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PERFECT POWERS IN SUM OF THREE FIFTH POWERS

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Abstract. In this paper we determine the perfect powers that are sums of three fifth powers in an arithmetic progression. More precisely, we completely solve the Diophantine equation

$$(x-d)^5 + x^5 + (x+d)^5 = z^n, \ n \geq 2,$$

where $d, x, z \in \mathbb{Z}$ and $d = 2^a 5^b$ with $a, b \geq 0$.

1. Introduction

In 1956, Schäffer [35] considered the equation

$$1^k + 2^k + \cdots + x^k = y^n. \tag{1.1}$$

He proved that if $k \geq 1$ and $n \geq 2$ are fixed, then (1.1) has only finitely many solutions except for the cases $(k,n) \in \{(1,2),(3,2),(3,4),(5,2)\}$. In the same paper Schäffer stated the following conjecture on the integral solution of (1.1).

Conjecture 1. [Schäffer, [35]**]**
Let $k \geq 1, n \geq 2$ be integers and $(k,n) \notin \{(1,2),(3,2),(3,4),(5,2)\}$. The equation

$$1^k + 2^k + \cdots + x^k = y^n,$$

has only one non-trivial solution, namely $(k,n,x,y) = (2,2,24,70)$.

The equation (1.1) and its generalizations have a long and rich history. Bennett, Győry, Pintér [5] proved the Conjecture 1 for arbitrary n and $k \leq 11$. Following and extending the approach of [5], Pintér [33] proved Conjecture 1 for odd values of k with $1 \leq k < 170$ and even values of n.

As a natural generalization of equation (1.1), Zhang and Bai [45] considered the equation

$$(x+1)^k + (x+2)^k + \cdots + (x+r)^k = y^n, \tag{1.2}$$

where $x, y \in \mathbb{Z}$, $r, k \in \mathbb{N}$ and $n \geq 2$.

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The equation (1.2) is comparatively more difficult than the equation (1.1). There has been some progress on equation (1.2) for particular values of k, n and r. Below we list down all results to the best of our knowledge obtained for equation (1.2) .

We first list down the results obtained for power sums in a range of values of r. Stroeker [38] completely solved the equation (1.2) for $k = 3$, $n = 2$ and $2 \leq r \leq 50$. Recently, Bennett, Patel and Siksek [8] extended the result of Stroeker for $n \geq 3$. Zhang and Bai [45] solved the equation (1.2) for $k = 2$ and $r = x$. Bartoli and Soydan [37, 4] extended the result of Zhang and Bai [45] for $k \geq 2$ and $r = lx$ with $l \geq 2$.

The equation (1.2) has also been studied for fixed r. Cassels [17] solved the equation (1.2) completely for $n = 2$, $r = 3$ and $k = 3$. Zhang [43] subsequently considered the equation (1.2) for $r = 3$ and he solved it completely for $k \in \{2, 3, 4\}$. Recently, Bennet, Patel and Siksek [7] extended Zhang's result, by completely solving equation (1.2) for $r = 3$ in the cases $k = 5$ and $k = 6$. Several authors have also studied equations (1.1), (1.2) and its variants using a variety of classical and modern techniques (see e.g. [5, 10, 14, 19, 20, 23, 24, 25]).

As a natural generalization of all the above results many mathematicians have recently studied power sums in arithmetic progression. They have considered the equation

$$(x+d)^k + (x+2d)^k + \cdots + (x+rd)^k = y^n, \ x, y, d \in \mathbb{Z}, \ r, k \in \mathbb{N}, \ n \geq 2. \qquad (1.3)$$

In this paper we are particularly interested in the case $r = 3$, in particular for the equation

$$(x-d)^k + x^k + (x+d)^k = y^n, \quad n \geq 2. \qquad (1.4)$$

We mention some related results for $k \leq 4$. Koutsianas [26] studied the equation (1.4) for $k = 2$, where d is of the form p^b with p a suitable prime. Koutsianas and Patel [27] completely solved the equation (1.4) for $k = 2$ and for all values of $1 \leq d \leq 5000$, using the characterization of primitive divisors in Lehmer sequences by Bilu-Hanrot-Voutier [11]. For $1 \leq d \leq 10^6$ and $k = 3$, Argáez-García and Patel [1, 2, 3] studied the equation (1.3) for $r \in \{3, 5, 7\}$. For $k = 4$ the equation (1.4) was solved by Zhang [44] for some particular choices of d and Langen [29] under the assumption $\gcd(x, d) = 1$. For further reference we include all known results on equation (1.3) in Table 1.

The general equation (1.4) for $k \geq 5$ is a difficult problem. In this paper, we study the Diophantine equation

$$(x-d)^5 + x^5 + (x+d)^5 = z^n, \quad n \geq 2, \ xz \neq 0. \qquad (1.5)$$

d	r	k	n	References
1	x	2	≥ 2	Zhang and Bai [45]
1	3	3	2	Cassels [17]
1	3	$\{2,3,4\}$	≥ 2	Zhang [43]
1	3	$\{5,6\}$	≥ 2	Bennett, Patel and Siksek [7]
1	$\{1,\cdots,50\}$	3	≥ 2	Bennett, Patel and Siksek [8]
1	$\{2,\cdots,10\}$	2	≥ 2	Patel [31]
a suitable set of prime powers	3	2	≥ 7	Koutsianas [26]
composed of a suitable set of prime powers	3	4	≥ 11	Zhang [44]
$\gcd(x,d)=1$	3	4	≥ 2	Langen [29]
$\{1,\cdots,10^4\}$	$\{2,\cdots,10\}$	2	≥ 2	Kundu and Patel [28]
$\{1,\cdots,10^6\}$	$3,7$	3	≥ 5	Argáez-Garciá and Patel [2, 3]
$\{1,\cdots,10^6\}$	5	3	≥ 5	Argáez-Garciá [1]
$\{1,\cdots,5000\}$	3	2	≥ 2	Koutsianas and Patel [27]

Table 1. Notable results on solutions of special cases of (1.3).

Recently, Bennett and Koutsianas [6] solved equation (1.5) with the natural assumption $\gcd(x,d)=1$. This assumption enables them to factorize the left-hand side of (1.5) and reduce the problem to the resolution of Fermat type equations of signature $(n,n,2)$ with coefficients independent of d. In the general case, the coefficients of the Fermat type equations have prime factors that divide $10\cdot\gcd(x,d)$. Therefore, without any restrictions to $\gcd(x,d)$ we are not able to solve (1.5) with the current techniques. The existence of infinite family of solutions for small exponents n, for instance the solutions $(x,d,y,n)=(ra^4,sa^4,a^3,7)$, $(ra^6,sa^6,a,31)$ and $(ra^8,sa^8,a,41)$ where $r,s\in\mathbb{Z}^*$ and[1] $a=(r-s)^5+r^5+(r+s)^5$, show that the complete resolution of (1.5) is a hard and challenging problem.

From the above we understand that if we want to resolve equation (1.5) without the assumption $\gcd(x,d)=1$ but with freedom in the choice of d, we have to fix the prime factors of d. Let us assume that d is divisible by primes that lie in a fixed finite set S. As we mentioned in the previous paragraph the coefficients of the Fermat type equations will depend on $S\cup\{2,5\}$. Because the primes 2 and 5 always show up for any choice of S it is very natural to study the resolution of (1.5) for $S=\{2,5\}$, i.e. when $d=2^a5^b$ with $a,b\geq 0$. We prove the following which is a generalization of [7, Theorem 1].

[1]For example, the quadruples $(x,d,y,n)=(33^4,33^4,33^3,7)$, $(2\cdot 276^6,276^6,276,31)$ and $(243^8,2\cdot 243^8,243,41)$ are the solutions to the equation (1.4) for the three smallest positive values of a.

