\frac{53}{210}} \left(1-\frac{2117}{5\,080\,320x^7}\right) < \Gamma(x+1)$$
$$< \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{12x^3+\frac{24}{7}x-\frac{1}{2}}\right)^{x^2+\frac{53}{210}} \left(1-\frac{2117}{5\,080\,320x^7}+\frac{1\,892\,069}{2\,347\,107\,840x^9}\right),$$

which deduces the following approximation for the gamma function:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{12x^3+\frac{24}{7}x-\frac{1}{2}}\right)^{x^2+\frac{53}{210}}, \qquad x \to \infty. \tag{1.9}$$

Moreover, we show that the formula (1.9) is more accurate than the Burnside, Gosper, Ramanujan, Windschitl, and Nemes formulas. We develop (1.9) to produce an asymptotic expansion.

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

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2. Lemmas

The following lemmas are required in our present investigation.

Lemma 2.1. *(See [2, Theorem 8].) Let $n \geq 0$ be an integer. The functions*

$$F_n(x) = \ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \frac{1}{2}\ln(2\pi) - \sum_{j=1}^{2n} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}$$

and

$$G_n(x) = -\ln\Gamma(x) + \left(x - \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{2n+1} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}$$

are strictly completely monotonic on $(0, \infty)$. Here B_n $(n \in \mathbb{N}_0)$ are the Bernoulli numbers.

Remark 2.1. Lemma 2.1 can be stated that for every $m \in \mathbb{N}_0$, the function

$$R_m(x) = (-1)^m \left[\ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \ln\sqrt{2\pi} - \sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}\right]$$

is completely monotonic on $(0, \infty)$.

In 2006, Koumandos [24] presented a simpler proof of the complete monotonicity property of $R_m(x)$. In 2009, Koumandos and Pedersen [25, Theorem 2.1] strengthened this result.

From $F_n(x) > 0$ and $G_n(x) > 0$ for $x > 0$, we obtain that for $x > 0$ and $n \in \mathbb{N}_0$,

$$\sum_{j=1}^{2n} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} < \ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \frac{1}{2}\ln(2\pi) < \sum_{j=1}^{2n+1} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}.$$

In particular, we have, for $x > 0$,

$$\begin{aligned} &\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9} - \frac{691}{360\,360x^{11}} \\ &\quad < \ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \frac{1}{2}\ln(2\pi) \\ &\quad < \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9} - \frac{691}{360\,360x^{11}} + \frac{1}{156x^{13}}. \end{aligned} \tag{2.1}$$

Lemma 2.2. *Let*

$$L(x) = \left(x^2 + \frac{53}{210}\right) \ln\left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right) + \ln\left(1 - \frac{2117}{5\,080\,320x^7}\right) \tag{2.2}$$

and

$$U(x) = \left(x^2 + \frac{53}{210}\right) \ln\left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right) + \ln\left(1 - \frac{2117}{5\,080\,320x^7} + \frac{1\,892\,069}{2\,347\,107\,840x^9}\right). \tag{2.3}$$

Then, for $x \geq 1$,

$$L(x) < \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{7601}{213\,373\,440x^9} \tag{2.4}$$

and

$$U(x) > \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9} - \frac{265}{49\,787\,136x^{11}}. \tag{2.5}$$

Proof. Define the function $M(x)$ by

$$M(x) = L(x) - \left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{7601}{213\,373\,440x^9}\right).$$

Direct computation yields

$$\begin{aligned}\frac{1}{x}M'(x) = {} & 2\ln\left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right) \\ & - \frac{N(x)}{5x^2(5\,080\,320x^7 - 2117)(168x^3 + 48x - 7)(168x^3 + 48x + 7)} \\ & + \frac{1}{x}\left(\frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6} - \frac{1}{240x^8} + \frac{7601}{23\,708\,160x^{10}}\right) := H(x),\end{aligned}$$

where

$$\begin{aligned}N(x) = {} & 179\,233\,689\,600x^{12} + 62\,305\,044\,480x^{10} + 4\,308\,111\,360x^8 - 2\,091\,257\,280x^6 \\ & - 74\,687\,760x^5 - 1\,195\,004\,160x^4 - 25\,962\,888x^3 - 170\,714\,880x^2 \\ & - 1\,795\,216x + 3\,630\,655.\end{aligned}$$