Theorem 1. *Let $n \geq 2$ be an integer and $d = 2^a 5^b$ with integers $a, b \geq 0$. Then, the equation*

$$(x-d)^5 + x^5 + (x+d)^5 = z^n,$$

is solvable in integers x, z with $xz \neq 0$ only if $n = 5$ and $a \geq 1$, in which case all integer solutions are given by $(x, z) = \pm(d/2, 3d/2)$.

The paper is organized as follows. In Section 2 we associate a solution (x, z) of (1.5) for $d = 2^a 5^b$ with two Fermat type equations of signature $(n, n, 2)$ with pairwise coprime terms. In Section 3 we resolve (1.5) for $n = 2$, 3 and 5 using a variety of elementary and advanced techniques. In Section 4 we explain how we can apply the modular method and the recipes in [9] to resolve (1.5) when $n \geq 7$ is a prime. Finally, in Section 5 we complete the proof of Theorem 1 for $n \geq 7$ and we give the necessary details of the computations.

Remark 2. In principle, the method we describe in this paper will work for any choice of the set S when S contains 2 and 5. However, the required computations for the spaces of newforms are beyond to the current computer power.

The computations of this paper have been accomplished in computer software `Magma` [12] and the code can be found at

`https://github.com/akoutsianas/5th_powers`.

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2. Preliminaries

We have the equation

$$(x-d)^5 + x^5 + (x+d)^5 = z^n, \quad d = 2^a 5^b,\ a, b \geq 0, \quad xz \neq 0. \tag{2.1}$$

Clearly, in order to prove Theorem 1 we may assume that n is a prime number. Since (2.1) can be rewritten as

$$x(3x^4 + 20d^2x^2 + 10d^4) = z^n, \tag{2.2}$$

it suffices to consider only the case in which both x and z are positive.

Let $\nu_p(N)$ denotes the p-adic valuation of an integer N, where p is a prime. We set

$$x = 2^\alpha 5^\beta x_1,\ \gcd(x_1, 10) = 1, \quad P = 3x^4+20d^2x^2+10d^4, \quad z = 2^u5^vZ,\ \gcd(Z, 10) = 1,$$

hence

$$2^\alpha 5^\beta x_1 P = 2^{nu}5^{nv}Z^n. \tag{2.3}$$

Since $\gcd(x, P) = \gcd(x, 10d^4)$, it follows that

$$\gcd(x, P) = 2^{\min\{4a+1,\alpha\}} \cdot 5^{\min\{4b+1,\beta\}} \quad \text{and} \quad \gcd(x_1, P) = 1.$$

Let us put $P = 2^{\nu_2(P)}5^{\nu_5(P)}P_1$. Clearly, $\gcd(x_1, P_1) = 1$ and $\gcd(P_1, 10) = 1$; therefore, $P_1 = 2^{-\nu_2(P)}5^{-\nu_5(P)}P$. Using these in (2.3) we obtain $2^{\alpha+\nu_2(P)}5^{\beta+\nu_5(P)}x_1P_1 = 2^{nu}5^{nv}Z^n$, where $\gcd(x_1P_1Z, 10) = 1$. It follows that

$$\alpha + \nu_2(P) = nu, \quad \beta + \nu_5(P) = nv,$$

and $x_1P_1 = Z^n$. From $\gcd(x_1, P_1) = 1$, it follows that

$$x_1 = z_1^n,\ P_1 = z_2^n,\ Z = z_1z_2,\ \gcd(z_1, z_2) = 1,\ \gcd(z_1z_2, 10) = 1. \tag{2.4}$$

We rewrite equation (2.2) equivalently in the following two ways:

$$10(x^2 + d^2)^2 - 7x^4 = P, \tag{2.5}$$
$$(3x^2 + 10d^2)^2 - 70d^4 = 3P. \tag{2.6}$$

Noting that

$$P = 3 \cdot 2^{4\alpha}5^{4\beta}x_1^4 + 2^{2a+2\alpha+2}5^{2b+2\beta+1}x_1^2 + 2^{4a+1}5^{4b+1}, \tag{2.7}$$

we consider four cases according to the values of a, α and b, β.

Case I. Suppose $4\alpha < 4a + 1$ and $4\beta < 4b + 1$. This is equivalent to $a \geq \alpha$ and $b \geq \beta$ and from (2.7) we get $\nu_2(P) = 4\alpha$ and $\nu_5(P) = 4\beta$. Because $x = 2^\alpha 5^\beta z_1^n$, $d = 2^a5^b$ and $P = 2^{4\alpha}5^{4\beta}z_2^n$, dividing equations (2.5) and (2.6) by $2^{4\alpha}5^{4\beta}$ we obtain the following two equations

$$z_2{}^n + 7z_1{}^{4n} = 10(z_1{}^{2n} + 2^{2(a-\alpha)}5^{2(b-\beta)})^2, \tag{2.8}$$
$$3z_2{}^n + 7 \cdot 2^{4(a-\alpha)+1}5^{4(b-\beta)+1} = (3z_1{}^{2n} + 2^{2(a-\alpha)+1}5^{2(b-\beta)+1})^2. \tag{2.9}$$

Case II. Suppose $4\alpha < 4a+1$ and $4\beta > 4b+1$. This is equivalent to $a \geq \alpha$ and $\beta \geq b+1$ and from (2.7) we get that $\nu_2(P) = 4\alpha$ and $\nu_5(P) = 4b+1$. Because $x = 2^\alpha 5^\beta z_1^n$, $d = 2^a 5^b$ and $P = 2^{4\alpha}5^{4b+1}z_2^n$, dividing equations (2.5) and (2.6) by $2^{4\alpha}5^{4b+1}$ we obtain the following two equations

$$z_2{}^n + 7\cdot 5^{4(\beta-b)-1}z_1{}^{4n} = 2(5^{2(\beta-b)}z_1{}^{2n} + 2^{2(a-\alpha)})^2, \tag{2.10}$$

$$3z_2{}^n + 7\cdot 2^{4(a-\alpha)+1} = 5(3\cdot 5^{2(\beta-b)-1}z_1{}^{2n} + 2^{2(a-\alpha)+1})^2. \tag{2.11}$$

Case III. Suppose $4\alpha > 4a+1$ and $4\beta < 4b+1$. This is equivalent to $\alpha \geq a+1$ and $b \geq \beta$ and from (2.7) we get that $\nu_2(P) = 4a+1$ and $\nu_5(P) = 4\beta$. Because $x = 2^\alpha 5^\beta z_1^n$, $d = 2^a 5^b$ and $P = 2^{4a+1}5^{4\beta}z_2^n$, dividing equations (2.5) and (2.6) by $2^{4a+1}5^{4\beta}$ we obtain the following two equations

$$z_2{}^n + 7\cdot 2^{4(\alpha-a)-1}z_1{}^{4n} = 5(2^{2(\alpha-a)}z_1{}^{2n} + 5^{2(b-\beta)})^2, \tag{2.12}$$

$$3z_2{}^n + 7\cdot 5^{4(b-\beta)+1} = 2(3\cdot 2^{2(\alpha-a)-1}z_1{}^{2n} + 5^{2(b-\beta)+1})^2. \tag{2.13}$$

Case IV. Suppose $4\alpha > 4a+1$ and $4\beta > 4b+1$. This is equivalent to $\alpha \geq a+1$ and $\beta \geq b+1$ and from (2.7) we get that $\nu_2(P) = 4a+1$ and $\nu_5(P) = 4b+1$. Because $x = 2^\alpha 5^\beta z_1^n$, $d = 2^a 5^b$ and $P = 2^{4a+1}5^{4b+1}z_2^n$, dividing equations (2.5) and (2.6) by $2^{4a+1}5^{4b+1}$ we obtain the following two equations

$$z_2^n + 7\cdot 2^{4(\alpha-a)-1}5^{4(\beta-b)-1}z_1{}^{4n} = (2^{2(\alpha-a)}5^{2(\beta-b)}z_1^{2n} + 1)^2, \tag{2.14}$$

$$3z_2^n + 7 = 10(3\cdot 2^{2(\alpha-a)-1}5^{2(\beta-b)-1}z_1{}^{2n} + 1)^2. \tag{2.15}$$

Equations (2.8)-(2.15) all have the general shape of a ternary generalized Fermat-type equation of signature $(n, n, 2)$, namely,

$$Aa^n + Bb^n = Cc^2. \tag{2.16}$$

where A, a, B, b, C, c are shown in Table 2. We need to view our equations as such in Sections 4 and 5 where we treat the case $n \geq 7$ by applying the recipes of [9]. For the application of the results therein we need that Aa, Bb, Cc be pairwise relatively prime.

In view of (2.16), Aa, Bb, Cc are pairwise coprime if and only if $\gcd(Aa, Bb) = 1$. Since we have already seen that $\gcd(z_1 z_2, 10) = 1$ and $\gcd(z_1, z_2) = 1$, the said requirement $\gcd(Aa, Bb) = 1$ is equivalent to $7 \nmid z_2$. This is true because, if $7 \mid z_2$, then, by (2.16) and the fact that $7 \mid B$ and $7 \nmid C$, it follows that $7 \mid c$. Thus, the valuation at 7 of the right-hand side of equations (2.8)-(2.15) is ≥ 2. On the other hand, because $n \geq 2$, $\gcd(z_1, z_2) = 1$ and $7 \mid z_2$, the valuation at 7 of the left-hand side of equations (2.8)-(2.15) is equal to 1, which is a contradiction.

Eq.	A	a	B	b	C	c
(2.8)	1	z_2	7	z_1^4	10	$z_1^{2n}+2^{2(a-\alpha)}5^{2(b-\beta)}$
(2.9)	3	z_2	$7\cdot 2^{4(a-\alpha)+1}5^{4(b-\beta)+1}$	1	1	$-3z_1^{2n}-2^{2(a-\alpha)+1}5^{2(b-\beta)+1}$
(2.10)	1	z_2	$7\cdot 5^{4(\beta-b)-1}$	z_1^4	2	$5^{2(\beta-b)}z_1^{2n}+2^{2(a-\alpha)}$
(2.11)	3	z_2	$7\cdot 2^{4(a-\alpha)+1}$	1	5	$-3\cdot 5^{2(\beta-b)-1}z_1^{2n}-2^{2(a-\alpha)+1}$
(2.12)	1	z_2	$7\cdot 2^{4(\alpha-a)-1}$	z_1^4	5	$2^{2(\alpha-a)}z_1^{2n}+5^{2(b-\beta)}$
(2.13)	3	z_2	$7\cdot 5^{4(b-\beta)+1}$	1	2	$3\cdot 2^{2(\alpha-a)-1}z_1^{2n}+5^{2(b-\beta)+1}$
(2.14)	1	z_2	$7\cdot 2^{4(\alpha-a)-1}5^{4(\beta-b)-1}$	z_1^4	1	$2^{2(\alpha-a)}5^{2(\beta-b)}z_1^{2n}+1$
(2.15)	3	z_2	7	1	10	$3\cdot 2^{2(\alpha-a)-1}5^{2(\beta-b)+1}z_1^{2n}+1$

Table 2. Parameters needed for the application of the recipes of [9].

3. Solving equation (2.1) when $n=2,3,5$

In this section we prove that equation (2.1) is impossible when $n=2,3$ and solve the equation when $n=5$.

3.1. **The case** $n=2$**.** In this case equation (2.1) becomes

$$(x-d)^5+x^5+(x+d)^5=z^2, \quad d=2^a5^b,\ a,b\geq 0, \quad xz\neq 0. \tag{3.1}$$

We have to consider each case (I)-(IV) (as defined in Section 2) separately.

Case (I): By (2.8) we have $z_2^2+7z_1^8\equiv 0 \pmod 5$. However, from $5\nmid z_1z_2$ it follows that $z_2^2+7z_1^8\equiv 1,3 \pmod 5$ and we get a contradiction.

Case (II): By (2.11) we have $3z_2^2+7\cdot 2^{4(a-\alpha)+1}\equiv 0 \pmod 5$. However, $3z_2^2+7\cdot 2^{4(a-\alpha)+1}\equiv 3z_2^2+14\not\equiv 0 \pmod 5$ (actually, $3z_2^2+14\equiv 1$ or $2 \pmod 5$ because $5\nmid z_2$) and we get a contradiction.

Case (III): By (2.12) we have $z_2^2+7\cdot 2^{4(\alpha-a)-1}\equiv 5 \pmod 8$ as $2\nmid z_1$. On the other hand, as $2\nmid z_2$ and $4(\alpha-a)-1\geq 3$, we have $z_2^2+7\cdot 2^{4(\alpha-a)-1}\equiv 1 \pmod 8$, which is a contradiction.

Case (IV): Putting $z_1^2=x_1$ in equation (2.14) leads us to the equation

$$3\cdot 2^{4(\alpha-a)-1}5^{4(\beta-b)-1}x_1^4+2^{2(\alpha-a)+1}5^{2(\beta-b)}x_1^2+1={z_2}^2.$$

We put $4(\alpha-a)-1=4k+3$ and $4(\beta-b)-1=4l+3$, where $k,l\geq 0$ so that the above equation becomes

$$3000(2^k5^lx_1)^4+200(2^k5^lx_1)^2+1={z_2}^2.$$

The elliptic curve defined by $3000X^4+200X^2+1=Y^2$ is isomorphic to the elliptic curve with Cremona label 134400ed1 which has rank zero and torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$, hence $(X,Y)=(0,\pm 1)$ are its only affine rational point. Clearly, these points do not provide us with an acceptable pair (x_1,z_2).

Thus we conclude that equation (3.1) has no solutions, hence we have proved the following:

Proposition 3. *Equation* (2.1) *with* $n = 2$ *is impossible.*

3.2. **The case** $n = 3$**.** In this case equation (2.1) becomes

$$(x-d)^5 + x^5 + (x+d)^5 = z^3, \quad d = 2^a 5^b,\ a, b \geq 0,\ xz \neq 0. \tag{3.2}$$

In accordance with Section 2 we examine each case (I) through (IV) separately. To these cases correspond equations (2.8), (2.10), (2.12) and (2.14), respectively, which by elementary symbolic computations are transformed into equivalent equations as follows:

Equation (2.8) becomes

$$3\left(\frac{z_1^6}{2^{2(a-\alpha)}5^{2(b-\beta)}}\right)^2 + 20\left(\frac{z_1^6}{2^{2(a-\alpha)}5^{2(b-\beta)}}\right) + 10 = \frac{1}{2^{a-\alpha}5^{b-\beta}}\left(\frac{z_2}{2^{a-\alpha}5^{b-\beta}}\right)^3.$$

We rewrite this as

$$3Y^2 + 20Y + 10 = \frac{1}{2^{a-\alpha}5^{b-\beta}}X^3, \quad Y = \frac{z_1^6}{2^{2(a-\alpha)}5^{2(b-\beta)}}, \quad X = \frac{z_2}{2^{a-\alpha}5^{b-\beta}}. \tag{3.3}$$

Equation (2.10) becomes

$$3\left(\frac{5^{2(\beta-b)}z_1^6}{2^{2(a-\alpha)}}\right)^2 + 20\left(\frac{5^{2(\beta-b)}z_1^6}{2^{2(a-\alpha)}}\right) + 10 = \frac{5}{2^{a-\alpha}}\left(\frac{z_2}{2^{a-\alpha}}\right)^3.$$