Differentiating $H(x)$ yields

$$H'(x) = -\frac{P_{24}(x-1)}{23\,708\,160x^{12}(5\,080\,320x^7 - 2117)^2(168x^3 + 48x + 7)^2(168x^3 + 48x - 7)^2}$$

with

$$P_{24}(x) = 371\,006\,809\,418\,272\,478\,330\,880\,000x^{24} + \cdots$$
$$+ 949\,087\,692\,150\,834\,513\,725\,439\,907,$$

where $P_{24}(x)$ is a polynomial of degree 24 with non-negative integer coefficients. Hence, we have, for $x \geq 1$,

$$H'(x) < 0 \Longrightarrow H(x) > \lim_{t\to\infty} H(t) = 0 \Longrightarrow M'(x) > 0 \Longrightarrow M(x) < \lim_{t\to\infty} M(t) = 0.$$

Define the function $V(x)$ by

$$V(x) = U(x) - \left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9} - \frac{265}{49\,787\,136x^{11}}\right).$$

Direct computation yields

$$\begin{aligned}\frac{1}{x}V'(x) = 2\ln\left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)\\ &- \frac{T(x)}{5x^2(168x^3+48x-7)(168x^3+48x+7)(2\,347\,107\,840x^9 - 978\,054x^2 + 1\,892\,069)}\\ &+ \frac{1}{x}\left(\frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6} - \frac{1}{240x^8} + \frac{1}{132x^{10}} - \frac{2915}{49\,787\,136x^{12}}\right) := Q(x),\end{aligned}$$

where

$$\begin{aligned}T(x) = {}& 82\,805\,964\,595\,200x^{14} + 28\,784\,930\,549\,760x^{12} + 1\,990\,347\,448\,320x^{10}\\ &- 966\,160\,863\,360x^8 - 34\,505\,745\,120x^7 + 1\,850\,987\,073\,600x^6 + 54\,757\,340\,064x^5\\ &+ 1\,294\,317,722\,880x^4 + 22\,374\,944\,424x^3 + 197\,847\,076\,530x^2 + 1\,604\,474\,512x\\ &- 4\,172\,012\,145.\end{aligned}$$

Differentiating $Q(x)$ yields

$$Q'(x) = \frac{P_{28}(x-1)}{248\,935\,680x^{14}(2\,347\,107\,840x^9 - 978\,054x^2 + 1\,892\,069)^2(168x^3+48x+7)^2(168x^3+48x-7)^2}$$

with

$$P_{28}(x) = 11\,190\,098\,038\,081\,591\,751\,580\,057\,600\,000x^{28} + \cdots$$
$$+ 78\,435\,120\,990\,199\,770\,500\,780\,040\,691\,795,$$

where $P_{28}(x)$ is a polynomial of degree 28 with non-negative integer coefficients. Hence, we have, for $x \geq 1$,

$$Q'(x) > 0 \Longrightarrow Q(x) < \lim_{t\to\infty} Q(t) = 0 \Longrightarrow V'(x) < 0 \Longrightarrow V(x) > \lim_{t\to\infty} V(t) = 0.$$

The proof of Lemma 2.2 is complete. □

Lemma 2.3. *(See [15, Lemma 5].) Let g be a function with asymptotical expansion (as $x \to \infty$):*

$$g(x) \sim \sum_{n=0}^{\infty} c_n x^{-n}, \qquad (c_0 \neq 0).$$

Then

$$[g(x)]^{-1} \sim \sum_{n=0}^{\infty} P_n x^{-n},$$

where

$$P_0 = \frac{1}{c_0}, \quad P_n = -\frac{1}{c_0}\sum_{k=1}^{n} c_k P_{n-k}, \qquad n \geq 1.$$

3. Main results

Theorem 3.1. *For $x \geq 2$,*

$$\begin{aligned}
&\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)^{x^2+\frac{53}{210}} \left(1 - \frac{2117}{5\,080\,320x^7}\right) < \Gamma(x+1) \\
&< \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)^{x^2+\frac{53}{210}} \left(1 - \frac{2117}{5\,080\,320x^7} + \frac{1\,892\,069}{2\,347\,107\,840x^9}\right).
\end{aligned} \tag{3.1}$$

Proof. The lower bound is obtained by considering the function $f(x)$ defined by

$$f(x) = \ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \ln\sqrt{2\pi} - L(x),$$

where $L(x)$ is defined in (2.2). Using the first inequality in (2.1) and the inequality (2.4), we have, for $x \geq 2$,

$$\begin{aligned}
f(x) &> \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9} - \frac{691}{360\,360x^{11}} \\
&\quad - \left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{7601}{213\,373\,440x^9}\right) \\
&= \frac{39\,879\,236 + 98\,387\,588(x-2) + 24\,596\,897(x-2)^2}{30\,512\,401\,920x^{11}} > 0.
\end{aligned}$$

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Table 1
Comparison among approximation formulas (3.2)–(3.4).

n	$\frac{c_n - n!}{n!}$	$\frac{n! - w_n}{n!}$	$\frac{n! - g_n}{n!}$
1	1.398×10^{-4}	3.417×10^{-4}	3.724×10^{-4}
10	4.088×10^{-11}	6.115×10^{-9}	6.470×10^{-9}
100	4.166×10^{-18}	6.172×10^{-14}	6.527×10^{-14}
1000	4.167×10^{-25}	6.172×10^{-19}	6.528×10^{-19}

The upper bound is obtained by considering the function $F(x)$ defined by

$$F(x) = \ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \ln\sqrt{2\pi} - U(x),$$

where $U(x)$ is defined in (2.3). Using the second inequality in (2.1) and the inequality (2.5), we have, for $x \geq 2$,

$$\begin{aligned} F(x) &< \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9} - \frac{691}{360\,360x^{11}} + \frac{1}{156x^{13}} \\ &\quad - \left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9} - \frac{265}{49\,787\,136x^{11}}\right) \\ &= -\frac{44\,090\,036 + 272\,281\,076(x-2) + 68\,070\,269(x-2)^2}{35\,597\,802\,240x^{13}} < 0 \end{aligned}$$

The proof of Theorem 3.1 is complete. □

We now give a comparison table to demonstrate the superiority of our formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n^3 + \frac{24}{7}n - \frac{1}{2}}\right)^{n^2 + \frac{53}{210}} := c_n \tag{3.2}$$

over Windschitl's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(n \sinh\frac{1}{n}\right)^{n/2} := w_n \tag{3.3}$$

and Nemes' formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n^2 - \frac{1}{10}}\right)^{n} := g_n. \tag{3.4}$$

It is observed from Table 1 that, among approximation formulas (3.2)–(3.4), for $n \in \mathbb{N}$, the formula (3.2) would be the best one.

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In fact, we have, as $n \to \infty$,

$$n! = \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}\left(1+O\left(\frac{1}{n}\right)\right), \tag{3.5}$$

$$n! = \sqrt{2\pi\left(n+\frac{1}{6}\right)}\left(\frac{n}{e}\right)^{n}\left(1+O\left(\frac{1}{n^2}\right)\right), \tag{3.6}$$

$$n! = \sqrt{\pi}\left(\frac{n}{e}\right)^{n}\left(8n^3+4n^2+n+\frac{1}{30}\right)^{1/6}\left(1+O\left(\frac{1}{n^4}\right)\right), \tag{3.7}$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\left(n\sinh\frac{1}{n}\right)^{n/2}\left(1+O\left(\frac{1}{n^5}\right)\right), \tag{3.8}$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\left(1+\frac{1}{12n^2-\frac{1}{10}}\right)^{n}\left(1+O\left(\frac{1}{n^5}\right)\right), \tag{3.9}$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\left(1+\frac{1}{12n^3+\frac{24}{7}n-\frac{1}{2}}\right)^{n^2+\frac{53}{210}}\left(1+O\left(\frac{1}{n^7}\right)\right). \tag{3.10}$$

Clearly, among approximation formulas (3.5)–(3.10), the formula (3.10) would be the best one.