We rewrite this as

$$3Y^2 + 20Y + 10 = \frac{5}{2^{a-\alpha}}X^3, \quad Y = \frac{5^{2(\beta-b)}}{2^{2(a-\alpha)}}z_1^6, \quad X = \frac{z_2}{2^{a-\alpha}}. \tag{3.4}$$

Equation (2.12) becomes

$$3\left(\frac{2^{2(\alpha-a)}z_1^6}{5^{2(b-\beta)}}\right)^2 + 20\left(\frac{2^{2(\alpha-a)}z_1^6}{5^{2(b-\beta)}}\right) + 10 = \frac{2}{5^{b-\beta}}\left(\frac{z_2}{5^{b-\beta}}\right)^3,$$

which we rewrite as

$$3Y^2 + 20Y + 10 = \frac{2}{5^{b-\beta}}X^3, \quad Y = \frac{2^{2(\alpha-a)}}{5^{2(b-\beta)}}z_1^6, \quad X = \frac{z_2}{5^{b-\beta}}. \tag{3.5}$$

Equation (2.14) becomes

$$3(2^{2(\alpha-a)}5^{2(\beta-b)}z_1^6)^2 + 20(2^{2(\alpha-a)}5^{2(\beta-b)}z_1^6) + 10 = z_2^3,$$

which we rewrite as

$$3Y^2 + 20Y + 10 = 10X^3, \quad Y = 2^{2(\alpha-a)}5^{2(\beta-b)}z_1^6, \quad X = z_2. \tag{3.6}$$

We note that in every equation (3.3)-(3.6), the change of variable $X = \mu X_1$, where μ is an appropriate explicit S-integer with $S \subseteq \{2, 5\}$, depending on the classes (mod 3) of $a - \alpha, b - \beta$, leads to an equation

$$3Y^2 + 20Y + 10 = cX_1^3, \tag{3.7}$$

where c runs through a "small" explicit set of S-integers with $S \subseteq \{2, 5\}$ (see below for each case separately). The elliptic curve defined by (3.7) is isomorphic to the elliptic curve

$$y_1^2 = x_1^3 + 630c^2, \quad x_1 = 3cX_1,\ y_1 = 3c(3Y + 10). \tag{3.8}$$

Remark 4. In all cases above, X and Y are S-integers with $S \subseteq \{2, 5\}$ and we see that X_1 is also an S-integer. Therefore, we have to compute all S-integral points (x_1, y_1) on the elliptic curve defined by (3.8). For certain values of c the rank of the corresponding elliptic curve is zero with trivial torsion subgroup, therefore no rational points exist. For all the remaining values of c the rank is 2 and we compute the S-integral points with the aid of the MAGMA [12] routines `SIntegralPoints` (when $S \neq \emptyset$) or `IntegralPoints` (when $S = \emptyset$); for the background of these routines we refer, respectively, to [32] and [39, 22] (see also [41]). We observe that the y-coordinate of an S-integral point corresponds to a Y that must be equal to the product of an S-integer times a sixth power of a non-zero integer. If that does not happen then no solutions of (3.2) arise from that point. We ask the reader to have these remarks in mind whenever we expose the solutions of (3.8) for the various values of c.

We consider equation (3.8) separately for each equation (3.3)-(3.6).

Equation (3.3): We put $(a - \alpha, b - \beta) = (3a_1 + i, 3b_1 + j)$, where $0 \le i, j \le 2$. Then, in (3.3) we have

$$Y = \frac{z_1^6}{2^{6a_1+2i}5^{6b_1+2j}}, \quad X = \frac{z_2}{2^{3a_1+i}5^{3b_1+j}}, \quad \frac{1}{2^{a-\alpha}5^{b-\beta}}X^3 = cX_1^3,$$

where

$$c = \frac{1}{2^i5^j}, \quad X_1 = \frac{z_2}{2^{4a_1+i}5^{4b_1+j}}.$$

Now we consider equation (3.8) with

$$c \in \{1,\ 1/2,\ 1/4,\ 1/5,\ 1/25,\ 1/10,\ 1/50,\ 1/20,\ 1/100\}.$$

When $c \in \{1,\ 1/2,\ 1/5,\ 1/10,\ 1/50,\ 1/20\}$, the curve defined by the equation (3.8) is of zero rank with trivial torsion subgroup, hence there are no rational points.

For the remaining values $c = 1/4, 1/25, 1/100$ equation (3.8) defines an elliptic curve of rank 2 and we have to compute all S-integral points on it, where $S = \{2, 5\}$.

When $c = 1/4$, equation (3.8) becomes $y_1^2 = x_1^3 + \frac{315}{8}$ and its S-integral points are

$$(x_1, y_1) = (-\frac{3}{2}, \pm 6),\ (-\frac{33}{50}, \pm\frac{1563}{250}),\ (\frac{9}{4}, \pm\frac{57}{8}),\ (\frac{849}{256}, \pm\frac{35673}{4096}),\ (\frac{23}{2}, \pm\frac{79}{2}).$$

From (3.8), the values of Y corresponding to the above solutions are respectively:

$$Y = -\frac{2}{3},\ -6,\ -\frac{2^4\,13}{3\cdot 5^3},\ -\frac{2^2\,191}{5^3},\ -\frac{1}{6},\ -\frac{13}{2},\ \frac{13\cdot 127}{2^{10}\,3},\ -\frac{3\cdot 2459}{2^{10}},\ \frac{2^7}{3^2},\ -\frac{2^2\,47}{3^2}.$$

Then from Remark 4 these values do not lead to a solution and hence for $c = 1/4$ equation (3.3) has no solutions.

When $c = 1/25$, equation (3.8) becomes $y_1^2 = x_1^3 + \frac{126}{125}$ and its S-integral points are

$$(x_1, y_1) = (-\frac{1}{5},\ \pm 1),\ (\frac{1009}{2500},\ \pm\frac{129527}{125000}),\ (\frac{69}{80},\ \pm\frac{411}{320}),\ (\frac{99}{25},\ \pm\frac{993}{125}),$$

with the corresponding values of Y being

$$Y = -\frac{5}{3^2},\ -\frac{5\cdot 11}{3^2},\ -\frac{59\cdot 347}{2^3\,3^2\,5^4},\ -\frac{31\cdot 71\cdot 127}{2^3\,3^2\,5^4},\ \frac{15}{2^6},\ -\frac{5^2\,53}{2^6\,3},\ \frac{281}{15},\ -\frac{127}{5}.$$

Again by Remark 4 these values do not lead to a solution.

Finally, if $c = 1/100$ then (3.8) becomes $y_1^2 = x_1^3 + \frac{63}{1000}$ and its S-integral points are

$$(x_1, y_1) = (-\frac{159}{400},\ \pm\frac{111}{8000}),\ (\frac{1}{100},\ \pm\frac{251}{1000}),\ (\frac{3}{10},\ \pm\frac{3}{10}),\ (\frac{81}{100},\ \pm\frac{771}{1000}),$$
$$(\frac{129921}{16\cdot 10^4},\ \pm\frac{49508031}{64\cdot 10^6}),\ (\frac{33}{10},\ \pm 6).$$

These respectively give

$$Y = -\frac{7\cdot 109}{2^4\,15},\ -\frac{3^2\,31}{2^4\,5},\ -\frac{7^2}{90},\ -\frac{19\cdot 29}{90},\ 0,\ -\frac{2^2\,5}{3},\ \frac{157}{30},\ -\frac{119}{10},\ \frac{13\cdot 83\cdot 3121}{2^{10}\,5^4},$$
$$-\frac{7\cdot 47\cdot 67\cdot 1039}{2^{10}\,5^4\,3},\ \frac{190}{3},\ -70.$$

As above these values do not lead to a solution.

Conclusion: No solutions to (3.3) *exist.*

Equation (3.4): We put $a - \alpha = 3a_1 + i$ with $0 \le i \le 2$. Then, in (3.4) we have

$$Y = \frac{5^{2(\beta-b)} z_1^6}{2^{6a_1+2i}},\quad X = \frac{z_2}{2^{3a_1+i}},\quad \frac{5}{2^{a-\alpha}} X^3 = cX_1^3,$$

where

$$c = \frac{5}{2^i},\quad X_1 = \frac{z_2}{2^{4a_1+i}}.$$

Thus we consider equation (3.8) with $c \in \{5, 5/2, 5/4\}$. Now X, Y and X_1 are S-integers with $S = \{2\}$, and (x_1, y_1) is S-integral solution to (3.8).

If $c = 5$, then (3.8) becomes $y_1^2 = x_1^3 + 15750$ and its S-integral solutions are

$$(x_1, y_1) = (-5,\ \pm 125),\ (\frac{345}{16},\ \pm\frac{10275}{64}),\ (99,\ \pm 993).$$

These furnish us with the following values of Y:

$$Y=-\frac{5}{9}\,,\ -\frac{55}{9}\,,\ \frac{15}{2^6}\,,\ -\frac{5^2\,53}{2^6\,3}\,,\ \frac{281}{15}\,,\ -\frac{127}{5}\,,$$

Again by Remark 4 the above values do not lead to a solution. Therefore, when $c=5$, (3.4) has no solutions.

When $c=5/2$, the elliptic curve defined by (3.8) is of zero rank with trivial torsion subgroup, hence there are no rational solutions.

When $c=5/4$, (3.8) becomes $y_1^2=x_1^3+\frac{7875}{8}$ and its S-integral solutions are

$$(x_1,y_1)=(-\frac{159}{16}\,,\pm\frac{111}{64})\,,\ (\frac{1}{4}\,,\pm\frac{251}{8})\,,\ (\frac{15}{2}\,,\pm\frac{75}{2})\,,\ (\frac{81}{4}\,,\pm\frac{771}{8})\,,\ (\frac{165}{2}\,,\pm 750),$$

with the corresponding values of Y being

$$Y=-\frac{7\cdot 109}{2^4\cdot 15}\,,,\ -\frac{3^2\cdot 31}{2^4\cdot 5}\,,\ -\frac{7^2}{90}\,,\ -\frac{19\cdot 29}{90}\,,\ 0,\ -\frac{2^2\,5}{3}\,,\ \frac{157}{30}\,,\ -\frac{7\cdot 17}{10}\,,\ \frac{10\cdot 19}{3}\,,\ -70.$$

Similar to the above case these values of Y do not lead to a solution of (3.4).

Conclusion: No solutions to (3.4) *exist.*

Equation (3.5): We put $b-\beta=3b_1+j$ with $0\le j\le 2$. Then, in (3.5) we have

$$Y=\frac{2^{2(\alpha-a)}z_1^6}{5^{6b_1+2j}}\,,\quad X=\frac{z_2}{5^{3b_1+j}}\,,\quad \frac{2}{5^{b-\beta}}X^3=cX_1^3\,,$$

where

$$c=\frac{2}{5^j}\,,\quad X_1=\frac{z_2}{2^{4b_1+j}}\,.$$

Thus we consider equation (3.8) with $c\in\{2,\,2/5,\,2/25\}$. Now X,Y and X_1 are S-integers with $S=\{5\}$, and (x_1,y_1) is S-integral solution to (3.8).

If $c=2$, then (3.8) becomes $y_1^2=x_1^3+2520$. Its S-integral solutions are

$$(x_1,y_1)=(-6,\,\pm 48),\ (-\frac{66}{25}\,,\pm\frac{6252}{125})\,,\ (9,\,\pm 57),\ (46,\,\pm 316),$$

and the corresponding values of Y are

$$Y=-\frac{2}{3}\,,\ -6,\ -\frac{2^4\,13}{3\cdot 5^3}\,,\ -\frac{2^2\,191}{5^3}\,,\ -\frac{1}{6}\,,,\ -\frac{13}{2}\,,\ \frac{2^7}{3^2}\,,\ -\frac{2^2\,47}{3^2},$$

and none of these values of Y lead to a solution of (3.5).

When $c=2/5$, the elliptic curve defined by (3.8) is of zero rank with trivial torsion subgroup, hence there are no rational solutions.

When $c = 2/25$, (3.8) becomes $y_1^2 = x_1^3 + \frac{504}{125}$, the S-integral solutions of which are

$$(x_1, y_1) = (\frac{1}{25}, \pm\frac{251}{125}), (\frac{6}{5}, \pm\frac{12}{5}), (\frac{81}{25}, \pm\frac{771}{125}), (\frac{66}{5}, \pm 48),$$

with the corresponding values of Y being

$$Y = -\frac{7^2}{90}, -\frac{19\cdot 29}{90}, 0, -\frac{2^2\,5}{3}, \frac{157}{30}, -\frac{7\cdot 17}{10}, \frac{10\cdot 19}{3}, -70.$$

However, no non-zero value of Y lead to a solution of (3.5).

Conclusion: No solutions to (3.5) *exist.*

Equation (3.6): Now, X, Y in equation (3.6) are integers and this equation is equivalent to $y_1^2 = x_1^3 + 63000$, where $x_1 = 30X$ and $y_1 = 30(3Y + 10)$. All integer solutions of this equation (if we forget the above special form of x_1, y_1) are

$$(x_1, y_1) = (1, \pm 251), (30, \pm 300), (81, \pm 771), (330, \pm 6000),$$

from which only the solution $(x_1, y_1) = (330, -6000)$ returns to a non-zero integral solution (X, Y), namely, $(X, Y) = (11, -70)$. But this value of Y is not of the form required by (3.6).

Conclusion: No solutions to (3.6) *exist.*

In view of our previous conclusions we have shown that equation (3.2) has no solutions which proves the following:

Proposition 5. *Equation* (2.1) *with* $n = 3$ *is impossible.*

3.3. **The case** $n = 5$**.** Suppose (x, z) is a solution of (2.1) for $n = 5$. Let $r = \gcd(x, d)$, then $x = x_1 r$ and $d = d_1 r$ with $\gcd(x_1, d_1) = 1$, hence $r^5 \mid z^5$ and consequently $r \mid z$. Setting $z = rz_1$ we obtain the equation

$$(x_1 - d_1)^5 + x_1^5 + (x_1 + d_1)^5 = z_1^5, \quad \gcd(x_1, d_1) = 1.$$

By Theorem 1.1 of [6], this equation has nonzero integer solutions only when $d_1 = 2$, in which case the only solution is $(x_1, z_1) = \pm(1, 3)$. It follows that $d = 2r$, which shows that, in $d = 2^a 5^b$ we have $a \geq 1$, hence $r = 2^{a-1}5^b$ and $(x, z) = \pm(2^{a-1}5^b, 3\cdot 2^{a-1}5^b) = \pm(d/2, 3d/2)$. Thus we have proved the following:

Proposition 6. *Equation* (2.1) *with* $n = 5$ *has integer solutions only if* $a \geq 1$*, in which case its integer solutions are given by* $(x, z) = \pm(d/2, 3d/2)$.

4. The modular method for $n \geq 7$

In this section we prove that there are no solutions of (2.1) when n is a prime greater or equal to 7. In the proof we make use of the modular method which has its origin in the proof of Fermat's Last Theorem [42]. The main idea in the modular method is to attach Frey-Hellegourach curves associated with our equations and using the modularity of elliptic curves [42, 40, 13], the work of Mazur [30] and Ribet's level-lowering theorem [34], to compare Galois representations. For the rest of the section we assume that $n \geq 7$ is a prime.

Before we study the equations (2.8)-(2.15) with the modular method, we have to recall some standard results and terminology. The reader can find a more detailed exposition of the techniques and ideas in, for example, [18, Chapter 15].