Theorem 3.2 develops (1.9) to produce an asymptotic expansion.

Theorem 3.2. *The gamma function has the following asymptotic formula:*

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}\left(1+\frac{1}{12x^3+\frac{24}{7}x-\frac{1}{2}+\frac{2117}{35\,280x^3}-\frac{94\,049}{970\,200x^5}+\cdots}\right)^{x^2+\frac{53}{210}},$$
$$x \to \infty. \tag{3.11}$$

Proof. Let

$$A(x) = \frac{1}{x^2+\frac{53}{210}}\ln\frac{\Gamma(x+1)}{\sqrt{2\pi x}\,(x/e)^x}.$$

Then, (3.11) can be written as

$$\frac{1}{e^{A(x)}-1} \sim 12x^3+\frac{24}{7}x-\frac{1}{2}+\frac{2117}{35\,280x^3}-\frac{94\,049}{970\,200x^5}+\cdots. \tag{3.12}$$

It is easy to see that

$$\frac{1}{x^2+\frac{53}{210}} = \frac{1}{x^2}\frac{1}{1+\frac{53}{210x^2}} = \sum_{k=0}^{\infty}(-1)^k\left(\frac{53}{210}\right)^k\frac{1}{x^{2k+2}}.$$

It follows from (1.2) that

$$\ln \frac{\Gamma(x+1)}{\sqrt{2\pi x}\,(x/e)^x} \sim \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}.$$

We then obtain

$$\begin{aligned} A(x) &\sim \sum_{k=0}^{\infty}(-1)^k \left(\frac{53}{210}\right)^k \frac{1}{x^{2k+2}} \sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \\ &\sim \sum_{j=1}^{\infty}\sum_{k=1}^{j} \frac{B_{2k}}{(2k-1)2k}(-1)^{j-k}\left(\frac{53}{210}\right)^{j-k} \frac{1}{x^{2j+1}}, \end{aligned}$$

namely,

$$\begin{aligned} A(x) \sim{}& \frac{1}{12x^3} - \frac{1}{42x^5} + \frac{1}{147x^7} - \frac{571}{246\,960x^9} + \frac{271\,031}{190\,159\,200x^{11}} \\ &- \frac{107\,472\,269}{47\,194\,056\,000x^{13}} + \cdots . \end{aligned} \tag{3.13}$$

By using $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$, we deduce from (3.13) that

$$\begin{aligned} e^{A(x)} - 1 \sim \frac{1}{x^3}\biggl(& \frac{1}{12} - \frac{1}{42x^2} + \frac{1}{288x^3} + \frac{1}{147x^4} - \frac{1}{504x^5} - \frac{39\,397}{17\,781\,120x^6} + \frac{1}{1176x^7} \\ &+ \frac{1\,531\,861}{1\,140\,955\,200x^8} - \frac{300\,973}{853\,493\,760x^9} - \frac{11\,693\,641}{5\,243\,784\,000x^{10}} + \cdots \biggr). \end{aligned} \tag{3.14}$$

By Lemma 2.3, from (3.14) we deduce (3.12). The proof of Theorem 3.2 is complete. □