Suppose f is a cuspidal newform of weight 2 and level N_f with q-expansion

$$f = q + \sum_{i=2}^{\infty} a_i(f) q^i.$$

We denote by K_f the *eigenvalue field* of f and say that f is *irrational* if $[K_f : \mathbb{Q}] > 1$ and *rational* otherwise. Suppose n is a rational prime and $\mathfrak{n} \mid n$ a prime ideal in K_f. Then, we can associate a continuous, semisimple Galois representation

$$\bar{\rho}_{f,\mathfrak{n}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{F}}_n),$$

that is unramified at all primes $\ell \nmid nN_f$ and $\mathrm{Tr}(\bar{\rho}_{f,\mathfrak{n}}(\mathrm{Frob}_\ell)) \equiv a_\ell(f) \pmod{\mathfrak{n}}$ where Frob_ℓ is a Frobenius element at ℓ.

Suppose E is an elliptic curve over $\mathbb{Q}$ with conductor N_E. For a prime ℓ of good reduction for E, we let $a_\ell(E) = \ell + 1 - \#\tilde{E}(\mathbb{F}_\ell)$, where $\tilde{E}$ is the reduction of E at ℓ. We denote by $\bar{\rho}_{E,n}$ the Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acting on the n-torsion subgroup of E.

The following proposition provides a standard technique that is used to bound n; its origin goes back to Serre [36].

Proposition 7. *Suppose f is a cuspidal newform of weight 2, level N_f and trivial character with eigenvalue field K_f. We assume that $\bar{\rho}_{f,\mathfrak{n}} \simeq \bar{\rho}_{E,n}$ where $\bar{\rho}_{f,\mathfrak{n}}$ is the residual representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ associated to f and $\mathfrak{n} \mid n$ is a prime ideal in K_f. Let $\ell \neq n$ be a prime, then*

- *if $\ell \nmid N_E N_f$ then $a_\ell(f) \equiv a_\ell(E) \pmod{\mathfrak{n}}$,*
- *if $\ell \nmid N_f$ and $\ell \| N_E$ then $a_\ell(f) \equiv \pm(\ell+1) \pmod{\mathfrak{n}}$,*

where $a_\ell(f)$ is the Hecke eigenvalue of f at ℓ.

4.1. **Frey-Hellegourach curves of signature** $(n, n, 2)$**.** We apply the recipes of Bennett and Skinner [9, Section 2] to equations (2.8)-(2.15). According to the different values of $a - \alpha$ and $b - \beta$ we attach the corresponding Frey-Hellegourach curve.

Case I. We consider the equations (2.8) and (2.9). We recall that $a \geq \alpha$ and $b \geq \beta$. For any value of $a - \alpha$, the equation (2.8) satisfies the conditions of case (ii) in [9, Section 2].

Case	Equation	Frey-Hellegourach Curve
$a \geq \alpha + 2$	(2.8)	$E_{I,1} : Y^2 = X^3 + 20({z_1}^{2n} + 2^{2(a-\alpha)}5^{2(b-\beta)})X^2 + 70{z_1}^{4n}X$
	(2.9)	$F_{I,1} : Y^2 + XY = X^3 - \frac{3z_1^{2n}+2^{2(a-\alpha)+1}5^{2(b-\beta)+1}+1}{4}X^2$ $+7 \cdot 2^{4(a-\alpha)-5}5^{4(b-\beta)+1}X$
$a = \alpha$	(2.8)	$E_{I,2} : Y^2 = X^3 + 20({z_1}^{2n} + 5^{2(b-\beta)})X^2 + 70{z_1}^{4n}X$
	(2.9)	$F_{I,2} : Y^2 = X^3 + 2(3{z_1}^{2n} + 2 \cdot 5^{2(b-\beta)+1})X^2 + 14 \cdot 5^{4(b-\beta)+1}X$
$a = \alpha + 1$	(2.8)	$E_{I,3} : Y^2 = X^3 + 20({z_1}^{2n} + 4 \cdot 5^{2(b-\beta)})X^2 + 70{z_1}^{4n}X$
	(2.9)	$F_{I,3} : Y^2 = X^3 - (3{z_1}^{2n} + 8 \cdot 5^{2(b-\beta)+1})X^2 + 56 \cdot 5^{4(b-\beta)+1}X$

Table 3. Frey-Hellegourach curves for the case I. It holds $a \geq \alpha$ and $\beta \leq b$.

Case	Equation	Frey-Hellegourach Curve
$a \geq \alpha + 2$	(2.10)	$E_{II,1} : Y^2 = X^3 + 4(5^{2(\beta-b)}z_1^{2n} + 2^{2(a-\alpha)})X^2 + 14 \cdot 5^{4(\beta-b)-1}z_1^{4n}X$
	(2.11)	$F_{II,1} : Y^2 + XY = X^3 - \frac{3 \cdot 5^{2(\beta-b)}z_1^{2n}+5 \cdot 2^{2(a-\alpha)+1}+1}{4}X^2 + 35 \cdot 2^{4(a-\alpha)-5}X$
$a = \alpha$	(2.10)	$E_{II,2} : Y^2 = X^3 + 4(5^{2(\beta-b)}z_1^{2n} + 1)X^2 + 14 \cdot 5^{4(\beta-b)-1}z_1^{4n}X$
	(2.11)	$F_{II,2} : Y^2 = X^3 + 10(3 \cdot 5^{2(\beta-b)-1}z_1^{2n} + 2)X^2 + 70X$
$a = \alpha + 1$	(2.10)	$E_{II,3} : Y^2 = X^3 + 4(5^{2(\beta-b)}z_1^{2n} + 4)X^2 + 14 \cdot 5^{4(\beta-b)-1}z_1^{4n}X$
	(2.11)	$F_{II,3} : Y^2 = X^3 - (3 \cdot 5^{2(\beta-b)}z_1^{2n} + 40)X^2 + 280X$

Table 4. Frey-Hellegourach curves for the case II. It holds $a \geq \alpha$ and $\beta > b$.

Next we focus on equation (2.9). Suppose that $a \geq \alpha + 2$, then we are in case (v) in [9, Section 2]. When $a = \alpha$ then we are in case (ii) and when $a = \alpha + 1$ then we are in case (iv) in [9, Section 2]. The Frey-Hellegouarch curves are represented in the Table 3.

Case II. We consider the equations (2.10) and (2.11). We recall that $a \geq \alpha$ and $\beta > b$. For any value of $a - \alpha$, the equation (2.10) satisfies the conditions of case (ii) in [9, Section 2].

Next we turn to equation (2.11). Suppose that $a \geq \alpha + 2$, then we are in case (v) in [9, Section 2]. When $a = \alpha$ then we are in case (ii) and when $a = \alpha + 1$ then we are in case (iv) of [9, Section 2]. The Frey-Hellegouarch curves are represented in the Table 4.

Case	Equation	Frey-Hellegourach Curve
$\alpha \geq (a+2)$	(2.12)	$E_{III,1}: Y^2 + XY = X^3 + \frac{5(2^{2(\alpha-a)}z_1^{2n}+5^{2(b-\beta)})-1}{4}X^2$ $+35 \cdot 2^{4(\alpha-a)-7}z_1^{4n}X$
	(2.13)	$F_{III,1}: Y^2 = X^3 + 4(3\cdot 2^{2(\alpha-a)-1}z_1^{2n} + 5^{2(b-\beta)+1})X^2 + 14\cdot 5^{4(b-\beta)+1}X$
$\alpha = a+1$	(2.12)	$E_{III,2}: Y^2 = X^3 + 5(4z_1^{2n} + 5^{2(b-\beta)})X^2 + 70\cdot z_1^{4n}X$
	(2.13)	$F_{III,2}: Y^2 = X^3 + 4(6z_1^{2n} + 5^{2(b-\beta)+1})X^2 + 14\cdot 5^{4(b-\beta)+1}X$

Table 5. Frey-Hellegourach curves for the case III. It holds $\alpha \geq a+1$ and $b \geq \beta$.

Case	Equation	Frey-Hellegourach Curve
$\alpha \geq (a+2)$	(2.14)	$E_{IV,1}: Y^2 + XY = X^3 + 2^{2(\alpha-a)-2}5^{2(\beta-b)}z_1^{2n}X^2$ $+7\cdot 2^{4(\alpha-a)-7}5^{4(\beta-b)-1}z_1^{4n}X$
	(2.15)	$F_{IV,1}: Y^2 = X^3 + 20(3\cdot 2^{2(\alpha-a)-1}5^{2(\beta-b)-1}z_1^{2n}+1)X^2 + 70X$
$\alpha = a+1$	(2.14)	$E_{IV,2}: Y^2 = X^3 + (4\cdot 5^{2(\beta-b)}z_1^{2n}+1)X^2 + 14\cdot 5^{4(\beta-b)-1}z_1^{4n}X$
	(2.15)	$F_{IV,2}: Y^2 = X^3 + 20(6\cdot 5^{2(\beta-b)-1}z_1^{2n}+1)X^2 + 70X$

Table 6. Frey-Hellegourach curves for the case IV. It holds $\alpha \geq a+1$ and $\beta \geq b+1$.

Case III. We consider the equations (2.12) and (2.13). We recall that $\alpha \geq a+1$ and $b \geq \beta$. For any value of $a-\alpha$, the equation (2.13) satisfies the conditions of case (ii) of [9, Section 2].

Now we focus on equation (2.12). If $\alpha \geq a+2$, then we are in case (v) of [9, Section 2]. When $\alpha = a+1$, we are in case (iv) of [9, Section 2]. The Frey-Hellegouarch curves are represented in the Table 5.

Case IV. We consider the equations (2.14) and (2.15). We recall that $\alpha \geq a+1$ and $\beta \geq b+1$. For any value of $a-\alpha$, the equation (2.15) satisfies the conditions of case (ii) in [9, Section 2].

We turn to equation (2.14) now. If $\alpha \geq a+2$, then we are in case (v) of [9, Section 2]. When $\alpha = a+1$, we are in case (iv) of [9, Section 2]. The Frey-Hellegouarch curves are represented in the Table 6.

Let $E = E_{i,k}$ or $F_{i,k}$ as above. We denote by $\bar{\rho}_{E,n}$ the Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acting on the n-torsion points of E.

Proposition 8. *The representation $\bar{\rho}_{E,n}$ is absolutely irreducible.*

Frey curve	Discriminant $\Delta(E)$	$N_n(E)$
$E_{I,1}$	$2^9 \cdot 5^3 \cdot 7^2 \cdot (z_2 z_1^8)^n$	$2^8 \cdot 5^2 \cdot 7$
$F_{I,1}$	$2^{8(a-\alpha)-10} \cdot 3 \cdot 5^{8(b-\beta)+2} \cdot 7^2 \cdot z_2^n$	$2 \cdot 3 \cdot 5 \cdot 7$
$E_{I,2}$	$2^9 \cdot 5^3 \cdot 7^2 \cdot (z_2 z_1^8)^n$	$2^8 \cdot 5^2 \cdot 7$
$F_{I,2}$	$2^8 \cdot 3 \cdot 5^{8(b-\beta)+2} \cdot 7^2 \cdot z_2^n$	$2^7 \cdot 3 \cdot 5 \cdot 7$
$E_{I,3}$	$2^9 \cdot 5^3 \cdot 7^2 \cdot (z_2 z_1^8)^n$	$2^8 \cdot 5^2 \cdot 7$
$F_{I,3}$	$2^{10} \cdot 3 \cdot 5^{8(b-\beta)+2} \cdot 7^2 \cdot z_2^n$	$2^3 \cdot 3 \cdot 5 \cdot 7$
$E_{II,1}$	$2^9 \cdot 5^{8(\beta-b)-2} \cdot 7^2 \cdot (z_2 z_1^8)^n$	$2^8 \cdot 5 \cdot 7$
$F_{II,1}$	$2^{8(a-\alpha)-10} \cdot 3 \cdot 5^3 \cdot 7^2 \cdot z_2^n$	$2 \cdot 3 \cdot 5^2 \cdot 7$
$E_{II,2}$	$2^9 \cdot 5^{8(\beta-b)-2} \cdot 7^2 \cdot (z_2 z_1^8)^n$	$2^8 \cdot 5 \cdot 7$
$F_{II,2}$	$2^8 \cdot 3 \cdot 5^3 \cdot 7^2 \cdot z_2^n$	$2^7 \cdot 3 \cdot 5^2 \cdot 7$
$E_{II,3}$	$2^9 \cdot 5^{8(\beta-b)-2} \cdot 7^2 \cdot (z_2 z_1^8)^n$	$2^8 \cdot 5 \cdot 7$
$F_{II,3}$	$2^{10} \cdot 3 \cdot 5^3 \cdot 7^2 \cdot z_2^n$	$2^3 \cdot 3 \cdot 5^2 \cdot 7$
$E_{III,1}$	$2^{8(\alpha-a)-14} \cdot 5^3 \cdot 7^2 \cdot (z_2 z_1^8)^n$	$2 \cdot 5^2 \cdot 7$
$F_{III,1}$	$2^9 \cdot 3 \cdot 5^{8(b-\beta)+2} \cdot 7^2 \cdot z_2^n$	$2^8 \cdot 3 \cdot 5 \cdot 7$
$E_{III,2}$	$2^6 \cdot 5^3 \cdot 7^2 \cdot (z_2 z_1^8)^n$	$2^5 \cdot 5^2 \cdot 7$
$F_{III,2}$	$2^9 \cdot 3 \cdot 5^{8(b-\beta)+2} \cdot 7^2 \cdot z_2^n$	$2^8 \cdot 3 \cdot 5 \cdot 7$
$E_{IV,1}$	$2^{8(\alpha-a)-14} \cdot 5^{8(\beta-b)-2} \cdot 7^2 \cdot (z_2 z_1^8)^n$	$2 \cdot 5 \cdot 7$
$F_{IV,1}$	$2^9 \cdot 3 \cdot 5^3 \cdot 7^2 \cdot z_2^n$	$2^8 \cdot 3 \cdot 5^2 \cdot 7$
$E_{IV,2}$	$2^6 \cdot 5^{8(\beta-b)-2} \cdot 7^2 \cdot (z_2 z_1^8)^n$	$2^5 \cdot 5 \cdot 7$
$F_{IV,2}$	$2^9 \cdot 3 \cdot 5^3 \cdot 7^2 \cdot z_2^n$	$2^8 \cdot 3 \cdot 5^2 \cdot 7$

TABLE 7. The discriminant and $N_n(E)$ of the Frey-Hellegouarch curves.

Proof. This is an immediate consequence of [9, Corollary 3.1], based on work by Mazur [30], and the fact that $z_1 z_2 \neq \pm 1$. $\square$

Let

$$N_n(E) = N(E) \Big/ \prod_{q \mid z_1 z_2} q.$$

Proposition 9. *Suppose $\bar{\rho}_{E,n}$ is as above. Then there exists a newform f of trivial character, weight 2 and level $N_n(E)$ and a prime ideal $\mathfrak{n} \mid n$ of K_f such that*

$$\bar{\rho}_{E,n} \simeq \bar{\rho}_{f,\mathfrak{n}}.$$

Proof. This is an immediate consequence of modularity of elliptic curves [42, 40, 13], Proposition 8, Table 7 and Ribet's level lowering [34]. $\square$

In Table 7, we have computed the $N_n(E)$ of $\bar{\rho}_{E,n}$ according to [9, Lemmas 2.1 and 3.3].