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing, Natl. Bur. Stand., Appl. Math. Ser., vol. 55, Dover, New York, 1972.
[2] H. Alzer, On some inequalities for the gamma and psi functions, Math. Comp. 66 (1997) 373–389.
[3] H. Alzer, On Ramanujan's double inequality for the gamma function, Bull. Lond. Math. Soc. 35 (2003) 601–607.
[4] H. Alzer, Sharp upper and lower bounds for the gamma function, Proc. Roy. Soc. Edinburgh Sect. A 139 (2009) 709–718.
[5] G.E. Andrews, B.C. Berndt, Ramanujan's Lost Notebook, Part IV, Springer, New York, 2013.
[6] N. Batir, Very accurate approximations for the factorial function, J. Math. Inequal. 4 (2010) 335–344.
[7] N. Batir, Improving Stirling's formula, Math. Commun. 16 (2011) 105–114.
[8] B.C. Berndt, Y.-S. Choi, S.-Y. Kang, The Problems Submitted by Ramanujan to the Journal of the Indian Mathematical Society, Contemp. Math., vol. 236, 1999, pp. 15–56.
[9] T. Burić, N. Elezović, New asymptotic expansions of the gamma function and improvements of Stirling's type formulas, J. Comput. Anal. Appl. 13 (2011) 785–795.
[10] W. Burnside, A rapidly convergent series for log $N!$, Messenger Math. 46 (1917) 157–159.
[11] C.-P. Chen, Unified treatment of several asymptotic formulas for the gamma function, Numer. Algorithms 64 (2013) 311–319.
[12] C.-P. Chen, Asymptotic expansions of the gamma function related to Windschitl's formula, Appl. Math. Comput. 245 (2014) 174–180.

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[13] C.-P. Chen, Inequalities and asymptotic expansions associated with the Ramanujan and Nemes formulas for the gamma function, Appl. Math. Comput. 261 (2015) 337–350.
[14] C.-P. Chen, A sharp version of Ramanujan's inequality for the factorial function, Ramanujan J. 39 (2016) 149–154.
[15] C.-P. Chen, N. Elezović, L. Vukšić, Asymptotic formulae associated with the Wallis power function and digamma function, J. Classical Anal. 2 (2013) 151–166.
[16] C.-P. Chen, L. Lin, Remarks on asymptotic expansions for the gamma function, Appl. Math. Lett. 25 (2012) 2322–2326.
[17] C.-P. Chen, J.-Y. Liu, Inequalities and asymptotic expansions for the gamma function, J. Number Theory 149 (2015) 313–326.
[18] C.-P. Chen, R.B. Paris, Inequalities, asymptotic expansions and completely monotonic functions related to the gamma function, Appl. Math. Comput. 250 (2015) 514–529.
[19] E.T. Copson, Asymptotic Expansions, Cambridge University Press, 1965.
[20] R.W. Gosper, Decision procedure for indefinite hypergeometric summation, Proc. Natl. Acad. Sci. USA 75 (1978) 40–42.
[21] M.D. Hirschhorn, A new version of Stirling's formula, Math. Gaz. 90 (2006) 286–291.
[22] M.D. Hirschhorn, M.B. Villarino, A refinement of Ramanujan's factorial approximation, Ramanujan J. 34 (2014) 73–81.
[23] E.A. Karatsuba, On the asymptotic representation of the Euler gamma function by Ramanujan, J. Comput. Appl. Math. 135 (2001) 225–240.
[24] S. Koumandos, Remarks on some completely monotonic functions, J. Math. Anal. Appl. 324 (2006) 1458–1461.
[25] S. Koumandos, H.L. Pedersen, Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler's gamma function, J. Math. Anal. Appl. 355 (2009) 33–40.
[26] L. Lin, C.-P. Chen, Asymptotic formulas for the gamma function by Gosper, J. Math. Inequal. 9 (2015) 541–551.
[27] D. Lu, A generated approximation related to Burnside's formula, J. Number Theory 136 (2014) 414–422.
[28] D. Lu, A new sharp approximation for the Gamma function related to Burnside's formula, Ramanujan J. 35 (2014) 121–129.
[29] D. Lu, J. Feng, C. Ma, A general asymptotic formula of the gamma function based on the Burnside's formula, J. Number Theory 145 (2014) 317–328.
[30] D. Lu, L. Song, C. Ma, A generated approximation of the gamma function related to Windschitl's formula, J. Number Theory 140 (2014) 215–225.
[31] D. Lu, L. Song, C. Ma, Some new asymptotic approximations of the gamma function based on Nemes' formula, Ramanujan's formula and Burnside's formula, Appl. Math. Comput. 253 (2015) 1–7.
[32] D. Lu, X. Wang, A generated approximation related to Gosper's formula and Ramanujan's formula, J. Math. Anal. Appl. 406 (2013) 287–292.
[33] D. Lu, X. Wang, A new asymptotic expansion and some inequalities for the gamma function, J. Number Theory 140 (2014) 314–323.
[34] C. Mortici, An ultimate extremely accurate formula for approximation of the factorial function, Arch. Math. (Basel) 93 (2009) 37–45.
[35] C. Mortici, Sharp inequalities related to Gosper's formula, C. R. Math. Acad. Sci. Paris 348 (2010) 137–140.
[36] C. Mortici, Product approximations via asymptotic integration, Amer. Math. Monthly 117 (2010) 434–441.
[37] C. Mortici, The asymptotic series of the generalized Stirling formula, Comput. Math. Appl. 60 (2010) 786–791.
[38] C. Mortici, Asymptotic expansions of the generalized Stirling approximations, Math. Comput. Modelling 52 (2010) 1867–1868.
[39] C. Mortici, A class of integral approximations for the factorial function, Comput. Math. Appl. 59 (2010) 2053–2058.
[40] C. Mortici, Ramanujan formula for the generalized Stirling approximation, Appl. Math. Comput. 217 (2010) 2579–2585.
[41] C. Mortici, Best estimates of the generalized Stirling formula, Appl. Math. Comput. 215 (2010) 4044–4048.

Please cite this article in press as: C.-P. Chen, A more accurate approximation for the gamma function, J. Number Theory (2016), http://dx.doi.org/10.1016/j.jnt.2015.11.007

[42] C. Mortici, On the gamma function approximation by Burnside, Appl. Math. E-Notes 11 (2011) 274–277.
[43] C. Mortici, On Gospers formula for the Gamma function, J. Math. Inequal. 5 (2011) 611–614.
[44] C. Mortici, Improved asymptotic formulas for the gamma function, Comput. Math. Appl. 61 (2011) 3364–3369.
[45] C. Mortici, Ramanujan's estimate for the gamma function via monotonicity arguments, Ramanujan J. 25 (2011) 149–154.
[46] C. Mortici, On Ramanujan's large argument formula for the gamma function, Ramanujan J. 26 (2011) 185–192.
[47] C. Mortici, An improvement of the Ramanujan formula for approximation of the Euler gamma function, Carpathian J. Math. 28 (2012) 301–304.
[48] C. Mortici, A continued fraction approximation of the gamma function, J. Math. Anal. Appl. 402 (2013) 405–410.
[49] C. Mortici, A new fast asymptotic series for the gamma function, Ramanujan J. 38 (2015) 549–559.
[50] G. Nemes, New asymptotic expansion for the Gamma function, Arch. Math. (Basel) 95 (2010) 161–169.
[51] G. Nemes, More accurate approximations for the gamma function, Thai J. Math. 9 (2011) 21–28.
[52] F. Qi, Integral representations and complete monotonicity related to the remainder of Burnside's formula for the gamma function, J. Comput. Appl. Math. 268 (2014) 155–167.
[53] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa Publishing House, New Delhi, 1988, with an introduction by George E. Andrews, Springer-Verlag, Berlin.
[54] W.D. Smith, The gamma function revisited, http://schule.bayernport.com/gamma/gamma05.pdf, 2006.
[55] http://www.rskey.org/gamma.htm.

Please cite this article in press as: C.-P. Chen, A more accurate approximation for the gamma function, J. Number Theory (2016), http://dx.doi.org/10.1016/j.jnt.2015.11.007