5. Proof of Theorem 1 for $n \geq 7$

Proof of Theorem 1. Suppose (x, z) is a solution of the equation (2.1) for some value of d, where $d = 2^a 5^b$ with $a, b \geq 0$ are integers and $n \geq 7$ is a prime. As we explain in Section 2, there exist integers z_1, z_2, u_1 and u_2 with $(z_1, z_2) = 1$, $(z_1 z_2, 10) = 1$ and u_1, u_2 are $\{2, 5\}$-units such that

$$x = u_1 z_1^n,$$
$$P = u_2 z_2^n,$$

where $P = 3x^4 + 20d^2x^2 + 10d^4$. According to the valuation of u_1, u_2 and d at 2 and 5 we have four possible cases (I)-(IV) and for each case we construct two Fermat type equations of signature $(n, n, 2)$ for the pair (z_1, z_2); the equations (2.8)-(2.15). From the work of Bennett and Skinner [9] we attach two Frey-Hellegouarch curves $E_{i,k}$ and $F_{i,k}$ for each case and pair (z_1, z_2), as we have explained in Section 4 (see Tables 3-6).

We denote by $\bar{\rho}_{E_{i,k}}$ and $\bar{\rho}_{F_{i,k}}$ the Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acting on the n-torsion points of $E_{i,k}$ and $F_{i,k}$, respectively. We denote by $N_n(E_{i,k})$ and $N_n(F_{i,k})$ the Serre level of $\bar{\rho}_{E_{i,k}}$ and $\bar{\rho}_{F_{i,k}}$, respectively (see Table 7). Then, from Proposition 9, we know that there exists a newform f (resp. g) of weight 2, trivial character and level $N_n(E_{i,k})$ (resp. $N_n(F_{i,k})$) such that $\bar{\rho}_{f,\mathfrak{n}} \simeq \bar{\rho}_{E_{i,k}}$ (resp. $\bar{\rho}_{g,\mathfrak{n}'} \simeq \bar{\rho}_{F_{i,k}}$) where $\mathfrak{n} \mid n$ (resp. $\mathfrak{n}' \mid n$) is a prime ideal of K_f (resp. K_g).

Because for each pair (z_1, z_2) we have attached two Frey curves we apply the powerful multi-Frey approach to get a bound for n [15, 16]. Suppose $\ell \neq 2, 3, 5, 7$ is a prime. We define $\Delta_a = |a - \alpha|$ and $\Delta_b = |b - \beta|$. We also define

$$R_\ell(f) = \begin{cases} N_{K_f/\mathbb{Q}}\left(a_\ell(E_{i,k}) - a_\ell(f)\right), & \ell \nmid \Delta(E_{i,k}), \\ N_{K_f/\mathbb{Q}}\left((\ell+1)^2 - a_\ell^2(f)\right), & \ell \mid \Delta(E_{i,k}), \end{cases}$$

where $\Delta(E_{i,k})$ is the discriminant of $E_{i,k}$. Similarly, we define

$$R'_\ell(g) = \begin{cases} N_{K_g/\mathbb{Q}}\left(a_\ell(F_{i,k}) - a_\ell(g)\right), & \ell \nmid \Delta(F_{i,k}), \\ N_{K_g/\mathbb{Q}}\left((\ell+1)^2 - a_\ell^2(g)\right), & \ell \mid \Delta(F_{i,k}), \end{cases}$$

where $\Delta(F_{i,k})$ is the discriminant of $F_{i,k}$. It is important to mention that both $R_\ell(f)$ and $R'_\ell(g)$ depend on the residue class of (z_1, z_2) modulo ℓ and (Δ_a, Δ_b) modulo $(\ell - 1)$. Now let

$$T_\ell(f, g) = \ell \cdot \prod_{\substack{(z_1,z_2)\in\mathbb{F}_\ell^2, \\ (\Delta_a,\Delta_b)\in(\mathbb{Z}/(\ell-1)\mathbb{Z})^2}} \gcd(R_\ell(f), R'_\ell(g)).$$

From Proposition 7, we know that if a solution (z_1, z_2) arises from the pair of newforms (f, g) then it should hold $n \mid T_\ell(f, g)$.

We have written a Magma script that computes $U(f, g) = \gcd_{\ell \leq B}(T_\ell(f, g))$ where B is a suitable positive integer. For the majority of the pairs (f, g) it is enough to consider $B = 19$ to deduce that $n \leq 5$. However, there are a few pairs (f, g) for which we have to increase B up to 59 to get $n \leq 5$. The total amount of time for the above computations was roughly 56 hours.

In Table 8 we give a summary of the data for the spaces of newforms that we had to compute, together with the amount of time Magma needed to compute the spaces. It is important to note that there are two ways of computing weight 2 modular forms in Magma, either the classical approach, or using the package of Hilbert newforms viewing classical newforms as Hilbert newforms over $\mathbb{Q}$ [21]. The package of Hilbert newforms is faster in current implementation of Magma (Magma V2.25-3) and the total amount of time was roughly 146 hours with the most expensive case to be the space of level 134400 and 107 hours to be computed. $\square$

Level	Dimension	#conjugacy classes	(d, #newforms of degree d)	Time
$2 \cdot 5 \cdot 7$	1	1	$(1,1)$	~ 1 sec
$2 \cdot 3 \cdot 5 \cdot 7$	5	5	$(1,5)$	~ 1 sec
$2 \cdot 5^2 \cdot 7$	10	8	$(1,6)$, $(2,4)$	~ 1 sec
$2^3 \cdot 3 \cdot 5 \cdot 7$	12	11	$(1,10)$, $(2,2)$	~ 3 sec
$2 \cdot 3 \cdot 5^2 \cdot 7$	18	18	$(1,18)$	~ 4 sec
$2^5 \cdot 5 \cdot 7$	24	20	$(1,16)$, $(2,8)$	~ 3 sec
$2^3 \cdot 3 \cdot 5^2 \cdot 7$	58	43	$(1,32)$, $(2,14)$, $(3,12)$	~ 1 min
$2^5 \cdot 5^2 \cdot 7$	114	52	$(1,22)$, $(2,32)$, $(3,12)$, $(4,16)$, $(5,20)$, $(6,12)$	~ 1 min
$2^8 \cdot 5 \cdot 7$	192	64	$(1,20)$, $(2,24)$, $(3,36)$ $(4,16)$, $(6,96)$	~ 22 sec
$2^7 \cdot 3 \cdot 5 \cdot 7$	192	112	$(1,64)$, $(2,56)$, $(3,36)$, $(4,16)$, $(5,20)$	~ 2 min
$2^8 \cdot 3 \cdot 5 \cdot 7$	384	128	$(1,48)$, $(2,32)$, $(3,48)$, $(4,112)$, $(6,48)$, $(8,96)$	~ 30 min
$2^8 \cdot 5^2 \cdot 7$	912	196	$(1,52)$, $(2,64)$, $(3,36)$, $(4,88)$, $(5,40)$, $(6,168)$ $(8,96)$, $(9,72)$, $(12,192)$, $(16,32)$, $(18,72)$	~ 2 h
$2^7 \cdot 3 \cdot 5^2 \cdot 7$	912	356	$(1,176)$, $(2,128)$, $(3,36)$ $(4,144)$, $(5,140)$, $(6,48)$ $(7,168)$, $(9,72)$	~ 36 h
$2^8 \cdot 3 \cdot 5^2 \cdot 7$	1824	396	$(1,124)$, $(2,120)$, $(3,60)$, $(4,208)$, $(5,40)$, $(6,240)$, $(8,224)$, $(9,72)$, $(10,80)$ $(11,88)$, $(12,192)$, $(13,104)$ $(16,192)$, $(20,80)$	~ 107 h

Table 8. Data for newform computations.

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